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Multistability and convergence in delayed neural networks

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Abstract

We present the existence of 2^n stable stationary solutions for a general n-dimensional delayed neural networks with several classes of activation functions. The theory is obtained through formulating parameter conditions motivated by a geometrical observation. Positively invariant regions for the flows generated by the system and basins of attraction for these stationary solutions are established. The theory is also extended to the existence of 2^n limit cycles for the n-dimensional delayed neural networks with time-periodic inputs. It is further confirmed that quasiconvergence is generic for the networks through justifying the strongly order preserving property as the self-feedback time lags are small for the neurons with negative self-connection weights. Our theory on existence of multiple equilibria is then incorporated into this quasiconvergence for the network. Four numerical simulations are presented to illustrate our theory.
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1. Introduction

Existence of many equilibria is a necessary feature in the applications of neural networks to associative memory storage or pattern recognition [1–4]. The notion of "multistability" of a neural network describes coexistence of multiple stable patterns such as equilibria or periodic orbits. Recently, further application potentials of multistability have been found in decision making, digital selection or analogy amplification [5]. "Quasiconvergence" for a system refers to that every solution tends to the set of stationary solutions, while "convergence" means that every solution tends to a single stationary solution, as time tends to infinity.

In this presentation, we address multistability and quasiconvergence for a general delayed neural network:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t))$$
$$+ \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t - \tau_{ij})) + I_i, \qquad (1.1)$$

where $i = 1, \ldots, n$; $\mu_i > 0$; α_{ij}, β_{ij} are connection weights from neuron j to neuron i; $g_j(\cdot)$ are activation functions; $0 \leq \tau_{ij} \leq \tau$ are time lags; I_i stands for an independent bias current source. System (1.1) reduces to the classical and delayed Hopfield neural networks [3,6], as $\beta_{ij} = 0$ and $\alpha_{ij} = 0$ for all i, j, respectively. It also represents the cellular neural networks (CNN) without delays [7] and with delays [8]. Indeed, a CNN system built in a multi-dimensional coupling fashion can always be rewritten in a one-dimensional coupling form, by renaming the indices [9]. Such an arrangement, however, suppresses the local connection representation.

In electronic implementation, time delays of neural network systems are unavoidable due to axonal conduction times, distances of interneurons and the finite switching speeds of amplifiers. The dynamics for differential equations with delays can be rather complicated. Although the stationary equations are identical for system (1.1) without delay ($\tau_{ij} = 0$ for all i, j) and with delay ($\tau_{ij} > 0$), the stability for the equilibrium points and dynamical behaviors of the systems can be very different. There have been papers [10–15] exploring the effects of delays in differential equations and neural network systems. For system (1.1), the theory of unique equilibrium and global convergence to the equilibrium has been studied

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extensively in [8,16–25]. These studies indicate a coincidence of dynamics between the systems with delays and without delays. This presentation moves the investigation in this direction by establishing the existence of multiple stationary solutions for system (1.1). More specifically, we construct 2^n stable stationary solutions for a general n-dimensional delayed neural network with several classes of activation functions. The theory is obtained through formulating parameter conditions based on a geometrical setting. We first derive conditions for the existence of 3^n equilibria for (1.1) with sigmoidal activation functions and saturated activation functions. Some regions containing these stationary solutions are shown to be positively invariant under the flows generated by (1.1) and the basins of attraction for these stationary solutions are also estimated. In fact, through a further subtle estimate, it can be justified that the basins of attraction are at least as large as the positively invariant sets. The theory is also extended to confirm the existence of 2^n limit cycles for system (1.1) with time-periodic inputs. We further discuss the property of strongly order preserving, hence quasiconvergent behaviors for (1.1). The dynamics scenario for system (1.1) is thus composed of multiple equilibria and quasiconvergence (or convergence) almost everywhere. Our investigations also illustrate different criteria and distinct dynamical behaviors between (1.1) with smooth sigmoidal activation functions and (1.1) with saturated activation functions.

The existence of multiple equilibria and their attractive domains for (1.1) with the standard activation function have been studied in [26]. The result therein strongly relies on the piecewise linearity and saturations of the standard activation function as well as subsequent partition of the phase space. Our geometrical approach can be applied to (1.1) with general sigmoidal activation functions. In addition, larger positively invariant sets and basins of attraction are established. Moreover, the criteria in our theory are weaker than those in [26]. The approach in this presentation is an extension from [27] which mainly treats multistability for the Hopfield neural networks with smooth sigmoidal activation functions.

The monotone dynamic property for the classical neural networks (without delays) was first exploited by Hirsch [28]. Such a property was then extended to the two-neuron and the n-neuron delayed Hopfield neural networks in [12] and [13] respectively; see also [29] for a comprehensive overview. The investigations in [12,13] are mainly concerned with global attractivity of a single equilibrium and the effect of delays upon such a dynamical scenario. Our study aims at incorporating the existence of multiple equilibria into the monotone dynamics and the quasiconvergence for (1.1).

The remaining part of this presentation is organized as follows. In Section 2, we consider two classes of activation functions which are commonly employed in neural networks. We then derive conditions for the existence of 3^n equilibria for the networks. In Section 3, we show that, with additional condition, there are 2^n regions in $\mathbb{R}^n$ which are positively invariant under the flow generated by system (1.1). Each of these regions contains one equilibrium out of those 3^n equilibria. Subsequently, it is argued that these 2^n equilibria

are asymptotically stable. Existence of multiple stable periodic orbits for system (1.1) with periodic inputs is demonstrated in Section 4. We discuss strongly order preserving property and quasiconvergence for system (1.1) in Section 5. Finally, in Section 6, we present four numerical simulations to illustrate the present theory and distinct dynamical behaviors for different activation functions.

2. Activation functions and multiple equilibria

Existence and stability of stationary patterns for neural networks certainly depend on the characteristics of activation functions. We shall consider the following two classes of activation functions g_i for (1.1):

$$\text{class } \mathcal{A} : g_i \in C^2, \begin{cases} u_i < g_i(\xi) < v_i, & g_i'(\xi) > 0, \\ (\xi - \sigma_i)g_i''(\xi) < 0, & \text{for all } \xi \in \mathbb{R}, \\ \lim_{\xi \to +\infty} g_i(\xi) = v_i, \\ \lim_{\xi \to -\infty} g_i(\xi) = u_i; \end{cases}$$

$$\text{class } \mathcal{B} : g_i \in C, \ g_i(\xi) = \begin{cases} u_i & \text{if } -\infty < \xi < p_i, \\ \overline{g}_i(\xi) & \text{if } p_i \leq \xi \leq q_i, \\ v_i & \text{if } q_i < \xi < \infty, \end{cases}$$

where, u_i, v_i, p_i, q_i and σ_i are constants with $u_i < v_i, p_i < q_i$, and $\overline{g}_i(\cdot), i = 1, \ldots, n$, are C^1 increasing functions. Class $\mathcal{A}$ contains general bounded smooth sigmoidal functions, and class $\mathcal{B}$ consists of nondecreasing functions with saturations, including the piecewise linear functions with two corner points at p_i, q_i:

$$\overline{g}_i(\xi) = u_i + \frac{v_i - u_i}{q_i - p_i}(\xi - p_i); \tag{2.1}$$

and, in particular, the standard activation function for the CNN:

$$g_i(\xi) = g^s(\xi) := \frac{1}{2}(|\xi + 1| - |\xi - 1|), \quad i = 1, \ldots, n. \tag{2.2}$$

Typical configurations of these functions are depicted in Figs. 1 and 2(a). Notably, in practical implementation, the transition from the linear regime to the saturated regime in the standard activation function is smooth. Thus, the theory developed for the dynamics of (1.1) should also be valid for the activation functions which are smooth at $\xi = \pm 1$, as demonstrated in Fig. 2(b). Our investigations have provided theoretical basis for all these activation functions. In Section 5, we will see some distinct dynamics between (1.1) with activation functions of class $\mathcal{A}$ and (1.1) with the ones of class $\mathcal{B}$.

Let us review some basic notion of delayed differential equations. We set $\tau = \max_{1 \leq i, j \leq n} \tau_{ij}$. The initial condition for (1.1) is $x_i(\theta) = \phi_i(\theta), -\tau \leq \theta \leq 0, i = 1, \ldots, n$, with $\phi = (\phi_1, \ldots, \phi_n) \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. We define the norm of ϕ as $\| \phi \| = \max_{1 \leq i \leq n} \{\sup_{\theta \in [-\tau, 0]} |\phi_i(\theta)|\}$. Let $\ell > 0$. For $\mathbf{x}(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot)) \in \mathcal{C}([-\tau, \ell], \mathbb{R}^n)$, and $t \in [0, \ell]$, we define

$$\mathbf{x}_t(\theta) = \mathbf{x}(t + \theta), \quad \theta \in [-\tau, 0]. \tag{2.3}$$
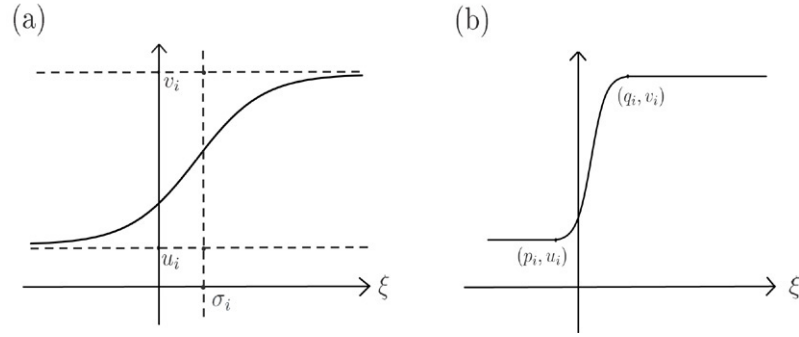
(a)

(b)

Fig. 1. The configurations of (a) typical smooth sigmoidal activation functions in class $\mathcal{A}$ and (b) saturated activation functions in class $\mathcal{B}$.
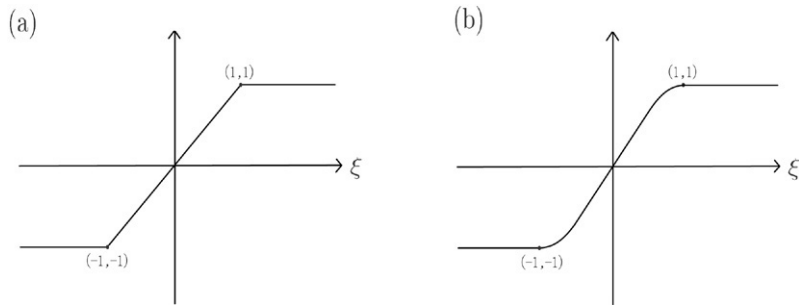
(a)

(b)

Fig. 2. The graphs for (a) the standard activation function $g^{\mathrm{s}}(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|)$, (b) saturated activation functions with smooth corners.

Let us denote $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_n)$, where $\tilde{F}_i$ is the right hand side of system (1.1),

$$\tilde{F}_i(\mathbf{x}_t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t))$$
$$+ \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t - \tau_{ij})) + I_i.$$

A function $\mathbf{x}(\cdot)$ is called a solution of (1.1) on $[-\tau, \ell)$ if $\mathbf{x}(\cdot) \in \mathcal{C}([-\tau, \ell), \mathbb{R}^n)$, and $\mathbf{x}_t$ defined as (2.3) lies in the domain of $\tilde{F}$ and satisfies (1.1) for $t \in [0, \ell)$. For a given $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, let us denote by $\mathbf{x}(t; \phi)$ the solution of (1.1) with $\mathbf{x}(\theta; \phi) = \phi(\theta)$, for $\theta \in [-\tau, 0]$.

Notably, the stationary equation for (1.1) is

$$F_i(\mathbf{x}) := -\mu_i x_i + \sum_{j=1}^{n} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0,$$
$$i = 1, \ldots, n. \tag{2.4}$$

Next, we shall consider the above activation functions and formulate sufficient conditions for the existence of multiple stationary solutions for (1.1). Our approach is based on a geometrical observation. The first condition for (1.1) with activation functions in classes $\mathcal{A}$, $\mathcal{B}$ is, respectively,

$(\mathrm{H}_1{}^{\mathcal{A}})$: $0 = \inf_{\xi \in \mathbb{R}} g_i'(\xi) < \dfrac{\mu_i}{\alpha_{ii} + \beta_{ii}}$
$< \max_{\xi \in \mathbb{R}} g_i'(\xi) (= g_i{}'(\sigma_i)), \quad i = 1, \ldots, n$

$(\mathrm{H}_1{}^{\mathcal{B}})$: $(\alpha_{ii} + \beta_{ii}) \max_{\xi \in \mathbb{R}} \overline{g}_i'(\xi) > \mu_i, \quad i = 1, \ldots, n.$

Condition $(\mathrm{H}_1{}^{\mathcal{B}})$ reduces to $(\alpha_{ii} + \beta_{ii}) \frac{v_i - u_i}{q_i - p_i} > \mu_i$, if piecewise linear activation functions (2.1) are adopted, and reduces to

$$\alpha_{ii} + \beta_{ii} > \mu_i, \quad i = 1, \ldots, n, \tag{2.5}$$

if the standard activation function $g^{\mathrm{s}}(\cdot)$ in (2.2) is employed, with $p_i = u_i = -1$, $q_i = v_i = 1$. We define, for $i = 1, \ldots, n$,

$$\hat{f}_i(\xi) = -\mu_i \xi + (\alpha_{ii} + \beta_{ii}) g_i(\xi) + k_i^+,$$
$$\check{f}_i(\xi) = -\mu_i \xi + (\alpha_{ii} + \beta_{ii}) g_i(\xi) + k_i^-, \tag{2.6}$$

where $k_i^+ = \sum_{j=1, j \neq i}^{n} \rho_j(|\alpha_{ij}| + |\beta_{ij}|) + I_i$, $k_i^- = -\sum_{j=1, j \neq i}^{n} \rho_j(|\alpha_{ij}| + |\beta_{ij}|) + I_i$, and $\rho_j := \max\{|u_j|, |v_j|\}$. It follows that $\check{f}_i(x_i) \leq F_i(\mathbf{x}) \leq \hat{f}_i(x_i)$, for all $\mathbf{x} = (x_1, \ldots, x_n)$ and $i = 1, \ldots, n$. We introduce a family of single neuron equations, for $i = 1, \ldots, n$

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = f_i(\xi) := -\mu_i \xi + (\alpha_{ii} + \beta_{ii}) g_i(\xi) + J_i,$$
$$\xi \in \mathbb{R}, \quad k_i^- \leq J_i \leq k_i^+.$$

Proposition 2.1. There exist two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i < \sigma_i < \tilde{q}_i$ (resp. $\tilde{p}_i \geq p_i$ and $\tilde{q}_i \leq q_i$) such that $f_i'(\tilde{p}_i) = f_i'(\tilde{q}_i) = 0$, $i = 1, \ldots, n$, under condition $(\mathrm{H}_1{}^{\mathcal{A}})$ (resp. $(\mathrm{H}_1{}^{\mathcal{B}})$), for activation functions of class $\mathcal{A}$ (resp. $\mathcal{B}$).

Proof. We only prove for class $\mathcal{A}$. For each i, since $f_i'(\xi) = -\mu_i + (\alpha_{ii} + \beta_{ii}) g_i'(\xi)$, we have $f_i'(\xi) = 0$ if and only if $g_i'(\xi) = \mu_i/(\alpha_{ii} + \beta_{ii})$. The graph of function $g_i'(\xi)$ is concave down and has its maximal value at σ_i. Note that $\lim_{\xi \to \pm\infty} g_i'(\xi) = 0$.
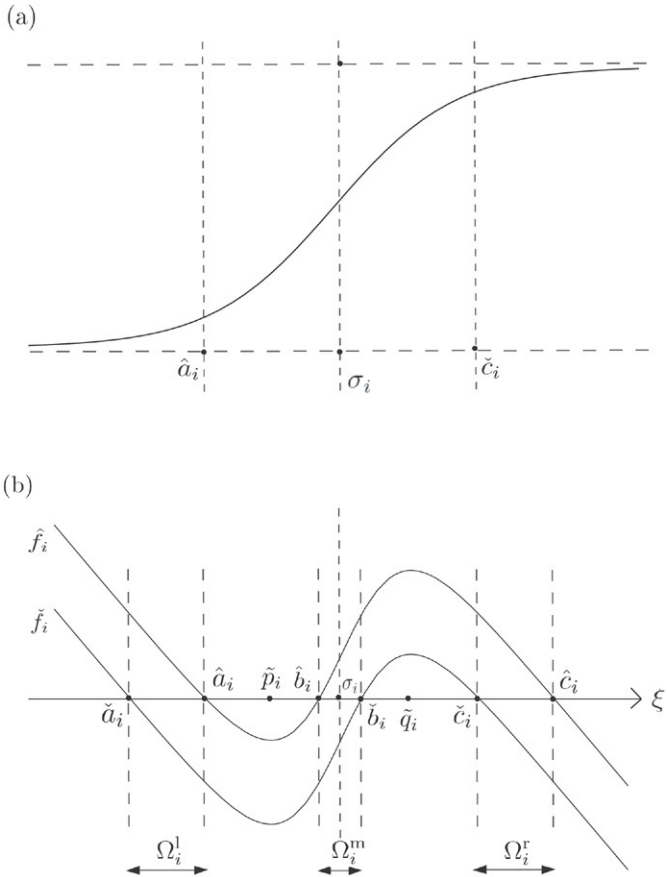
(a)

(b)

Fig. 3. (a) The graph of activation function g_i in class $\mathcal{A}$, (b) configurations of functions $\hat{f}_i$ and $\check{f}_i$.

Hence, since each g_i' is continuous, if

$$
0 = \inf_{\xi \in \mathbb{R}} g_i'(\xi) < \frac{\mu_i}{\alpha_{ii} + \beta_{ii}}
$$
$$
< \max_{\xi \in \mathbb{R}} g_i'(\xi) \; (= g_i'(\sigma_i)), \quad i = 1, \ldots, n,
$$

there exist two points $\tilde{p}_i$, $\tilde{q}_i$, with $\tilde{p}_i < \sigma_i < \tilde{q}_i$, such that $g_i'(\tilde{p}_i) = g_i'(\tilde{q}_i) = \mu_i/(\alpha_{ii} + \beta_{ii})$. This completes the proof. $\square$

For (1.1) with piecewise linear activation functions, f_i attains its local minimum at $\tilde{p}_i = p_i$, and local maximum at $\tilde{q}_i = q_i$, under assumption $(H_1{}^{\mathcal{B}})$. In particular, for the standard activation function g^s, $\tilde{p}_i = -1, \tilde{q}_i = 1, i = 1, \ldots, n$. A consequence of Proposition 2.1 is that f_i is strictly increasing
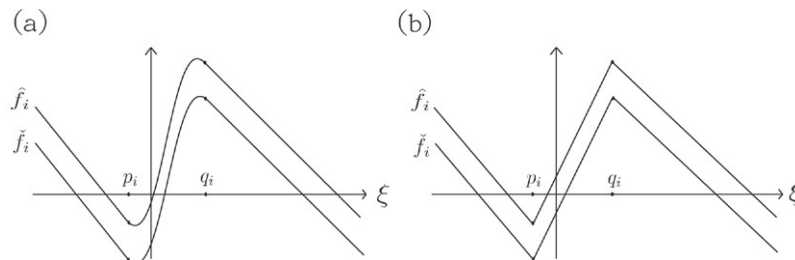
on $(-\infty, \tilde{p}_i]$, decreasing on $[\tilde{q}_i, \infty)$, under condition $(H_1{}^*)$. Note that condition $(H_1{}^*)$, $* = \mathcal{A}, \mathcal{B}$, implies $\alpha_{ii} + \beta_{ii} > 0$ for each $i = 1, \ldots, n$, since μ_i is already assumed positive.

We consider the second parameter condition which is used to establish existence of multiple equilibria for (1.1):

(H_2): $\hat{f}_i(\tilde{p}_i) < 0$, $\check{f}_i(\tilde{q}_i) > 0$, $i = 1, \ldots, n$.

The configuration that motivates (H_2) is depicted in Figs. 3 and 4. Under assumptions $(H_1{}^*)$ and (H_2), $* = \mathcal{A}, \mathcal{B}$, there exist points $\hat{a}_i$, $\hat{b}_i$, $\hat{c}_i$ with $\hat{a}_i < \hat{b}_i < \hat{c}_i$ such that $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$ as well as points $\check{a}_i, \check{b}_i, \check{c}_i$ with $\check{a}_i < \check{b}_i < \check{c}_i$, such that $\check{f}_i(\check{a}_i) = \check{f}_i(\check{b}_i) = \check{f}_i(\check{c}_i) = 0$.

Theorem 2.2. *There exist 3^n equilibria for system* (1.1) *with activation functions of class $*$, $* = \mathcal{A}, \mathcal{B}$, under conditions $(H_1{}^*)$ and (H_2).*

Proof. We only prove the case of class $\mathcal{A}$. The equilibria of system (1.1) are roots of (2.4). Conditions $(H_1{}^{\mathcal{A}})$ and (H_2) induce a configuration depicted in Fig. 3. Accordingly, there are 3^n disjoint closed regions in $\mathbb{R}^n$, namely,

$$
\boldsymbol{\Omega}^{\mathbf{w}} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in \Omega_i{}^{w_i}\},
$$
$$
\mathbf{w} = (w_1, \ldots, w_n), w_i = \text{“l”}, \text{“m”}, \text{“r”}, \tag{2.7}
$$

where $\Omega_i{}^1 = [\check{a}_i, \hat{a}_i]$, $\Omega_i{}^{\mathrm{m}} = [\hat{b}_i, \check{b}_i]$, $\Omega_i{}^{\mathrm{r}} = [\check{c}_i, \hat{c}_i]$ are intervals. Herein, “l”, “m”, “r” mean respectively “left”, “middle” and “right”. Let $\boldsymbol{\Omega}^{\mathbf{w}}$ be one of these regions. For any given $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \boldsymbol{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$
h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i)
$$
$$
+ \sum_{j=1, j \neq i}^{n} (\alpha_{ij} + \beta_{ij}) g_j(\tilde{x}_j) + I_i = 0, \tag{2.8}
$$

$i = 1, \ldots, n$. Note that h_i is a vertical shift and lies between $\hat{f}_i$ and $\check{f}_i$, due to (2.6). Accordingly, one can always find three solutions to (2.8), which lie in regions $\Omega_i{}^1, \Omega_i{}^{\mathrm{m}}, \Omega_i{}^{\mathrm{r}}$ respectively, for each i. We define a mapping $\mathbf{H_w} : \boldsymbol{\Omega}^{\mathbf{w}} \to \boldsymbol{\Omega}^{\mathbf{w}}$ by $\mathbf{H_w}(\tilde{\mathbf{x}}) = \underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_n)$ where $\underline{x}_i$ is the solution of (2.8) lying in $\Omega_i{}^{w_i}$. The mapping $\mathbf{H_w}$ as defined is continuous, since g_i is continuous. It follows from Brouwer's fixed point theorem that there exists one fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ of $\mathbf{H_w}$ in $\boldsymbol{\Omega}^{\mathbf{w}}$, which is also a zero of F in (2.4). Consequently, there exist 3^n equilibria for system (1.1) and each of them lies in one of the 3^n regions $\boldsymbol{\Omega}^{\mathbf{w}}$. $\square$

(a) (b)

Fig. 4. (a) The graphs of $\hat{f}_i$ and $\check{f}_i$ induced from the activation function of class $\mathcal{B}$. (b) The graphs of $\hat{f}_i$ and $\check{f}_i$ induced from the standard activation function g^s.

3. Stability of equilibria and basins of attraction

In this section, we first establish some positively invariant sets for system (1.1) and investigate stability of the equilibrium in each invariant set. As a result, we also obtain a basin of attraction for each of the asymptotically stable equilibria.

We consider the following 2^n subsets of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Let $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_i =$ "l" or "r", and set

$$\tilde{\mathbf{\Lambda}}^{\mathbf{w}} = \{\varphi = (\varphi_1, \ldots, \varphi_n) \mid \varphi_i \in \tilde{\Lambda}_i^{\mathrm{l}}$$
$$\text{if } w_i = \text{"l"}, \varphi_i \in \tilde{\Lambda}_i^{\mathrm{r}} \text{ if } w_i = \text{"r"}\}, \qquad (3.1)$$

where $\tilde{\Lambda}_i^{\mathrm{l}} = \{\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) < \hat{b}_i \text{ for all } \theta \in [-\tau, 0]\}$, $\tilde{\Lambda}_i^{\mathrm{r}} = \{\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) > \check{b}_i \text{ for all } \theta \in [-\tau, 0]\}$.

Theorem 3.1. *Assume that* $(\mathrm{H_1}^*)$, $(\mathrm{H_2})$, $* = \mathcal{A}, \mathcal{B}$, *and* $\beta_{ii} > 0, i = 1, \ldots, n$, *then each* $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ *is positively invariant under the solution flow generated by system* (1.1) *with activation functions of class* $*$.

Proof. We only prove the $\mathcal{A}$ case. Let $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ be a subset defined in (3.1). Consider any initial condition $\phi = (\phi_1, \ldots, \phi_n) \in \tilde{\mathbf{\Lambda}}^{\mathbf{w}}$; there exists a sufficiently small constant $\varepsilon_0 > 0$ such that $\phi_i(\theta) \geq \check{b}_i + \varepsilon_0$ for all $\theta \in [-\tau, 0]$, if $w_i =$ "r", and $\phi_i(\theta) \leq \hat{b}_i - \varepsilon_0$ for all $\theta \in [-\tau, 0]$, if $w_i =$ "l". We claim that the solution $\mathbf{x}(t; \phi)$ remains in $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ for all $t \geq 0$. If this is not true, there exists a component of $\mathbf{x}(t; \phi)$ which is the first one (or one of the first ones) decreasing across the value $\check{b}_i + \varepsilon_0$ or increasing across the value $\hat{b}_i - \varepsilon_0$; i.e., there exists some $i \in \{1, \ldots, n\}$ and $t_1 > 0$ such that either $x_i(t_1) = \check{b}_i + \varepsilon_0$, $(\mathrm{d}x_i/\mathrm{d}t)(t_1) \leq 0$, and $x_i(t) > \check{b}_i + \varepsilon_0$ for $-\tau \leq t < t_1$ or $x_i(t_1) = \hat{b}_i - \varepsilon_0$, $(\mathrm{d}x_i/\mathrm{d}t)(t_1) \geq 0$ and $x_i(t) < \hat{b}_i - \varepsilon_0$ for $-\tau \leq t < t_1$. For the first case, we derive from (1.1) that

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}(t_1) = -\mu_i(\check{b}_i + \varepsilon_0) + \alpha_{ii}g_i(\check{b}_i + \varepsilon_0) + \beta_{ii}g_i(x_i(t_1 - \tau_{ii}))$$
$$+ \sum_{j=1, j\neq i}^{n} \alpha_{ij}g_j(x_j(t_1))$$
$$+ \sum_{j=1, j\neq i}^{n} \beta_{ij}g_j(x_j(t_1 - \tau_{ij})) + I_i \leq 0. \qquad (3.2)$$

On the other hand,

$$-\mu_i(\check{b}_i + \varepsilon_0) + \alpha_{ii}g_i(\check{b}_i + \varepsilon_0) + \beta_{ii}g_i(x_i(t_1 - \tau_{ii}))$$
$$+ \sum_{j=1, j\neq i}^{n} \alpha_{ij}g_j(x_j(t_1)) + \sum_{j=1, j\neq i}^{n} \beta_{ij}g_j(x_j(t_1 - \tau_{ij})) + I_i$$
$$\geq -\mu_i(\check{b}_i + \varepsilon_0) + (\alpha_{ii} + \beta_{ii})g_i(\check{b}_i + \varepsilon_0)$$
$$- \sum_{j=1, j\neq i}^{n} \rho_j(|\alpha_{ij}| + |\beta_{ij}|) + I_i$$
$$= \check{f}_i(\check{b}_i + \varepsilon_0) > 0, \qquad (3.3)$$

due to $(\mathrm{H_2})$, $\beta_{ii} > 0$, $|g_j(\cdot)| \leq \rho_j$, and $g_i(x_i(t_1 - \tau_{ii})) \geq g_i(\check{b}_i + \varepsilon_0)$, from the monotonicity of g_i and the definition of t_1. This yields a contradiction to (3.2). Hence, $x_i(t) \geq \check{b}_i + \varepsilon_0$

for all $t > 0$. Similar arguments can be employed to show that $x_i(t) \leq \hat{b}_i - \varepsilon_0$, for all $t > 0$ for the situation that $x_i(t_1) = \hat{b}_i - \varepsilon_0$ and $(\mathrm{d}x_i/\mathrm{d}t)(t_1) \geq 0$. Therefore, $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ is positively invariant under the flow generated by system (1.1). The proof is completed. $\square$

Next, we consider the following criterion concerning stability of the equilibria for the system with activation functions in class $\mathcal{A}$. Let $\eta_j, j = 1, \ldots, n$, be real numbers satisfying

$$\max\{g_j'(\xi) | \xi = \check{c}_j, \hat{a}_j\} < \eta_j < \min\{g_j'(\xi) | \xi = \tilde{p}_i, \tilde{q}_i\}.$$

Consider

$$(\mathrm{H_3}): \quad \mu_i > \sum_{j=1}^{n} \eta_j(|\alpha_{ij}| + |\beta_{ij}|), \quad i = 1, \ldots, n.$$

For activation functions $g_j(\cdot)$ in class $\mathcal{A}$, we define $\underline{d}_j$ and $\bar{d}_j$ as

$$\underline{d}_j = \min\{\xi \mid g_j'(\xi) = \eta_j\}, \bar{d}_j = \max\{\xi \mid g_j'(\xi) = \eta_j\}. \quad (3.4)$$

Then $\underline{d}_j > \hat{a}_j, \bar{d}_j < \check{c}_j$. For the activation functions g_j in class $\mathcal{B}$, $\bar{g}_i$ in (2.1), and g^{s} in (2.2), we define, respectively,

$$\underline{d}_j = \tilde{p}_j, \bar{d}_j = \tilde{q}_j; \underline{d}_j = p_j, \bar{d}_j = q_j; \underline{d}_j = -1, \bar{d}_j = 1. \quad (3.5)$$

We consider the following 2^n subsets of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Let $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_i =$ "l" or "r", and set

$$\mathbf{\Lambda}^{\mathbf{w}} = \{\varphi = (\varphi_1, \ldots, \varphi_n) \mid \varphi_i \in \Lambda_i^{\mathrm{l}}$$
$$\text{if } w_i = \text{"l"}, \varphi_i \in \Lambda_i^{\mathrm{r}} \text{ if } w_i = \text{"r"}\}, \qquad (3.6)$$

where $\Lambda_i^{\mathrm{l}} = \{\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) \leq \underline{d}_i, \forall \theta \in [-\tau, 0]\}$, $\Lambda_i^{\mathrm{r}} = \{\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) \geq \bar{d}_i, \forall \theta \in [-\tau, 0]\}$. In the following, we will derive that each of these 2^n subsets $\mathbf{\Lambda}^{\mathbf{w}}$ of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ lies in the basin of attraction for the respective equilibrium and justify that these 2^n equilibria are exponentially stable.

Theorem 3.2. *Under conditions* $(\mathrm{H_1}^{\mathcal{A}})$, $(\mathrm{H_2})$, $(\mathrm{H_3})$, *and* $\beta_{ii} > 0, i = 1, \ldots, n$, *there exist* 2^n *exponentially stable equilibria for system* (1.1) *with activation functions of class* $\mathcal{A}$. *The same assertion holds for activation functions of class* $\mathcal{B}$, *under conditions* $(\mathrm{H_1}^{\mathcal{B}})$, $(\mathrm{H_2})$.

Proof. We only prove the case of class $\mathcal{A}$. Let $\mathbf{\Lambda}^{\mathbf{w}}$ be a subset defined in (3.6) and $\bar{\mathbf{x}}$ be an equilibrium lying in $\mathbf{\Lambda}^{\mathbf{w}}$. For each $i = 1, \ldots, n$, we consider the single-variable function $G_i(\zeta) = \mu_i - \zeta - \sum_{j=1}^{n} \eta_j|\alpha_{ij}| - \sum_{j=1}^{n} \eta_j|\beta_{ij}|\mathrm{e}^{\zeta\tau_{ij}}$. Then, $(\mathrm{H_3})$ implies $G_i(0) > 0$, and there exists a constant $\lambda > 0$ such that $G_i(\lambda) > 0$, for all $i = 1, \ldots, n$, due to continuity of G_i. Let $\mathbf{x}(t) = \mathbf{x}(t; \phi)$ be the solution to system (1.1) with initial condition $\phi \in \mathbf{\Lambda}^{\mathbf{w}}$. With translation $\mathbf{y}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$, system (1.1) becomes

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -\mu_i y_i(t) + \sum_{j=1}^{n} \alpha_{ij}[g_j(x_j(t)) - g_j(\bar{x}_j)]$$
$$+ \sum_{j=1}^{n} \beta_{ij}[g_j(x_j(t - \tau_{ij})) - g_j(\bar{x}_j)], \qquad (3.7)$$

where $\mathbf{y} = (y_1, \ldots, y_n)$. Now, consider functions $z_i(\cdot)$ defined by $z_i(t) = e^{\lambda t}|y_i(t)|, i = 1, \ldots, n$. Let $\delta > 1$ and let $K = \max_{1 \le i \le n}\{\sup_{\theta \in [-\tau, 0]} |x_i(\theta) - \bar{x}_i|\} > 0$. It follows that $z_i(t) < K\delta$, for $t \in [-\tau, 0]$ and $i = 1, \ldots, n$. We shall justify that

$$z_i(t) < K\delta, \quad \text{for all } t > 0, i = 1, \ldots, n. \tag{3.8}$$

Suppose (3.8) does not hold, then there is a $k \in \{1, \ldots, n\}$ and a $t_1 > 0$ for the first time such that $z_i(t) \le K\delta, t \in [-\tau, t_1], i = 1, \ldots, n, i \ne k, z_k(t) \le K\delta, t \in [-\tau, t_1)$, and $z_k(t_1) = K\delta$, with $\dot{z}_k(t_1) \ge 0$. Note that $|y_k(t)|$ and $z_k(t)$ are differentiable at $t = t_1$, since $z_k(t_1) = K\delta > 0$ implies $y_k(t_1) \ne 0$. From (3.7), we compute that

$$\frac{\mathrm{d}}{\mathrm{d}t}|y_k(t_1)| \le -\mu_k|y_k(t_1)| + \sum_{j=1}^{n}|\alpha_{kj}g'_j(\xi_j)y_j(t_1)|$$
$$+ \sum_{j=1}^{n}|\beta_{kj}g'_j(\varsigma_j)y_j(t_1 - \tau_{kj})|,$$

for some ξ_j between $x_j(t_1)$ and $\bar{x}_j$ as well as ς_j between $x_j(t_1 - \tau_{kj})$ and $\bar{x}_j$. Hence,

$$\frac{\mathrm{d}z_k(t_1)}{\mathrm{d}t} \le \lambda e^{\lambda t_1}|y_k(t_1)| + e^{\lambda t_1}\Bigg[-\mu_k|y_k(t_1)|$$
$$+ \sum_{j=1}^{n}|\alpha_{kj}g'_j(\xi_j)y_j(t_1)|$$
$$+ \sum_{j=1}^{n}|\beta_{kj}g'_j(\varsigma_j)y_j(t_1 - \tau_{kj})|\Bigg]$$
$$= \lambda z_k(t_1) - \mu_k z_k(t_1) + \sum_{j=1}^{n}|\alpha_{kj}|g'_j(\xi_j)z_j(t_1)$$
$$+ \sum_{j=1}^{n}|\beta_{kj}|g'_j(\varsigma_j)e^{\lambda \tau_{kj}}z_j(t_1 - \tau_{kj})$$
$$\le -(\mu_k - \lambda)z_k(t_1) + \sum_{j=1}^{n}|\alpha_{kj}|\eta_j z_j(t_1)$$
$$+ \sum_{j=1}^{n}|\beta_{kj}|\eta_j e^{\lambda \tau_{kj}}[\sup_{\theta \in [t_1 - \tau, t_1]} z_j(\theta)].$$

Herein, the positive invariance property of $\mathbf{\Lambda^w}$ can be verified using the same treatment as the proof of Theorem 3.1, under condition $\beta_{ii} > 0, i = 1, \ldots, n$, for activation functions in class $\mathcal{A}$. Due to $G_k(\lambda) > 0$, we obtain a contradiction that

$$0 \le \frac{\mathrm{d}z_k(t_1)}{\mathrm{d}t} \le -\Bigg\{\mu_k - \lambda - \sum_{j=1}^{n}\eta_j|\alpha_{kj}|$$
$$- \sum_{j=1}^{n}\eta_j|\beta_{kj}|e^{\lambda \tau_{kj}}\Bigg\}K\delta < 0.$$

Hence assertion (3.8) holds and $z_i(t) \le K$ for all $t > 0$, $i = 1, \ldots, n$, by taking $\delta \to 1^+$. We thus obtain $|x_i(t) - \bar{x}_i| \le e^{-\lambda t}\max_{1 \le j \le n}\{\sup_{\theta \in [-\tau, 0]}|x_j(\theta) - \bar{x}_j|\}$, for $t > 0$ and $i = 1, \ldots, n$. Therefore, $\mathbf{x}(t)$ converges to $\bar{\mathbf{x}}$ exponentially. This completes the proof. $\quad\square$

In the above theorem, we have imposed a restriction: $\beta_{ii} > 0, i = 1, \ldots, n$ (positive self-feedback delays) for the activation functions in class $\mathcal{A}$. The situation is different for the activation functions in class $\mathcal{B}$, thanks to the zero slopes of these functions in the saturated parts. In addition, for the piecewise linear functions $\bar{g}_i$ in (2.1), since the slopes $v_i := (v_i - u_i)/(q_i - p_i)$ in the middle parts are fixed, there cannot exist parameters $\mu_i, \alpha_{ij}, \beta_{ij}$, and η_i satisfying both (H₃) and (H₁$^{\mathcal{B}}$). Indeed, a contradiction arises in $\mu_i > v_i(\sum_{j=1}^{n}|\alpha_{ij}| + |\beta_{ij}|)$ versus $v_i(\alpha_{ii} + \beta_{ii}) > \mu_i$. Thus, the definition of $\mathbf{\Lambda^w}$ for the activation functions in $\mathcal{B}$ and the standard activation function g^s are as indicated in (3.5) and every $\mathbf{\Lambda^w}$ lies in the saturated parts corresponding to the activation functions.

Corollary 3.3. *Each of these 2^n subsets $\mathbf{\Lambda^w}$ of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$, defined in (3.6), lies in the basin of attraction for the unique equilibrium in $\mathbf{\Lambda^w}$, under the assumptions of Theorem 3.2.*

Corollary 3.4. *Under condition $\alpha_{ii} + \beta_{ii} - \sum_{j=1, j \ne i}^{n}(|\alpha_{ij}| + |\beta_{ij}|) - |I_i| > \mu_i, i = 1, \ldots, n$, there exist 2^n exponentially stable equilibria for (1.1) with activation function g^s in (2.2).*

Proof. The condition yields (2.5), and (H₂) with $\tilde{p}_i = -1$ and $\tilde{q}_i = 1$ for all $i = 1, \ldots, n$. The assertion hence holds. $\quad\square$

Remark 3.1. (i) There exists a globally attracting set for system (1.1), according to [30]. Therefore, every solution of system (1.1) is bounded in forward time.

(ii) System (1.1) with $\mu_i = 1, i = 1, \ldots, n$, and the standard activation function g^s was investigated in [26]. It was proved therein that under condition

$$\alpha_{ii} - \sum_{j=1, j \ne i}^{n}|\alpha_{ij}| - \sum_{j=1}^{n}|\beta_{ij}| - |I_i| > 1,$$
$$i = 1, \ldots, n, \tag{3.9}$$

there exist exactly 2^n exponentially stable isolated equilibria. It is obvious that our condition in Corollary 3.4 is weaker than condition (3.9). In addition, it was shown that the set $\{\mathbf{x} \mid \mathbf{x} = (x_1, \ldots, x_n), x_i < -1 \text{ or } x_i > 1\}$ is positively invariant. Our Theorem 3.1 has exploited a larger positively invariant set $\tilde{\mathbf{\Lambda}}^\mathbf{w}$. The computations in deriving the results in [26] heavily depend on the saturation of the activation functions. Restated, as $x_j(t - \tau_{ij})$ lies in $\{\xi < -1\}$ or $\{\xi > 1\}$, $g^s(x_j(t - \tau_{ij}))$ is either -1 or 1, and thus the delays in (1.1) do not have any actual effect in these regions. The numerical simulations therein thus dealt with ordinary differential equations. As mentioned in Section 2, the transition from the linear regime to the saturated regime in the standard activation function is smooth in a practical situation. Our theory is based on a geometrical observation and has been established with these practical considerations being taken into account.

(iii) It can be further justified that the basins of attraction for the equilibria are actually larger than $\mathbf{\Lambda^w}$, through additional derivations and estimates. In fact, they are at least as large as the positively invariant sets $\tilde{\mathbf{\Lambda}}^\mathbf{w}$. The justification can be found in [31]. We have shown

(Theorem 3.2) that the solutions lying entirely in $\boldsymbol{\Lambda}^{\mathbf{w}}$ converge exponentially to the respective equilibrium in $\boldsymbol{\Lambda}^{\mathbf{w}}$. However, the convergence for the solutions lying entirely in $\tilde{\boldsymbol{\Lambda}}^{\mathbf{w}}$ may not be of exponential rate.

4. Periodic orbits for systems with periodic inputs

In this section, we study the periodic solutions of the delayed neural networks with periodic input:

$$
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t))
$$

$$
+ \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t - \tau_{ij})) + J_i(t), \qquad (4.1)
$$

where $i = 1, \ldots, n$, each $J_i : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a continuous function of period T, i.e., $J_i(t + T) = J_i(t)$ for all $t \geq 0$. There have been investigations on the existence of a single periodic solution for system (4.1), cf. [18]. The results in this direction of study can be achieved by constructing a suitable Lyapunov functional or Poincaré mapping. In this section, we establish existence of multiple stable periodic solutions via constructing suitable Poincaré mapping.

Theorem 4.1. *Under conditions* $(H_1{}^{\mathcal{A}})$, (H_2), (H_3), *and* $\beta_{ii} > 0, i = 1, \ldots, n$, *there exist* 2^n *exponentially stable T-period solutions for system* (4.1) *with activation functions of class $\mathcal{A}$. The same conclusion holds for* (4.1) *with activation functions of class $\mathcal{B}$, under conditions* $(H_1{}^{\mathcal{B}})$, (H_2).

Proof. Recall the notations in Section 2: $\mathbf{x}(t; \phi) = (x_1(t; \phi), \ldots, x_n(t; \phi))$, the solution of (4.1) with $\mathbf{x}(\theta; \phi) = \phi(\theta), \theta \in [-\tau, 0]$, and $\mathbf{x}_t(\theta; \phi) = \mathbf{x}(t + \theta; \phi), \theta \in [-\tau, 0], t \geq 0$. Consider $\varphi, \psi \in \boldsymbol{\Lambda}^{\mathbf{w}}$, for some $\mathbf{w} = (w_1, \ldots, w_n)$, with $w_i = $ "l" or "r", defined in (3.6). Then $\mathbf{x}(t; \varphi), \mathbf{x}(t; \psi) \in \boldsymbol{\Lambda}^{\mathbf{w}}$ for all $t \geq 0$, by positive invariance of $\boldsymbol{\Lambda}^{\mathbf{w}}$. From (4.1) we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}[x_i(t; \varphi) - x_i(t; \psi)] = -\mu_i[x_i(t; \varphi) - x_i(t; \psi)]
$$

$$
+ \sum_{j=1}^{n} \alpha_{ij} \left[g_j(x_j(t; \varphi)) - g_j(x_j(t; \psi)) \right]
$$

$$
+ \sum_{j=1}^{n} \beta_{ij} \left[g_j(x_j(t - \tau_{ij}; \varphi)) - g_j(x_j(t - \tau_{ij}; \psi)) \right],
$$

for $t \geq 0, i = 1, 2, \ldots, n$. Similar to the proof of Theorem 3.2, we obtain

$$
|x_i(t; \varphi) - x_i(t; \psi)| \leq \mathrm{e}^{-\bar{\lambda}t} \max_{1 \leq j \leq n} \left(\sup_{\theta \in [-\tau, 0]} |x_j(\theta; \varphi) - x_j(\theta; \psi)| \right),
$$

for $t \geq 0, i = 1, \ldots, n$, where $\bar{\lambda} > 0$ is a small constant. Therefore,

$$
\|x_i(t + \theta; \varphi) - x_i(t + \theta; \psi)\| \leq \mathrm{e}^{-\bar{\lambda}(t+\theta)} \|\varphi - \psi\|
$$

$$
\leq \mathrm{e}^{-\bar{\lambda}(t-\tau)} \|\varphi - \psi\|,
$$

for $\theta \in [-\tau, 0]$ and $t \geq \tau$, and then

$$
\|\mathbf{x}_t(\theta; \varphi) - \mathbf{x}_t(\theta; \psi)\| \leq \mathrm{e}^{-\bar{\lambda}(t-\tau)} \|\varphi - \psi\|, \quad \text{for all } t \geq 0.
$$
$$(4.2)$$

We choose a positive integer m such that $\mathrm{e}^{-\bar{\lambda}(mT-\tau)} = \kappa < 1$. Define a Poincaré mapping $P : \boldsymbol{\Lambda}^{\mathbf{w}} \to \boldsymbol{\Lambda}^{\mathbf{w}}$ by $P(\varphi) = \mathbf{x}_T(\cdot, \varphi)$. Then we can derive from (4.2) that

$$
\|P^m(\varphi) - P^m(\psi)\| \leq \kappa \|\varphi - \psi\|.
$$

This inequality implies that P^m is a contraction mapping, hence there exists a unique fixed point $\overline{\phi} \in \boldsymbol{\Lambda}^{\mathbf{w}}$ such that $P^m(\overline{\phi}) = \overline{\phi}$. Note that

$$
P^m(P\overline{\phi}) = P(P^m\overline{\phi}) = P(\overline{\phi}).
$$

Thus $P(\overline{\phi}) \in \boldsymbol{\Lambda}^{\mathbf{w}}$ is also a fixed point of P^m, and so $P(\overline{\phi}) = \overline{\phi}$, i.e. $\mathbf{x}_T(\cdot; \overline{\phi}) = \overline{\phi}$. Let $\mathbf{x}(t; \overline{\phi})$ be the solution of (4.1) with initial condition $\overline{\phi}$ at $t = 0$, then $\mathbf{x}(t + T; \overline{\phi})$ is also a solution of (4.1). Note that $\mathbf{x}_{t+T}(\cdot; \overline{\phi}) = \mathbf{x}_t(\cdot; \mathbf{x}_T(\cdot; \overline{\phi})) = \mathbf{x}_t(\cdot; \overline{\phi})$, for all $t \geq 0$; therefore

$$
\mathbf{x}(t + T; \overline{\phi}) = \mathbf{x}(t; \overline{\phi}), \quad \text{for all } t \geq 0.
$$

This shows that $\mathbf{x}(t; \overline{\phi})$ is exactly a T-period solution of (4.1) in $\boldsymbol{\Lambda}^{\mathbf{w}}$, and all other solutions of (4.1) in $\boldsymbol{\Lambda}^{\mathbf{w}}$ converge exponentially to it as $t \to +\infty$. Thus, there exist 2^n exponentially stable T-period solutions for system (4.1). $\quad\square$

We remark that the above result can be extended to the existence of 2^n stable periodic solutions of the delayed neural network (4.1) with periodic coefficients and connection weights, i.e., the situation that $\mu_i = \mu_i(t), \alpha_{ij} = \alpha_{ij}(t), \beta_{ij} = \beta_{ij}(t)$ are T-periodic. The extension can be confirmed by modifying the assumptions to adapt to the considered circumstances and by similar arguments as the proof of Theorem 4.1. The investigation can also be generalized to almost periodic coefficients and solutions.

5. Quasiconvergence

We shall discuss the monotone property for system (1.1) with activation functions of class $\mathcal{A}$ in this section. The derivation can be adapted to activation functions of class $\mathcal{B}$. Let us first recall the following definition.

Definition 5.1. Let $\mathcal{E}$ be the set of all equilibrium points for a system with phase space $\mathcal{C}$. We say that $\phi \in \mathcal{C}$ is a quasiconvergent point if its ω-limit set $\omega(\phi) \subset \mathcal{E}$. The set of such points is denoted by Q. A point $\phi \in \mathcal{C}$ is called a convergent point, if $\omega(\phi)$ consists of a single point of $\mathcal{E}$.

Note that quasiconvergence yields convergence for continuous-time dynamical systems, if all equilibria are isolated. In order to apply the theory of monotone dynamical systems, we need the following notations and definitions. Consider the standard componentwise partial order "$\leq$" and inequality "$<$" on $\mathbb{R}^n$:

$$
\mathbf{x} \leq \mathbf{y} \Leftrightarrow x_i \leq y_i, \quad \text{for all } i,
$$

$$
\mathbf{x} < (\ll)\mathbf{y} \Leftrightarrow \mathbf{x} \leq \mathbf{y} \quad \text{and} \quad x_i < y_i \quad \text{for some (all) } i.
$$

Then the partial order "$\leq$", called the standard order, and the inequality "$<$" on $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ are defined by

$$\phi \leq \psi \Leftrightarrow \phi(\theta) \leq \psi(\theta) \quad \text{for all } \theta \in [-\tau, 0],$$
$$\phi < \psi \Leftrightarrow \phi \leq \psi \quad \text{and} \quad \phi \neq \psi,$$
$$\phi \ll \psi \Leftrightarrow \phi(\theta) \ll \psi(\theta) \quad \text{for all } \theta \in [-\tau, 0].$$

Definition 5.2. Let "$<$" be a partial order. (i) A semiflow Φ is said to be monotone provided $\Phi_t(\phi) \leq \Phi_t(\psi)$ whenever $\phi \leq \psi$ and $t \geq 0$. (ii) Φ is called strongly order preserving (SOP), if it is monotone and whenever $\phi < \psi$, there exist open subsets U, V of $\mathcal{C}$ with $\phi \in U$ and $\psi \in V$ and $t_0 > 0$ such that $\Phi_{t_0}(U) \leq \Phi_{t_0}(V)$.

It has been shown in [32] that if the phase space can be approximated from below or above, then $\text{Int}\,Q$ is dense in $\mathcal{C}$ for a SOP system, under a compactness assumption. The assumptions of this theorem can all be justified in our situation herein.

Notably, the one-dimensional delayed equation

$$\frac{dx(t)}{dt} = -ax(t) + bg(x(t - \tau)), \quad a > 0, b < 0,$$

fails to be monotone under the standard ordering in $\mathcal{C}$; so do the higher-dimensional cases [32]. We shall adopt a special order introduced in [33] to conclude the monotone behavior for system (1.1). Let M be an $n \times n$ essentially nonnegative matrix, which means that $M + \lambda I$ is entrywise nonnegative for all sufficiently large λ. Define

$$K_M = \{\psi \in \mathcal{C} | \psi \geq 0 \text{ and } e^{-tM}\psi(t) \geq e^{-sM}\psi(s),$$
$$\text{for } -\tau \leq s \leq t \leq 0\}. \tag{5.1}$$

Then K_M is a cone in the space $\mathcal{C}$, that is, under addition and scalar multiplication by nonnegative scalars, K_M is closed in $\mathcal{C}$ and $K_M \cap (-K_M) = \emptyset$. Moreover, K_M is a normal cone, which means that every order interval is a bounded set in $\mathcal{C}$. According to [33], K_M induces a partial order on $\mathcal{C}$.

Definition 5.3. If $\phi, \psi \in \mathcal{C}$, we say $\phi \leq_M \psi$ whenever $\psi - \phi \in K_M$. We write $\phi <_M \psi$ to indicate that $\phi \leq_M \psi$ and $\phi \neq \psi$.

Consider the functional differential equation, with $\tilde{F} \in C^1(\mathcal{C}, \mathbb{R}^n)$

$$\frac{d\mathbf{x}(t)}{dt} = \tilde{F}(\mathbf{x}_t). \tag{5.2}$$

Theorem 5.1 ([13,33]). *The semiflow Φ generated by (5.2) is SOP on $\mathcal{C}$ under order "$\leq_M$", if the following conditions hold:*

(i) *$d\tilde{F}(\phi)\psi - M\psi(0) \gg 0$ for every $\phi \in \mathcal{C}$ and every $\psi \in K_M$ with $\psi \gg 0$,*

(ii) *If $\phi \in \mathcal{C}$, $\psi \in K_M$ and L is a (nonempty) proper subset of $\{1, \ldots, n\}$ such that $\psi_j \gg 0$ for $j \in L$ and $\psi_k(0) = 0$ for $k \notin L$, then $(d\tilde{F}(\phi)\psi)_i > 0$, for some $i \notin L$.*

Herein, we set the $n \times n$ matrix $M = \text{diag}(-\mu_1 - \nu_1, \ldots, -\mu_n - \nu_n)$, where $\nu_i > 0$ will be chosen later. Indeed,

the matrix M is essentially nonnegative. An $n \times n$ matrix $A = [A_{ij}]$ is called *irreducible* if whenever the set $\{1, \ldots, n\}$ is expressed as the union of two disjoint proper subsets S, S', then for every $i \in S$ there exists $j, k \in S'$ such that $A_{ij} \neq 0$, $A_{ki} \neq 0$. Let $\gamma_i = \sup_{\xi \in \mathbb{R}} g'_i(\xi)$.

Proposition 5.2. *Assume that one of the matrices A and B is irreducible, where $A = [\alpha_{ij}]$, $B = [\beta_{ij}]$, $\alpha_{ij} \geq 0$, $\beta_{ij} \geq 0$ for all $i \neq j$, $\alpha_{ii} + \beta_{ii} > 0$ for all i, and the time lags $\{\tau_{ij}\}$ satisfy*

$$\tau_{ii} \leq 1/(\mu_i + e|\beta_{ii}|\gamma_i), \tag{5.3}$$

for all i with $\beta_{ii} < 0$. Then the semiflow Φ generated by the solutions of system (1.1) is SOP under order "$\leq_M$".

Proof. Recall the previous definition of $\tilde{F}$ defined in (1.1):

$$\tilde{F}_i(\phi) = -\mu_i\phi_i(0) + \sum_{j=1}^{n} \alpha_{ij}g_j(\phi_j(0))$$

$$+ \sum_{j=1}^{n} \beta_{ij}g_j(\phi_j(-\tau_{ij})) + I_i, \quad i = 1, \ldots, n.$$

For any $\phi = (\phi_1, \ldots, \phi_n) \in \mathcal{C}$ and $\psi = (\psi_1, \ldots, \psi_n) \in K_M$ with $\psi \gg 0$, we have

$$(d\tilde{F}(\phi)\psi)_i - (M\psi(0))_i$$

$$= \nu_i\psi_i(0) + \sum_{j=1}^{n} \alpha_{ij}g'_j(\phi_j(0))\psi_j(0)$$

$$+ \sum_{j=1}^{n} \beta_{ij}g'_j(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}) \tag{5.4}$$

$$\geq [(\nu_i e^{-\tau_{ii}(\mu_i + \nu_i)} + \beta_{ii}g'_i(\phi_i(-\tau_{ii}))]\psi_i(-\tau_{ii})$$

$$+ \alpha_{ii}g'_i(\phi_i(0)))\psi_i(0)$$

$$+ \sum_{j=1, j\neq i}^{n} \alpha_{ij}g'_j(\phi_j(0))\psi_j(0)$$

$$+ \sum_{j=1, j\neq i}^{n} \beta_{ij}g'_j(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}), \tag{5.5}$$

since $\psi_i(0) \geq e^{-\tau_{ii}(\mu_i + \nu_i)}\psi(-\tau_{ii})$, from $\psi \in K_M$, and $\psi(0) \geq e^{-sM}\psi(s)$, for all $s \in [-\tau, 0]$. Here, we take $\nu_i > 0$ satisfying $\nu_i = e|\beta_{ii}|\gamma_i$. If $\beta_{ii} < 0$, then $\alpha_{ii} > 0$, and the assumption $\tau_{ii} \leq 1/(\mu_i + e|\beta_{ii}|\gamma_i)$ yields $\nu_i \exp[-\tau_{ii}(\mu_i + \nu_i)] + \beta_{ii}g'_i(\phi_i(-\tau_{ii})) > 0$. Thus $(d\tilde{F}(\phi)\psi)_i - (M\psi(0))_i > 0$, from (5.5). When $\beta_{ii} \geq 0$, $(d\tilde{F}(\phi)\psi)_i - (M\psi(0))_i > 0$ follows from $\nu_i + \alpha_{ii}\gamma_i > 0$ and (5.4). Next, we will prove that condition (ii) in Theorem 5.1 holds. For any $\phi \in \mathcal{C}$ and $\psi \in K_M$, let L be a (nonempty) proper subset of $\{1, \ldots, n\}$ such that $\psi_j \gg 0$ for $j \in L$ and $\psi_k(0) = 0$ for $k \notin L$. Then $\psi_i(-\tau_{ii}) = 0$ for each $i \notin L$, due to $\psi_i(-\tau_{ii}) \leq \exp[\tau_{ii}(\mu_i + \nu_i)]\psi_i(0)$. Since one of matrices A and B is irreducible, there is some $i \notin L$ such that

$$(d\tilde{F}(\phi)\psi)_i = -\mu_i\psi_i(0) + \sum_{j=1}^{n} \alpha_{ij}g'_j(\phi_j(0))\psi_j(0)$$

$$+ \sum_{j=1}^{n} \beta_{ij}g'_j(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij})$$

$$
\begin{aligned}
&= \sum_{j=1, j\neq i}^{n} \alpha_{ij} g'_j(\phi_j(0))\psi_j(0) \\
&\quad + \sum_{j=1, j\neq i}^{n} \beta_{ij} g'_j(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}) \\
&= \sum_{j\in L} \alpha_{ij} g'_j(\phi_j(0))\psi_j(0) \\
&\quad + \sum_{j\in L} \beta_{ij} g'_j(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}) > 0.
\end{aligned}
$$

Hence, it follows from Theorem 5.1 that the semiflow Φ generated by the solutions of (1.1) is SOP under order "$\leq_M$".
□

Notably, condition (H_1^A) yields $\alpha_{ii} + \beta_{ii} > 0$ for all i. Thus, under conditions (H_1^A) and (H_2), and the assumptions in Proposition 5.2, there exist 3^n equilibria for (1.1) and $\mathrm{int}\,Q$ is dense in $\mathcal{C}$. In fact, the assumptions of irreducibility of A, B and *non-inhibitory interactions*, $\alpha_{ij}, \beta_{ij} \geq 0$ for all $i \neq j$, can be removed via a *decomposition approach* in competitive–cooperative systems after imbedding the network into a larger system. Such a technique was adopted to study global convergence to an unique equilibrium in [34,35]. It was previously employed by Cosner [36] and Wu and Zhao [37] in the study of population dynamics. This decomposition approach fits in with our formulation for multiple equilibria pertinently and skillfully.

Theorem 5.3. *Assume that* (H_1^A) *and* (H_2) *hold and the delay time* $\{\tau_{ij}\}$ *satisfy* (5.3). *Then system* (1.1) *has* 3^n *equilibria and* $\mathrm{int}\,Q$ *is dense in* $\mathcal{C}$.

Proof. Define matrices $A^+ = [a_{ij}^+]$, $A^- = [a_{ij}^-]$, $B^+ = [b_{ij}^+]$ and $B^- = [b_{ij}^-]$ by

$$
a_{ij}^+ = \begin{cases} \alpha_{ii}, & \text{for } j = i \\ \alpha_{ij}^+ + s, & \text{for } j \neq i, \end{cases} \quad a_{ij}^- = \begin{cases} 0, & \text{for } j = i \\ \alpha_{ij}^- + s, & \text{for } j \neq i, \end{cases}
$$

$$
b_{ij}^+ = \begin{cases} \beta_{ii}, & \text{for } j = i \\ \beta_{ij}^+ + s, & \text{for } j \neq i, \end{cases} \quad b_{ij}^- = \begin{cases} 0, & \text{for } j = i \\ \beta_{ij}^- + s, & \text{for } j \neq i, \end{cases}
$$

where $\alpha_{ij}^+ = \max\{\alpha_{ij}, 0\}$, $\alpha_{ij}^- = \max\{-\alpha_{ij}, 0\}$, similarly for $\beta_{ij}^+, \beta_{ij}^-$; $s > 0$ will be suitably chosen. Since $\alpha_{ij} = a_{ij}^+ - a_{ij}^-$ and $\beta_{ij} = b_{ij}^+ - b_{ij}^-$, system (1.1) becomes

$$
\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -\mu_i x_i(t) + \sum_{j=1}^{n} a_{ij}^+ g_j(x_j(t)) \\
&\quad - \sum_{j=1}^{n} a_{ij}^- g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}^+ g_j(x_j(t-\tau_{ij})) \\
&\quad - \sum_{j=1}^{n} b_{ij}^- g_j(x_j(t-\tau_{ij})) + I_i,
\end{aligned}
\tag{5.6}
$$

$i = 1, \ldots, n$. Define $y_i = -x_i$, and set $\tilde{g}_i(\xi) = -g_i(-\xi)$, $i = 1, \ldots, n$. Then (5.6) is embedded into the following system:

$$
\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -\mu_i x_i(t) + \sum_{j=1}^{n} a_{ij}^+ g_j(x_j(t)) \\
&\quad + \sum_{j=1}^{n} a_{ij}^- \tilde{g}_j(y_j(t)) + \sum_{j=1}^{n} b_{ij}^+ g_j(x_j(t-\tau_{ij})) \\
&\quad + \sum_{j=1}^{n} b_{ij}^- \tilde{g}_j(y_j(t-\tau_{ij})) + I_i
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} &= -\mu_i y_i(t) + \sum_{j=1}^{n} a_{ij}^- g_j(x_j(t)) \\
&\quad + \sum_{j=1}^{n} a_{ij}^+ \tilde{g}_j(y_j(t)) + \sum_{j=1}^{n} b_{ij}^- g_j(x_j(t-\tau_{ij})) \\
&\quad + \sum_{j=1}^{n} b_{ij}^+ \tilde{g}_j(y_j(t-\tau_{ij})) - I_i,
\end{aligned}
\tag{5.7}
$$

$i = 1, \ldots, n$. Note that each $\tilde{g}_i$ also admits the characteristics of g_i. We define $z_k(t)$ and $h_k(\xi)$ by $z_i(t) = x_i(t)$, $z_{n+i}(t) = y_i(t)$, and $h_i(\xi) = g_i(\xi)$, $h_{n+i}(\xi) = \tilde{g}_i(\xi)$, for $i = 1, \ldots, n$. Then (5.7) can be written as

$$
\begin{aligned}
\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} &= -\tilde{\mu}_i z_i(t) + \sum_{j=1}^{2n} \tilde{a}_{ij} h_j(z_j(t)) \\
&\quad + \sum_{j=1}^{2n} \tilde{b}_{ij} h_j(z_j(t-\tilde{\tau}_{ij})) + \tilde{I}_i,
\end{aligned}
\tag{5.8}
$$

$i = 1, \ldots, 2n$, where the $2n \times 2n$ matrices $\tilde{A}$ and $\tilde{B}$ are defined by

$$
\tilde{A} = [\tilde{a}_{ij}] = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}, \qquad \tilde{B} = [\tilde{b}_{ij}] = \begin{bmatrix} B^+ & B^- \\ B^- & B^+ \end{bmatrix},
$$

and $\tilde{\mu}_i$, $\tilde{I}_i$, $\tilde{\tau}_{ij}$ are given by $\tilde{\mu}_i = \mu_i$, $\tilde{\mu}_{n+i} = \mu_i$; $\tilde{I}_i = I_i$, $\tilde{I}_{n+i} = -I_i$, $i = 1, \ldots, n$; $\tilde{\tau}_{ij} = \tilde{\tau}_{n+i,j} = \tilde{\tau}_{i,n+j} = \tilde{\tau}_{n+i,n+j} = \tau_{ij}$, $i, j = 1, \ldots, n$.

Note that $\tilde{A}, \tilde{B}$ as defined are both irreducible; in addition, $\tilde{a}_{ij} > 0, \tilde{b}_{ij} > 0$, for all $i \neq j$. System (5.8) thus satisfies the assumptions except that $\tilde{a}_{ii} + \tilde{b}_{ii} > 0$, for all i, in Proposition 5.2. It can be justified that conditions (H_1^A) and (H_2) for system (1.1) yield conditions analogous to (H_1^A) and (H_2) for system (5.8), by choosing sufficiently small $s > 0$. Thus, $\tilde{a}_{ii} + \tilde{b}_{ii} > 0$, for all i, and the semiflow generated by the solutions of system (5.8) is SOP. Therefore, there exist 3^{2n} equilibria and $\mathrm{Int}\,Q$ is dense in $\mathcal{C}([-\tau, 0], \mathbb{R}^{2n})$ for system (5.8). On the other hand, one also observes that if $x_i(0) + y_i(0) = 0$, then $x_i(t) + y_i(t) = 0$ for all $t \geq 0$ for solutions $(x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t))$ of system (5.7). Restated, the dynamics of system (5.8) on the positively invariant regions $\{x_1 = -y_1, \ldots, x_n = -y_n\}$ are exactly the dynamics for system (1.1). Thereafter, there exist 3^n equilibria for (1.1) and $\mathrm{Int}\,Q$ is dense in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ for system (1.1), if (H_1^A), (H_2) and condition (5.3) hold. □
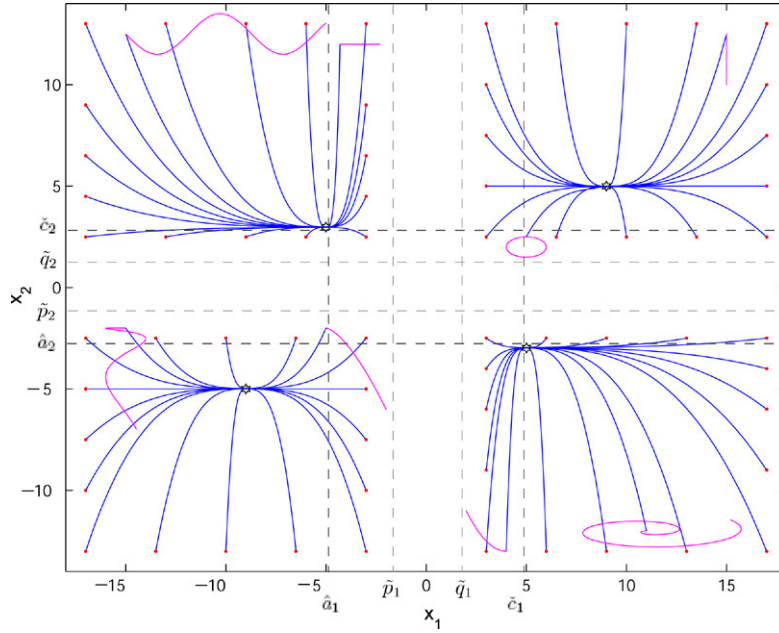
Fig. 5. Illustration for the dynamics in Example 6.1. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

6. Numerical illustrations

In this section, four two-dimensional examples of system (1.1) are presented to illustrate our theory. In particular, Example 6.2 demonstrates the multistability of system (1.1) with the standard activation function (2.2). This example adopts parameters satisfying the criteria in our theory but not the one in [26]. Example 6.3 demonstrates Theorem 4.1. The parameters in Example 6.4 satisfy conditions (H$_1$*), $* = \mathcal{A}, \mathcal{B}$, and (H$_2$), but not (H$_3$).

Example 6.1. Consider the following system with activation functions $g_1(\xi) = g_2(\xi) = \tanh(\xi)$, which belongs to class $\mathcal{A}$:

$$\frac{dx_1(t)}{dt} = -x_1(t) + 4g_1(x_1(t)) + g_2(x_2(t))$$
$$+ 3g_1(x_1(t-10)) + g_2(x_2(t-10))$$
$$\frac{dx_2(t)}{dt} = -3x_2(t) + 2g_1(x_1(t)) + 7g_2(x_2(t))$$
$$+ g_1(x_1(t-10)) + 5g_2(x_2(t-10)).$$

Direct computation gives $\hat{f}_1(x_1) = -x_1 + 7g(x_1) + 2$, $\check{f}_1(x_1) = -x_1 + 7g(x_1) - 2$, $\hat{f}_2(x_2) = -3x_2 + 12g(x_2) + 3$, $\check{f}_2(x_2) = -3x_2 + 12g(x_2) - 3$. Herein, the parameters satisfy our conditions in Theorem 3.2:

Condition (H$_1$$^\mathcal{A}$):

$$0 < \mu_1/(\alpha_{11} + \beta_{11}) = 1/7 < 1,$$
$$0 < \mu_2/(\alpha_{22} + \beta_{22}) = 3/12 < 1.$$

Condition (H$_2$):

$$\hat{f}_1(p_1) = -2.8524 < 0, \qquad \check{f}_1(q_1) = 2.8524 > 0,$$
$$\hat{f}_2(p_2) = -3.4414 < 0, \qquad \check{f}_2(q_2) = 3.4414 > 0.$$

Condition(H$_3$):

$$\mu_1 = 1 > 0.98 = (|\alpha_{11}| + |\beta_{11}|)\eta_1 + (|\alpha_{12}| + |\beta_{12}|)\eta_2,$$
$$\mu_2 = 3 > 1.98 = (|\alpha_{21}| + |\beta_{21}|)\eta_1 + (|\alpha_{22}| + |\beta_{22}|)\eta_2,$$

where $\eta_1 = 0.1$ and $\eta_2 = 0.14$ are chosen in (H$_3$); $\hat{a}_1 = -4.9994$, $\underline{d}_1 = -1.8184$, $p_1 = -1.6283$, $\hat{b}_1 = -0.3491$, $q_1 = 1.6283$, $\overline{d}_1 = 1.8184$, $\hat{c}_1 = 9.0000$, $\check{a}_1 = -9.0000$, $\check{b}_1 = 0.3491$, $\check{c}_1 = 4.9993$, $\hat{a}_2 = -2.9793$, $\underline{d}_2 = -1.6392$, $p_2 = -1.3170$, $\hat{b}_2 = -0.3518$, $q_2 = 1.3170$, $\overline{d}_2 = 1.6392$, $\hat{c}_2 = 4.9996$, $\check{a}_2 = -4.9996$, $\check{b}_2 = 0.3518$, $\check{c}_2 = 2.9793$. The dynamics of this system are illustrated in Fig. 5, where evolutions of 72 initial conditions have been tracked. The constant initial conditions are plotted in red color, and the time-dependent initial conditions are plotted in purple. There are four exponentially stable equilibria in the system, as confirmed by our theory. The simulation demonstrates convergence to these four equilibria from initial functions ϕ lying in the respective basin for the equilibrium.

Example 6.2. Consider the following system with the standard activation function (2.2):

$$\frac{dx_1(t)}{dt} = -x_1(t) + 2g_1(x_1(t)) + g_2(x_2(t))$$
$$+ 3g_1(x_1(t-5)) + g_2(x_2(t-5))$$
$$\frac{dx_2(t)}{dt} = -x_2(t) - g_1(x_1(t)) + 4g_2(x_2(t))$$
$$+ 2g_1(x_1(t-5)) + 5g_2(x_2(t-5)) + 1,$$

where $g_1(\xi) = g_2(\xi) = g^s(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|)$. The parameters satisfy the criterion in Corollary 3.4: $\alpha_{11} + \beta_{11} - (|\alpha_{12}| + |\beta_{12}|) - |I_1| = 3 > 1 = \mu_1$, $\alpha_{22} + \beta_{22} - (|\alpha_{21}| + |\beta_{21}|) - |I_2| = 5 > 1 = \mu_2$. Therefore, there exist 2^n exponentially stable equilibria. The parameters herein do not
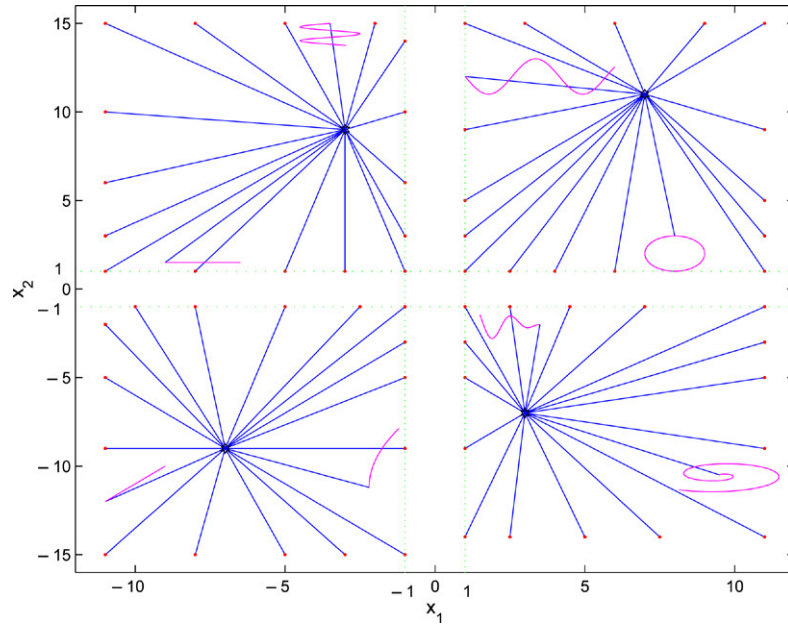
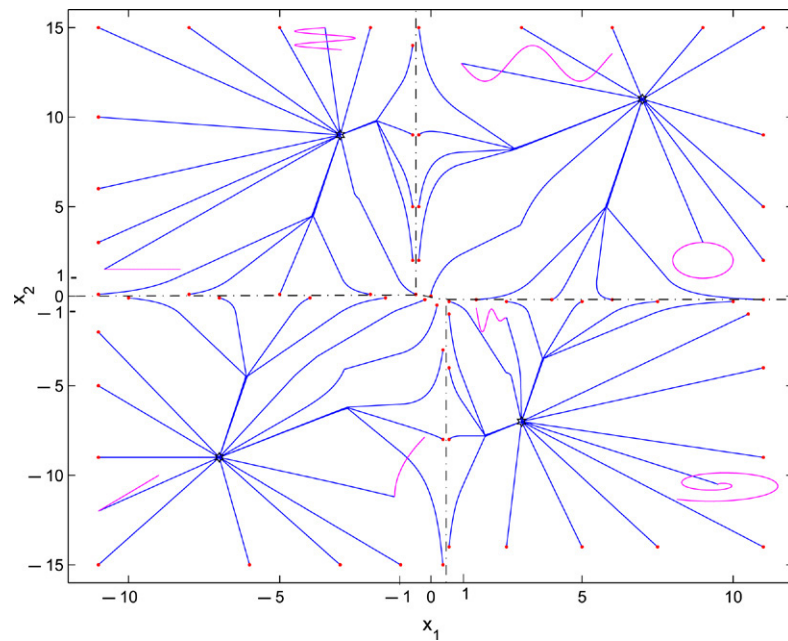Fig. 6. Illustration for the dynamics in Example 6.2 ($\tilde{p}_i = -1$, $\tilde{q}_i = 1$).

Fig. 7. Illustration for the dynamics in Example 6.2 ($\tilde{p}_i = -1$, $\tilde{q}_i = 1$).

satisfy the criterion (3.9) for the theory in [26]: $\alpha_{11} - |\alpha_{12}| - (|\beta_{11}| + |\beta_{12}|) - |I_1| = -3$ which is not greater than $\mu_1 = 1$. The dynamics of the system are illustrated in Fig. 6. We allow initial conditions from larger basins of attraction in Fig. 7, to demonstrate the assertion in Remark 3.1(iii).

Example 6.3. Consider the following system with periodic inputs and the standard activation function (2.2):

$$\frac{dx_1(t)}{dt} = -x_1(t) + 2g_1(x_1(t)) + g_2(x_2(t)) + 3g_1(x_1(t - 2)) + g_2(x_2(t - 2)) + \cos(t)$$

$$\frac{dx_2(t)}{dt} = -x_2(t) - g_1(x_1(t)) + 4g_2(x_2(t)) + 2g_1(x_1(t - 2)) + 5g_2(x_2(t - 2)) + 1 + \sin(t),$$

where $g_1(\xi) = g_2(\xi) = g^s(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|)$. The existence of four limit cycles for the system is illustrated in Fig. 8.

Example 6.4. Consider the following system with activation functions $g_1(\xi) = g_2(\xi) = \tanh(\xi)$, which belongs to class $\mathcal{A}$,

$$\frac{dx_1(t)}{dt} = -x_1(t) + 7g_1(x_1(t)) + 0.5g_2(x_2(t)) - 4g_1(x_1(t - \tau_{11})) + 0.5g_2(x_2(t - \tau_{12}))$$
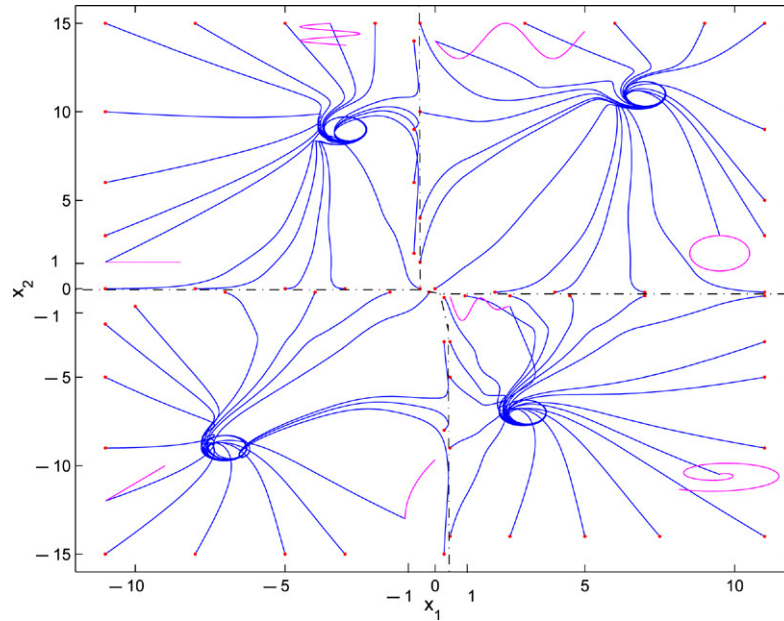
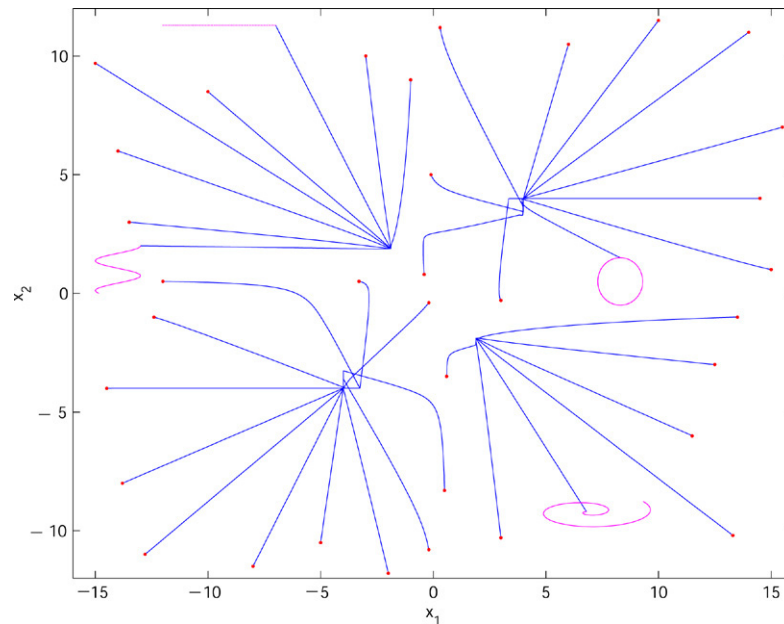Fig. 8. Illustration for the dynamics in Example 6.3 ($\tilde{p}_i = -1$, $\tilde{q}_i = 1$).

Fig. 9. Illustration for the dynamics in Example 6.4 with activation function $g_i(\xi) = \tanh(\xi)$ and $\tau_{11} = 0.08$, $\tau_{12} = 10$, $\tau_{21} = 10$, $\tau_{22} = 0.08$.

$$\frac{dx_2(t)}{dt} = -x_2(t) + 0.5g_1(x_1(t)) + 7g_2(x_2(t)) + 0.5g_1(x_1(t - \tau_{21})) - 4g_2(x_2(t - \tau_{22})).$$

Direct computation gives $\hat{f}_1(x_1) = -x_1 + 7g(x_1) + 2$, $\check{f}_1(x_1) = -x_1 + 7g(x_1) - 2$, $\hat{f}_2(x_2) = -3x_2 + 12g(x_2) + 3$, $\check{f}_2(x_2) = -3x_2 + 12g(x_2) - 3$. Herein, the parameters satisfy condition $(H_1{}^{\mathcal{A}}) : 0 < \mu_1/(\alpha_{11} + \beta_{11}) = \mu_2/(\alpha_{22} + \beta_{22}) = 1/3 < 1$, and condition (H_2) : $\hat{f}_1(p_1) = -2.8524 < 0$, $\check{f}_1(q_1) = 2.8524 > 0$, $\hat{f}_2(p_2) = -3.4414 < 0$, $\check{f}_2(q_2) = 3.4414 > 0$. In addition, $\hat{a}_1 = -1.8572$, $p_1 = -1.1462$, $\hat{b}_1 = -0.5902$, $q_1 = 1.1462$, $\hat{c}_1 = 3.9980$, $\check{a}_1 = -3.9980$, $\check{b}_1 = 0.5902$, $\check{c}_1 = 1.8572$, $\hat{a}_2 = -1.8572$, $p_2 = -1.1462$, $\hat{b}_2 = -0.5902$,

$q_2 = 1.1462$, $\hat{c}_2 = 3.9980$, $\check{a}_2 = -3.9980$, $\check{b}_2 = 0.5902$, $\check{c}_2 = 1.8572$. Note that $g'(\xi)$ is decreasing for $\xi > 0$ and increasing for $\xi < 0$. Condition (H_3) does not hold since $\mu_1 = 1 < (|\alpha_{11}| + |\beta_{11}|)g'(\hat{a}_1) + (|\alpha_{12}| + |\beta_{12}|)g'(\hat{a}_1) \simeq 11 \times 0.0929 + 1 \times 0.0929 = 1.1148$. We choose $\tau_{11} = 0.08$, $\tau_{12} = 10$, $\tau_{21} = 10$, $\tau_{22} = 0.08$ to satisfy (5.3): $\tau_{11} = \tau_{22} = 0.08 < 1/(1 + 4e) \simeq 0.08475$. The dynamics of this system are illustrated in Fig. 9.

Fig. 10 depicts the dynamics for the system with the same parameters but with time lags $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = 10$, which do not satisfy criterion (5.3). It appears that two of the four equilibria become unstable. The dynamics are apparently different if we replace the activation function $\tanh(\xi)$ by the
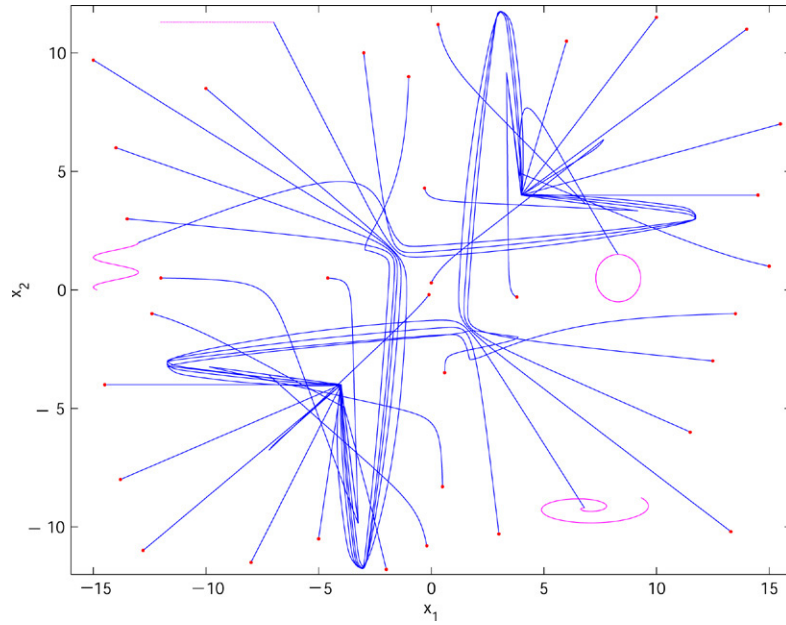
Fig. 10. Illustration for the dynamics in Example 6.4 with activation function $g_i(\xi) = \tanh(\xi)$ and $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = 10$.
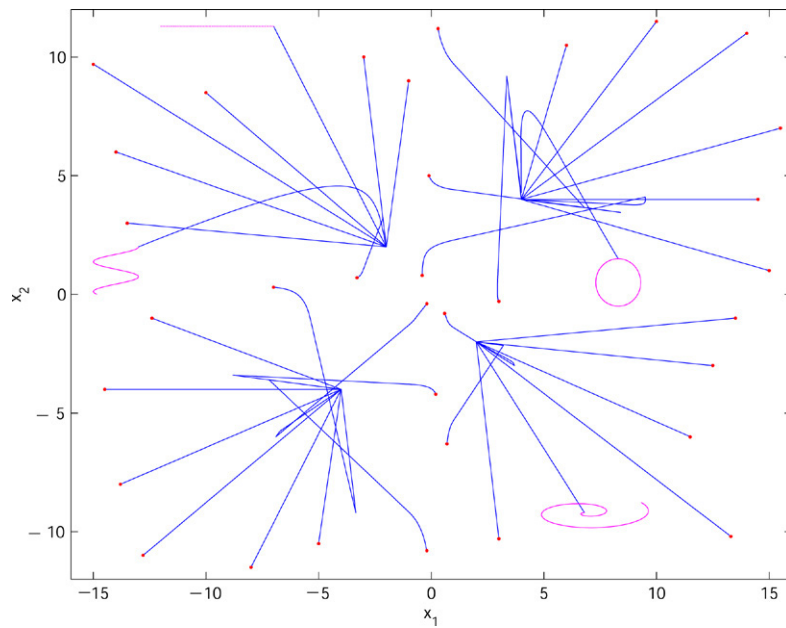

Fig. 11. Illustration for the dynamics in Example 6.4 with the standard activation function $g_i(\xi) = g^s(\xi) = \frac{1}{2}(|\xi+1| - |\xi-1|)$ and $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = 10$.

standard activation function $g^s(\xi) = \frac{1}{2}(|\xi+1| - |\xi-1|)$. There still exist four stable equilibria, as illustrated in Fig. 11.

7. Conclusions

With a geometrical observation, parameter conditions (H$_1$*) and (H$_2$) assuring the existence of 3^n stationary solutions for a general n-dimensional neural network with delays, have been derived. Under the same conditions, 2^n out of these 3^n equilibria are stable if the activation functions of class $\mathcal{B}$ are employed for the system. Additional assumptions (H$_3$) and $\beta_{ii} > 0$ are required to guarantee the same assertion of multiple stable equilibria, if activation functions of class $\mathcal{A}$ are adopted.

Further analysis has been performed to establish existence of 2^n limit cycles for the network with time-periodic inputs. The derived parameter conditions are concrete and can be examined easily. We have also applied the theory of monotone dynamics to confirm the strongly order preserving property for the networks. Subsequently, that generic points in the phase space are quasiconvergent as well as existence of 3^n equilibria comprise the phase structure for the system, under conditions (H$_1$*), (H$_2$), and that delays τ_{ii} are small enough for those neurons i with $\beta_{ii} < 0$. We have provided several numerical simulations to illustrate these theories for the two-neuron cases. It is interesting to see the distinct dynamics between system (1.1) with the activation functions in class $\mathcal{A}$ and with the

ones in class $\mathcal{B}$. When we take parameters satisfying conditions $(H_1{}^*)$, (H_2), and τ_{ii} not satisfying condition (5.3), there exist 3^2 equilibria for system (1.1) with activation functions in classes $\mathcal{A}$ and $\mathcal{B}$. There are still four stable equilibria if the activation function of class $\mathcal{B}$ is employed, as illustrated in Example 6.4, and confirmed by our Theorem 3.2. However, if we adopt the activation function in class $\mathcal{A}$, two of these four equilibria become unstable.

Acknowledgments

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