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# Robust Control of the Robot Manipulator via an Improved Sliding Mode Scheme

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***Abstract*** **- In this paper, an improved Sliding Mode Control (SMC) scheme, reformed from the conventional SMC approaches, is developed for position tracking of a robot manipulator with parameter uncertainties and external disturbances. The improved SMC scheme, which uses saturation-type SMC laws, is shown to be able to surpass the level of uniformly ultimate boundedness to achieve asymptotic stability. In addition, the presented first-order SMC laws are continuous; as a result, this scheme can also alleviate the undesirable chattering behavior that is characteristic of sign-type SMC laws. Simulation results demonstrate the benefits of the proposed scheme.**

***Index Terms - Asymptotic stability, Nonlinear systems, Robot manipulator control, Sliding mode control, Tracking.***

## I. INTRODUCTION

Challenges in the development of accurate systems for robotic control have recently attracted a lot of attention (see e.g., [1], [7], [9]-[11], [15], [16]-[19], [22]-[23], and the references therein), especially in the field of industrial automation. As is well known, robot dynamics are highly nonlinear and depend on various payloads during operation, the properties of which are generally unknown. Therefore, conventional designs requiring an accurate model are inadequate for control missions. To meet the efficiency and robustness requirements, the Sliding Mode Control (SMC) technique has been applied to controller design in light of its rapid response and high sensitivity to system parameter uncertainties and disturbances (see e.g., [3]-[8], [10]-[15], [17]-[21], [23]). For instance, Slotine and Sastry [18] employed the SMC technique to develop a control algorithm that achieves accurate tracking of robot manipulators in the presence of disturbances and parameter variations. However, their method requires solving quite complicated algebraic inequalities. In search of a less complicated approach, some of researchers (see e.g., [19]) used linear parameterization to decompose the manipulator dynamics into the product of a known nonlinear function of manipulator dynamics, called a regressor matrix, and a constant unknown vector of manipulator parameters. Based on this model representation, Su and Leung [19] established an adaptive algorithm for the unknown parameters to improve the tracking performance.

Although many existing studies perform tracking relatively well, most of them have adopted either a conventional saturation- or sign-type controller. It is known that the performance of saturation-type SMC controllers can only be guaranteed to reach uniformly ultimate boundedness (for definition, see e.g., [2]) rather than asymptotic stability, especially for systems with uncertainties and/or disturbances. On the other hand, the sign-type SMC schemes often incur chattering behavior because of their discontinuous nature. The drawbacks of such behavior include damage to the mechanisms, excitation of dynamics that have not been modelled, and wasted energy for a system state near the sliding surface [17]. As a result, chatter is undesired for many practical applications. In the last decade, many efforts have been made to suppress or reduce the chatter produced by sign-type controllers (see e.g., [6], [16]-[17], [21]). For instance, one approach uses a continuous function to approximate the switched term of the control inside the boundary layer [17], [21]. Although this approach alleviates the chatter, it compromises the robustness of the SMC controller. In this paper, we present a class of first-order continuous SMC designs that do not require estimation of any differentiated variables. These improved SMC laws are shown not only to alleviate the chatter produced by sign-type controllers but to also improve the performance of the saturation-type SMC design from the level of uniformly ultimate boundedness to that of asymptotic stability.

## II. PROBLEM FORMULATION

Consider a rigid *n*-link manipulator described by [19], [22]

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + g(\mathbf{q}) = \mathbf{u}(t) + \mathbf{d}(t) \qquad (1)$$

where $\mathbf{q}(t) \in \Re^n$ and $\mathbf{u}(t) \in \Re^n$ denote joint positions and input joint torques, respectively, $M(\mathbf{q}) \in \Re^{n\times n}$ is the symmetric, bounded and positive definite inertia matrix, $C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} \in \Re^n$ contains the centripetal and the Coriolis torques, $g(\mathbf{q}) \in \Re^n$ is the vector of gravity effect and $\mathbf{d}(t) \in \Re^n$ denotes the vector of uncertainties involving friction torques (e.g., viscous and Coulomb friction) and external disturbances. In the subsequent discussions, we make use of the following two well-known properties (see e.g., [11]), which are important to the stability analysis.

*Property 1:* The matrix $\dot{M}(\mathbf{q}) - 2C(\mathbf{q},\dot{\mathbf{q}})$ is skew-symmetric in the sense that $\mathrm{x}^T\left[\dot{M}(\mathrm{q}) - 2C(\mathrm{q},\dot{\mathrm{q}})\right]\mathrm{x} = \mathbf{0}$ for all $\mathbf{x} \in \Re^n$.

*Property 2:* The left-hand side of Eq. (1) is linear in terms of suitable selection of manipulator parameters in the form of

$$M(\mathbf{q}) = \sum_{k=1}^{m} D_k(\mathbf{q}) a_k \ , \quad C(\mathbf{q},\dot{\mathbf{q}}) = \sum_{k=1}^{m} E_k(\mathbf{q},\dot{\mathbf{q}}) a_k \ ,$$

$$\text{and } g(\mathbf{q}) = \sum_{k=1}^{m} h_k(\mathbf{q}) a_k \qquad (2)$$

where $D_k(\mathbf{q})$, $E_k(\mathbf{q},\dot{\mathbf{q}}) \in \Re^{n\times n}$ and $h_k(\mathbf{q}) \in \Re^n$ are the known functions of the manipulator structural dynamics, and $a_k \in \Re$, $k = 1,\cdots,m$, ,denote the unknown manipulator parameters including masses, payloads and moments of inertia of the links.

The main goal of this paper is then to synthesize a control law to achieve the position trajectory tracking performance under external disturbances and model uncertainties. That is, to achieve $\mathbf{q}(t) \to \mathbf{q}_d(t)$ as $t \to \infty$, where $\mathbf{q}_d(t)$ is the desired trajectory.

## III. Main Results

To achieve this goal, we will employ the Sliding Mode Control (SMC) technique. It is known that the SMC design procedure consists of two main steps (see e.g., [12]-[13]). The first step is to select an appropriate sliding surface, which consists of the system state and the desired trajectory. The selected surface should have a property that the tracking performance can be achieved if the system state keeps staying on the sliding surface. The second step is to design a proper controller that forces the closed-loop system state reaching the sliding surface in a finite amount of time and makes the sliding surface an invariant manifold.

For the first step, we choose the sliding surface in the form of (3) below.

$$\mathrm{s}(t) := \dot{\mathrm{e}}(t) + W\mathrm{e}(t) = 0, \quad \mathrm{e}(t) = \mathrm{q}(t) - \mathrm{q}_d(t) \qquad (3)$$

where $W = \mathrm{diag}(W_{11},\cdots,W_{nn}) > 0$. Clearly, if the system state keeps staying on the sliding surface, then the tracking performance $\mathbf{q}(t) \to \mathbf{q}_d(t)$ as $t \to \infty$ can be achieved.

The second step of SMC design is to synthesize a control law in the form of

$$\mathbf{u} = \mathbf{u}^{re} + \mathbf{u}^{eq} \qquad (4)$$

where $\mathbf{u}^{re}$ plays a role of forcing the system state reaching the selected sliding surface in a finite time, while $\mathbf{u}^{eq}$ keeps the sliding surface an invariant set. To this end, we note from Eqs. (1)-(3) and Property 2 that

$$\begin{aligned} M(\mathbf{q})\dot{\mathbf{s}} &= \mathbf{u} + \mathbf{d} - C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q}) - M(\mathbf{q})\ddot{\mathbf{q}}_d \\ &\quad + M(\mathbf{q})W\dot{\mathbf{q}} - M(\mathbf{q})W\dot{\mathbf{q}}_d \\ &= \mathbf{u} + \mathbf{d} - \sum_{k=1}^{m} \{E_k(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{h}_k(\mathbf{q}) a_k \\ &\quad + D_k(\mathbf{q})[\ddot{\mathbf{q}}_d - W\dot{\mathbf{q}} + W\dot{\mathbf{q}}_d]\} a_k \\ &= \mathbf{u} + \mathbf{d} - Y(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d)\mathbf{a}. \end{aligned} \qquad (5)$$

Here, $\mathbf{a} = (a_1,\cdots,a_m)^T$ and $Y = (\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d) \in \Re^{n\times m}$ has been introduced consisting of known functions of manipulator's state and the desired trajectory. Rewrite the uncertain parameter $\mathbf{a}$ as

$$\mathbf{a} = \mathbf{a}^* + \Delta\mathbf{a} \qquad (6)$$

where $\mathbf{a}^*$ denotes the nominal value of $\mathbf{a}$.

Following the SMC design procedure [13], we choose

$$\mathbf{u}^{eq} = Y(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d)\mathbf{a}^* \qquad (7)$$

It follows from (4)-(7) that

$$M(\mathbf{q})\dot{\mathbf{s}} = \mathbf{u}^{re} + \mathbf{d} - Y(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d)\Delta\mathbf{a} \qquad (8)$$

To compensate for the effects of disturbances and uncertainties, we impose the following assumption:

*Assumption 1:* There exist $2n$ non-negative scalar known functions $\xi_i(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d)$ and $\rho_i(\mathbf{q},\dot{\mathbf{q}},t)$ such that

$$\left|\left(Y(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d)\Delta a\right)_i\right| \le \xi_i(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d)$$

$$\text{and } |\mathbf{d}_i(t)| \le \rho_i(\mathbf{q},\dot{\mathbf{q}},t) \qquad (9)$$

for all $i = 1,\cdots,n$, where $(\cdot)_i$ denote the $i$th entry of a vector.

After these settings, two classes of conventional SMC laws to achieve the objective of the paper have the form [10]

$$\mathbf{u} = Y\bar{z} - [k_1 \,\mathrm{sgn}(s_1),\cdots,k_n \,\mathrm{sgn}(s_n)]^T \qquad (10)$$

$$\bar{z}_i = -\bar{a}_i \cdot \mathrm{sgn}\left(\sum_{j=1}^{n} s_j Y_{ji}\right) \quad i = 1,\cdots,m \qquad (11)$$

$$k_i = \sum_{j=1}^{n} F_{ij}(q,\dot{q})|s_j| + \rho_i + \delta_i \quad i = 1,\cdots,n \qquad (12)$$

or

$$\mathbf{u} = Y\bar{z} - [k_1 \mathrm{sat}(s_1/\phi_1),\cdots,k_n \mathrm{sat}(s_n/\phi_n)]^T \qquad (13)$$

$$\bar{z}_i = -\bar{a}_i \cdot \mathrm{sat}\left(\sum_{j=1}^{n} s_j Y_{ji}\right) \quad i = 1,\cdots,m \qquad (14)$$

$$k_i = \sum_{j=1}^{n} F_{ij}(q,\dot{q})|s_{\Delta j}| + \rho_i + \delta_i \quad i = 1,\cdots,n \qquad (15)$$

$$s_{\Delta j} = s_j - \phi_j \cdot \mathrm{sat}(s_j/\phi_j) \quad j = 1,\cdots,n \qquad (16)$$

Here, $\mathrm{sgn}(\cdot)$ and $\mathrm{sat}(\cdot)$ denote the sign and the saturation functions, respectively. For $i = 1,\cdots,m$ and $j = 1,\cdots,n$, $\bar{a}_i, \phi_j$ ,and $\delta_j$ denote the estimation of upper bound of $|a_i|$, the boundary layer widths of the saturation-type control and the selected positive constants, respectively, while $F_{ij}$, $1 \le i, j \le n$, are chosen to satisfy

$$|C_{ij}(\mathbf{q},\dot{\mathbf{q}})| < F_{ij}(\mathbf{q},\dot{\mathbf{q}}). \qquad (17)$$

Although possibly achieving the desired performance, the two classes of conventional SMC laws have their drawbacks. First, the discontinuity of sign-type controls as given by (10)-(12) leads to chattering, which is generally undesirable in practical applications since it involves extremely high control activity and might further excite high-frequency dynamics neglected in the course of modeling. Second, although a continuous one, the saturation-type controllers as given by (13)-(16) can only ensure that the system state enters the boundary layer but does not provide asymptotic stability in the presence of disturbances [16]. Instead of sign- and saturation-type controllers as given by (10)-(12) and (13)-(16), in the following we present a continuous control law to alleviate the chattering behavior while retaining the system performance of asymptotic stability. To introduce the improved control law, we define the following function:

$$\psi(s_i)=\frac{2s_i}{|s_i|+\varepsilon e^{-\gamma t}} \qquad i=1,\cdots,n\,, \tag{18}$$

with $\varepsilon>0$ and $\gamma>0$. The values of $\in$ and $\gamma$ are selected by the designer and their interpretations will be clear later. It is noted that $\psi(s_i)$ behaves like sign and saturation functions when $|s_i|>>\varepsilon e^{-\gamma t}$ and $|s_i|<<\varepsilon e^{-\gamma t}$, respectively. The improved SMC laws then has the form of (4) with $|s_i|>>\varepsilon e^{-\gamma t}$ given by (7)

$$u_i^{re}=-(k_i+\xi_i)\psi(s_i) \tag{19}$$

where $k_i$ is given by (12) and $\xi_i$ satisfies Assumption 1. Clearly, the improved laws given above are defined and continuous everywhere including the sliding surface. We now show that the main goal of the paper can be achieved by using the improved laws (19).

The idea behind the improved control laws is to construct a time-varying region $\Phi(\mathbf{q},\dot{\mathbf{q}},t)$ of the sliding surface as defined by

$$\Phi(\mathbf{q},\dot{\mathbf{q}},t)=\{(\mathbf{q},\dot{\mathbf{q}})\mid |s_i|\le \varepsilon e^{-\gamma t}, 1\le i\le n\} \tag{20}$$

By defining the width of the region $\Phi(\mathbf{q},\dot{\mathbf{q}},t)$ to be $\varepsilon e^{-\gamma t}$, it is clear that the width of $\Phi(\mathbf{q},\dot{\mathbf{q}},t)$ exponentially converges to zero. Let $\phi(t)=\varepsilon e^{-\gamma t}$ and

$$\mathbf{s}_\pi=(s_{\pi 1},\cdots,s_{\pi n})^T,\, s_{\pi i}=s_i-\phi(t)\cdot \mathrm{sat}(s_i/\phi(t)). \tag{21}$$

It is noted that the function $s_{\pi i}$ is different from $s_{\Delta i}$, given by (16), in that the boundary layers of the saturation functions given in (16) are fixed while those in (21) are time-varying. Since $\mathrm{sat}(s_i/\phi(t))=s_i/\phi(t)$ if $|s_i|<\phi(t)$, we have from (21) that

$$s_{\pi i}=0 \quad \text{and} \quad \dot{s}_{\pi i}=0 \text{ if } |s_i|<\phi(t) \tag{22}$$

This implies that $\mathbf{s}_\pi=0$ if the system state is inside the region $\Phi(\mathbf{q},\dot{\mathbf{q}},t)$. In like manner, it is not difficult to show that:

$$\text{the system state is inside the region } \Phi(\mathrm{q},\dot{\mathrm{q}},t)\Leftrightarrow \mathrm{s}_\pi=0. \tag{23}$$

The next result verifies that the system state will enter the region $\Phi(\mathbf{q},\dot{\mathbf{q}},t)$ in a finite time and then keep staying inside there.

*Theorem 1:* Suppose that disturbance $\mathbf{d}$ and uncertain parameter $\Delta\mathbf{a}$ satisfy Assumption 1. Then, under the improved control law (4), (7) and (19), the system state of System (1) will enter the region $\Phi(\mathbf{q},\dot{\mathbf{q}},t)$ in a finite time, and thenkeep staying inside there.

*Proof:* From (23), we only need to show that $\|\mathbf{s}_\pi\|$ decays to zero in a finite time and keeps non-increasing for all time. It is observed from (18) and (21) that, when $|s_i|>\phi(t)$, $|\psi(s_i)|>1$ and $s_{\pi i}=s_i-\phi(t)\cdot\mathrm{sgn}(s_i)$. It follows that, when $|s_i|>\phi(t)$,

$$\mathrm{sgn}(s_i)=\mathrm{sgn}(s_i)=\mathrm{sgn}(\psi(s_i)), \tag{24}$$

$$-s_{\pi i}^T\psi(s_i)<-|s_{\pi i}|, \tag{25}$$

$$\dot{s}_{\pi i}=\dot{s}_i+\gamma\varepsilon e^{-\gamma t}\,\mathrm{sgn}(s_i). \tag{26}$$

Now we consider the function $V=\frac{1}{2}\mathbf{s}_\pi^T M\mathbf{s}_\pi$. It follows that $\dot{V}=\mathbf{s}_\pi^T M\dot{\mathbf{s}}_\pi+\frac{1}{2}\mathbf{s}_\pi^T\dot{M}\mathbf{s}_\pi$. From Eqs. (26) and Property 1 we have

$$\begin{aligned}\dot{V}&=\mathbf{s}_\pi^T\left(M\dot{\mathbf{s}}+\gamma\varepsilon e^{-\gamma t}M\,\mathrm{sgn}(\mathbf{s})+C\mathbf{s}_\pi\right)\\ &\overset{(8)}{=}\mathbf{s}_\pi^T\left[\mathbf{u}^{re}+\mathbf{d}-Y\Delta\mathbf{a}+\gamma\varepsilon e^{-\gamma t}M\,\mathrm{sgn}(\mathbf{s})+C\mathbf{s}_\pi\right]\end{aligned} \tag{27}$$

Since, by (22), $s_{\pi i}=0$ if $|s_i|<\phi(t)$, thus to investigate $\dot{V}$ we only need to consider those $i$ for which $|s_i|\ge\phi(t)$. Without loss of any generality, we assume that $|s_i|\ge\phi(t)$ for $i=1,\ldots,n_0$ and $n_0\le n$. It then follows from (19), (25) and (27) that

$$\begin{aligned}\dot{V}<&-\sum_{i=1}^{n_0}(k_i+\xi_i)|s_{\pi i}|+\sum_{i=0}^{n_0}s_{\pi i}\left[d_i-(Y\Delta\mathbf{a})_i+\sum_{j=1}^{n_0}C_{ij}s_{\pi j}\right]\\ &+\gamma\varepsilon e^{-\gamma t}\mathbf{s}_\pi^T M\,\mathrm{sgn}(\mathbf{s})\end{aligned}$$

Moreover, by Eqs. (12), (17), Assumption 1 and the fact that $|s_j|>|s_{\pi j}|$ if $|s_j|>\phi(t)$ we have

$$\begin{aligned}\dot{V}&<-\sum_{i=1}^{n_0}\delta_i|s_{\pi i}|+\gamma\varepsilon e^{-\gamma t}\mathbf{s}_\pi^T M\,\mathrm{sgn}(\mathbf{s})\\ &<-\delta_{\min}\cdot\|\mathbf{s}_\pi\|+\sqrt{n}\gamma\varepsilon e^{-\gamma t}\cdot\|M\mathbf{s}_\pi\|\end{aligned}$$

where $\delta_{\min}:=\min\{\delta_1\ldots\delta_{n_0}\}$ and we have used that facts that $\sum_{i=1}^{n_0}|s_{\pi i}|\ge\|\mathbf{s}_\pi\|$ and $\|\mathrm{sgn}(\mathbf{s})\|=\sqrt{n}$. Since $M(\mathbf{q})$ is a bounded positive function, say $\|M^{1/2}\|\le v$ and $v>0$, then $-\|\mathbf{s}_\pi\|\le-\frac{1}{v}\|M^{1/2}\mathbf{s}_\pi\|$ and thus

$$\dot{V}<-\frac{\delta_{\min}}{v}\cdot\|M^{1/2}\mathbf{s}_\pi\|+\sqrt{n}v\gamma\varepsilon e^{-\gamma t}\cdot\|M^{1/2}\mathbf{s}_\pi\| \tag{28}$$

Since $e^{-\gamma t}\to 0$ exponentially, (28) then implies that, after a short time transient,

$$\dot{V}<-\delta^*\cdot\|M^{1/2}\mathbf{s}_\pi\| \tag{29}$$

for some $0<\delta^*<\frac{\delta_{\min}}{v}$. It is also noted from the definition of $V$ that $\dot{V}=\frac{1}{2}\frac{d}{dt}\|M^{1/2}\mathbf{s}_\pi\|^2=\|M^{1/2}\mathbf{s}_\pi\|\cdot\frac{d}{dt}\|M^{1/2}\mathbf{s}_\pi\|$. Equation (29) then implies that

$$\frac{d}{dt}\|M^{1/2}\mathbf{s}_\pi\|<-\delta^* \tag{30}$$

Thus, $\left\|M^{1/2}\mathbf{s}_\pi\right\|$ or $\left\|\mathbf{s}_\pi\right\|$ will approach zero in a finite time and keep non-increasing. That is, the system state will enter $\Phi(\mathbf{q},\dot{\mathbf{q}},t)$ in a finite time and then keep staying inside there.

(Q.E.D.)

Theorem 1 enables us to conclude the asymptotic performance as given in Theorem 2 below.

*Theorem 2:* Suppose that disturbance d and uncertain parameter $\Delta a$ satisfy Assumption 1. Then the tracking performance for system (1) with control law given by (4), (7) and (19) can be achieved asymptotically if $\gamma > \lambda_{\max}(W) - \lambda_{\min}(W)$ ,where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalues of a matrix, respectively.

*Proof:* Theorem 1 claims that the system state will enter the region $\Phi(\mathbf{q},\dot{\mathbf{q}},t)$ in a finite time and then keep staying inside there. It remains to show that $\mathbf{q} \to \mathbf{q}_d$ as $t \to \infty$ for any $(\mathbf{q},\dot{\mathbf{q}}) \in \Phi(\mathbf{q},\dot{\mathbf{q}},t)$. From (3) we have

$$e(t) = e^{-W\,t}e(0) + \int_0^t e^{-W(t-\tau)}s(\tau)d\tau \qquad (31)$$

Since W>0, we have $e^{-W\,t}e(0) \to 0$ exponentially as $t \to \infty$. Moreover, since $(\mathbf{q},\dot{\mathbf{q}}) \in \Phi(\mathbf{q},\dot{\mathbf{q}},t)$, we have from (20) that $\|s\| \le \sum_{i=1}^n |s_i| \le n\varepsilon e^{-\gamma t}$ and

$$\left\|\int_0^t e^{-W(t-\tau)}s(\tau)d\tau\right\| \le \left\|e^{-Wt}\right\| \cdot \int_0^t \left\|e^{-W\tau}\right\| \left\|s(\tau)\right\| d\tau$$
$$\le e^{-\lambda_{\min}(W)\cdot t} \cdot \int_0^t e^{-\lambda_{\max}(W)\cdot\tau} \cdot n\varepsilon e^{-\gamma\tau} d\tau \le n\varepsilon\eta(t) \qquad (32)$$

with

$$\eta(t) = \begin{cases} te^{-\lambda_{\min}(W)t} & \text{if } \gamma = \lambda_{\max}(W) \\ e^{-\lambda_{\min}(W)t}[\dfrac{1}{\lambda_{\max}(W)-\gamma}(e^{(\lambda_{MAX}(W)-\gamma)t}-1)] & \\ & \text{if } \gamma \ne \lambda_{\max}(W) \end{cases}. \qquad (33)$$

Since $\gamma > \lambda_{\max}(W) - \lambda_{\min}(W)$, it is clear that $\eta(t) \to 0$ as $t \to \infty$. This together with Eqs. (31) and (32) give $\|e(t)\| \to 0$, or $q(t) \to q_d(t)$ as $t \to \infty$. This completes the proof. (Q.E.D.)

## IV. An Illustrative Example

Consider a two-link robot manipulator described by Fig. 1. The governing equation has the form of (1) with [9]

$$M(\mathbf{q}) = \begin{pmatrix} a_1 + a_2 + 2a_3\cos(q_2) & a_2 + a_3\cos(q_2) \\ a_2 + a_3\cos(q_2) & a_2 \end{pmatrix},$$

$$C(\mathbf{q},\dot{\mathbf{q}}) = \begin{pmatrix} -a_3\dot{q}_2\sin(q_2) & -a_3(\dot{q}_1+\dot{q}_2)\sin(q_2) \\ a_3\dot{q}_1\sin(q_2) & 0 \end{pmatrix}$$

and $g(\mathbf{q}) = \begin{pmatrix} a_4 g\cos(q_1) + a_5 g\cos(q_1+q_2) \\ a_5 g\cos(q_1+q_2) \end{pmatrix}$.

Here, g is the acceleration of gravity while the parameters $a_i$, for $i = 1,\cdots,5,$, depend on uncertain physical parameters. They are $a_1 = m_1 l_{c1}{}^2 + I_1 + m_2 l_1{}^2, a_2 = m_2 l_{c2}{}^2 + I_2, a_3 = m_2 l_1 l_{c2}$, $a_4 = m_1 l_{c1} + m_2 l_1$, and $a_5 = m_2 l_{c2}$, where the parameters $m_i, I_i, l_i$ and $l_{ci}$ are described in Fig. 1. In this example, the nominal values for these uncertain parameters are adopted from [18] as $m_1 = m_2 = 1, l_1 = l_2 = 1, l_{c1} = l_{c2} = 0.5$ and $I_1 = I_2$. $= 0.0833$ According to the chosen parameters, we have $a_1^* = 1.33, a_2^* = 0.33, a_3^* = 0.5, a_4^* = 1.5$ and $a_5^* = 0.5$. Due to the uncertainty of the parameters, a set of upper bound for $|a_i|$ are chosen as $\bar{a}_1 = 1.6, \bar{a}_2 = 0.9, \bar{a}_3 = 0.8, \bar{a}_4 = 2$, and $\bar{a}_5 = 1$. The upper bounds $F_{ij}$ for $|C_{ij}|$ is determined as $F_{11} = \bar{a}_3|\dot{q}_2|$, $F_{12} = \bar{a}_3|\dot{q}_1 + \dot{q}_2|, F_{21} = \bar{a}_3|\dot{q}_1|$ and $F_{22} = 0$. By direct calculation, $Y_{ij}$ for i=1,2 and $j = 1,\cdots,5$, in Eqs. (5) are found to be

$Y_{11} = -W_{11}(\dot{q}_1 - \dot{q}_{d1}) + \ddot{q}_{d1}$,
$Y_{12} = -W_{11}(\dot{q}_1 - \dot{q}_{d1}) - W_{22}(\dot{q}_2 - \dot{q}_{d2}) + (\ddot{q}_{d1} + \ddot{q}_{d2})$,
$Y_{13} = [-2W_{11}(\dot{q}_1 - \dot{q}_{d1}) - W_{22}(\dot{q}_2 - \dot{q}_{d2}) +$
$(2\ddot{q}_{d1} + \ddot{q}_{d2})]\cos(q_2) - (\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)\sin(q_2)$,
$Y_{14} = g\cos(q_1)$, $Y_{15} = g\cos(q_1 + q_1)$, $Y_{21} = 0$,
$Y_{22} = -W_{11}(\dot{q}_1 - \dot{q}_{d1}) - W_{22}(\dot{q}_2 - \dot{q}_{d2}) + (\ddot{q}_{d1} + \ddot{q}_{d2})$,
$Y_{23} = (-W_{11}\dot{q}_1 + W_{11}\dot{q}_{d1} + \ddot{q}_{d1})\cos(q_2) + \dot{q}_1^2\sin(q_2)$,
$Y_{24} = 0$ *and* $Y_{25} = g\cos(q_1 + q_2)$. Suppose that $\mathbf{q}_d(t) = (q_{d1(t)}(t), q_{d2(t)}(t))^T$ and $\mathbf{d}(t)$ have the form [18]

$$q_{d1}(t) = \begin{cases} -1.571 + 0.916\cdot(1-\cos(1.26t)) & \text{if } t \le 2.5 \\ 0.262 & \text{if } t > 2.5 \end{cases},$$
$$q_{d2}(t) = \begin{cases} 2.967 - 1.047\cdot(1-\cos(1.26t)) & \text{if } t \le 2.5 \\ 0.873 & \text{if } t > 2.5 \end{cases},$$

and $\mathbf{d}(t) = [0.25\sin(8t),\ 0.25\cdot\sin(8t)]^T$.

Then an upper bound for $\mathbf{d}(t)$ to satisfy Assumption 1 is $\rho_1 = \rho_2 = 0.25$. In addition, the initial state and the control parameters in this example are elected as $\mathbf{q}(0) = [-1.671, 3.067]^T$, $\dot{\mathbf{q}}(0) = [0,0]^T$, $\delta_1 = \delta_2 = 0.55$, $W_{11} = W_{22} = 5$, $\varepsilon = 1, \gamma = 1$ and $\varphi_1 = \varphi_2 = 0.05$, where $\varphi_1$ and $\varphi_2$ are the boundary layer widths given in Eq. (13).

Numerical simulations are summarized in Figs. 2-6. In these simulations, we use three different-type SMC controllers to perform the tracking task. They are the sign-type one given by (10)-(12), the saturation-type one given by (13)-(16), and the improved version given by (4), (7) and (19). Fig. 2 shows the time histories of the system states and the desired trajectory. The associated tracking errors and sliding variables are given in Figs. 3 and 4, respectively. To inspect the variation of sliding variables when system state is near the sliding surface, Figs. 5 exhibits their magnified scale from $t = 3$ to $t = 10$. Finally, Fig. 6 displays the required control forces for performing the tracking task. It is observed from Fig. 2 that the tracking performance seems to be successfully achieved by all of the three schemes, which can also be seen from Figs. 3(a) and (c); however, from Figs. 3(b) and (d), only the tracking error by improved scheme exhibits continuously decay tendency. As for the sliding variables, it is seen from Fig. 4 that these variables appear to decay to zero rapidly and then remain near the sliding surface; however, from Fig. 5, the sign-type one experiences chattering and only the one by improved scheme presents exponential decay feature (i.e., $|s_i| \le \varepsilon e^{-\gamma t}$) near the sliding surface, which agree with the theoretical results. The required control forces for sign-type scheme are observed, as expected, from Fig. 6 to experience chattering, while those for saturation-type and improved schemes are very close. By direct calculation, the consumed energy and the required maximum control magnitude have the following relations: $\left(\int \mathbf{u}^T \mathbf{u}\right)_{\text{improved}} = 4.85 < \left(\int \mathbf{u}^T \mathbf{u}\right)_{\text{sat}} = 5.09 < \left(\int \mathbf{u}^T \mathbf{u}\right)_{\text{sgn}} = 28.25$ and $\left(\|u\|_\infty\right)_{\text{imroved}} = 2.55 < \left(\|\mathbf{u}\|_\infty\right)_{\text{imroved}} = 2.94 < \left(\|u\|_\infty\right)_{\text{sgn}} = 4.54$. Clearly, because of the chattering behavior, the energy consumption by the use of sign-type SMC law is obviously higher than those of the other two SMC schemes, especially when system state is near the sliding surface. It is worth noting that there is a jump for all of the control curves near $t$ = 2.5. These jumps correspond to applying effective control to compensating the corner effect (abruptly change) of the desired trajectory at $t$ = 2.5, which can be seen from Fig. 2. From this example, it is concluded that the improved scheme not only can achieve asymptotic stability performance but also can alleviate the chattering. Besides, it does not create extra burden on control effort, compared with the sign- and saturation-type SMC schemes.

## V. Conclusion

An improved SMC scheme has been presented in this paper to improve the tracking performance of a robot manipulator. The improved SMC laws are continuous everywhere, which behave like conventional sign-type and saturation-type SMC laws when the system state is far from and close to the selected sliding surface, respectively. The improved scheme is shown to be able to alleviate the chatter resulting from conventional sign-type SMC laws and improve the performance of conventional saturation-type SMC laws from a state of uniformly ultimate boundedness to that of asymptotic stability. Simulation results clearly demonstrate the benefits of the proposed scheme compared with the conventional SMC designs.

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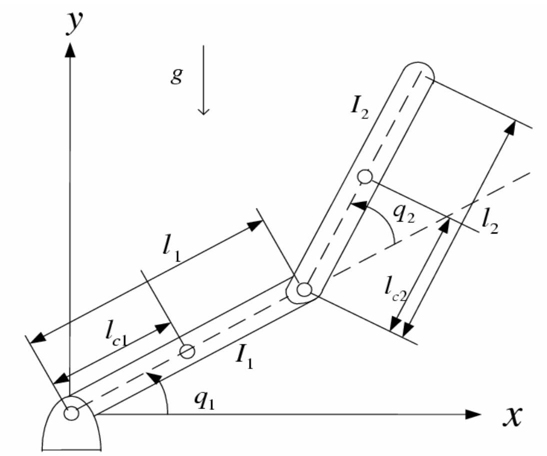


Fig. 1. A two-link robot manipulator.

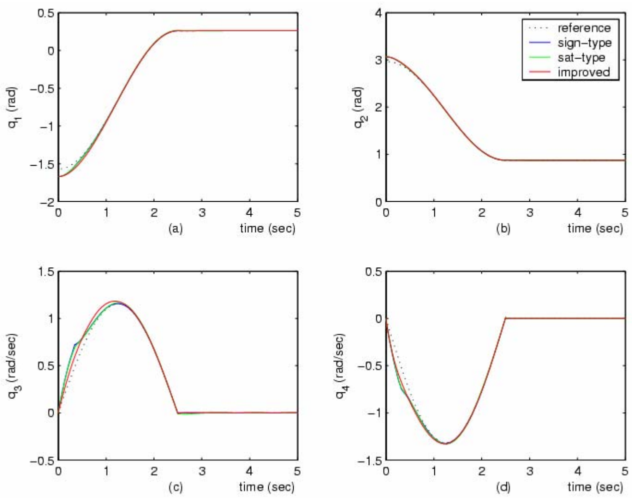


Fig. 2. Time histories of system states and reference trajectory.

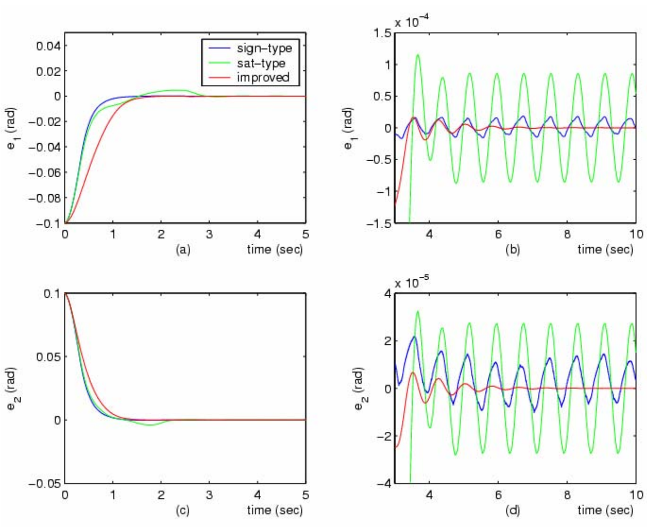


Fig. 3. Time histories of tracking errors, and their magnified scales from $t = 3$ to $t = 10$.

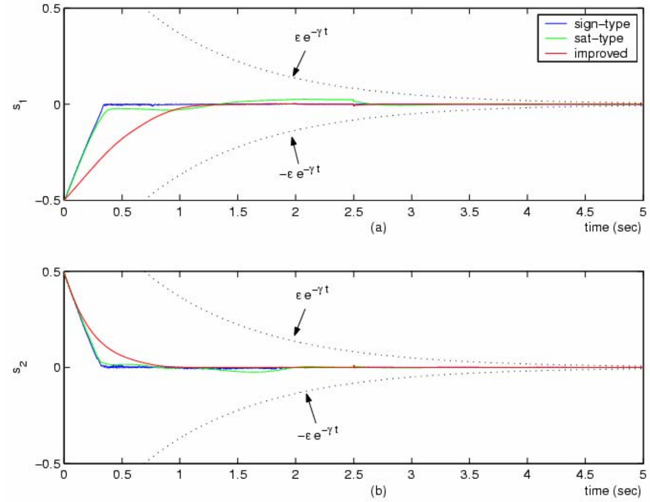


Fig. 4. Time histories of sliding variables.

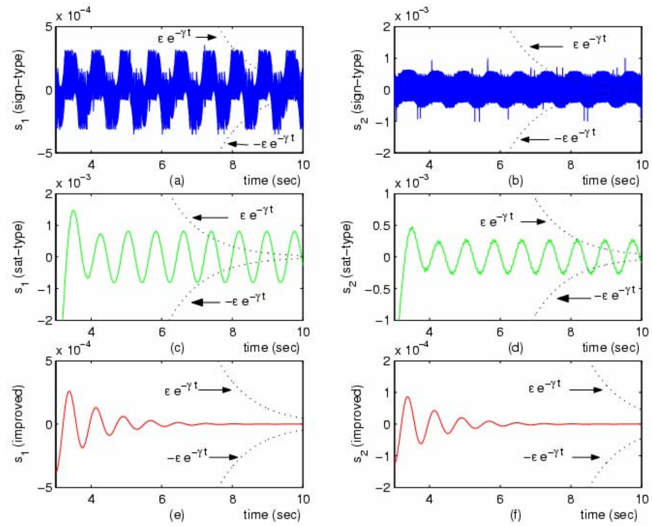


Fig. 5. Time histories of sliding variables with magnified scale from $t = 3$ to $t = 10$.

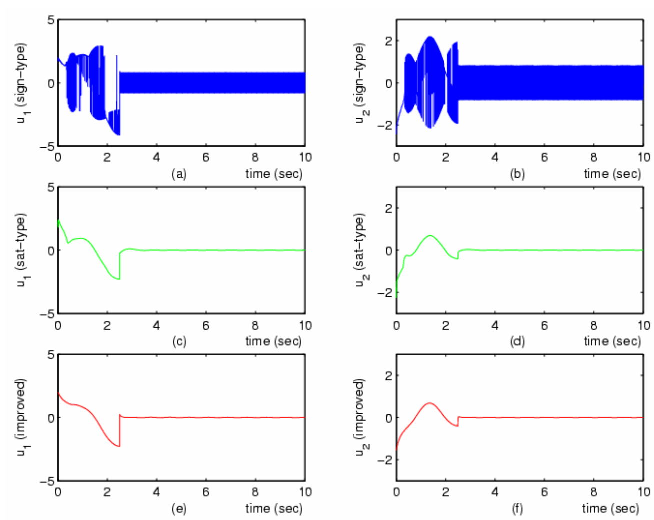


Fig. 6. Time histories of control inputs.