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Multivariate Capability Indices: Distributional and Inferential Properties

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ABSTRACT *Process capability indices have been widely used in the manufacturing industry for measuring process reproduction capability according to manufacturing specifications. Properties of the univariate processes have been investigated extensively, but are comparatively neglected for multivariate processes where multiple dependent characteristics are involved in quality measurement. In this paper, we consider two commonly used multivariate capability indices* MC_p *and* MC_{pm}*, to evaluate multivariate process capability. We investigate the statistical properties of the estimated* MC_p *and obtain the lower confidence bound for* MC_p*. We also consider testing* MC_p*, and provide critical values for testing if a multivariate process meets the preset capability requirement. In addition, an approximate confidence interval for* MC_{pm} *is derived. A simulation study is conducted to ascertain the accuracy of the approximation. Three examples are presented to illustrate the applicability of the obtained results.*

KEY WORDS: Multivariate capability index, lower confidence bound, hypothesis testing, critical value

Introduction

Process capability indices have been widely used in the manufacturing industry for measuring process reproduction capability according to its manufacturing specifications. In current practice, suppliers are often required to provide their process capability of the product to the customers in the supply chain partnership. Process capability indices can also be used as the benchmarking for quality improvement activities. Capability indices, C_p, C_{pk} and C_{pm}, have been proposed to evaluate process performance but restricted to cases with single engineering specification. A large number of papers have dealt with the statistical properties and the estimation of these univariate indices. Kotz & Lovelace (1998) provided a review of these indices in their textbook. Kotz & Johnson (2002) provided a compact survey and commented on some 170 publications on process capability indices during the years 1992 to 2000.

Process capability is defined to be the range over which the measurements of a process vary when the process variation is due to random causes only. Process capability indices provide an effective measure of process performance. Engineers can realize whether the product

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meets its specifications using process capability indices. In real applications, manufactured products often have multiple quality characteristics. That is, the process capability analysis involves more than one engineering specification. For this reason, multivariate methods for assessing process capability are proposed. Wang *et al.* (2000) reviewed three multivariate methods (Taam *et al.*, 1993; Chen 1994; and Shahriari *et al.*, 1995) and compared their capability for four problems. Although some multivariate capability indices have been proposed, research on the statistical properties of those multivariate capability indices has received little attention. It is needed to investigate the statistical properties for the multivariate process capability indices for practical purposes. In this paper, the statistical properties for the multivariate process indices MC_p and MC_{pm} are investigated. The next section describes the processes with multiple characteristics. The estimator of MC_p and its properties are presented in the third section. Confidence interval, lower confidence bound, and hypothesis testing for MC_p are presented in the fourth section. An approximate confidence interval and lower confidence bound for MC_{pm} and a simulation study to ascertain the accuracy of the approximate confidence interval for MC_{pm} are presented in the fifth section. Three examples are chosen to illustrate the proposed methodology in the sixth section. Conclusions are made in the final section.

Multivariate Capability Indices

Traditionally, process capability is confined to single characteristic of the product. In many cases, a manufactured product is described by more than one characteristic. That is, manufactured items are confined to several different characteristics for adequate description of their quality. Each of those characteristics must satisfy certain specifications. The assessed quality of a product depends on the combined effects of these characteristics, rather than on their individual values. For example, automobile paint needs to have a range of light reflective abilities and a range of adhesion abilities. A paint that satisfies one criterion but not the other may be undesirable. Those characteristics are related through the compositions of the paint. It is therefore only natural to consider a bivariate characterization of this paint.

As for the tolerance region about multiple characteristics, we often take an ellipsoidal region or a rectangular region. For more complex engineering specifications, the tolerance region will be very complicated. For instance, a drawing of a connecting rod in a combustion engine consists of crank-bore inner diameter, pin-bore inner diameter, rod length, bore true-location and so on. In multivariate processes, we usually assume that the observations X have a multivariate normal distribution $N_v(\mu, \Sigma)$, where v is the dimension of variables, μ is the mean vector and Σ represents the variance–covariance matrix of X. In addition, $\boldsymbol{T}$ is the target vector, $\overline{\boldsymbol{X}}$ is the sample mean vector and $\boldsymbol{S}$ is the sample covariance matrix.

We will focus on the multivariate process indices MC_p and MC_{pm} (Taam *et al.*, 1993). The multivariate capability index MC_p is defined as

$$MC_p = \frac{vol.(\text{modified tolerance region})}{vol.[(X-\mu)'\Sigma^{-1}(X-\mu) \le k(q)]} = \frac{vol.(\text{modified tolerance region})}{(\pi\chi^2_{v,0.9973})^{v/2}|\Sigma|^{1/2}[\Gamma(v/2+1)]^{-1}} \tag{1}$$

where $k(q)$ is the 99.73th percentile of the χ^2 distribution with v degrees of freedom, $|\Sigma|$ is the determinant of Σ, and $\Gamma(\cdot)$ is the gamma function. Also the multivariate capability index MC_{pm} is defined in the following, where $D = [1 + (\mu - T)'\Sigma^{-1}(\mu - T)]^{1/2}$.

$$MC_{pm} = \frac{MC_p}{D} \tag{2}$$

Note that, MC_{pm} is less than 1, the process is not close to the specified tolerance region. On the other hand, MC_p is larger than 1, which only indicates that the process variation is smaller than the specified range of variation.

Estimation of MC_p

An estimator of MC_p can be expressed as

$$\hat{M}C_p = \frac{vol.(\text{modified tolerance region})}{vol.(\textit{estimated } 99.73\% \textit{ process region})} = \frac{vol.(\text{modified tolerance region})}{(\pi \chi^2_{v,0.9973})^{v/2}|S|^{1/2}[\Gamma(v/2+1)]^{-1}} \tag{3}$$

where S is the sample variance-covariance matrix and $|S|$ is the determinant of S. According to the equation (1), $\hat{M}C_p$ can be expressed as $MC_p \times (|S|/|\Sigma|)^{-1/2}$. Let $X = (X_1, X_2, \ldots, X_n)'$ represents an n-dimensional vector of measurements from a multivariate normal distribution with mean vector $\mu = (\mu_1, \mu_2, \ldots, \mu_v)'$, T is the vector of target values, Σ is the process variance-covariance matrix. Using the following theorem, we can derive the distribution of $\hat{M}C_p$.

Theorem 1

The distribution of the generalized variance $|S|$ of a sample $X_1, X_2, \ldots, X_n$ from $N_v(\mu, \Sigma)$ is the same as the distribution of $|\Sigma|/(n-1)^v$ times the product of v independent factors, the distribution of the ith factor being the χ^2 distribution with $n-i$ degrees of freedom.

For the proof of Theorem 1, see Anderson (2003, p. 268). From the above theorem, we can have that $|S|/\Sigma$ is the distribution of $\chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}/(n-1)^v$. Let $y = \chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}$. Then, we have

$$\frac{MC_p^2}{\hat{M}C_p^2} \sim \frac{y}{(n-1)^v}$$

Using the transformation method, the probability density function of $\hat{M}C_p$ can be expressed as

$$f(x) = f_Y[g^{-1}(x)] \times \left|\frac{\mathrm{d}}{\mathrm{d}x} g^{-1}(x)\right| = f_Y\left[\frac{MC_p^2(n-1)^v}{x^2}\right] \times \frac{2(n-1)^v MC_p^2}{x^3} \quad \text{for } x > 0 \tag{4}$$

When $v = 1$, the probability density function of $\hat{M}C_p$ is equivalent to the pdf of $\hat{C}_p$. So, we have

$$\frac{C_p^2}{\hat{C}_p^2} \sim \frac{y}{(n-1)} \text{ where } y = \chi^2_{n-1}.$$

From equation (4) and the pdf of y is

$$f_Y(y) = \frac{(1/2)^{(n-1)/2} y^{(n-1)/2-1} e^{-y/2}}{\Gamma(n-1/2)}$$

the pdf of $\hat{C}_p$ is given by

$$f(x) = \frac{(n-1)^{(n-1)/2}}{C_p \Gamma(n-1/2) 2^{(n-3)/2}} \times \left(\frac{C_p}{x}\right) \times e^{-[(n-1)/2]\times(C_p/x)^2} \quad \text{for } x > 0 \tag{5}$$

When $v = 2$, we have

$$\frac{MC_p^2}{\hat{M}C_p^2} \sim \frac{y}{(n-1)^2} \text{ where } y = \chi_{n-1}^2 \times \chi_{n-2}^2.$$

It can be shown that $\chi_{n-1}^2 \times \chi_{n-2}^2 \sim (\chi_{2n-4}^2)^2/4$ (see Corollary 1 in Appendix). Thus, we have

$$\frac{MC_p^2}{\hat{M}C_p^2} \sim \frac{z}{4(n-1)^2} \text{ where } z = (\chi_{2n-4}^2)^2.$$

Using the transformation method, the pdf of z is

$$f(z) = \frac{(1/2)^{(n-1)} z^{n/2-2} e^{-\sqrt{z}/2}}{\Gamma(n-2)} \text{ for } z > 0$$

Similarly, from equation (4) and the pdf of z, the pdf of $\hat{M}C_p$ is given by

$$f(x) = \frac{(n-1)^{(n-2)}}{MC_p\Gamma(n-2)} \times \left(\frac{MC_p}{x}\right)^{n-1} \times e^{-(n-1)\times(MC_p/x)} \quad \text{for } x > 0 \tag{6}$$

When $v = 3$, we have

$$\frac{MC_p^2}{\hat{M}}C_p^2 \sim \frac{y}{(n-1)^3} \quad \text{where } y = \chi_{n-1}^2 \times \chi_{n-2}^2 \times \chi_{n-3}^2.$$

Let $z_1 = \chi_{n-1}^2 \times \chi_{n-2}^2 \sim (\chi_{2n-4}^2)^2/4$ and $z_2 = \chi_{n-3}^2$. Then, we have

$$\frac{MC_p^2}{\hat{M}C_p^2} \sim \frac{z_1 z_2}{(n-1)^3}$$

Let $u = z_1 \times z_2$ and $v = z_1$. Using the transformation method, the joint pdf of $u \times v$ is

$$f(u, v) = \frac{(1/2)^{(n-1)/2} \times v^{-1/2} \times u^{(n-5)/2} \times e^{-(\sqrt{v}+u/2v)}}{\Gamma(n-2)\Gamma(n-3/2)} \quad \text{for } u, v > 0$$

Then, the pdf of u is given by

$$f_U(u) = \int_0^\infty \frac{(1/2)^{(n-1)/2} \times v^{-1/2} \times u^{(n-5)/2} \times e^{-(\sqrt{v}+u/2v)}}{\Gamma(n-2)\Gamma(n-3/2)} \mathrm{d}v \quad \text{for } u > 0$$

Similarly, from equation (4) and the pdf of u, the pdf of $\hat{M}C_p$ is given by

$$f(x) = \frac{(n-1)^3 \times (1/2)^{(n/2)-(3/2)}}{MC_p\Gamma(n-2)\Gamma(n-3/2)} \times \left(\frac{MC_p}{x}\right)^{n-2} \times \int_0^\infty v^{-1/2} \times e^{-[\sqrt{v}+(n-1)^3(MC_p/x)^2/(2v)]} dv \text{ for } x > 0 \tag{7}$$

Since the hth moment of a χ^2 distribution with v degrees of freedom is $2^h\Gamma(v/2 + h)/\Gamma(v/2)$ and the moment of a product of independent variables is the product of the moments of the variables, the hth moment of $|S|/|\Sigma|$ can be obtained as

$$E(|S|/|\Sigma|)^h = \frac{2^{vh}\prod_{i=1}^{v}\Gamma[1/2(n-i)+h]}{(n-1)^{vh}\prod_{i=1}^{v}\Gamma[1/2(n-i)]} \tag{8}$$

Now we can derive the rth moment of $\hat{M}C_p$ according to the equation (8) and $\hat{M}C_p = MC_p \times (|S|/\Sigma)^{-1/2}$. Thus, we have

$$E(\hat{M}C_p^r) = E(MC_p^r \times (|S|/\Sigma)^{-r/2}) = MC_p^r \times E(|S|/\Sigma)^{-r/2} \tag{9}$$

Now, we can substitute $r = 1$ and $h = -1/2$ into the equations (8) and (9), respectively. Then, we have

$$\begin{aligned} E(\hat{M}C_p) &= MC_p \times \frac{2^{-v/2}\prod_{i=1}^{v}\Gamma[1/2(n-i)-1/2]}{(n-1)^{-v/2}\prod_{i=1}^{v}\Gamma[1/2(n-i)]} \\ &= MC_p \times \left(\frac{n-1}{2}\right)^{v/2}\frac{\Gamma[1/2(n-v)-1/2]}{\Gamma[1/2(n-1)]} = \frac{1}{b_v} \times MC_p \end{aligned} \tag{10}$$

where $b_v = \left(\frac{2}{n-1}\right)^{v/2}\Gamma[1/2(n-1)]\,\Gamma[1/2(n-v)-1/2]$ is a correction factor, so that $b_v \times \hat{M}C_p$ is an unbiased estimator of MC_p. Again, we can substitute $r = 2$ and $h = -1$ into the equations (8) and (9), respectively. Then, we have

$$E(\hat{M}C_p^2) = MC_p^2 \times \frac{2^{-v}\prod_{i=1}^{v}\Gamma[1/2(n-i)-1]}{(n-1)^{-v}\prod_{i=1}^{v}\Gamma[1/2(n-i)]} \tag{11}$$

From the equations (10) and (11), we have the variance of $\hat{M}C_p$ as

$$\text{Var}(\hat{M}C_p) = MC_p^2 \times \frac{2^{-v}\prod_{i=1}^{v}\Gamma[1/2(n-i)-1]}{(n-1)^{-v}\prod_{i=1}^{v}\Gamma[1/2(n-i)]} - \left[\frac{1}{b_v} \times MC_p\right]^2 \tag{12}$$

When $v = 1$, according to the equations (10) and (12), we can find that the expectation and variance of $\hat{M}C_p$ are equal to those of $\hat{C}_p$ (see Kotz & Lovelace, 1998). That is, the expectation and variance of $\hat{C}_p$ can be obtained as

$$E(\hat{C}_p) = MC_p \times \sqrt{\frac{n-1}{2}}\frac{\Gamma[1/2(n-2)]}{\Gamma[1/2(n-1)]} = \frac{1}{b_1} \times C_p$$

and

$$Var(\hat{C}_p) = \left[\frac{n-1}{n-3} - \frac{1}{b_1^2}\right] \times C_p^2$$

where $b_1 = \sqrt{2/(n-1)}\,[\Gamma[(1/2)(n-1)]/\Gamma[(1/2(n-2))]]$ is a correction factor, so that $b_1 \times \hat{C}_p$ is an unbiased estimator of C_p. When $v = 2$, according to the equations (10) and

(12), the expectation and variance of $\hat{M}C_p$ can be obtained as

$$E(\hat{M}C_p) = MC_p \times \frac{n-1}{n-3} = \frac{1}{b_2} \times MC_p$$

and

$$\mathrm{Var}(\hat{M}C_p) = MC_p^2 \times \frac{(n-1)^2}{(n-3)^2(n-4)} = \frac{1}{n-4} \times \frac{1}{b_2^2} \times MC_p^2$$

where $b_2 = (n-3)/(n-1)$ is a correction factor, so that $b_2 \times \hat{M}C_p$ is an unbiased estimator of MC_p. When $v = 3$, according to the equations (10) and (12), the expectation and variance of $\hat{M}C_p$ can be obtained as

$$E(\hat{M}C_p) = MC_p \times \left(\frac{n-1}{2}\right)^{3/2} \times \frac{\Gamma\,[1/2(n-3) - 1/2]}{\Gamma\,[1/2(n-1)]} = \frac{1}{b_3} \times MC_p$$

and

$$\begin{aligned}\mathrm{Var}(\hat{M}C_p) &= MC_p^2 \times \left[\frac{(n-1)^3}{(n-3)(n-4)(n-5)} - \frac{(n-1)^3}{2(n-3)^2} \times \frac{\Gamma^2\,(1/2n-2)}{\Gamma^2\,(1/2n-3/2)}\right] \\ &= \left[\frac{(n-1)^3}{(n-3)(n-4)(n-5)} - \frac{1}{b_3^2}\right] \times MC_p^2\end{aligned}$$

where $b_3 = (n-3)/(n-1) \times \sqrt{2/(n-1)} \times (\Gamma(1/2n - \frac{3}{2}))/(\Gamma(\frac{1}{2}n-2))$ is a correction factor, so that $b_3 \times \hat{M}C_p$ is an unbiased estimator of MC_p.

Confidence Intervals and Hypothesis Testing for MC_p

Confidence Interval and Lower Confidence Bound for MC_p

Since $\hat{M}C_p = MC_p \times (|S|/|\Sigma|)^{-1/2}$, like other statistics, is subject to the sampling variation, it is critical to compute an interval to provide a range that includes the true MC_p with high probability. Based on the definition, a $100(1-\alpha)\%$ confidence interval for MC_p can be established as:

$$\begin{aligned}&P\{L \le MC_p \le U\} = 1-\alpha \longrightarrow P\{L \le \hat{M}C_p \times \left(\frac{|S|}{|\Sigma|}\right)^{1/2} \le U\} = 1-\alpha \\ &\longrightarrow P\left\{\frac{L}{\hat{M}C_p} \le \left(\frac{|S|}{|\Sigma|}\right)^{1/2} \le \frac{U}{\hat{M}C_p}\right\} = 1-\alpha \\ &\longrightarrow P\left\{\frac{L}{\hat{M}C_p} \le \sqrt{\frac{\chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}}{(n-1)^v}} \le \frac{U}{\hat{M}C_p}\right\} = 1-\alpha \\ &\longrightarrow P\left\{\frac{L^2(n-1)^v}{\hat{M}C_p^2} \le \chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v} \le \frac{U^2(n-1)^v}{\hat{M}C_p^2}\right\} \\ &= 1-\alpha\end{aligned}$$

Let $y = \chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}$, then we have $\int_{L^2(n-1)^v/\hat{M}C_p^2}^{U^2(n-1)^v/\hat{M}C_p^2} f_Y(y)\mathrm{d}y = 1-\alpha$

So, we have
$F_Y^{-1}(\alpha/2) = (L^2(n-1)^v)/\hat{M}C_p^2$ and $F_Y^{-1}(1-\alpha/2) = (U^2(n-1)^v)/\hat{M}C_p^2$, where $F_Y(z) = \int_0^z f(y)\mathrm{d}y$.

Thus, a $100(1-\alpha)\%$ confidence interval for MC_p can be obtained as:

$$\left[\hat{M}C_p\sqrt{\frac{F_Y^{-1}(\alpha/2)}{(n-1)^v}}, \hat{M}C_p\sqrt{\frac{F_Y^{-1}(1-\alpha/2)}{(n-1)^v}}\right] \tag{13}$$

Furthermore, a $100(1-\alpha)\%$ lower confidence bound for MC_p can be obtained as

$$\left[\hat{M}C_p\sqrt{\frac{F_Y^{-1}(\alpha)}{(n-1)^v}}\right] \tag{14}$$

When $v = 1$, a $100(1-\alpha)\%$ confidence interval and lower confidence bound for C_p are given by

$$\left[\hat{C}_p\sqrt{\frac{\chi^2_{n-1,\,\alpha/2}}{(n-1)}}, \hat{C}_p\sqrt{\frac{\chi^2_{n-1,\,1-\alpha/2}}{(n-1)}}\right] \text{and} \left[\hat{C}_p\sqrt{\frac{\chi^2_{n-1,\,\alpha}}{(n-1)}}\right] \tag{15}$$

When $v = 2$, we can find that the distribution of $\chi^2_{n-1} \times \chi^2_{n-2}$ is equal to $(\chi^2_{2n-4})^2/4$. Thus, a $100(1-\alpha)\%$ confidence interval and lower confidence bound for MC_p are given by

$$\left[\hat{M}C_p\sqrt{\frac{(\chi^2_{2n-4,\alpha/2})^2}{4\times(n-1)^2}}, \hat{M}C_p\sqrt{\frac{(\chi^2_{2n-4,1-\alpha/2})^2}{4\times(n-1)^2}}\right] \text{and} \left[\hat{M}C_p\sqrt{\frac{(\chi^2_{2n-4,\alpha})^2}{4\times(n-1)^2}}\right] \tag{16}$$

When $v = 3$, we can find that the distribution of $\chi^2_{n-1} \times \chi^2_{n-2} \times \chi^2_{n-3}$ can be expressed as $y = (\chi^2_{2n-4})^2/4 \times \chi^2_{n-3}$.

Now, let $y_1 \sim (\chi^2_{2n-4})^2/4$ and $y_2 \sim \chi^2_{n-3}$. So, we have $y = y_1 y_2$. Let $w = y_1$. Using the transformation method, the probability density function of y is given by

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_{y_1,y_2}(y_1, y_2)\mathrm{d}y_1 \\ &= \int_0^\infty \frac{(1/2)^{(n-1)/2} \times y_1^{-1/2} \times y_2^{(n-5)/2} \times e^{-y_1^{1/2}-(y_2/2y_1)}}{\Gamma(n-2)\times\Gamma[(n-3)/2]}\mathrm{d}y_1, 0 \le y < \infty \end{aligned}$$

Thus, a $100(1-\alpha)\%$ confidence interval and lower confidence bound for MC_p are given by

$$\left[\hat{M}C_p\sqrt{\frac{F_Y^{-1}(\alpha/2)}{(n-1)^3}}, \hat{M}C_p\sqrt{\frac{F_Y^{-1}(1-\alpha/2)}{(n-1)^3}}\right] \text{and} \left[\hat{M}C_p\sqrt{\frac{F_Y^{-1}(\alpha)}{(n-1)^3}}\right] \tag{17}$$

where

$$F_Y(y) = \int_0^y \int_0^\infty \frac{(1/2)^{(n-1)/2} \times x^{-1/2} \times y^{(n-5)/2} \times e^{-x^{1/2}-(y/2x)}}{\Gamma(n-2)\times\Gamma[(n-3)/2]}\mathrm{d}x\,\mathrm{d}y$$

Efficient Mathematica programs are developed to obtain equation (17). These programs are available by sending an e-mail request to Wang.

Hypothesis Testing for MC_p

In hypothesis testing, we determine whether or not a hypothesized value of a parameter is true or not, based on the sample taken and the parameter estimate derived from it. That is, we are trying to find out where the estimated capability is relative to either true capability, hypothesized capability, or how different the estimated and true capabilities are. To do this, we estimate an index value, compare it to a lower bound c_0, and compute the so-called *p-value*. The quantity p refers to the actual risk of incorrectly concluding that the process is capable for a particular test. In general, we want *p-value* to be no greater than 0.05. To test whether a given process is capable, we may consider the following statistical hypothesis testing:

$$\mathrm{H}_0 : MC_p \leqq c_0 \text{(process is not capable)}$$
$$\mathrm{H}_1 : MC_p > c_0 \text{(process is capable)}$$

where c_0 is the standard minimal criteria for MC_p. The critical value, c, can be determined as:

$$P\left\{\hat{M}C_p > c|MC_p = c_0\right\} = \alpha \longrightarrow P\left\{\frac{MC_p}{\sqrt{(\chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v})/(n-1)^v}} > c|MC_p = c_0\right\} = \alpha.$$

Let $y = \chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}$, then we have

$$P\left\{\frac{c_0}{\sqrt{y/(n-1)^v}} > c\right\} = \alpha \longrightarrow P\left\{\frac{c_0^2}{y/(n-1)^v} > c^2\right\}$$
$$= \alpha \longrightarrow P\left\{y < \frac{(n-1)^v c_0^2}{c^2}\right\} = \alpha \longrightarrow \int_0^{(n-1)^v c_0^2/c^2} f_Y(y)\mathrm{d}y$$
$$= \alpha \longrightarrow F_Y^{-1}(\alpha) = \frac{(n-1)^v c_0^2}{c^2}.$$

Thus, the critical value can be expressed as

$$c = c_0\sqrt{\frac{(n-1)^v}{F_Y^{-1}(\alpha)}} \tag{18}$$

When $v = 1$ and $y = \chi^2_{n-1}$, the critical value is equal to

$$c_0\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}.$$

When $v = 2$ and $y = \chi^2_{n-1} \times \chi^2_{n-2} \sim (\chi^2_{2n-4})^2/4$, the critical value is equal to

$$c_0\sqrt{\frac{4(n-1)^2}{(\chi^2_{2n-4,\alpha})^2}}.$$

When $v = 3$ and $y = \chi^2_{n-1} \times \chi^2_{n-2} \times \chi^2_{n-3}$, the critical value is equal to

$$c_0\sqrt{\frac{(n-1)^3}{F_Y^{-1}(\alpha)}},\ \text{where } F_Y(y) = \int_0^y\int_0^\infty \frac{(1/2)^{(n-1)/2} \times x^{(-1/2)} \times y^{(n-5)/2} \times e^{-x^{1/2}-y/2x}}{\Gamma(n-2)\times\Gamma((n-3)/2)}\mathrm{d}x\mathrm{d}y$$

From the definition of c, it is obvious that the value of $\hat{M}C_p$ must be higher than the original target value for the true MC_p. The amount of difference required depends on the sample size, n. The power of the test, β, is given by

$$\begin{aligned}\beta(MC_p) &= P\{\hat{M}C_p > c|MC_p\}\\ &= P\left\{\frac{MC_p}{\sqrt{(\chi^2_{n-1}\times\chi^2_{n-2}\times\cdots\times\chi^2_{n-v})/(n-1)^v}} > c|MC_p\right\}\\ &= P\left\{y < \frac{(n-1)^v MC_p^2}{c^2}|MC_p\right\} \end{aligned} \tag{19}$$

where y is the probability density function of $\chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}$. From the above equation, we can obtain the operating characteristic (OC) curve for MC_p, which plots the true value of $1 - \beta(MC_p)$ against MC_p for two situations: (1) $n = 30, c = 1.33$; (2) $n = 70$, $c = 1.46$. When $v = 1, 2$ and 3, several operating characteristic (OC) curves for MC_p are shown in Figures 1–5. From these operating characteristic curves, we found that some interesting results are as follows.

(i) From these graphs, it is obvious that when the c_0 is larger, the chance of incorrectly concluding the process is not capable is smaller.
(ii) When n and c of the two are the same, then their OC curves are similar regardless of $v = 1$, $v = 2$ or $v = 3$.
(iii) When c is smaller, the chance of incorrectly concluding the process is not capable will be smaller.

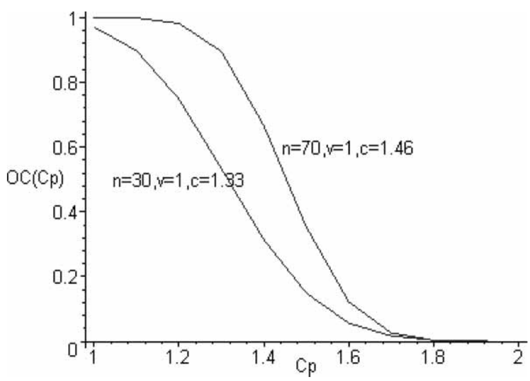

Figure 1. Operating characteristic curve for MC_p when $v = 1$

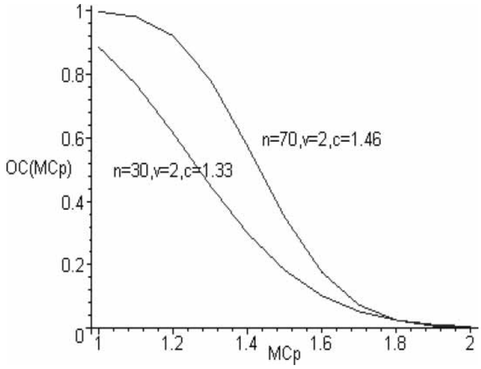

Figure 2. Operating characteristic curve for MC_p when $v = 2$

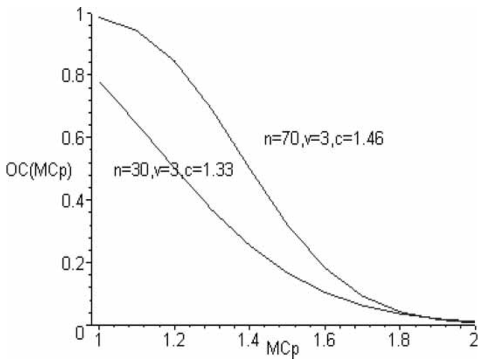

Figure 3. Operating characteristic curve for MC_p when $v = 3$

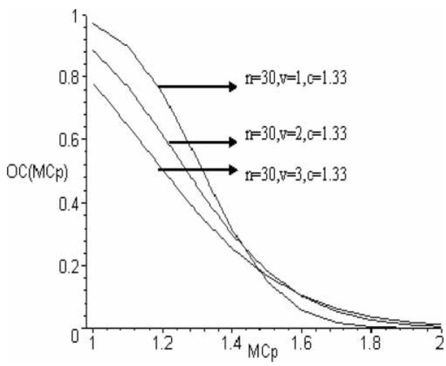

Figure 4. Operating characteristic curve for MC_p when $n = 30$

Approximate Confidence Intervals for MC_{pm}

An Approximate Confidence Interval and Lower Confidence Bound for MC_{pm}

An estimator of MC_{pm} can be expressed as

$$\hat{M}C_{pm} = \frac{\hat{M}C_p}{\hat{D}} = \frac{MC_p \times (|S|/|\Sigma|)^{-1/2}}{\hat{D}} = \frac{MC_p \times D \times (|S|/|\Sigma|)^{-1/2}}{D \times \hat{D}}$$
$$= MC_{pm} \times \frac{D}{\hat{D}} \times \left(\frac{|S|}{|\Sigma|}\right)^{-1/2} \tag{20}$$

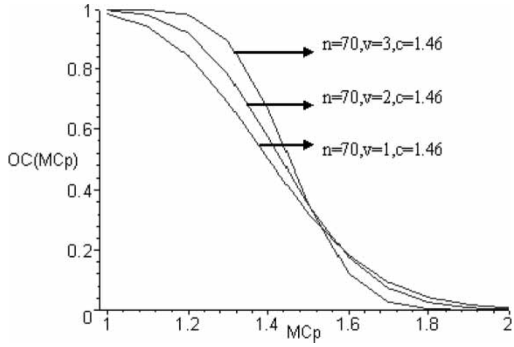

Figure 5. Operating characteristic curve for MC_p when $n = 70$

where

$$\hat{D} = \left[1 + \left(\frac{n}{n-1}\right)(\overline{X} - T)'S^{-1}(\overline{X} - T)\right]^{1/2}$$

Since

$$\hat{M}C_{pm} = MC_{pm} \times \frac{D}{\hat{D}} \times \left(\frac{|S|}{|\Sigma|}\right)^{-1/2} \quad \text{and} \quad \frac{|S|}{|\Sigma|} \sim \frac{\chi^2_{n-1}\chi^2_{n-2}\cdots\chi^2_{n-v}}{(n-1)^v}$$

we have

$$\left(MC_{pm}/\hat{M}C_{pm}\right) \times D = \hat{D} \times \left(\frac{|S|}{|\Sigma|}\right)^{1/2}$$

$$\Longrightarrow \left(MC_{pm}/\hat{M}C_{pm}\right)^2 \times D^2 = \hat{D}^2 \times \left(\frac{|S|}{|\Sigma|}\right) = \hat{D}^2 \times \frac{\chi^2_{n-1}\chi^2_{n-2}\cdots\chi^2_{n-v}}{(n-1)^v}$$

where

$$D^2 = [1 + (\mu - T)'\Sigma^{-1}(\mu - T)], \quad \hat{D}^2 = \left[1 + \left(\frac{n}{n-1}\right)(\overline{X} - T)'S^{-1}(\overline{X} - T)\right]$$

Based on the definition, a $100(1-\alpha)\%$ confidence interval forMC_{pm} is given as follows:

$$P\left\{L \le MC_{pm} \le U\right\} = 1 - \alpha \longrightarrow P\left\{\left(\frac{L}{\hat{M}C_{pm}}\right)^2 D^2 \le \left(\frac{MC_{pm}}{\hat{M}C_{pm}}\right)^2 D^2 \le \left(\frac{U}{\hat{M}C_{pm}}\right)^2 D^2\right\} = 1 - \alpha$$

$$\longrightarrow P\left\{\left(\frac{L}{\hat{M}C_{pm}}\right)^2 D^2 \le \hat{D}^2 \times \left(\frac{\chi^2_{n-1}\chi^2_{n-2}\cdots\chi^2_{n-v}}{(n-1)^v}\right) \le \left(\frac{U}{\hat{M}C_{pm}}\right)^2 D^2\right\} = 1 - \alpha$$

$$\longrightarrow P\left\{\left(\frac{L}{\hat{M}C_{pm}}\right)^2 D^2(n-1)^v \le \hat{D}^2 \times \chi^2_{n-1}\chi^2_{n-2}\cdots\chi^2_{n-v} \le \left(\frac{U}{\hat{M}C_{pm}}\right)^2 D^2(n-1)^v\right\} = 1-\alpha$$

Let $z = \hat{D}^2 \times \chi^2_{n-1}\chi^2_{n-2}\ldots\chi^2_{n-v}$. The above equation can be rewritten as

$$\int_{L^2(n-1)^v D^2/\hat{M}C^2_{pm}}^{U^2(n-1)^v D^2/\hat{M}C^2_{pm}} f_Z(z)\mathrm{d}z = 1-\alpha.$$

So, we have

$$F_Z^{-1}(\alpha/2) = \frac{L^2(n-1)^v D^2}{\hat{M}C^2_{pm}} \text{ and } F_Z^{-1}(1-\alpha/2) = \frac{U^2(n-1)^v D^2}{\hat{M}C^2_{pm}}$$

where $F_Z(z) = \int_0^z f_Z(z)\mathrm{d}z$. Thus, a $100(1-\alpha)\%$ confidence interval and lower confidence bound for MC_{pm} are given by

$$\left[\hat{M}C_{pm}\sqrt{\frac{F_Z^{-1}(\alpha/2)}{(n-1)^v D^2}}, \hat{M}C_{pm}\sqrt{\frac{F_Z^{-1}(1-\alpha/2)}{(n-1)^v D^2}}\right] \text{ and } \left[\hat{M}C_{pm}\sqrt{\frac{F_Z^{-1}(\alpha)}{(n-1)^v D^2}}\right] \quad (21)$$

In fact, D and τ^2 are unknown values, we can use $\hat{D}$ and $\hat{\tau}^2$ to estimate the values of D and τ^2, where

$$\hat{D} = \left[1 + \left(\frac{n}{n-1}\right)(\overline{X}-T)'S^{-1}(\overline{X}-T)\right]^{1/2} \text{ and } \hat{\tau}^2 = n(\overline{X}-T)'S^{-1}(\overline{X}-T)$$

Thus, an approximate $100(1-\alpha)\%$ confidence interval and lower confidence bound for MC_{pm} are given by

$$\left[\hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha/2)}{(n-1)^v \hat{D}^2}}, \hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(1-\alpha/2)}{(n-1)^v \hat{D}^2}}\right] \text{ and } \left[\hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha)}{(n-1)^v \hat{D}^2}}\right] \quad (22)$$

From Corollary 2 in the Appendix, the pdf of z can be found. When $v = 2$, an approximate $100(1-\alpha)\%$ confidence interval and lower confidence bound for MC_{pm} are given by

$$\left[\hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha/2)}{(n-1)^2 \hat{D}^2}}, \hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(1-\alpha/2)}{(n-1)^2 \hat{D}^2}}\right] \text{ and } \left[\hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha)}{(n-1)^2 \hat{D}^2}}\right] \quad (23)$$

where

$$\hat{F}_Z(z) = \int_0^z \int_1^\infty \frac{1/2e^{(-1/2(\hat{\tau}^2))}}{x\Gamma(n-2)\Gamma[(n-2)/2]}(z/x)^{(n-4)/2}\mathrm{e}^{-\sqrt{z/x}} \times \sum_{i=0}^\infty \frac{(\hat{\tau}^2/2)^i(x-1)^i\Gamma(n/2+i)}{i!\Gamma(i+1)x^{1/2n+i}}\mathrm{d}x\mathrm{d}z \quad \text{for } z \ge 0$$

When $v = 3$, an approximate $100(1-\alpha)\%$ confidence interval and lower confidence bound for MC_{pm} are given by

$$\left[\hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha/2)}{(n-1)^3\hat{D}^2}},\ \hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(1-\alpha/2)}{(n-1)^3\hat{D}^2}}\right] \text{ and } \left[\hat{M}C_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha)}{(n-1)^3\hat{D}^2}}\right] \quad (24)$$

where

$$\hat{F}_Z(z) = \int_0^z \int_1^\infty \int_0^\infty \frac{(1/2)^{(n-1)/2} x^{-1/2} e^{-\sqrt{x}-z/(2wx)}}{\Gamma(n-2)\Gamma[(n-3)/2]} \frac{e^{(-(1/2)\hat{\tau})^2}(z/w)^{(n-5)/2}}{w\Gamma[(n-3)/2]} \times \sum_{i=0}^{\infty} \frac{(\hat{\tau}^2/2)^i (w-1)^{i+1/2}\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{n/2+i}} dx\,dw\,dz$$

Simulation Study

In order to ascertain the performance of the confidence interval and the lower confidence bound in equation (23), a simulation study was conducted. In this study, random samples of size 25, 45 and 65 were generated from the multivariate normal distribution with a plethora of combinations of μ, Σ and MC_{pm}. The specification limits were assumed to be, without loss of generality, $\text{LSL}_1 = 10$, $\text{T}_1 = 13$, $\text{USL}_1 = 16$, $\text{LSL}_2 = 12$, $\text{T}_2 = 13$, $\text{USL}_2 = 14$. For each combination, 1000 random samples were generated and, for each of these samples, the corresponding confidence intervals and lower confidence bound were assessed. The proportion of times that each of these limits contains the actual value of the index was recorded. The frequency of coverage for the limit is a binomial random variable with $p = 0.95$ and $N = 1000$. Thus, a 99% confidence interval for the coverage proportion is $0.95 \pm 2.576\sqrt{0.95 \times 0.05/1000}$. Hence, the limits from 0.932 to 0.968 are the critical values for a statistical test at the 99% confidence level of the hypothesis that the $p =$ value is 0.95.

The obtained results are summarized in Table 1. More specifically, Table 1 presents that the observed coverage of 95% confidence interval as well as the observed coverage of the lower confidence bound are within the nominal interval at 99% confidence level. Thus, we can ascertain the performance of the confidence interval and the lower confidence bound in equation (23).

Illustration Examples

Three examples were chosen to illustrate the proposed methodology. The first example was used to illustrate a process with two variables. The other two examples were used to illustrate a process with three variables.

Example 1

Chen (1994) discussed a bivariate normal example and employed Sultan (1986) bivariate process data ($n = 25$). Of particular interest were the Brinell hardness (H) and the tensile strength (S) of a process. The specification limits for H and S were set at (112.7, 241.3) and (32.7, 73.3), respectively. The center of the specifications was $\mu_0^T = [177, 53]$. The sample

Table 1. Observed coverage of 95% confidence limits

μ	Σ	MC_{pm}	$n=25$		$n=45$		$n=65$	
			CI	LCB	CI	LCB	CI	LCB
$\begin{bmatrix}13\\13\end{bmatrix}$	$\begin{bmatrix}\frac{5}{0.75\chi^2_{2,0.9973}} & \frac{4}{0.75\chi^2_{2,0.9973}}\\ \frac{4}{0.75\chi^2_{2,0.9973}} & \frac{5}{0.75\chi^2_{2,0.9973}}\end{bmatrix}$	0.75	0.968	0.959	0.965	0.969	0.952	0.952
$\begin{bmatrix}13\\13\end{bmatrix}$	$\begin{bmatrix}\frac{5}{\chi^2_{2,0.9973}} & \frac{4}{\chi^2_{2,0.9973}}\\ \frac{4}{\chi^2_{2,0.9973}} & \frac{5}{\chi^2_{2,0.9973}}\end{bmatrix}$	1	0.968	0.959	0.966	0.970	0.952	0.952
$\begin{bmatrix}13\\13\end{bmatrix}$	$\begin{bmatrix}\frac{5}{1.25\chi^2_{2,0.9973}} & \frac{4}{1.25\chi^2_{2,0.9973}}\\ \frac{4}{1.25\chi^2_{2,0.9973}} & \frac{5}{1.25\chi^2_{2,0.9973}}\end{bmatrix}$	1.25	0.968	0.959	0.966	0.970	0.952	0.952
$\begin{bmatrix}13\\13\end{bmatrix}$	$\begin{bmatrix}\frac{5}{1.5\chi^2_{2,0.9973}} & \frac{4}{1.5\chi^2_{2,0.9973}}\\ \frac{4}{1.5\chi^2_{2,0.9973}} & \frac{5}{1.5\chi^2_{2,0.9973}}\end{bmatrix}$	1.5	0.968	0.959	0.966	0.970	0.952	0.952

Note: CI = confidence interval; LCB = lower confidence bound.

mean vector and sample covariance matrix were

$$\overline{X}^T = [177.2, 52.32] \text{ and } S = \begin{bmatrix}337.8000 & 85.3308\\ 85.3308 & 33.6247\end{bmatrix}$$

Using the discussion in the second and third sections, we can derive the estimated value, expectation, variance, confidence interval, lower confidence bound and critical value for MC_p. From the data, we have $\chi^2_{2,0.9973} = 11.829$ and $|S| = 4077.0782$. Then, we have

$$\hat{M}C_p = \frac{\pi \times [(241.3 - 112.7)/2] \times [(73.3 - 32.7)/2]}{\pi \times |S|^{1/2}\chi^2_{2,0.9973}} = 1.7282$$

Since $v = 2$ and $n = 25$, the expectation and variance of $\hat{M}C_p$ can be calculated as $(25/22) \times MC_p$, $(48/847) \times MC_p^2$, respectively. According to the equation (16), a 95% confidence interval for MC_p is calculated as

$$\left[1.7282\sqrt{\frac{(\chi^2_{46,0.025})^2}{4 \times 24^2}}, 1.7282\sqrt{\frac{(\chi^2_{46,0.975})^2}{4 \times 24^2}}\right] = [1.0499, 2.3985]$$

In addition, a 95% lower confidence bound for MC_p is

$$1.7282\sqrt{\frac{(\chi^2_{46,0.05})^2}{4 \times 24^2}} = 1.1319$$

To judge whether this process meets the present capability requirement, we consider a statistical hypothesis testing for MC_p: H$_0$: $MC_p \leqq 1$ versus H$_1$: $MC_p > 1$. According to equation (18), the critical value is obtained as

$$c = 1 \times \sqrt{\frac{4(25-1)^2}{(\chi^2_{46,0.05})^2}} = 1.5268$$

Since $\hat{M}C_p = 1.7282 > 1.5268$, we can conclude that the MC_p is larger than 1 at 95% confidence level. It implies that this process variation is smaller than the specified range of variation. From the data, we have $\hat{D} = 1.0228$ and $\hat{\tau}^2 = 1.1084$. Thus, the $\hat{M}C_{pm}$ can be calculated as $1.7282/1.0228 = 1.6896$. According to equation (23), an approximate 95% confidence interval for MC_{pm} is calculated as

$$\left[1.6896\sqrt{\frac{236.417}{(24)^2 \times 1.0228^2}},\, 1.6896\sqrt{\frac{1320.09}{(24)^2 \times 1.0228^2}}\right] = [1.0583, 2.5008]$$

Also, according to equation (23), an approximate 95% lower confidence bound for MC_{pm} is

$$1.6896\sqrt{\frac{275.491}{(24)^2 \times 1.0228^2}} = 1.1424$$

Therefore, we can conclude that the MC_{pm} is larger than 1 at 95% confidence level. It implies that this process is close to the specified target.

Example 2

A previous study (Wang & Chen, 1998) presented a trivariate quality control involving the joint control of the depth (D), the length (L), and the width (W) of a plastic product from a multivariate normality. Fifty observations were collected from a plastic production line. The specified limits for D, L, and W were set at [2.1, 2.3], [304.5, 305.1] and [304.5, 305.1], respectively. The 3D tolerance region is illustrated in Figure 6. The specification of the target value is $T^T = [2.2, 304.8, 304, 8]$.

The sample mean vector and sample covariance matrix for 50 observations were

$$\overline{X}^T = [2.16, 304.72, 304.77] \text{ and } S = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}$$

From the data, we have $\chi^2_{3,0.9973} = 14.1563$ and $|S| = 0.3347 \times 10^{-8}$. Then we have

$$\hat{M}C_p = \frac{4/3\pi \times (0.2/2) \times (0.6/2) \times (0.6/2)}{|S|^{1/2}(\pi \times \chi^2_{3,0.9973})^{3/2}[\Gamma(2.5)]^{-1}} = 2.9208$$

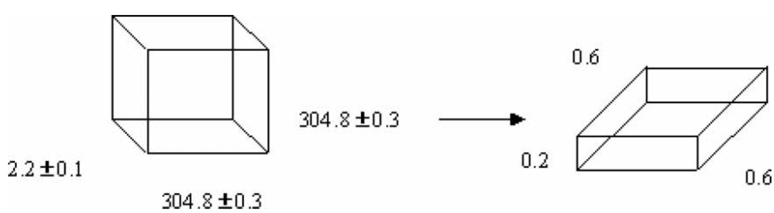

Figure 6. Tolerance region for example 2

Since $v = 3$ and $n = 50$, the expectation of $\hat{M}C_p$ can be calculated as $1.0819 \times MC_p$ and $0.038805 \times MC_p^2$, respectively. According to equation (17), a 95% confidence interval for MC_p is calculated as

$$\left[2.9208\sqrt{\frac{50504.6}{49^3}}, 2.9208\sqrt{\frac{204926}{49^3}}\right] = [1.9136, 3.8547]$$

In addition, a 95% lower confidence bound is

$$2.9208\sqrt{\frac{56994.6}{49^3}} = 2.0329$$

To judge whether this process meets the present capability requirement, we consider a statistical hypothesis testing for MC_p: H_0: $MC_p \leqq 1$ versus H_1: $MC_p > 1$. According to the equation (18), the critical values

$$c = 1 \times \sqrt{\frac{49^3}{56994.8}} = 1.4367$$

Since $\hat{M}C_p = 2.9208 > 1.4367$, we can conclude that the MC_p is larger than 1 at 95% confidence level. It implies that this process variation is smaller than the specified range of variation. From the data, we have $\hat{D} = 2.3408$ and $\hat{\tau}^2 = 219.495$. Thus, the $\hat{M}C_{pm}$ can be calculated as $2.9208/2.3408 = 1.2478$. According to equation (24), an approximate 95% confidence interval for MC_{pm} is calculated as

$$\left[1.2478\sqrt{\frac{266563}{(49)^3 \times 2.3408^2}}, 1.2478\sqrt{\frac{1341790}{(49)^3 \times 2.3408^2}}\right] = [0.8024, 1.8002]$$

Also, according to equation (24), an approximate 95% lower confidence bound for MC_{pm} is

$$1.2478\sqrt{\frac{304913}{(49)^3 \times 2.3408^2}} = 0.8582$$

Therefore, we can conclude that the lower bound for MC_{pm} is not larger than 1 at 95% confidence level. It implies that this process is not close to the specified target.

Example 3

Taam *et al.* (1993) and Karl *et al.* (1994) discussed a geometric dimensioning and tolerancing (GD&T) drawing that specifies a target value for a pin diameter corresponding to the midpoint of allowable pin sizes and allowable perpendicularity of the pin depending on its size. The specifications require a pin diameter between 9 and 11 tenths of an inch (all units in tenths of an inch) and the center line of the pin to be within a cylinder of diameter 0.5 at maximum material condition (MMC, i.e., maximum pin diameter), increasing to a cylinder 2.5 diameter at Least Material Condition (LMC, i.e. minimum pin diameter). Therefore, the tolerance of perpendicularity depends on the pin diameter: for a pin with a diameter of 9, the allowable perpendicular tolerance zone is a 2.5 diameter cylinder, whereas for a pin diameter of 11, the allowable perpendicular tolerance is a 0.5 diameter cylinder. A pin meeting this specification will fit a gage with an 11.5 diameter hole. These GD&T specifications result in a three-dimensional tolerance region in the shape of a frustum, as illustrated

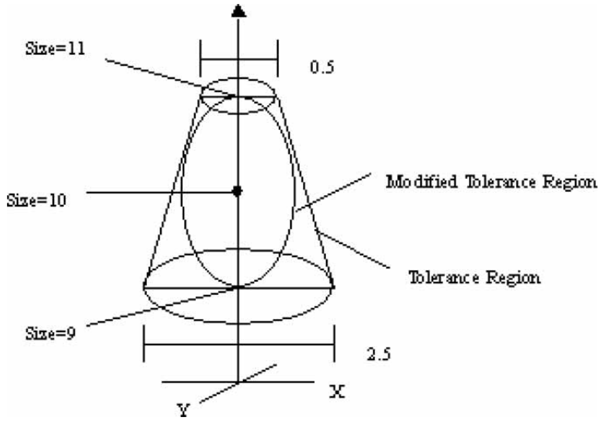

Figure 7. Tolerance region and modified tolerance region for example 3

in Figure 7. The center of the specification is $T^T = [0, 0, 10]$. The sample mean vector and sample covariance matrix for 70 observations were

$$\overline{X}^T = [-0.0124, -0.0062, 10.0586] \text{ and } S = \begin{bmatrix} 0.01313 & -0.00371 & 0.00884 \\ -0.00371 & 0.01618 & -0.01031 \\ 0.00884 & -0.01031 & 0.06473 \end{bmatrix}$$

From the data, we have $\chi^2_{3,0.9973} = 14.1563$ and $|S| = 0.00001092$. Then we have

$$\hat{M}C_p = \frac{4/3\pi \times [(11-9)/2] \times (2.5/2) \times (0.5/2)}{|S|^{1/2}(\pi \times \chi^2_{3,0.9973})^{3/2}[\Gamma(2.5)]^{-1}} = 1.7752$$

Now, $v = 3$ and $n = 70$, the expectation and variance of $\hat{M}C_p$ can be calculated as $1.0570 \times MC_p$ and $0.025685 \times MC_p^2$, respectively. According to equation (17), a 95% confidence interval for MC_p is calculated as

$$\left[1.7752\sqrt{\frac{164939}{69^3}}, 1.7752\sqrt{\frac{533052}{69^3}}\right] = [1.2579, 2.2613]$$

In addition, a 95% lower confidence bound is

$$1.7752\sqrt{\frac{182304}{69^3}} = 1.3224$$

To judge whether this process meets the present capability requirement, we consider a statistical hypothesis testing for MC_p: $H_0: MC_p \leqq 1$ versus $H_1: MC_p > 1$. According to the equation (18), the critical values

$$c = 1 \times \sqrt{\frac{69^3}{182304}} = 1.3423$$

Since $\hat{M}C_p = 1.7752 > 1.3423$, we can conclude that the MC_p is larger than 1 at 95% confidence level. It implies that this process variation is smaller than the specified range of variation. From the data, we have $\hat{D} = 1.0437$ and $\hat{\tau}^2 = 6.1608$. Thus, the $\hat{M}C_{pm}$ can be calculated as $1.7752/1.0437 = 1.7009$. According to equation (24), an approximate 95% confidence interval for MC_{pm} is calculated as

$$\left[1.7009\sqrt{\frac{184666}{(69)^3 \times 1.0437^2}},\ 1.7009\sqrt{\frac{619698}{(69)^3 \times 1.0437^2}}\ \right] = [1.2219, 2.2383]$$

Also, according to equation (24), an approximate 95% lower confidence bound for MC_{pm} is

$$1.7009\sqrt{\frac{204521}{(69)^3 \times 1.0437^2}} = 1.2859$$

Therefore, we can conclude that the MC_{pm} is larger than 1 at 95% confidence level. It implies that this process is close to the specified target.

Conclusions

Processes with multiple quality characteristics often occur in manufacturing industries. Multivariate capability indices such as MC_p and MC_{pm} have been proposed to measure process reproduction capability according to the corresponding multiple specifications. In this paper, we obtained the probability density function of the estimated $\hat{M}C_p$. We constructed lower confidence bounds for $\hat{M}C_p$, and developed the corresponding hypothesis testing for MC_p. In addition, we derived an approximate lower confidence bound for MC_{pm} for $v = 2, 3$. A simulation study was conducted to ascertain the accuracy of the approximation. Three examples are given to illustrate the obtained results. Although we only provided the distribution of MC_p for $v = 1, 2, 3$, using the variable transformation technique we could obtain the sampling distribution of MC_p for $v \geq 4$. Practitioners can use the proposed procedure to determine whether their process meets the preset capability requirement, and so make reliable decisions.

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Appendix

Corollary 1

If χ^2_{n-1} and χ^2_{n-2} are independently distributed, then $\chi^2_{n-1} \times \chi^2_{n-2}$ is distributed as $(\chi^2_{2n-4})^2/4$.

Proof

Let $x_1 \sim \chi^2_{n-1}$ and $x_2 \sim \chi^2_{n-2}$. The joint pdf of x_1 and x_2 is given by

$$f_{x_1,x_2}(x_1, x_2) = \frac{(1/2)^{(n-1)/2} x_1^{n/2-3/2} \mathrm{e}^{-x_1/2}}{\Gamma[(n-1)/2]} \times \frac{(1/2)^{(n-2)/2} x_2^{n/2-2} \mathrm{e}^{-x_2/2}}{\Gamma[(n-2)/2]}$$

Let $z_1 = x_1$ and $z_2 = 2\sqrt{x_1 x_2}$. Using the transformation method, the solution is $x_1 = z_1$ and $x_2 = z_2^2/4z_1$, and the Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 0 \\ -\dfrac{z_2^2}{4\sqrt{z_1}} & \dfrac{z_2}{2z_1} \end{vmatrix} = \frac{z_2}{2z_1}.$$

So, we find that the joint pdf of $z_1 z_2$ is

$$f_{z_1,z_2}(z_1, z_2) = \frac{(1/2)^{(n-1)/2} z_1^{n/2-3/2} \mathrm{e}^{-z_1/2}}{\Gamma[(n-1)/2]} \times \frac{(1/2)^{(n-2)/2} (z_2^2/4z_1)^{n/2-2} \mathrm{e}^{-(z_2^2/4z_1)/2}}{\Gamma[(n-2)/2]} \times \frac{z_2}{2z_1}, \quad 0 \le z_1, z_2 \le \infty$$

Then, the marginal density function of z_2 is obtained as follows:

$$f_{z_2}(z_2) = \int_0^\infty \frac{(1/2)^{(n-1)/2} z_1^{n/2-3/2} \mathrm{e}^{-z_1/2}}{\Gamma[(n-1)/2]} \times \frac{(1/2)^{(n-2)/2} (z_2^2/4z_1)^{n/2-2} \mathrm{e}^{-(z_2^2/4z_1)/2}}{\Gamma[(n-2)/2]} \times \frac{z_2}{2z_1} \mathrm{d}z_1$$

$$= C_1 \times z_2^{n-3} \times \int_0^\infty z_1^{-1/2} \times \mathrm{e}^{-z_1/2-(z_2^2/4z_1)/2} \mathrm{d}z_1, \quad 0 \le z_2 \le \infty$$

where

$$C_1 = \frac{(1/2)^{2n-9/2}}{\Gamma[(n-1)/2] \times \Gamma[(n-2)/2]}$$

Let $h(z_2) = \int_0^\infty z_1^{-1/2} \times \mathrm{e}^{-z_1/2-(z_2^2/4z_1)/2} \mathrm{d}z_1$. Hence, $h^{'}(z_2) = (-z_2/4z_1) \times \int_0^\infty z_1^{-1/2} \times \mathrm{e}^{-z_1/2-(z_2^2/4z_1)/2} \mathrm{d}z_1$. Now, let $z_2^2/4z_1 = w$. Using the transformation method, we find that

$$h^{'}(z_2) = \left(-\frac{1}{2}\right) \times \int_0^\infty w^{-1/2} \times \mathrm{e}^{-w/2-(z_2^2/4w)/2} \mathrm{d}w = \left(-\frac{1}{2}\right) \times h(z_2)$$

The above equation gives $h(z_2) = \mathrm{e}^{(-z_2/2+C_2)}$, where C_2 is a constant. Thus, the pdf of z_2 is given as the following, where $C_3 = C_1 \times e^{-C_2}$. Therefore, we have $z_2 \sim \chi^2_{2n-4}$, $f_{z_2}(z_2) = C_1 \times \mathrm{e}^{-C_2} \times z_2^{n-3} \times \mathrm{e}^{-z_2/2} = C_3 \times z_2^{(2n-4)/2-1} \times \mathrm{e}^{-z_2/2}$, $0 \le z_2 \le \infty$.

Theorem 2

Let $T^2 = n(\overline{X} - \mu_0)' S^{-1}(\overline{X} - \mu_0)$, where $X = (x_1, x_2, \ldots, x_n)'$ be a sample from $N(\mu, \Sigma)$ with mean vector $\mu = (\mu_1, \mu_2, \ldots, \mu_v)'$ and covariance matrix $\Sigma_{v\times v}$, and μ_0 is the vector of target values. The distribution of

$$\frac{T^2}{n-1} \times \frac{n-v}{v}$$

is a non-central F with v and $n - v$ degrees of freedom and non-centrality parameter $\tau^2 = n(\overline{X} - \mu_0)' \Sigma^{-1}(\overline{X} - \mu_0)$.

Proof

See Anderson (2003, pp. 174–176).

It can be shown that the pdf of T^2 is given by

$$\frac{e^{-(1/2)\tau^2}}{(n-1)\Gamma[1/2(n-v)]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i \left[t^2/(n-1)\right]^{(1/2)v+i-1} \Gamma(1/2n+i)}{i!\Gamma(1/2v+i)\left[1+t^2/(n-1)\right]^{(1/2)n+i}} \tag{A1}$$

(See Anderson, 2003, p. 186.)

Corollary 2

(1) For $v = 2$, if $z = \chi^2_{n-1} \times \chi^2_{n-2} \times \hat{D}^2$, where $\hat{D}^2 = [1 + n/n - 1(\overline{X} - \mu_0)' S^{-1}(\overline{X} - \mu_0)]$, then the pdf of z is

$$f_Z(z) = \int_1^{\infty} \frac{(1/2)\mathrm{e}^{-(1/2)\tau^2}}{w\Gamma(n-2)\Gamma[(n-2)/2]} (z/w)^{(n-4)/2} \mathrm{e}^{-\sqrt{z/w}} \times \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1)w^{(1/2)n+i}} \mathrm{d}w \quad \text{for } z \geq 0$$

(2) For $v = 3$, if $z = \chi^2_{n-1} \times \chi^2_{n-2} \times \chi^2_{n-3} \times \hat{D}^2$, where $\hat{D}^2 = [1 + n/n - 1(\overline{X} - \mu_0)' S^{-1}(\overline{X} - \mu_0)]$, then the pdf of z is

$$f_Z(z) = \int_1^{\infty} \int_0^{\infty} \frac{(1/2)^{(n-1)/2} x_1^{-1/2} \mathrm{e}^{-\sqrt{x_1} - z/(2wx_1)}}{\Gamma(n-2)\Gamma[(n-3)/2]} \frac{\mathrm{e}^{-(1/2)\tau^2} (z/w)^{(n-5)/2}}{\Gamma[(n-3)/2]} \times \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^i \Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{1/2n+i}} \mathrm{d}x_1 \mathrm{d}w \quad \text{for } z \geq 0$$

Proof

(1) From $\hat{D}^2 = [1 + n/(n-1)(\overline{X} - \mu_0)' S^{-1}(\overline{X} - \mu_0)]$ and Theorem 2, we find that $\hat{D}^2 = 1 + (1/(n-1))T^2$.

Let $y = 1 + (1/(n-1))T^2$. Using the transformation method and the equation (A1), the pdf of y is obtained as follows:

$$\begin{aligned} f_Y(y) &= f_{T^2}((n-1)(y-1)) \times |(n-1)| \\ &= \frac{e^{(-1/2)\tau^2}}{(n-1)\Gamma[(n-2)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (y-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1)y^{(1/2)n+i}} \times (n-1) \\ &= \frac{e^{-(1/2)\tau^2}}{\Gamma[(n-2)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (y-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1)y^{(1/2)n+i}} \quad \text{for } y \geq 1 \end{aligned}$$

Let $x = \chi^2_{n-1} \times \chi^2_{n-2}$. From Corollary 1, we find that $x = (\chi^2_{2n-4})^2/4$. That is, the pdf of x is

$$f_X(x) = \frac{(1/2)x^{(n-4)/2}\mathrm{e}^{-\sqrt{x}}}{\Gamma(n-2)} \quad \text{for } x > 0$$

Now, let $z = xy$ and $w = y$. Using the transformation method, the solution is $x = z/w$ and $y = w$, and the Jacobian of the transformation is $J = \begin{vmatrix} 1/w & -z/w^2 \\ 0 & 1 \end{vmatrix} = 1/w$. So, we find that the joint pdf of zw is

$$\begin{aligned} f_{ZW}(z,w) &= f_{XY}(z/w, w) \times (1/w) = \frac{(1/2)(z/w)^{(n-4)/2}\mathrm{e}^{-\sqrt{z/w}}}{w\Gamma(n-2)} \\ &\times \frac{\mathrm{e}^{-(1/2)\tau^2}}{\Gamma[(n-2)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1)w^{1/2n+i}} \quad \text{for } z \geq 0, w > 1 \end{aligned}$$

Then, the marginal density function of z is obtained as follows:

$$\begin{aligned} f_Z(z) &= \int_1^{\infty} \frac{(1/2)(z/w)^{(n-4)/2}\mathrm{e}^{-\sqrt{z/w}}}{w\Gamma(n-2)} \times \frac{\mathrm{e}^{-(1/2)\tau^2}}{\Gamma[(n-2)/2]} \\ &\quad \times \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1)w^{1/2n+i}} \mathrm{d}w \\ &= \int_1^{\infty} \frac{1/2\mathrm{e}^{-(1/2)\tau^2}}{w\Gamma(n-2)\Gamma[(n-2)/2]} (z/w)^{(n-4)/2}\mathrm{e}^{-\sqrt{z/w}} \\ &\quad \times \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1)w^{(1/2)n+i}} \mathrm{d}w \quad \text{for } z \geq 0 \end{aligned}$$

(2) From $\hat{D}^2 = [1 + n/(n-1)(\overline{X} - \mu_0)' S^{-1}(\overline{X} - \mu_0)]$ and Theorem 2, we find that $\hat{D}^2 = 1 + (1/(n-1))T^2$.

Let $y = 1 + (1/(n-1)T)^2$. Using the transformation method and the equation (A1), the pdf of y is obtained as follows:

$$\begin{aligned} f_Y(y) &= f_{T^2}((n-1)(y-1)) \times |(n-1)| \\ &= \frac{\mathrm{e}^{-(1/2)\tau^2}}{(n-1)\Gamma[(n-3)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (y-1)^{i+1/2} \Gamma(n/2+i)}{i!\Gamma(i+3/2)y^{(1/2)n+i}} \times (n-1) \\ &= \frac{\mathrm{e}^{-(1/2)\tau^2}}{\Gamma[(n-3)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (y-1)^{i+1/2} \Gamma(n/2+i)}{i!\Gamma(i+3/2)y^{1/2n+i}} \quad \text{for } y \geq 1 \end{aligned}$$

Let $x_1 = \chi^2_{n-1} \times \chi^2_{n-2}$ and $x_2 = \chi^2_{n-3}$. From Corollary 1, we find that $x_1 = (\chi^2_{2n-4})^2/4$. That is, the pdf of x_1 is

$$f_{x_1}(x_1) = \frac{(1/2)x_1^{(n-4)/2}\mathrm{e}^{-\sqrt{x_1}}}{\Gamma(n-2)} \quad \text{for } x_1 > 0$$

Now, let $x = x_1x_2$ and $u = x_1$. Using the transformation method, the solution is $x_1 = u$ and $x_2 = x/u$, and the Jacobian of the transformation is $J = \begin{vmatrix} 0 & 1 \\ 1/u & -x/u^2 \end{vmatrix} = 1/u$. So, we find that the joint pdf of xu is

$$\begin{aligned} f_{XU}(x, u) &= f_{X_1X_2}(u, x/u) \times \frac{1}{u} \\ &= \frac{(1/2)^{(n-1)/2} \times u^{-1/2} \times x^{(n-5)/2} \times \mathrm{e}^{-\sqrt{u}-x/2u}}{\Gamma(n-2) \times \Gamma[(n-3)/2]} \quad 0 \le x, u < \infty \end{aligned}$$

Then, the marginal density function of x is

$$f_X(x) = \int_0^\infty \frac{(1/2)^{(n-1)/2} \times u^{-1/2} \times x^{(n-5)/2} \times \mathrm{e}^{-\sqrt{u}-x/2u}}{\Gamma(n-2) \times \Gamma[(n-3)/2]}\mathrm{d}u \quad , 0 \le x$$

Again, let $z = xy$ and $w = y$. Using the transformation method, the solution is $x = z/w$ and $y = w$, and the Jacobian of the transformation is the $J = \begin{vmatrix} 1/w & -z/w^2 \\ 0 & 1 \end{vmatrix} = 1/w$. So, the joint pdf of zw is obtained as follows:

$$\begin{aligned} f_{ZW}(z, w) &= f_{XY}(z/w, w) \times (1/w) \\ &= \int_0^\infty \frac{(1/2)^{(n-1)/2} \times u^{-1/2} \times (z/w)^{(n-5)/2} \times \mathrm{e}^{-\sqrt{u}-z/2wu}}{\Gamma(n-2) \times \Gamma[(n-3)/2]}\mathrm{d}u \\ &\quad \times \frac{\mathrm{e}^{-(1/2)\tau^2}}{\Gamma[(n-3)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^{i+1/2}\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{(1/2)n+i}} \times \frac{1}{w} \\ &= \int_0^\infty \frac{(1/2)^{(n-1)/2}u^{-1/2}\mathrm{e}^{-\sqrt{u}-z/(2wu)}}{\Gamma(n-2)\Gamma[(n-3)/2]}\mathrm{d}u \frac{\mathrm{e}^{-1/2\tau^2}(z/w)^{(n-5)/2}}{w\Gamma[(n-3)/2]} \\ &\quad \times \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^{i+1/2}\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{1/2n+i}} \quad \text{for } z \ge 0, w > 1 \end{aligned}$$

Then, the marginal density function of z is

$$\begin{aligned} f_Z(z) = \int_1^\infty \int_0^\infty & \frac{(1/2)^{(n-1)/2}u^{-1/2}\mathrm{e}^{-\sqrt{u}-z/(2wu)}}{\Gamma(n-2)\Gamma[(n-3)/2]} \frac{e^{-1/2\tau^2}(z/w)^{(n-5)/2}}{w\Gamma[(n-3)/2]} \\ & \times \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^{i+(1/2)}\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{(1/2)n+i}}\mathrm{d}u\mathrm{d}w \quad \text{for } z \ge 0. \end{aligned}$$