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Distributions and Applications

On the Sampling Distributions of the Estimated Process Loss Indices with Asymmetric Tolerances

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Pearn et al. (2006a) proposed a new generalization of expected loss index L''_e to handle processes with both symmetric and asymmetric tolerances. Putting the loss in relative terms, a user needs only to specify the target and the distance from the target at which the product would have zero worth to quantify the performance of a process. The expected loss index L''_e may be expressed as $L''_e = L''_{ot} + L''_{pe}$, which provides an uncontaminated separation between information concerning the process accuracy and the process precision. In order to apply the theory of testing statistical hypothesis to test whether a process is capable or not under normality assumption, in this paper we first derive explicit form for the cumulative distribution function and the probability density function of the natural estimator of the three indices L''_{ot}, L''_{pe}, and L''_e. We have proved that the sampling distributions of $\widehat{L}''_{pe}$ and $\widehat{L}''_{ot}$ may be expressed as the chi-square distribution and the normal distribution, respectively. And the distribution of $\widehat{L}''_e$ can be described in terms of a mixture of the chi-square distribution and the normal distribution. Then, we develop a decision-making rule based on the estimated index $\widehat{L}''_e$. Finally, an example of testing L''_e is also presented for illustrative purpose.

Keywords Asymmetric tolerances; Decision-making rule; Process capability indices; Process loss indices; Sampling distributions.

Mathematics Subject Classification Primary 62E15; Secondary 62P30.

1. Introduction

Process capability indices (PCIs), including C_p, C_a, C_{pk}, C_{pm}, and C_{pmk} (see Chan et al., 1988; Kane, 1986; Pearn et al., 1992, 1998), are convenient and powerful tools

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for measuring performance from many different perspectives. Those indices convey critical information regarding whether a process is capable of reproducing items satisfying customers' requirement. In recent years, PCIs have received substantial research attention in quality assurance and statistical literatures as well. The use of PCIs in industry did not begin in the United States until the early 1980's. Soon after, this explosion of use expanded into other industries such as automated, semiconductor, and IC assembly manufacturing industries, to measure product qualities that meet specification. Based on analyzing the PCIs, a production department can trace and improve a poor process so that the quality level can be enhanced and the requirements of the customers can be satisfied. These well-known PCIs have been defined respectively as:

$$C_p = \frac{USL - LSL}{6\sigma}, \quad C_a = 1 - \frac{|\mu - m|}{d}, \tag{1}$$

$$C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\}, \quad C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{2}$$

$$C_{pmk} = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\}, \tag{3}$$

where μ is the process mean, σ is the process standard deviation, USL is the upper specification limit, LSL is the lower specification limit, $m = (USL + LSL)/2$ is the mid-point of the specification interval, T is the target value, and $d = (USL - -LSL)/2$ is half length of the specification interval.

1.1. *Loss Measure with Symmetric Tolerances*

Johnson (1992) developed the so-called relative expected loss L_e for symmetric case, which is defined as the ratio of the expected quadratic loss and the square of the half specification width:

$$L_e = \int_{-\infty}^{\infty} \left[\frac{(x - T)^2}{d^2}\right] \mathrm{d}F(x) = \left(\frac{\mu - T}{d}\right)^2 + \left(\frac{\sigma}{d}\right)^2, \tag{4}$$

where $F(x)$ is the cumulative distribution function (cdf) of the measured characteristic. If we denote the first term $[(\mu - T)/d]^2$ by L_{ot} and the second term $(\sigma/d)^2$ by L_{pe}, then L_e can be rewritten as $L_e = L_{ot} + L_{pe}$. Unfortunately, the index L_e inconsistently measures process capability in many cases, particularly for processes with asymmetric tolerances, and thus reflects process potential and performance inaccurately.

1.2. *Loss Measure with Asymmetric Tolerances*

To remedy for this, Pearn et al. (2006a) proposed a modification of expected loss index, which referred to as L_e'', to handle processes with both symmetric and asymmetric tolerances. Regardless of whether the tolerances are symmetric or asymmetric, the new index obtains the minimal value at $\mu = T$. Additionally, the half specification width d is substituted by d^*. This generalization of expected loss index may be expressed as follows:

$$L_e'' = \left(\frac{A}{d^*}\right)^2 + \left(\frac{\sigma}{d^*}\right)^2, \tag{5}$$

where $A = \max\{(\mu - T) \cdot d/D_u, (T - \mu) \cdot d/D_l\}$, $D_u = USL - T$, $D_l = T - LSL$, and $d^* = \min\{D_u, D_l\}$. Note that L_e'' is sensitive to target value T and it obtains larger value when T is away from the mid-point between the upper and the lower specification limits. We denoted $(A/d^*)^2$ by L_{ot}'', $(\sigma/d^*)^2$ by L_{pe}'', and hence $L_e'' = L_{ot}'' + L_{pe}''$. Obviously, if the tolerances are symmetric ($T = m$), then L_{pe}'' reduces to the original index L_e.

A process is said to have a symmetric tolerance if the target value T is set to be the mid-point of the specification interval (LSL, USL). In general, asymmetric tolerances ($T \neq m$) simply reflect that deviations from target are less tolerable in one direction than the other (see Wu and Tang, 1998). Recent research and advances made in this subject are Boyles (1994), Vännman (1997), Jessenberger and Weihs (2000), and the more recent Pearn et al. (2006a,b). Asymmetric tolerances can also arise from a situation where the tolerances are symmetric to begin with, but the process follows a non normal distribution and the data is transformed to achieve approximate normality as shown by Chou et al. (1998).

For statistical inferences problems, in order to develop a successfully decision-making rule based on the estimated index $\widehat{L}_e''$ to test whether a normally distributed process is capable or not, the cdf of $\widehat{L}_e''$ is needed. In this article, we first derive explicit forms for the cdf and probability density function (pdf) of the natural estimators of L_{ot}'', L_{pe}'', and L_e'' when sampling is drawn from a normal distributed data. Those sampling distribution results greatly simplify the complexity on analyzing the statistical properties of the estimated indices. Then, we develop a reliable decision-making rule based on the estimated index $\widehat{L}_e''$, which can be used to test whether the process is capable or not.

2. Contour Plots of L_e''

We investigate some effects of the process mean μ and the process variance σ^2 on L_e'' when the specification tolerances are symmetric or asymmetric. From the inequality $L_e'' \geq (A/d^*)^2$, it is not difficult to show a necessary condition for $L_e'' \leq C$ is

$$T - \frac{D_\ell d^* \sqrt{C}}{d} \leq \mu \leq T + \frac{D_u d^* \sqrt{C}}{d}, \tag{6}$$

where C is a constant. If the tolerances are symmetric, the necessary condition for $L_e'' \leq C$ reduces to

$$m - d\sqrt{C} \leq \mu \leq m + d\sqrt{C}. \tag{7}$$

Figures 1 and 2 display the contours of L_e'' in the (μ, σ) plane for the symmetric case (LSL, T, USL) = (20, 30, 40) and the asymmetric case (LSL, T, USL) = (20, 35, 40), respectively. Contours are shown for $L_e'' = 0.11, 0.06, 0.05, 0.04$, and 0.03 from top to bottom in each plot.

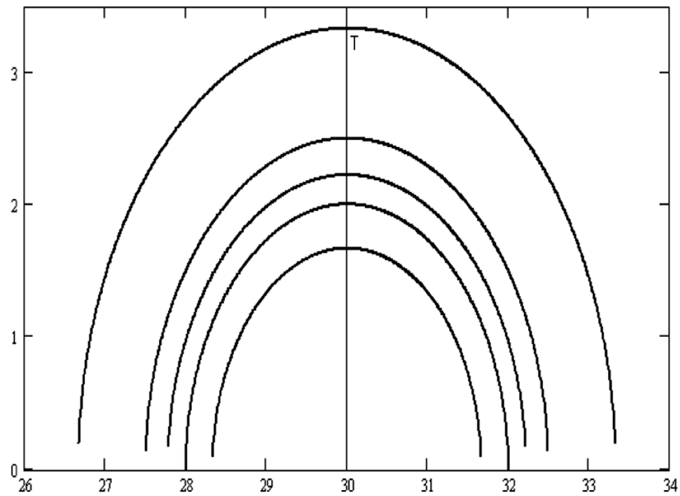

Figure 1. Contours of L_e'' in (μ, σ) plane with $L_e'' = 0.11$, 0.06, 0.05, 0.04, 0.03 (top to bottom) for symmetric case $D_u = D_l$.

To obtain the estimated value of L_e'', sample data must be collected, and a great degree of uncertainty may be introduced into capability assessments owing to the sampling errors. The approach by simply looking at the calculated values of the estimated indices and then make a conclusion on whether the given process is capable, is highly unreliable since the sampling errors have been ignored. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their sampling distributions on constructing confidence intervals and performing hypothesis testing.

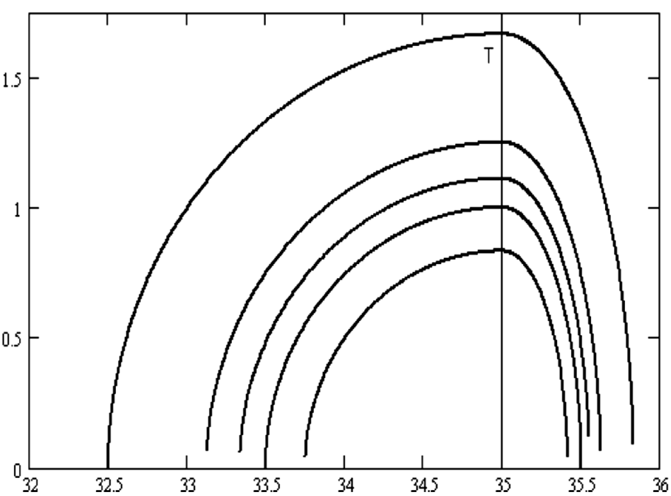

Figure 2. Contours of L_e'' in (μ, σ) plane with $L_e'' = 0.11$, 0.06, 0.05, 0.04, 0.03 (top to bottom) for asymmetric case $3D_u = D_l$.

3. Sampling Distributions of the Estimated Process Loss Indices

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normally distributed process $N(\mu, \sigma^2)$ with mean μ and standard deviation σ. To estimate the generalization L''_e, Pearn et al. (2006a) proposed the natural estimator $\widehat{L}''_e$, which is defined as:

$$\widehat{L}''_e = \left(\frac{\widehat{A}}{d^*}\right)^2 + \left(\frac{S_n}{d^*}\right)^2, \tag{8}$$

where $\widehat{A} = \max\{(\overline{X} - T) \cdot d/D_u, (T - \overline{X}) \cdot d/D_l\}$, the mean μ is estimated by the sample mean, $\overline{X} = \sum_{i=1}^n X_i/n$, and the variance σ^2 by $S_n^2 = \sum_{i=1}^n (X_i - \overline{X})^2/n$, the maximum likelihood estimator. By letting $\widehat{L}''_{ot} = (\widehat{A}/d^*)^2$ and $\widehat{L}''_{pe} = (S_n/d^*)^2$, the relationship $\widehat{L}''_e = \widehat{L}''_{ot} + \widehat{L}''_{pe}$ may be established.

For the case where the production tolerance is symmetric, $\widehat{A}$ may be simplified as $|\overline{X} - T|$. Therefore, the estimator $\widehat{L}''_e$ reduces to $\widehat{L}_e = (n^{-1}d^{-2}) \cdot \sum_{i=1}^n (X_i - T)^2$, the natural estimator of L_e discussed in Johnson (1992). Consequently, we may view the estimator $\widehat{L}''_e$ as a direct extension of $\widehat{L}_e$. In the following, we focus on sampling distributions of the estimated process loss indices $\widehat{L}''_{pe}$, $\widehat{L}''_{ot}$, and $\widehat{L}''_e$.

In attempt to derive the cdf and pdf of $\widehat{L}''_{pe}$, $\widehat{L}''_{ot}$, and $\widehat{L}''_e$, we first introduce the following notation:

(1) $B = (nd^{*2})/\sigma^2$;
(2) $K = (nS_n^2)/(\sigma^2)$, which is distributed as χ^2_{n-1};
(3) $Z = n^{1/2}(\overline{X} - T)/\sigma$, which is distributed as $N(\delta, 1)$, where $\delta = n^{1/2}(\mu - T)/\sigma$;
(4) $Y = \max^2\{d_u Z, -d_l Z\}$, where $d_u = d/D_u$, $d_l = d/D_l$.

After some algebraic manipulations, the following expressions $\widehat{L}''_{pe} = K/B$, $\widehat{L}''_{ot} = Y/B$, and $\widehat{L}''_e = (Y + K)/B$ can be established.

3.1. *Sampling Distribution of $\widehat{L}''_{pe}$*

Theorem 3.1. *Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normally distributed process $N(\mu, \sigma^2)$. Then $\widehat{L}''_{pe}$ is distributed as $\sigma^2/(nd^{*2})$ times a chi-square distribution with $n - 1$ degrees of freedom. And the pdf and cdf of $\widehat{L}''_{pe}$ can be expressed respectively as*:

$$f_{\widehat{L}''_{pe}}(x) = \frac{nd^{*2}}{\sigma^2} f_K\left(\frac{nd^{*2}}{\sigma^2} x\right) = Bf_K(Bx), \tag{9}$$

$$F_{\widehat{L}''_{pe}}(x) = F_K\left(\frac{nd^{*2}}{\sigma^2} x\right) = F_K(Bx), \quad \text{for } x > 0, \tag{10}$$

where $F_K(\cdot)$ and $f_K(\cdot)$, respectively, denote the cdf and the pdf of K, which is distributed as χ^2_{n-1}.

Proof. Since $\widehat{L}''_{pe} = K/B$, the cdf of $\widehat{L}''_{pe}$ can be obtained directly as:

$$F_{\widehat{L}''_{pe}}(x) = P(\widehat{L}''_{pe} \le x) = P(K \le Bx), \quad \text{for } x > 0. \tag{11}$$

Differentiate (11) with respect to x, the pdf of $\widehat{L}''_{pe}$ may be derived.

If the specification tolerance is symmetric (i.e., $T = m$), then $d^* = d$ and the inconsistency loss index estimator $\widehat{L}''_{pe}$ reduces to $\widehat{L}_{pe}$. Therefore, the pdf and the cdf of $\widehat{L}''_{pe}$ can be simplified, respectively, as:

$$f_{\widehat{L}_{pe}}(x) = \frac{nd^2}{\sigma^2} f_K\left(\frac{nd^2}{\sigma^2}x\right), \tag{12}$$

$$F_{\widehat{L}_{pe}}(x) = F_K\left(\frac{nd^2}{\sigma^2}x\right), \quad \text{for } x > 0. \tag{13}$$

Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cdf and the pdf of the standardized normal distribution $N(0, 1)$, respectively. Then we can express the cdf of Z as $F_Z(z) = \Phi(z - \delta)$ and the pdf of Z as $f_Z(z) = \phi(z - \delta)$, respectively. The cdf of Y can be written as:

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(Y \le y, Z < 0) + P(Y \le y, Z \ge 0) \\ &= P(-d_\ell^{-1}\sqrt{y} \le Z < 0) + P(0 \le Z \le d_u^{-1}\sqrt{y}) \\ &= \Phi(d_\ell^{-1}\sqrt{y} + \delta) + \Phi(d_u^{-1}\sqrt{y} - \delta) - 1, \quad \text{for } y > 0. \end{aligned} \tag{14}$$

Taking the derivative of $F_Y(y)$ with respect to y to obtain the pdf of Y as:

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(d_\ell^{-1}\phi(d_\ell^{-1}\sqrt{y} + \delta) + d_u^{-1}\phi(d_u^{-1}\sqrt{y} - \delta)\right), \quad \text{for } y > 0. \tag{15}$$

If the tolerance is symmetric, then $d_u = d_l = 1$, and the corresponding pdf of Y can be simplified as:

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\phi(\sqrt{y} + \delta) + \phi(\sqrt{y} - \delta)\right). \tag{16}$$

For symmetric case, since $Y = Z^2$ reduces to the non central chi-square distribution with one degree of freedom and non centrality parameter $\delta = n^{1/2}(\mu - T)/\sigma$. The pdf of Y, an alternative form of (16), can be expressed as:

$$f_Y(y) = \sum_{j=0}^{\infty} P_j(\lambda/2) f_{Y_j}(y), \quad y > 0, \tag{17}$$

where Y_j is distributed as χ^2_{1+2j}, $P_j(\lambda/2) = e^{-\lambda/2}(\lambda/2)^j/(j!) = P(W = j)$, and W follows a Poisson distribution with expected value $\lambda/2$, where $\lambda = \delta^2$.

3.2. *Sampling Distribution of $\widehat{L}''_{ot}$*

Theorem 3.2. *Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normally distributed process $N(\mu, \sigma^2)$. Then the pdf and the cdf of $\widehat{L}''_{ot}$ can be expressed, respectively, as:*

$$f_{\widehat{L}''_{ot}}(x) = \frac{\sqrt{B/x}}{2}\left(d_\ell^{-1}\phi\left(d_\ell^{-1}\sqrt{Bx} + \delta\right) + d_u^{-1}\phi\left(d_u^{-1}\sqrt{Bx} - \delta\right)\right), \tag{18}$$

$$F_{\widehat{L}''_{ot}}(x) = \Phi\left(d_\ell^{-1}\sqrt{Bx} + \delta\right) + \Phi\left(d_u^{-1}\sqrt{Bx} - \delta\right) - 1, \quad \text{for } x > 0, \tag{19}$$

where $\Phi(\cdot)$ *and* $\phi(\cdot)$ *are the cdf and the pdf of the standardized normal distribution* $N(0, 1)$, *respectively.*

Proof. Since $\widehat{L}''_{ot} = Y/B$, the cdf of $\widehat{L}''_{ot}$ can be derived easily by (14):

$$\begin{aligned} F_{\widehat{L}''_{ot}}(x) &= P\left(\widehat{L}''_{ot} \le x\right) = P(Y \le Bx) \\ &= \Phi\left(d_\ell^{-1}\sqrt{Bx} + \delta\right) + \Phi\left(d_u^{-1}\sqrt{Bx} - \delta\right) - 1, \quad \text{for } x > 0. \end{aligned} \tag{20}$$

The pdf of $\widehat{L}''_{ot}$ follows by differentiate (20) with respect to x.

If the specification tolerance is symmetric, then $D_u = D_l = d$, $d_u = d_l = 1$, and the off-target loss index estimator $\widehat{L}''_{ot}$ reduces to $\widehat{L}_{ot}$. The pdf and the cdf of $\widehat{L}''_{ot}$ therefore can be simplified, respectively, as:

$$f_{\widehat{L}_{ot}}(x) = \frac{\sqrt{B/x}}{2}\left(\phi\left(\sqrt{Bx} + \delta\right) + \phi\left(\sqrt{Bx} - \delta\right)\right), \tag{21}$$

$$F_{\widehat{L}_{ot}}(x) = \Phi\left(\sqrt{Bx} + \delta\right) + \Phi\left(\sqrt{Bx} - \delta\right) - 1, \quad \text{for } x > 0. \tag{22}$$

3.3. *Sampling Distribution of* $\widehat{L}''_e$

Theorem 3.3. *Let* $X_1, X_2, \ldots, X_n$ *be a random sample of size* n *from a normally distributed process* $N(\mu, \sigma^2)$. *Then the pdf and the cdf of* $\widehat{L}''_e$ *can be expressed, respectively, as*:

$$\begin{aligned} f_{\widehat{L}''_e}(x) &= \int_0^1 \frac{\sqrt{B^3x/t}}{2} f_K(Bx(1-t)) \\ &\quad \times \left(d_\ell^{-1}\phi\left(d_\ell^{-1}\sqrt{Bxt} + \delta\right) + d_u^{-1}\phi\left(d_u^{-1}\sqrt{Bxt} - \delta\right)\right)\mathrm{d}t, \end{aligned} \tag{23}$$

$$\begin{aligned} F_{\widehat{L}''_e}(x) &= \int_0^{Bx} F_K(Bx - y)\frac{1}{2\sqrt{y}} \\ &\quad \times \left(d_\ell^{-1}\phi\left(d_\ell^{-1}\sqrt{y} + \delta\right) + d_u^{-1}\phi\left(d_u^{-1}\sqrt{y} - \delta\right)\right)\mathrm{d}y, \quad \textit{for } x > 0, \end{aligned} \tag{24}$$

where $f_K(\cdot)$ *and* $F_K(\cdot)$, *respectively, denote the pdf and the cdf of* K, *which is distributed as* χ^2_{n-1}.

Proof. Since $\widehat{L}''_e = (Y + K)/B$, the cdf of $\widehat{L}''_e$ can be expressed as:

$$\begin{aligned} F_{\widehat{L}''_e}(x) &= P\left(\widehat{L}''_e \le 0x\right) = P(Y + K \le Bx) \\ &= \int_0^\infty P(K \le Bx - Y | Y = y) f_Y(y)\mathrm{d}y \\ &= \int_0^{Bx} F_K(Bx - y) f_Y(y)\mathrm{d}y, \quad \text{for } x \ge 0. \end{aligned} \tag{25}$$

The last equality is valid since $(Bx - y) \ge 0$ for $0 \le y \le Bx$, and $(Bx - y) < 0$ for $y > Bx$. Thus, $F_K(Bx - y) = 0$ for $y > Bx$. Using (15) the distribution function $F_{\widehat{L}''_e}$(x) can be expressed as (24).

On the other hand, since

$$\frac{d}{dx}\left(\int_0^{u(x)} f(x,t)\mathrm{d}t\right) = \int_0^{u(x)} \frac{\partial}{\partial x} f(x,t)\mathrm{d}t + f(x,u(x))u'(x)$$

(see Varberg and Purcell, 1992) and

$$f_{\widehat{L}_e''}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{\widehat{L}_e''}(x),$$

we may obtain the pdf of $\widehat{L}_e''$ as:

$$f_{\widehat{L}_e''}(x) = \int_0^{Bx} Bf_K(Bx-y)f_Y(y)\mathrm{d}y + BF_K(0)f_Y(Bx).$$

Note that K is distributed as χ^2_{n-1}, so $F_K(0) = 0$. By changing variable $t = y/(Bx)$ in the above integral, we have $y = Bxt$ and $\mathrm{d}y = Bx\,\mathrm{d}t$. Hence,

$$f_{\widehat{L}_e''}(x) = \int_0^1 B^2 x f_K(Bx(1-t))f_Y(Bxt)\mathrm{d}t.$$

Using (15) the pdf $f_{\widehat{L}_e''}(x)$ can be expressed as (23).

For the case when the specification tolerance is symmetric, then $D_u = D_l = d$, $d_u = d_l = 1$, and estimator of the expected loss index $\widehat{L}_e''$ reduces to $\widehat{L}_e$. The pdf (23) and the cdf (24) of $\widehat{L}_e''$ reduce to those of $\widehat{L}_e$. Hence, the cdf and the pdf of $\widehat{L}_e$ can be expressed, respectively, as:

$$f_{\widehat{L}_e}(x) = \int_0^1 \frac{\sqrt{B^3x/t}}{2} f_K(Bx(1-t))\left(\phi\left(\sqrt{Bxt}+\delta\right) + \phi\left(\sqrt{Bxt}-\delta\right)\right)\mathrm{d}t, \qquad (26)$$

$$F_{\widehat{L}_e}(x) = \int_0^{Bx} F_K(Bx-y)\frac{1}{2\sqrt{y}}\left(\phi\left(\sqrt{y}+\delta\right) + \phi\left(\sqrt{y}-\delta\right)\right)\mathrm{d}y, \quad \text{for } x > 0. \qquad (27)$$

Furthermore, for symmetric tolerances since Y follows a non central chi-square distribution with one degree of freedom and non centrality parameter δ, we can substitute the pdf of Y as expressed in (17) into (25). Hence, the cdf of $\widehat{L}_e$ can be expressed in an alternative form as:

$$F_{\widehat{L}_e}(x) = \sum_{j=0}^{\infty} P_j(\lambda/2)\int_0^{Bx} F_K(Bx-y)f_{Y_j}(y)\mathrm{d}y, \quad \text{for } x > 0, \qquad (28)$$

where Y_j is distributed as χ^2_{1+2j}, $\lambda = \delta^2$, and $P_j(\lambda/2) = e^{-\lambda/2}(\lambda/2)^j/(j!)$.

Taking the derivative of $F_{\widehat{L}_e}(x)$ in (28) with respect to x to obtain the pdf of $\widehat{L}_e$ as:

$$f_{\widehat{L}_e}(x) = \sum_{j=0}^{\infty} P_j(\lambda/2)\int_0^{Bx} Bf_K(Bx-y)f_{Y_j}(y)\mathrm{d}y, \quad \text{for } x > 0. \qquad (29)$$

Now by changing variable $t = y/(Bx)$ in the above integral, we have $y = Bxt$ and $\mathrm{d}y = Bx\mathrm{d}t$. Hence, the result follows:

$$f_{\widehat{L}_e}(x) = \sum_{j=0}^{\infty} P_j(\lambda/2) \int_0^1 B^2 x f_K(Bx(1-t)) f_{Y_j}(Bxt)\mathrm{d}t, \quad \text{for } x > 0. \tag{30}$$

Since

$$f_K(Bx(1-t)) = \frac{2^{-(n-1)/2}}{\Gamma((n-1)/2)}(Bx(1-t))^{(n-3)/2} e^{-Bx(1-t)/2},$$
$$f_{Y_j}(Bxt) = \frac{2^{-(2j+1)/2}}{\Gamma((2j+1)/2)}(Bxt)^{(2j-1)/2} e^{-Bxt/2},$$

we have

$$\begin{aligned} f_{\widehat{L}_e}(x) &= \frac{2^{-n/2}B^{n/2}x^{n/2-1}}{\Gamma((n-1)/2)} \sum_{j=0}^{\infty} P_j\left(\frac{\lambda}{2}\right) \frac{e^{-Bx/2}(Bx/2)^j}{\Gamma((2j+1)/2)} \int_0^1 t^{(2j+1)/2-1}(1-t)^{(n-1)/2-1}\mathrm{d}t \\ &= \frac{(B/2)^{n/2}\, x^{n/2-1}}{\Gamma((n-1)/2)} \sum_{j=0}^{\infty} P_j\left(\frac{\lambda}{2}\right) P_j\left(\frac{Bx}{2}\right) \frac{\Gamma(j+1)}{\Gamma((2j+1)/2)} \frac{\Gamma((2j+1)/2)\Gamma((n-1)/2)}{\Gamma((2j+n)/2)} \\ &= (B/2)^{n/2}\, x^{n/2-1} \sum_{j=0}^{\infty} P_j\left(\frac{\lambda}{2}\right) P_j\left(\frac{Bx}{2}\right) \frac{\Gamma(j+1)}{\Gamma((2j+n)/2)}, \quad \text{for } x > 0, \end{aligned} \tag{31}$$

where $P_j(\lambda/2) = e^{-\lambda/2}(\lambda/2)^j/(j!)$.

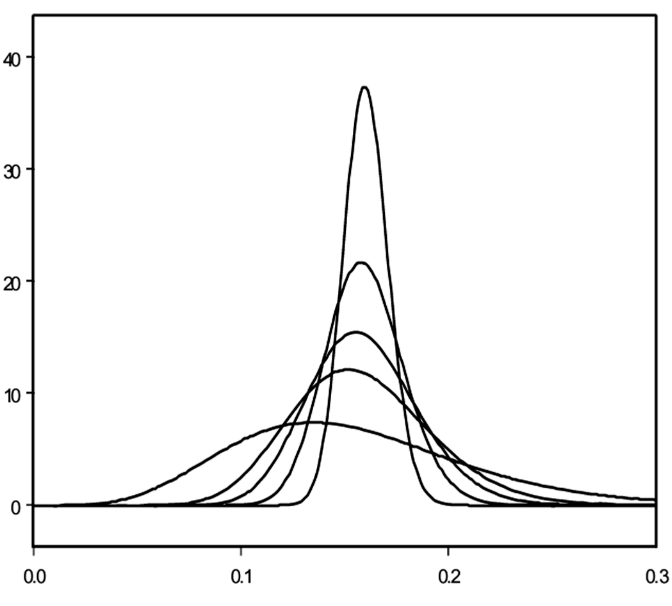

Figure 3. The pdf of $\widehat{L}_e''$ with $a = -1$, $b = 3$, and $n = 10, 30, 50, 100, 300$ (bottom to top in plot) for $3D_u = D_l$.

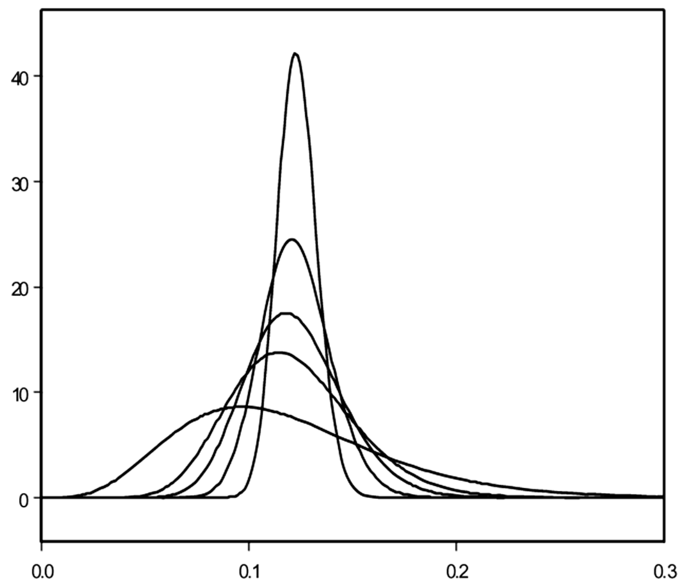

Figure 4. The pdf of $\widehat{L}''_e$ with $a = -0.5$, $b = 3$, and $n = 10, 30, 50, 100, 300$ (bottom to top in plot) for $3D_u = D_l$.

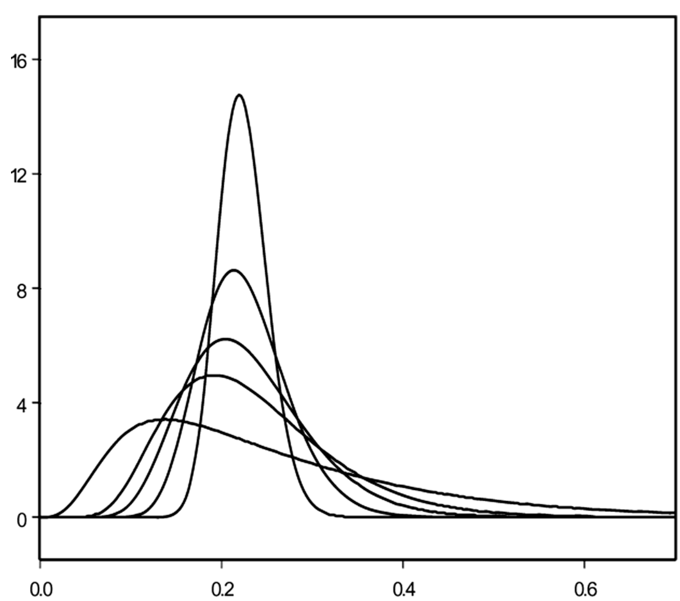

Figure 5. The pdf of $\widehat{L}''_e$ with $a = 0.5$, $b = 3$, and $n = 10, 30, 50, 100, 300$ (bottom to top in plot) for $3D_u = D_l$.

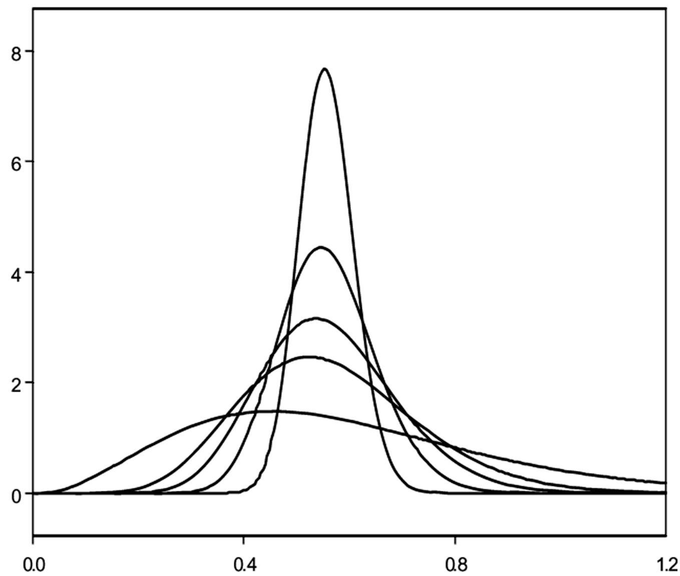

Figure 6. The pdf of $\widehat{L}_e''$ with $a = 1$, $b = 3$, and $n = 10, 30, 50, 100, 300$ (bottom to top in plot) for $3D_u = D_l$.

4. Distribution Plots of $\widehat{L}_e''$

We plot the pdf of $\widehat{L}_e''$ when the underlying process is normal for several selected cases. Figures 3–6 depict the plots of the pdf of $\widehat{L}_e''$ for four levels of L_e'' index value with parameter a set to $a = (\mu - T)/\sigma = -1.0$ $(L_e'' = 0.16)$, $a = -0.5$ $(L_e'' = 0.12)$, $a = 0.5(L_e'' = 0.22)$, and $a = 1.0(L_e'' = 0.56)$, respectively. The asymmetric case is considered by setting $3(USL - T) = T - LSL$, $b = d^*/\sigma = 3$, and sample size $n = 10$, 30, 50, 100, and 300 from bottom to top in each figure.

From Figures 3–6, we discover that as the value of L_e'' increases, the spread of the distribution also increases. For small sample size $n = 10$ as example, the distributions are skew to the right (have positive skewness) and have large spread. As sample size n increases, the spread decreases and so does the skewness. We also observe that $\widehat{L}_e''$ is approximately unbiased and bell-shaped for sample size n greater than 50.

5. A Decision-Making Rule for Testing L_e''

Under normality assumption, we proved that the cdf and the pdf of $\widehat{L}_e''$ can be represented in terms of a mixture of the central chi-square distribution and the normal distribution. Using the index L_e'', the engineers can assess the process performance and monitor the manufacturing processes on routine basis. To obtain an effective decision-making rule, we consider a testing hypothesis with the null hypothesis and the alternative hypothesis, respectively, as

$$H_0 : L_e'' \geq C \text{ (incapable)} \quad \text{versus} \quad H_1 : L_e'' < C \text{ (capable)}.$$

The null hypothesis H_0 will be rejected if $\widehat{L}''_e < c_\alpha$, where the constant c_α, called the critical value, is determined so that the significance level of the test is α, i.e., $P\big(\widehat{L}''_e < c_\alpha \,|\, L''_e = C\big) = \alpha$. The decision-making rule to be used is then stated as follows: for given α (the probability of wrongly reject null hypothesis when it is true) and sample size n, the process will be considered capable if $\widehat{L}''_e < c_\alpha$ and incapable if $\widehat{L}''_e \geq c_\alpha$.

We note that, by setting $a = (\mu - T)/\sigma$ and $b = d^*/\sigma$, the indices L''_{ot} and L''_{pe} can be rewritten as $L''_{ot} = (d_l a/b)^2$ for $a < 0$, $L''_{pe} = (d_u a/b)^2$ for $a > 0$, and $\widehat{L}''_{pe} = (1/b)^2$, where $d_u = d/D_u$, $d_l = d/D_l$. Hence, the value of $L''_e = L''_{ot} + L''_{pe}$ can be calculated when values of a, b, d_u, and d_l are given. For example, if $(a, b, d_u, d_l) = (1, 3, 2/3, 2)$ then $L''_e = (2 \times 1/3)^2 + (1/3)^2 = 5/9$. If $L''_e = C$, from $L''_e = L''_{ot} + L''_{pe}$, we have

$$C = \frac{(d_\ell a)^2}{b^2} + \frac{1}{b^2} \quad \text{for } a \leq 0 \quad \text{and} \quad C = \frac{(d_u a)^2}{b^2} + \frac{1}{b^2} \quad \text{for } a > 0,$$

then $b^2 = \big[(d_\ell a)^2 + 1\big]/C$ for $a \leq 0$ and $b^2 = \big[(d_u a)^2 + 1\big]/C$ for $a > 0$. In addition, we have $B = nd^{*2}/\sigma^2 = nb^2$. Therefore, if $L''_e = C$, then

$$B = n\big[(d_\ell a)^2 + 1\big]/C \quad \text{for } a \leq 0 \quad \text{and} \quad B = n\big[(d_u a)^2 + 1\big]/C \quad \text{for } a > 0. \tag{32}$$

Furthermore, we have the equality $\delta = n^{1/2}(\mu - T)/\sigma = n^{1/2}a$. From the result of Theorem 3.3, we can use the central chi-square distribution and the normal distribution to find the critical value c_α satisfying $P\big(\widehat{L}''_e < c_\alpha \,|\, L''_e = C\big) = \alpha$, i.e., $F_{\widehat{L}''_e}(c_\alpha) = \alpha$ given $L''_e = C$, or equivalent to

$$\int_0^{Bc_\alpha} F_K(Bc_\alpha - y)\frac{1}{2\sqrt{y}}\big(d_\ell^{-1}\phi\big(d_\ell^{-1}\sqrt{y} + \sqrt{na}\big) + d_u^{-1}\phi\big(d_u^{-1}\sqrt{y} - \sqrt{na}\big)\big)\mathrm{d}y = \alpha, \tag{33}$$

where B is given in (32).

We note that the distribution characteristic parameter $a = (\mu - T)/\sigma$ in (33) is usually unknown, which has to be estimated in real applications, naturally by substituting μ and σ^2 by the sample mean $\overline{X}$ and the maximum likelihood estimator $S_n^2 = \sum_{i=1}^n \big(X_i - \overline{X}\big)^2/n$. To realize the relationship between c_α and a, we examine the behavior of c_α against $a = -3(0.05)3$, which covers a wide range of applications with process capability analysis. The results indicate that the critical value reaches its minimum at $a = 0.5$ in all cases with accuracy up to 10^{-3}. Figures 7–10 plot the curves of c_α vs. a for some selected cases.

5.1. *Making Decision by Critical Value*

Tables 1–2 display the critical values c_α for $3D_u = D_l$, $3D_u = 2D_l$, respectively, with $C = 0.05$, sample size $n = 100$, $a = -3(0.2)3$, and $\alpha = 0.01, 0.05, 0.10$. To test if the process meets the capability requirement, we first determine the value of C and α, then estimate the index L''_e and parameter a from the collected sample. If the calculated value of $\widehat{L}''_e$ is smaller than the critical value c_α $\big(\widehat{L}''_e < c_\alpha\big)$, then we conclude that the process meets the capability requirement $\big(L''_e < C\big)$. Otherwise, we do not have sufficient information to conclude whether the process meets the preset capability requirement. In this case, we would believe that $L''_e \geq C$ (the process is incapable).

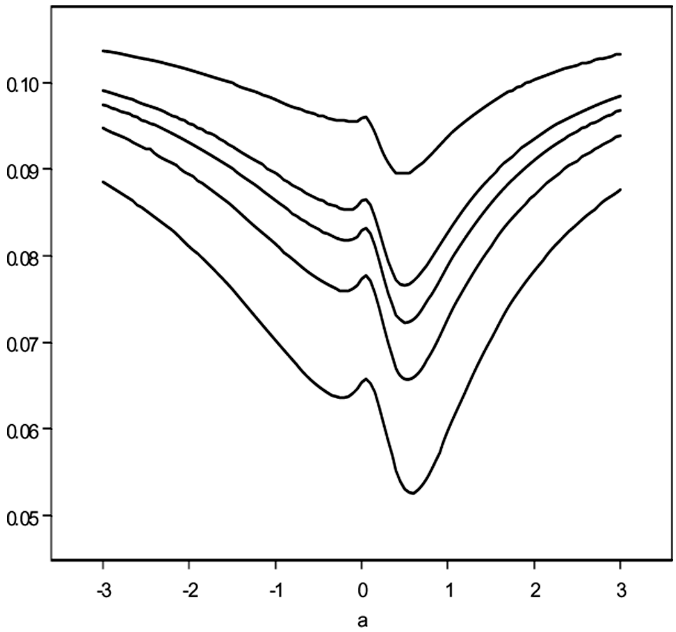

Figure 7. Plots of c_α vs. a for $L_e'' = 0.11$, $3D_u = D_l$, $n = 25, 50, 75, 100, 300$ (bottom to top in plot).

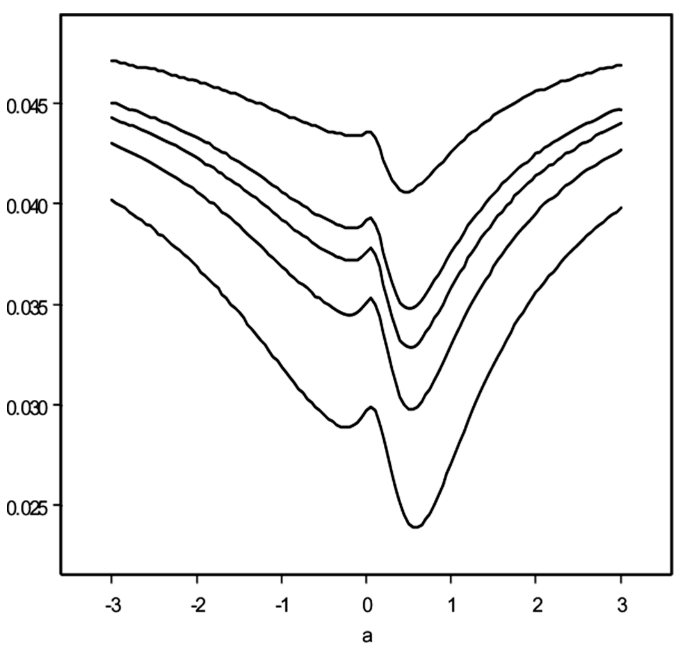

Figure 8. Plots of c_α vs. a for $L_e'' = 0.05$, $3D_u = D_l$, $n = 25, 50, 75, 100, 300$ (bottom to top in plot).

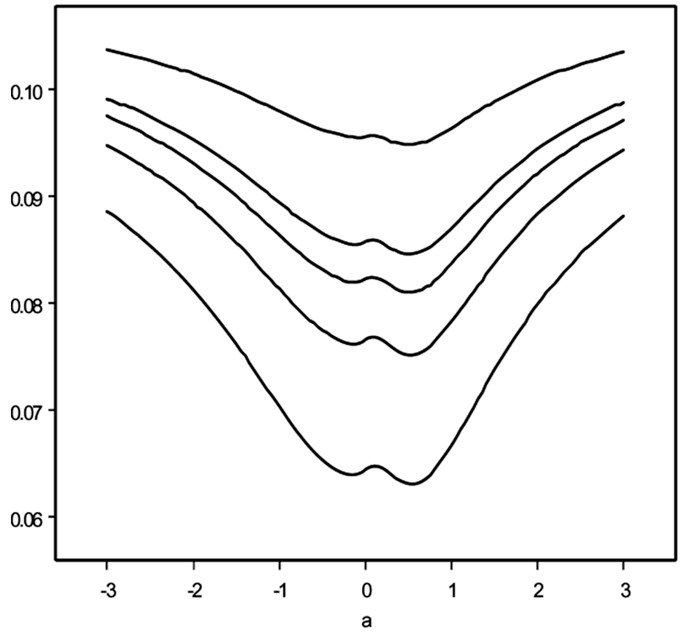

Figure 9. Plots of c_α vs. a for $L''_e = 0.11$, $3D_u = 2D_l$, $n = 25, 50, 75, 100, 300$ (bottom to top in plot).

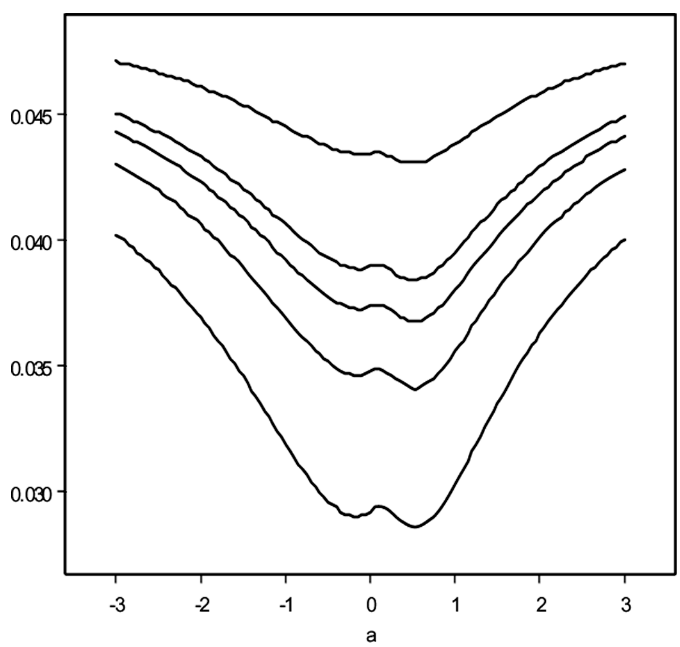

Figure 10. Plots of c_α vs. a for $L''_e = 0.05$, $3D_u = 2D_l$, $n = 25, 50, 75, 100, 300$ (bottom to top in plot).

Table 1
Critical values c_α for $C = 0.05$ with $n = 100$, $a = -3(0.2)3$, and $\alpha = 0.01, 0.05, 0.10$ under $3D_u = D_l$

a	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
−3.0	0.0431	0.0450	0.0461
−2.8	0.0427	0.0447	0.0458
−2.6	0.0423	0.0444	0.0456
−2.4	0.0418	0.0441	0.0453
−2.2	0.0413	0.0437	0.0450
−2.0	0.0408	0.0433	0.0447
−1.8	0.0402	0.0428	0.0443
−1.6	0.0395	0.0423	0.0439
−1.4	0.0388	0.0418	0.0435
−1.2	0.0380	0.0412	0.0430
−1.0	0.0373	0.0406	0.0425
−0.8	0.0365	0.0401	0.0420
−0.6	0.0358	0.0395	0.0416
−0.4	0.0353	0.0391	0.0412
−0.2	0.0349	0.0388	0.0410
0.0	0.0352	0.0392	0.0414
0.2	0.0334	0.0376	0.0400
0.4	0.0304	0.0351	0.0379
0.6	0.0300	0.0350	0.0379
0.8	0.0313	0.0362	0.0389
1.0	0.0331	0.0376	0.0401
1.2	0.0348	0.0389	0.0412
1.4	0.0363	0.0400	0.0421
1.6	0.0375	0.0410	0.0429
1.8	0.0386	0.0418	0.0435
2.0	0.0396	0.0425	0.0441
2.2	0.0404	0.0431	0.0445
2.4	0.0411	0.0436	0.0449
2.6	0.0417	0.0440	0.0453
2.8	0.0422	0.0444	0.0456
3.0	0.0427	0.0447	0.0459

5.2. *Making Decision by p-value*

We also can calculate the p-value, i.e., the probability that $\widehat{L}''_e$ does not exceed the observed index given the values of C, d_u, d_l, a, and sample size n, and then compare this probability with the significance level α. If the estimated index value is l_0, then the p-value can be calculated as:

$$p\text{-value} = \int_0^{Bl_0} F_K(Bl_0 - y)\frac{1}{2\sqrt{y}}\left(d_\ell^{-1}\phi\left(d_\ell^{-1}\sqrt{y} + \sqrt{n}a\right) + d_u^{-1}\phi\left(d_u^{-1}\sqrt{y} - \sqrt{n}a\right)\right)dy, \tag{34}$$

Table 2
Critical values c_α for $C = 0.05$ with $n = 100$, $a = -3(0.2)3$, and $\alpha = 0.01, 0.05, 0.10$ under $3D_u = 2D_l$

a	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
−3.0	0.0431	0.0450	0.0461
−2.8	0.0427	0.0447	0.0458
−2.6	0.0423	0.0444	0.0456
−2.4	0.0418	0.0441	0.0453
−2.2	0.0413	0.0437	0.0450
−2.0	0.0407	0.0433	0.0447
−1.8	0.0401	0.0428	0.0443
−1.6	0.0394	0.0423	0.0439
−1.4	0.0387	0.0418	0.0435
−1.2	0.0379	0.0412	0.0430
−1.0	0.0372	0.0406	0.0425
−0.8	0.0364	0.0400	0.0420
−0.6	0.0358	0.0395	0.0416
−0.4	0.0353	0.0391	0.0413
−0.2	0.0350	0.0389	0.0411
0.0	0.0350	0.0390	0.0412
0.2	0.0349	0.0389	0.0411
0.4	0.0344	0.0385	0.0408
0.6	0.0344	0.0385	0.0408
0.8	0.0349	0.0389	0.0411
1.0	0.0357	0.0395	0.0417
1.2	0.0367	0.0403	0.0423
1.4	0.0377	0.0410	0.0429
1.6	0.0386	0.0417	0.0435
1.8	0.0394	0.0423	0.0439
2.0	0.0402	0.0429	0.0444
2.2	0.0408	0.0434	0.0448
2.4	0.0415	0.0438	0.0451
2.6	0.0420	0.0442	0.0454
2.8	0.0424	0.0446	0.0457
3.0	0.0429	0.0449	0.0460

where B is given in (32). The numerical calculations can be easily carried out using the computer software, to integrate the function based on the chi-square distribution and the normal distribution. If the p-value is smaller than α, then we conclude that the process meets the capability requirement.

5.3. *An Example of Testing L''_e*

Due to low-power consumption, high reliability, and high brightness, Light Emitting Diode (LED) lamps have many applications in traffic signals, full-color displays, etc. As an illustrative example, we consider an LED manufacturing process. Suppose a customer has told his LED supplier that, in order to quality for business with his company, the supplier must demonstrate that his process capability L''_e is less than 0.05. This problem may be formulated as a hypothesis-testing problem:

$$H_0 : L''_e \geq 0.05 \text{ (incapable)} \quad \text{versus} \quad H_1 : L''_e < 0.05 \text{ (capable)}.$$

In statistical hypothesis testing, rejection of H_0 is always a strong conclusion. The supplier would like to reject H_0, thereby demonstrating that his process is capable. Moreover, he wants to be sure that if the process capability is below 0.05 there will be a high probability of judging the process capable (say, 0.95).

With a focus on the critical characteristic, the luminous intensity of LED sources, we examine a particular LED product model to test whether the production process of LED is capable or not. Historical data based on routine process monitoring shows that the process is under statistical control and the process distribution is shown to be fairly close to the normal distribution. The upper and the lower specification limits of luminous intensity are set to $USL = 40$ mcd, $LSL = 20$ mcd, and the target value is set to $T = 35$ mcd. Some calculations are made to obtain $d = 10$, $D_u = 5$, $D_l = 15$, $d^* = 5$, $d_u = 2$, $d_l = 2/3$. A random sample of size $n = 100$ is taken, and calculated statistics are $\overline{X} = 35.25$, $S_n = 0.3125$, $\widehat{A} = 0.5$, $\hat{a} = 0.8$, and $\widehat{L}''_e = 0.0325$. Using Table 1 based on $n = 100$, we obtain $c_\alpha = 0.0362$. Since the calculated $\widehat{L}''_e \leq 0.0362$, we may claim that the process is capable at the significant level $\alpha = 0.05$.

Alternatively, we obtain the p-value $= 0.015$ via (34). We would conclude that the process meets the capability requirement if α is set to be larger than 0.015. Otherwise, we do not have sufficient information to make a conclusion. We note that in the illustrative example, the estimated off-target loss index $\widehat{L}''_{ot} = 0.02$ which occupies 61.5% of $\widehat{L}''_e$ value, and the estimated inconsistency loss index $\widehat{L}''_{pe} = 0.0125$ which occupies 38.5% of $\widehat{L}''_e$ value. Obviously, it can be seen that the variability is contributed mainly by the process departure in this case.

6. Conclusions

Pearn et al. (2006a) introduced a new generalization of expected loss index L''_e to handle processes with both symmetric and asymmetric tolerances. The relative expected loss $L''_e = L''_{ot} + L''_{pe}$, which provides an uncontaminated separation between information concerning the relative inconsistency loss (L''_{pe}) and the relative off-target loss (L''_{ot}). In this article, we considered the three indices, and derive the sampling distributions of their natural estimators under normality assumption. In addition, the theory of testing statistical hypothesis was used to determine whether a process is capable or not. For illustrative purpose, we demonstrated the use of derived results by presenting a case study on LED manufacturing process, to evaluate the process performance.

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