

Transition temperature of the interacting dipolar Bose gas

Yee-Mou Kao¹ and T. F. Jiang²

¹*Department of Physics, National Changhua University of Education, Changhua 50058, Taiwan*

²*Institute of Physics, National Chiao-Tung University, Hsinchu 30010, Taiwan*

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We investigate the effects of long-ranged dipole-dipole potential on the transition temperature of a weakly interacting Bose gas. We apply the two-fluid model to derive the energy spectra of the thermal and the condensate parts. From the interaction modified spectra of the system, the formula for the shift of transition temperature was derived. Compared to the conventional weakly interacting Bose system with contact potential only where thermal effect is larger, we find that the condensate effect is about two times that of the thermal part in the dipolar system. Due to the relative smallness of dipole-dipole interaction with respect to the contact interaction in current dipolar Bose-Einstein condensation, we suggest to measure the dipolar effect by tuning the scattering length to negligible small by the Feshbach resonance technique.

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I. INTRODUCTION

In a previous paper [1], we derived the effects of atomic contact interaction on the transition temperature (T_c) of weakly interacting Bose gas by using the two-fluid model [2]. The effects contain contributions from both the condensate and the thermal part and the effects on transition temperature can be calculated through the shifts of energy spectra. We found that the effect of the condensate part that was often neglected is in fact 34% of the usually treated thermal part interaction effect [3]. With both the effects considered together with the trap potential effect and the finite-size effect, our transition temperature formula fits the experimental measurement very well [4]. The inclusion of the interaction effect from condensate explains the discrepancy before our work. For completeness, we make a brief summary of results in current Bose gas [5]. First, the correction of transition temperature to the ideal Bose gas is due to the trap potential of cylindrical symmetry, $V_{ext}(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$ where $\omega_x = \omega_y = \omega_z/\kappa$, and m is the atomic mass. The transition temperature in the thermodynamic limit (T_c^0) becomes one-third power in number density instead of two-thirds power of the ideal gas without trap potential, and the expression is

$$T_c^0 = \frac{\hbar\omega}{k_B} \left(\frac{N}{\zeta(3)} \right)^{1/3} \approx 0.9405 \frac{\hbar\omega}{k_B} N^{1/3}, \quad (1)$$

where $\omega = (\omega_1 \omega_2 \omega_3)^{1/3}$ is the geometrical mean of the frequencies, N is the number of particles, and ζ is the Riemann's zeta function.

Next, the shift of transition temperature due to the finite number of particles can be expressed by the energy shift

$$\frac{\delta T_c^{fin}}{T_c^0} = \frac{\zeta(2)}{3[\zeta(3)]^{2/3}} \frac{(\Delta\epsilon_T - \Delta\epsilon_c)}{\hbar\omega} N^{-1/3}, \quad (2)$$

where $\Delta\epsilon_T$ is the shift of thermal energy level and $\Delta\epsilon_c$ is the shift of the condensate energy level. This so-called finite-size effect is due to $\Delta\epsilon_T = 0$ and $\Delta\epsilon_c = 3\hbar\bar{\omega}/2$, where $\bar{\omega} = (\omega_1 + \omega_2 + \omega_3)/3$ is the arithmetic mean of the trap frequencies. Then

$$\frac{\delta T_c^{fin}}{T_c^0} = -0.73 \frac{\bar{\omega}}{\omega} N^{-1/3}. \quad (3)$$

The shift is downward and vanishes in the large- N limit. And finally, the shift of transition temperature due to the atomic contact interaction is [1]

$$\frac{\delta T_c^{int}}{T_c^0} = \frac{\delta T_c^T + \delta T_c^c}{T_c^0} = -1.78 \frac{a_0}{a_{ho}} N^{1/6}, \quad (4)$$

where a_0 is the s -wave scattering length and $a_{ho} = \sqrt{\hbar/m\omega}$ is the harmonic oscillator length scale which is usually in the order of μm .

In 2005, the Bose-Einstein condensation of dipolar Bose gas was first realized by using the ⁵²Cr atoms [6,7]. Besides the well-known atomic contact potential term in the mean-field equation, there is an additional dipole-dipole interaction potential which is long ranged and anisotropic in the dipolar system. The prominent double-hump feature of the condensate has been recently investigated [8]. The statistical mechanical properties of the new system are certainly of great interest [9]. So, now we have two kinds of interaction in the system. And each interaction will affect the energy spectra and henceforth the transition temperature. The ⁵²Cr atom used in current experiment has a magnetic dipole moment of 6 Bohr magneton, and the scattering wave length is about 105 bohr. The contact interaction energy is about thirty times larger than the dipolar energy. Hence the effects of the dipole-dipole interaction is only a perturbation to the system. Unless the very high precision measurement of T_c is feasible, to identify the dipolar effect on T_c would be rather difficult. Fortunately, the technique of Feshbach resonance can tune the scattering length. Thus the study of the system with dipolar interaction only is still meaningful and can provide a way to identify the new features of the ultracold atomic system with long-range interactions. With the success of the two-fluid model in the interacting Bose gas [1,2], we already solved the effects of contact interaction. Our task is now to study the effects on transition temperature coming from the newly realized long-ranged dipole-dipole interaction only.

Our approach of the method to the dipolar Bose gas will provide a reliable result of the statistical properties.

The paper is organized as follows: In Sec. II, we derive the mean-field equations for the condensate and thermal parts through the two-fluid model. In Sec. III, we calculate the energy spectra to the first order of interaction parameter g_d . In Sec. IV, we obtain the transition temperature by the shift of energy spectra. Discussions are given in Sec. V. Finally, some mathematical derivations are given in the Appendixes.

II. TWO-FLUID MODEL

In the following treatment, we study the dipole-dipole interaction by assuming the contact scattering has been turned off with the Feshbach resonance technique. The Heisenberg equation of motion of the Bose field operator $\hat{\psi}(\mathbf{r})$ is

$$i\frac{\partial\hat{\psi}(\mathbf{r},t)}{\partial t} = \left(-\frac{\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu\right)\hat{\psi}(\mathbf{r},t) + g_d\left(\int d^3r'\hat{\psi}^\dagger(\mathbf{r}',t)\hat{\psi}(\mathbf{r}',t)V_d(\mathbf{r},\mathbf{r}')\right)\hat{\psi}(\mathbf{r},t), \quad (5)$$

where $g_d = \frac{\mu_0\mu_{Cr}^2}{4\pi}$, and

$$V_d(\mathbf{r},\mathbf{r}') = \frac{1 - 3\cos^2\theta}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (6)$$

In these equations, the magnetic dipole moments μ_{Cr} are aligned and θ is the angle between the relative position vector $\mathbf{r} - \mathbf{r}'$ and the magnetic dipoles. μ_{Cr} is the atomic magnetic dipole moment.

Separating out the condensate part in the usual way, we have

$$\hat{\psi}(\mathbf{r},t) = \Phi(\mathbf{r}) + \psi(\mathbf{r},t), \quad (7)$$

where the averaged $\Phi(\mathbf{r}) \equiv \langle \hat{\psi}(\mathbf{r},t) \rangle$ plays the role of a spatially varying macroscopic Bose field. The time average of fluctuation $\langle \psi^\dagger(\mathbf{r},t) \rangle = \langle \psi(\mathbf{r},t) \rangle = 0$.

Follow the method of Griffin, the field operator part in the interaction term of Eq. (5) can be written as

$$\begin{aligned} \hat{\psi}^\dagger(\mathbf{r}',t)\hat{\psi}(\mathbf{r}',t)\hat{\psi}(\mathbf{r},t) &= |\Phi(\mathbf{r}')|^2\Phi(\mathbf{r}) + |\Phi(\mathbf{r}')|^2\psi(\mathbf{r},t) \\ &+ \Phi^*(\mathbf{r}')\psi(\mathbf{r}',t)\Phi(\mathbf{r}) \\ &+ \Phi(\mathbf{r}')\psi^\dagger(\mathbf{r}',t)\Phi(\mathbf{r}) \\ &+ \Phi^*(\mathbf{r}')\psi(\mathbf{r}',t)\psi(\mathbf{r},t) \\ &+ |\psi(\mathbf{r}',t)|^2\Phi(\mathbf{r}) \\ &+ \psi^\dagger(\mathbf{r}',t)\Phi(\mathbf{r}')\psi(\mathbf{r},t) \\ &+ |\psi(\mathbf{r}',t)|^2\psi(\mathbf{r},t). \end{aligned} \quad (8)$$

The last term under mean-field approximation can be written as

$$\begin{aligned} \psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r}',t)\psi(\mathbf{r},t) &= \langle \psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r}',t) \rangle \psi(\mathbf{r},t) \\ &+ \langle \psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t) \rangle \psi(\mathbf{r}',t) + \langle \psi(\mathbf{r}',t)\psi(\mathbf{r},t) \rangle \psi^\dagger(\mathbf{r}',t). \end{aligned} \quad (9)$$

With the above expressions, the stationary part of Eq. (5) gives the order-parameter equation. Subtraction of Eq. (5) with the stationary part, the equation of excitations will be obtained.

A. Order parameter

The time independent Bose order parameter $\Phi(\mathbf{r})$ is given directly by taking the average over Eq. (5)

$$\begin{aligned} \left(-\frac{\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu\right)\Phi(\mathbf{r}) \\ + g_d \int d^3r' \langle \hat{\psi}^\dagger(\mathbf{r}',t)\hat{\psi}(\mathbf{r}',t)\hat{\psi}(\mathbf{r},t) \rangle V_d(\mathbf{r},\mathbf{r}') = 0. \end{aligned} \quad (10)$$

We introduce the local densities

$$\Phi^*(\mathbf{r})\Phi(\mathbf{r}) \equiv n_c(\mathbf{r}), \quad (11a)$$

$$\Phi(\mathbf{r})\Phi(\mathbf{r}) \equiv m_c(\mathbf{r}), \quad (11b)$$

$$\langle \psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t) \rangle \equiv n_T(\mathbf{r}), \quad (11c)$$

$$\langle \psi(\mathbf{r},t)\psi(\mathbf{r},t) \rangle \equiv m_T(\mathbf{r}); \quad (11d)$$

and the correlation functions

$$\Phi^*(\mathbf{r})\Phi(\mathbf{r}') \equiv n_c(\mathbf{r},\mathbf{r}'), \quad (12a)$$

$$\Phi(\mathbf{r})\Phi(\mathbf{r}') \equiv m_c(\mathbf{r},\mathbf{r}'), \quad (12b)$$

$$\langle \psi^\dagger(\mathbf{r},t)\psi(\mathbf{r}',t) \rangle \equiv n_T(\mathbf{r},\mathbf{r}'), \quad (12c)$$

$$\langle \psi(\mathbf{r},t)\psi(\mathbf{r}',t) \rangle \equiv m_T(\mathbf{r},\mathbf{r}'). \quad (12d)$$

We obtain the Gross-Pitaevskii equation (GPE) for the condensate part:

$$\begin{aligned} \left(-\frac{\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu\right)\Phi(\mathbf{r}) + g_d \left(\int d^3r' [n_c(\mathbf{r}') \right. \\ \left. + n_T(\mathbf{r}')] V_d(\mathbf{r},\mathbf{r}') \right) \Phi(\mathbf{r}) \\ + g_d \left(\int d^3r' n_T(\mathbf{r}',\mathbf{r}) V_d(\mathbf{r},\mathbf{r}') \Phi(\mathbf{r}') \right) \\ + g_d \left(\int d^3r' m_T(\mathbf{r}',\mathbf{r}) V_d(\mathbf{r},\mathbf{r}') \Phi^*(\mathbf{r}') \right). \end{aligned} \quad (13)$$

B. Excitation equation

The excitations of the condensate are described by $\psi(\mathbf{r},t)$ and are given by subtraction of the time average part of Eq. (5):

$$\begin{aligned}
i \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = & \left(-\frac{\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu \right) \psi(\mathbf{r}, t) \\
& + g_d \int d^3 r' [\hat{\psi}^\dagger(\mathbf{r}', t) \hat{\psi}(\mathbf{r}', t) \hat{\psi}(\mathbf{r}, t) V_d(\mathbf{r}, \mathbf{r}')] \\
& - g_d \int d^3 r' [\langle \hat{\psi}^\dagger(\mathbf{r}', t) \hat{\psi}(\mathbf{r}', t) \hat{\psi}(\mathbf{r}, t) \rangle V_d(\mathbf{r}, \mathbf{r}')].
\end{aligned} \tag{14}$$

Under the mean-field approximation

$$\begin{aligned}
\psi^\dagger(\mathbf{r}', t) \psi(\mathbf{r}, t) & \simeq \langle \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}) \rangle = n_T(\mathbf{r}', \mathbf{r}), \\
\psi(\mathbf{r}', t) \psi(\mathbf{r}, t) & \simeq \langle \psi(\mathbf{r}') \psi(\mathbf{r}) \rangle = m_T(\mathbf{r}', \mathbf{r}),
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
& \hat{\psi}^\dagger(\mathbf{r}', t) \hat{\psi}(\mathbf{r}', t) \hat{\psi}(\mathbf{r}, t) - \langle \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) \rangle \\
& \simeq \langle \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \rangle \psi(\mathbf{r}, t) + \langle \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}) \rangle \psi(\mathbf{r}', t) \\
& + \langle \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) \rangle \psi^\dagger(\mathbf{r}', t).
\end{aligned} \tag{16}$$

Define the self-consistent densities

$$\begin{aligned}
n(\mathbf{r}) & \equiv \langle \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \rangle = n_c(\mathbf{r}) + n_T(\mathbf{r}), \\
m(\mathbf{r}) & \equiv \langle \hat{\psi}(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \rangle = \Phi^2(\mathbf{r}) + m_T(\mathbf{r}).
\end{aligned} \tag{17}$$

Then, Eq. (14) reduces to

$$\begin{aligned}
i \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = & \left(-\frac{\nabla^2}{2m} + V_{ext}(\mathbf{r}) - \mu \right) \psi(\mathbf{r}, t) \\
& + g_d \left(\int d^3 r' n(\mathbf{r}') V_d(\mathbf{r}, \mathbf{r}') \right) \psi(\mathbf{r}, t) \\
& + g_d \left(\int d^3 r' n_c(\mathbf{r}', \mathbf{r}) V_d(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}', t) \right) \\
& + g_d \left(\int d^3 r' m_c(\mathbf{r}', \mathbf{r}) V_d(\mathbf{r}, \mathbf{r}') \psi^\dagger(\mathbf{r}', t) \right)
\end{aligned} \tag{18}$$

In high temperature $m_T(\mathbf{r}) \ll n_T(\mathbf{r})$ [2], we can also assume that the off-diagonal density is much smaller than the diagonal one, that is, $m_c(\mathbf{r}) \ll n_c(\mathbf{r})$ and the correlation functions can be neglected near T_c .

III. ENERGY SPECTRA

In a previous paper [1], we showed that the transition temperature is determined by the energy spectrum, or more precisely, determined by the shifts of the thermal and condensate states. Thus the main task for the study of the transition temperature is to find the shifts due to the new dipole-dipole interaction from previous derived equations for order parameter and excitations.

A. Dipolar effect on the thermal gas

Near transition temperature, the condensate component is localized around the trap center ($\mathbf{r}=0$) and its size is much

smaller than the thermal component. So in calculating the energy shift of the thermal component, we can neglect the condensate density n_c . We define an effective density \bar{n}_{TT} to be found and write the energy spectrum as the shift of the thermal gas:

$$\Delta \epsilon_T = g_d \bar{n}_{TT}. \tag{19}$$

To determine \bar{n}_{TT} , consider the phase space Bose-Einstein distribution

$$f(\mathbf{p}, \mathbf{r}) = \frac{1}{\exp[\epsilon(\mathbf{p}, \mathbf{r})/k_B T] - 1}, \tag{20}$$

where $\epsilon(\mathbf{p}, \mathbf{r})$ is the semiclassical excitation spectrum. By Eq. (18)

$$\epsilon(\mathbf{p}, \mathbf{r}) \simeq \epsilon^{ide}(\mathbf{p}, \mathbf{r}) + g_d n_{eff}(\mathbf{r}) - \mu, \tag{21}$$

where $n_{eff}(\mathbf{r}) \equiv \int d^3 r' V_d(\mathbf{r}, \mathbf{r}') n(\mathbf{r}')$ is an effective density with dipole-dipole interaction V_d . Expand Eq. (20) to the first order of g_d ,

$$\begin{aligned}
f(\mathbf{p}, \mathbf{r}) & = f(\mathbf{p}, \mathbf{r})|_{g_d=0} + g_d \left(\frac{\partial f(\mathbf{p}, \mathbf{r})}{\partial g_d} \right)_{g_d=0} \\
& = f_0(\mathbf{p}, \mathbf{r}) - g_d n_{eff}(\mathbf{r}) \frac{\partial f_0(\mathbf{p}, \mathbf{r})}{\partial \mu},
\end{aligned} \tag{22}$$

where $f_0(\mathbf{p}, \mathbf{r})$ is the distribution function of the noninteracting Bose gas. Integrate over the momentum variables, and we get the modification of $n_T(\mathbf{r})$ due to the interaction:

$$n_T^{int}(\mathbf{r}) \simeq n_T - g_d n_{eff} \frac{\partial n_T}{\partial \mu}, \tag{23}$$

where $n_T(\mathbf{r})$ is the thermal density distribution of noninteracting Bose gas. Further integrate over the coordinates of Eq. (23), and we get the number of thermal particles,

$$N_T^{int} = \int n_T d^3 r - g_d \int n_{eff} \frac{\partial n_T}{\partial \mu} d^3 r. \tag{24}$$

On the other hand, by the Taylor's expansion to the first order of g_d ,

$$N_T^{int} \simeq N_T + g_d \frac{\partial N_T}{\partial g_d}. \tag{25}$$

With our assumption

$$\epsilon = \epsilon_T^{ide} + g_d \bar{n}_{TT} - \mu, \tag{26}$$

we obtain

$$\frac{\partial N_T}{\partial g_d} = -\bar{n}_{TT} \frac{\partial N_T}{\partial \mu} = -\bar{n}_{TT} \int \frac{\partial n_T}{\partial \mu} d^3 r. \tag{27}$$

Put Eq. (27) into Eq. (25) and compare with Eq. (24), the effective density is derived as

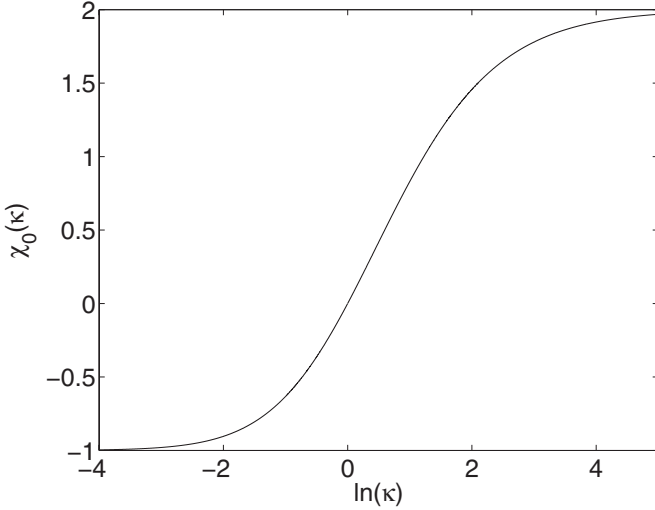


FIG. 1. $\chi_0(\kappa)$ vs the aspect parameter. The function is smooth and monotonic. In the cigar-shaped trap, $\kappa \rightarrow 0$ and $\chi_0(\kappa) \rightarrow -1$ and in the pancake-shaped trap, $\kappa \rightarrow \infty$ and $\chi_0(\kappa) \rightarrow 2$.

$$\begin{aligned} \bar{n}_{TT} &= \frac{\int d^3r \frac{\partial n_T}{\partial \mu} n_{eff}(\mathbf{r})}{\int d^3r \frac{\partial n_T}{\partial \mu}} \\ &= \frac{\int d^3r d^3r' \{[\partial n_T(\mathbf{r})/\partial \mu] n_T(\mathbf{r}') V_d(\mathbf{r}, \mathbf{r}')\}}{\int d^3r \frac{\partial n_T}{\partial \mu}} \equiv S_{TT} n_T(0). \end{aligned} \quad (28)$$

By the local density approximation (LDA) [10], the spatial distribution of the thermal state n_T can be written as

$$n_T(\mathbf{r}) = \frac{1}{\lambda_T} g_{3/2} e^{-[V_{ext}(\mathbf{r}) - \mu]/k_B T}, \quad (29)$$

where $\lambda_T = \hbar(2\pi/mk_B T)^{1/2}$ is the thermal wavelength, and $g_\nu(x) = \sum_{n=1}^{\infty} x^n/n^\nu$ is the Bose-Einstein function. With $\mu=0$, we obtain for the \hat{z} polarized atoms

$$S_{TT} = \frac{-4\pi}{3} \chi_0(\kappa) S, \quad (30)$$

where $\chi_0(\kappa)$ is a function of κ and is derived in Eq. (A9) and $S = \sum_{j,k=1}^{\infty} \{1/[j^{1/2} k^{3/2} (k+j)^{3/2}]\} / [\zeta(2)\zeta(3/2)] \approx 0.281$. $\chi_0(\kappa)$ is monotonically increasing, in cigar-shaped potential, $\kappa \rightarrow 0$ and $\chi_0(\kappa) \rightarrow -1$; and in pancake-shaped potential, $\kappa \rightarrow \infty$ and $\chi_0(\kappa) \rightarrow 2$. The function is plotted in Fig. 1. Finally, the energy shift of thermal part is found to be $\Delta\epsilon_T = g_d \bar{n}_{TT}$ and caused only by thermal density itself.

B. Dipolar effect on the condensate gas

Near T_c , the density of condensate is still very small, the contact potential is hence negligible. We can estimate the condensate wave function by using the eigenfunction of the harmonic oscillator,

$$\phi(\mathbf{r}) = \left(\frac{1}{\pi a_{ho}^2}\right)^{3/4} e^{-[x^2/(2a_{hox}^2) + y^2/(2a_{hoz}^2) + z^2/(2a_{hoz}^2)]}, \quad (31)$$

where $a_{hoi} = \sqrt{\hbar/m\omega_i}$ is the harmonic oscillator length in the x_i direction and $a_{ho} = \sqrt[3]{a_{hox} a_{hoz} a_{hoz}}$ is their geometrical mean, and the condensate density can be written as $n_c(\mathbf{r}) = N_c |\phi(\mathbf{r})|^2$. For the thermal part in the trap with energy $k_B T$, the classical oscillation amplitude of a particle is $\xi^T = (2\pi m \omega^2 / k_B T)^{-1/2}$, where k_B is the Boltzmann constant. This represents the length scale of thermal density $n_T(\mathbf{r})$. As $\xi^c = a_{ho}$ is the size of the condensate, we find that

$$\frac{\xi^T}{\xi^c} = \left(\frac{k_B T}{2\pi \hbar \omega}\right)^{1/2} \sim O(N^{1/6}) \gg 1. \quad (32)$$

This justifies our previous assumption that the size of the condensate component n_c is negligible in calculating the thermal spectrum shift.

Now we will use these concepts to find the shift of condensate energy. The expression of the condensate energy is

$$E = N_c \epsilon_c^{ide} + N_c g_d \bar{n}_{cT} + \frac{N_c}{2} g_d \bar{n}_{cc}, \quad (33)$$

where $\epsilon_c^{ide} = 3\hbar\bar{\omega}/2$ is the energy level of ideal Bose gas in the ground state, and define

$$\bar{n}_{cT} = \iint d^3r d^3r' [n_T(\mathbf{r}) n_c(\mathbf{r}') V_d(|\mathbf{r} - \mathbf{r}'|)] / N_c \equiv S_{cT} n_T(0) \quad (34)$$

to describe is the condensate-thermal effect. Use

$$\bar{n}_{cc} = \iint d^3r d^3r' [n_c(\mathbf{r}) \cdot n_c(\mathbf{r}') V_d(|\mathbf{r} - \mathbf{r}'|)] / N_c \equiv S_{cc} n_c(0) \quad (35)$$

to describe the condensate-condensate effect. The energy level can be obtained through $\epsilon_c = \partial E / \partial N_c$. And we find that the modification to the condensate energy is the shift of the condensate energy:

$$\Delta\epsilon_c = g_d \bar{n}_{cT} + g_d \bar{n}_{cc}. \quad (36)$$

This is the shift of condensate energy relative to the non-interacting Bose gas in the trap potential. We obtain in Appendix B that for the \hat{z} polarized atoms

$$S_{cT} = S_{cc} = \frac{-4\pi}{3} \chi_0(\kappa). \quad (37)$$

The shift is affected by both thermal density and condensate density.

IV. SHIFT OF TRANSITION TEMPERATURE

We can divide the effects on the transition temperature due to the dipole-dipole interaction result into two parts, the thermal effect and the condensate effect.

A. Shift of T_c by the thermal part

The shift of transition temperature due to the energy shift of the thermal effect can be calculated follows our method [1]. The result is

$$\frac{\delta T_c^T}{T_c^0} = (2\pi/3)\chi_0(\kappa) \frac{\zeta(2)}{3[\zeta(3)]^{2/3}} \frac{2g_d(1-S)n_T(0)}{\hbar\omega} N^{-1/3}. \quad (38)$$

For convenience, we define an effective dipole-dipole scattering length a_d through $g_d = 4\pi\hbar^2 a_d/m$, then [11]

$$\frac{\delta T_c^T}{T_c^0} = 1.33(2\pi/3)\chi_0(\kappa) \frac{a_d}{a_{ho}} N^{1/6} = 2.78 \frac{a_d}{a_{ho}} N^{1/6} \chi_0(\kappa). \quad (39)$$

B. Shift of T_c by the condensate part

The shift of transition temperature due to the energy shift of the condensate effect is found to be

$$\begin{aligned} \frac{\delta T_c^c}{T_c^0} &= (4\pi/3)\chi_0(\kappa) \frac{\zeta(2)}{3[\zeta(3)]^{2/3}} \frac{g_d n_c(0)}{\hbar\omega} N^{-1/3}, \\ &= 5.46 \frac{a_d}{a_{ho}} N^{1/6} \chi_0(\kappa). \end{aligned} \quad (40)$$

The total shift due to dipole-dipole interaction for the \hat{z} polarized atoms is

$$\frac{\delta T_c^{(z)}}{T_c^0} = \frac{\delta T_c^c + \delta T_c^T}{T_c^0} = 8.24 \frac{a_d}{a_{ho}} N^{1/6} \chi_0(\kappa). \quad (41)$$

We found that the shift of T_c by the condensate part in the long-ranged dipolar system is about two times that of the thermal part, while in the system with contact potential, the condensate shift of T_c is about one-third of thermal part. This is an interesting characteristic of the dipolar system.

Let α be the angle of aligned dipoles with the z axis; by the angular property of the dipole-dipole interaction, the total shift of transition temperature is

$$\frac{\delta T_c}{T_c^0} = 4.12 \frac{a_d}{a_{ho}} (3 \cos^2 \alpha - 1) N^{1/6} \chi_0(\kappa). \quad (42)$$

In Ref. [9], a similar formula was presented but considered the thermal effect only.

V. DISCUSSIONS

We derived the effects of dipole-dipole interaction on the transition temperature for cylindrical trap potential. The effect of trap potential is described by the function $\chi_0(\kappa)$. We find that in the dipolar system, the effect from the condensate part is about two times that of the thermal part. For the current realized dipolar Bose-Einstein condensate [6,7], the magnetic dipole moment of atomic chromium is $\mu_{Cr} = 6\mu_B$. It gives the effective dipolar scattering length $a_d = 1.93 \text{ \AA}$. The s -wave scattering length of chromium is $a_0 = 105a_B = 55.7 \text{ \AA}$.

We have the ratio $a_d/a_0 = 0.035$. From our previous paper [1], the shift of transition temperature is linear in scattering length. We can see that the measurement of dipolar effect on transition temperature would be difficult with the presence of both the contact and dipolar interactions, due to the relative smallness of a_d/a_0 . However, it will be feasible if the contact interaction is tuned to negligible by the Feshbach resonance technique, and the unique characteristics of dipolar can be explored.

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APPENDIX A: INTEGRAL RELATED TO THE DIPOLAR INTERACTION

Consider the dipolar interaction in the following form:

$$V_d(\mathbf{r}, \mathbf{r}') = \frac{1 - 3 \cos^2 \theta}{|\mathbf{r} - \mathbf{r}'|^3} = -2 \frac{P_2(\cos \theta)}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (A1)$$

where θ is the angle between $\mathbf{r} - \mathbf{r}'$ and the z direction. Let the two cylindrical symmetrical Gaussian distribution functions be

$$A(\mathbf{r}) = \exp[-\alpha_a r^2 \Phi(\kappa, \phi)],$$

$$B(\mathbf{r}) = \exp[-\alpha_b r^2 \Phi(\kappa, \phi)], \quad (A2)$$

where ϕ is the angle with the z direction, κ is the aspect parameter, then the angular function will be

$$\Phi(\kappa, \phi) = \frac{\sin^2 \phi}{\kappa^2} + \cos^2 \phi. \quad (A3)$$

By using the identity

$$\begin{aligned} &\int \int d^3 r_1 d^3 r_2 A(\mathbf{r}_1) V_d(|\mathbf{r}_1 - \mathbf{r}_2|) B(\mathbf{r}_2) \\ &= (2\pi)^{3/2} \int d^3 k A(-\mathbf{k}) V_d(\mathbf{k}) B^*(-\mathbf{k}), \end{aligned} \quad (A4)$$

where

$$f(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3 r f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (A5)$$

we obtain

$$V_d(\mathbf{k}) = \sqrt{\frac{2}{\pi}} (1 - 3 \cos^2 \theta_k) F(ka), \quad (A6)$$

where $\cos \theta_k = \hat{k} \hat{e}_{k_z}$ and

$$F(x) = \frac{\sin x}{x^3} - \frac{\cos x}{x^2}. \quad (A7)$$

Here a is a cutoff distance corresponding to the atomic length scale (a few Bohr radius). Since a is much smaller

than any significant length scale of the system, one can set $a \rightarrow 0$, and then $F(a) \rightarrow 1/3$. Thus we arrive at the approximation form

$$\int \int d^3 r_1 d^3 r_2 A(\mathbf{r}_1) \frac{1 - 3 \cos^2 \theta}{|\mathbf{r}_1 - \mathbf{r}_2|^3} B(\mathbf{r}_2) = -\frac{4}{3} \frac{(\pi)^{3/2} \kappa^2}{(\alpha_a + \alpha_b)^{3/2}} \chi_0(\kappa), \quad (\text{A8})$$

where

$$\chi_0(\kappa) = \left(\frac{1 + 2\kappa^2 - 3\kappa^2 H(\kappa)}{\kappa^2 - 1} \right), \quad (\text{A9})$$

and

$$H(\kappa) = \frac{\tanh^{-1} \sqrt{1 - \kappa^2}}{\sqrt{1 - \kappa^2}}. \quad (\text{A10})$$

APPENDIX B: EFFECTIVE INTERACTION DENSITIES

1. Approximate densities in anisotropic trap potential

Consider the dilute atoms in the trap potential,

$$V_{\text{ext}}(\mathbf{r}) = \frac{1}{2} m \omega_z^2 r^2 \Phi(\kappa, \phi); \quad (\text{B1})$$

we can approximate the ground state by dropping the non-linear term. And we write the density as the harmonic oscillator ground state:

$$n_c(\mathbf{r}) = \frac{N_c}{\pi^{3/2} a_{\text{hoz}}^3 \lambda^2} e^{-r^2 \Phi(\lambda, \phi) / a_{\text{hoz}}^2}, \quad (\text{B2})$$

where $a_{\text{hoz}} = \sqrt{\hbar / m \omega_z}$ and $\lambda = \sqrt{\kappa}$. But for convenience, let all the densities have the same angular distribution $\Phi(\kappa, \theta)$. The density of the thermal state will be

$$n_T(\mathbf{r}) = \frac{1}{\lambda_T^3} \sum_{j=1}^{\infty} \frac{z e^{-j V_{\text{ext}}(\mathbf{r}) / k_B T}}{j^{3/2}}. \quad (\text{B3})$$

2. Effect from the thermal-thermal interaction

Define the coefficient

$$S_{TT} = \frac{\int d^3 r d^3 r' \{ [\partial n_T(\mathbf{r}) / \partial \mu] n_T(\mathbf{r}') V_d(|\mathbf{r}_1 - \mathbf{r}_2|) \}}{n_T(0) \int d^3 r (\partial n_T / \partial \mu)}, \quad (\text{B4})$$

where

$$\begin{aligned} \int d^3 r \frac{\partial n_T}{\partial \mu} &= \frac{1}{\lambda_T^3} \sum_{j=1}^{\infty} \frac{1}{j^{1/2} (k_B T)} \int d^3 r (z^j e^{-j V_{\text{ext}}(\mathbf{r}) / k_B T}) \\ &= \frac{1}{\lambda_T^3} \sum_{j=1}^{\infty} \frac{1}{j^{1/2} (k_B T)} \frac{\pi^{3/2} (k_B T)^{3/2}}{j^{3/2} [(m/2) \omega^2]^{3/2}} \\ &= \frac{\pi^{3/2} (k_B T)^{1/2}}{\lambda_T^3 [(m/2) \omega^2]^{3/2}} \zeta(2). \end{aligned} \quad (\text{B5})$$

With Eq. (B3), and designate $\alpha_1 = j m \omega_z^2 / (2 k_B T)$ and $\alpha_2 = k m \omega_z^2 / (2 k_B T)$, then

$$\begin{aligned} S_{TT} &= \left(\frac{m \omega^2}{2 \pi k_B T} \right)^{3/2} \frac{1}{\zeta(3/2) \zeta(2)} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j^{1/2} k^{3/2}} \\ &\quad \times \int \int d^3 r_1 d^3 r_2 [e^{-\alpha_1 r_1^2 \Phi(\lambda, \phi)} e^{-\alpha_2 r_2^2 \Phi(\lambda, \phi)} V_d(|\mathbf{r}_1 - \mathbf{r}_2|)] \\ &= \left(\frac{m \omega^2}{2 \pi k_B T} \right)^{3/2} \frac{1}{\zeta(3/2) \zeta(2)} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j^{1/2} k^{3/2}} \\ &\quad \times \frac{-4\pi}{3} \frac{\pi^{3/2} \kappa^2 \chi_0(\kappa)}{[(k m \omega_z^2 / 2 k_B T) + (j m \omega_z^2 / 2 k_B T)]^{3/2}} \end{aligned} \quad (\text{B6})$$

$$= \frac{-4\pi}{3} \chi_0(\kappa) \frac{1}{\zeta(3/2) \zeta(2)} \sum_{k=1}^{\infty} \frac{1}{j^{1/2} k^{3/2} (k+j)^{3/2}}, \quad (\text{B7})$$

where $\sum_{j,k=1}^{\infty} \{1 / [j^{1/2} k^{3/2} (k+j)^{3/2}]\} / [\zeta(2) \zeta(3/2)] \approx 0.281$.

3. Effect from ground-thermal interaction

Define the coefficient

$$S_{cT} = \frac{\int \int d^3 r d^3 r' [n_T(\mathbf{r}) n_c(\mathbf{r}') V_d(|\mathbf{r}_1 - \mathbf{r}_2|)]}{n_T(0) N_c}, \quad (\text{B8})$$

substitute Eq. (B2) in, and designate $\alpha_1 = j m \omega_z^2 / (2 k_B T)$ and $\alpha_2 = 1 / a_{\text{hoz}}^2$,

$$\begin{aligned} S_{cT} &= \frac{1}{\pi^{3/2} a_{\text{hoz}}^3 \kappa^2 \zeta(3/2)} \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \\ &\quad \times \left(\int \int d^3 r_1 d^3 r_2 e^{-\alpha_1 r_1^2 \Phi(\lambda, \phi)} e^{-\alpha_2 r_2^2 \Phi(\lambda, \phi)} V_d(|\mathbf{r}_1 - \mathbf{r}_2|) \right) \\ &= \frac{1}{\pi^{3/2} a_{\text{hoz}}^3 \kappa^2 \zeta(3/2)} \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \frac{-4\pi}{3} \frac{\pi^{3/2} \kappa^2 \chi_0(\kappa)}{[(1/a_{\text{hoz}}^2) + j m \omega_z^2 / 2 k_B T]^{3/2}} \\ &= \frac{-4\pi}{3 \zeta(3/2)} y_0^{3/2} \chi_0(\kappa) \sum_{j=1}^{\infty} \frac{1}{j^{3/2} (y_0 + j)^{3/2}}, \end{aligned} \quad (\text{B9})$$

where $y_0 = 2 k_B T / (m \omega_z^2 a_{\text{hoz}}^2) \gg 1$, we find

$$S_{cT} = \frac{-4\pi}{3} \chi_0(\kappa). \quad (\text{B10})$$

4. Effect from the condensate-condensate interaction

Define the coefficient

$$S_{cc} = \frac{\int \int d^3r d^3r' [n_c(\mathbf{r})n_c(\mathbf{r}')V_d(|\mathbf{r}_1 - \mathbf{r}_2|)]}{n_c(0)N_c}. \quad (\text{B11})$$

Substitute Eq. (B2) in and let $\alpha_1 = \alpha_2 = 1/a_{hoz}^2$, and we arrive at

$$S_{cc} = \frac{1}{\pi^{3/2}a_{hoz}^3\kappa^2} - \frac{4\pi}{3} \frac{(\pi)^{3/2}\kappa^2}{(1/a_{hoz}^2)^{3/2}} \chi_0(\kappa) = -\frac{4\pi}{3} \chi_0(\kappa). \quad (\text{B12})$$

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- [1] Y. M. Kao and T. F. Jiang, Phys. Rev. A **73**, 043604 (2006).
- [2] For a brief historical review, see A. Griffin, in *Bose-Einstein Condensation in Atomic Gases*, Proceedings of the International School of Physics ‘‘Enrico Fermi,’’ edited by M. Inguscio, S. Stringari, and C. E. Wieman (IOS Press, Amsterdam, 1999); Phys. Rev. B **53**, 9341 (1996).
- [3] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A **54**, R4633 (1996); Phys. Rev. Lett. **78**, 3987 (1997).
- [4] F. Gerbier, J. H. Thywissen, S. Richard, M. Hugbart, P. Bouyer, and A. Aspect, Phys. Rev. Lett. **92**, 030405 (2004); Phys. Rev. A **70**, 013607 (2004).
- [5] For a general review, see F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999), and references therein.
- [6] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, Phys. Rev. Lett. **94**, 160401 (2005); J. Werner, A. Griesmaier, S. Hensler, J. Stuhler, T. Pfau, A. Simoni, and E. Tiesinga, *ibid.* **94**, 183201 (2005); J. Stuhler, A. Griesmaier, T. Koch, M. Fattori, T. Pfau, S. Giovanazzi, P. Pedri, and L. Santos, *ibid.* **95**, 150406 (2005).
- [7] A. Griesmaier, J. Stuhler, and T. Pfau, Appl. Phys. B: Lasers Opt. **82**, 211 (2006).
- [8] T. F. Jiang and W. C. Su, Phys. Rev. A **74**, 063602 (2006).
- [9] K. Glaum, A. Pelster, H. Kleinert, and T. Pfau, Phys. Rev. Lett. **98**, 080407 (2007).
- [10] T. T. Chou, Chen Ning Yang, and L. H. Yu, Phys. Rev. A **53**, 4257 (1996).
- [11] Note that, with the definitions of C_δ in Ref. [9], our result becomes $\frac{\delta T_c^0}{T_c^0} = 2\chi_0(\kappa) \frac{\mu m^2 M C_\delta}{48 \pi \hbar^2 \lambda_c^{(0)}}$. This is exactly the result of dipolar effect in their Eq. (10) with dipoles aligned along the z axis.