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α-labeling number of trees☆

Chin-Lin Shiue[a], Hung-Lin Fu[b]

[a] *Department of Applied Mathematics, Chung Yuan Christian University, Chung Li 32023, Taiwan*
[b] *Department of Applied Mathematics, National Chiao Tung University, Hsin Chu 30050, Taiwan*

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Abstract

In this paper, we prove that the α-labeling number of trees T, $T_\alpha \leqslant \lceil r/2 \rceil n$ where $n = |E(T)|$ and r is the radius of T. This improves the known result $T_\alpha \leqslant e^{O(\sqrt{n \log n})}$ tremendously and this upper bound is very close to the upper bound $T_\alpha \leqslant n$ conjectured by Snevily. Moreover, we prove that a tree with n edges and radius r decomposes K_t for some $t \leqslant (r+1)n^2 + 1$.

Keywords: α-labeling number; Tree decomposition

1. Introduction and preliminaries

Throughout this paper, all graphs we consider are finite and simple, i.e., multiple edges and loops are not allowed. For terms and notations used in this paper we refer to the textbook by West [10]. Let G be a graph. For any two vertices u and v, the distance from u to v, denoted by $d(u, v)$, is the least length of a u, v-path. If G has no such path, then $d(u, v) = \infty$. The eccentricity of a vertex u, written $e(u)$, is $\max_{v \in V(G)} d(u, v)$. The radius of a graph G is $\min_{u \in V(G)} e(u)$. Clearly, a tree of n edges has radius at most $\lceil n/2 \rceil$.

Let G be a graph with q edges. An injective function $f: V(G) \to S$, S is a set of nonnegative integers, has been called a vertex labeling, a valuation, or a vertex numbering of G. For convenience, we denote the set $\{0, 1, 2, \ldots, q\}$ by $[0, q]$. A vertex labeling f of G is called a β-labeling if f is an injection from $V(G)$ into $[0, q]$ such that the values $|f(u) - f(v)|$ for the q pairs of adjacent vertices u and v are distinct. A β-labeling is also known as a graceful labeling. If f is a β-labeling of G such that there exists an integer λ so that each edge $uv \in E(G)$ either $f(u) \leqslant \lambda < f(v)$ or $f(v) \leqslant \lambda < f(u)$, then f is called an α-labeling of G. Clearly, if G has an α-labeling, then G must be a bipartite graph. The following theorem is folklore now.

Theorem 1.1 (*Rosa [7]*). *Let G be a graph with q edges, and let G have an α-labeling. Then, the complete graph K_{2pq+1} can be decomposed into isomorphic copies of G, where p is an arbitrary positive integer.*

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E-mail address: hlfu@math.nctu.edu.tw (H.-L. Fu).

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To obtain an α-labeling of a graph is not easy at all. Due to the close relation with graph decomposition, Snevily [9] introduced the following notion.

H is called a "host" graph of G if H has an α-labeling and H can be decomposed into copies of G. The α-labeling number of G is defined by $G_\alpha = \min\{t\colon \exists$ a "host" graph H of G with $|E(H)| = t|E(G)|\}$.

Then Snively conjectured that $G_\alpha < +\infty$ in [9]. Later, the conjecture was proved by El-Zanati et al. [1]. For trees, we can do even better. By the fact that an n-cube has an α-labeling [9] for each $n \geqslant 1$ and an n-cube can be decomposed into copies of an arbitrary tree with n edges [2], we conclude that $T_\alpha \leqslant 2^{n-1}$. Furthermore, Shiue [8] proved that $T_\alpha \leqslant e^{O(\sqrt{n \log n})}$. Note that Snevily [9] conjectured that $T_\alpha \leqslant n$. Clearly, this can be proved by showing that $K_{n,n}$ tree decomposition conjecture holds and $K_{n,n}$ has an α-labeling.

$K_{n,n}$ **tree decomposition conjecture** (*Ringel [6]*). For each tree T with n edges, $K_{n,n}$ can be decomposed into isomorphic copies of T.

In this paper, we manage to prove that a special regular bipartite graph H with $\lceil r/2 \rceil n^2$ edges can be decomposed into isomorphic copies of arbitrary tree T with n edges, where r is the radius of T. Moreover, by showing this special bipartite graph has an α-labeling, we conclude that the α-labeling number of T, $T_\alpha \leqslant \lceil r/2 \rceil n$. Since $r \leqslant n$, this improves the exponential upper bound for T_α to a quadratic upper bound, and it is very close to the upper bound $T_\alpha \leqslant n$ especially when r is a constant.

2. The main result

We start with introducing the special bipartite graphs mentioned in Section 1. A bipartite graph defined on $A \cup B$ where $A \cap B = \emptyset$, $A = \{a_0, a_1, \ldots, a_{l-1}\}$, and $B = \{b_0, b_1, \ldots, b_{l-1}\}$ is called an (n, l)-crown if for each $i \in [0, l-1]$, a_i is adjacent to b_j, $j \in \{i, i+1, \ldots, i+n-1\} \pmod{l}$. Clearly, an (n, l)-crown is an n-regular bipartite graph with $2l$ vertices and nl edges.

Proposition 2.1. *Let n and l be two positive integers such that $n > 1$ and $n|l$. Then an (n, l)-crown has an α-labeling.*

Proof. Let $G = (A, B)$ be an (n, l)-crown such that $A = \{a_0, a_1, \ldots, a_{l-1}\}$ and $B = \{b_0, b_1, \ldots, b_{l-1}\}$ and a_i is adjacent to b_j if and only if $j \in \{i, i+1, \ldots, i+n-1\} (\bmod\, l)$. First, we partition A into $n (= l/k)$ sets such that $A_0 = \{a_0, a_1, \ldots, a_{k-1}\}$ and for each $1 \leqslant h \leqslant n-1$, $A_h = \{a_t | t = l - h - j(n-1), j = 0, 1, 2, \ldots, k-1\}$. Define a vertex labeling f of G as follows:

1. $f(x) = i$ if $x = b_i$, $i = 0, 1, 2, \ldots, l-1$;
2. $f(x) = nl - jn + j$ if $x = a_j$, $j = 0, 1, \ldots, k-1$; and
3. $f(x) = hl + j$ if $x = a_t$ where $t = l - h - jn + j$, $j = 0, 1, 2, \ldots, k-1$.

Then, it is a routine matter to check that f is an injective function from $V(G)$ into $[0, nl]$ and λ can be chosen as $l - 1$. So, it is left to verify that $\{f(a_x) - f(b_y) | a_x \in A, b_y \in B$ and $a_x b_y \in E(G)\} = [1, |E(G)|] = [1, nl]$.

Let E_i denote the set of edges which are incident to the vertices in A_i, $i = 0, 1, 2, \ldots, n-1$, and let $f(E_i) = \{f(a_x) - f(b_y) | a_x \in A_i, b_y \in B$ and $a_x b_y \in E_i\}$. Now, by the definition of f, we have

$$
\begin{aligned}
f(E_0) &= \bigcup_{j=0}^{k-1} \{f(a_j) - f(b_i) | i = j, j+1, j+2, \ldots, j+n-1\} \\
&= \bigcup_{j=0}^{k-1} \{nl - jn + j - i | i = j, j+1, j+2, \ldots, j+n-1\} \\
&= \bigcup_{j=0}^{k-1} [(l-1)n - jn + 1, ln - jn] = [(n-1)l + 1, nl]. \qquad (1)
\end{aligned}
$$

Note that the vertex a_{l-h} is adjacent to the n vertices $b_{l-h}, b_{l-h+1}, \ldots, b_{l-1}, b_0, b_1 \ldots, b_{n-h-1}$ for each $1 \leqslant h \leqslant n-1$. By the definition of f, we have

$$\begin{aligned}
f(E_h) &= \bigcup_{j=0}^{k-1} \{f(a_t) - f(b) | b \in N(a_t), t = l - h - jn + j\} \\
&= \bigcup_{j=1}^{k-1} \{hl + j - i | i = l - h - jn + j, l - h - jn + j + 1, \ldots, l - h - jn + j + (n-1)\} \\
&\cup \{hl - i | i = l - h, l - h + 1, \ldots, l - 1\} \cup \{hl - i | i = 0, 1, \ldots, n - h - 1\} \\
&= \bigcup_{j=1}^{k-1} [(h-1)l + h + (j-1)n + 1, (h-1)l + h + jn] \\
&\cup [(h-1)l + 1, (h-1)l + h] \cup [lh + h - n + 1, lh] \\
&= [(h-1)l + h + 1, (h-1)l + h + kn - n] \cup [(h-1)l + 1, (h-1)l + h] \\
&\cup [lh + h - n + 1, lh] \\
&= [(h-1)l + h + 1, lh + h - n] \cup [(h-1)l + 1, (h-1)l + h] \cup [lh + h - n + 1, lh] \\
&= [(h-1)l + 1, hl], \quad h = 1, 2, \ldots, n-1.
\end{aligned} \tag{2}$$

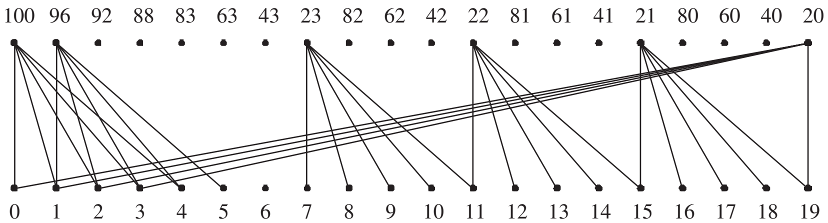
By combining (1) and (2), we obtain

$$\begin{aligned}
f(E) &= \left\{ \bigcup_{h=1}^{n-1} f(E_h) \right\} \cup f(E_0) = \left\{ \bigcup_{h=1}^{n-1} [(h-1)l + 1, hl] \right\} \cup [(n-1)l + 1, nl] \\
&= [1, nl].
\end{aligned}$$

This concludes the proof. □

For clearness, we give an example to show the idea of our α-labeling.

Example. $n = 5, l = 20$.



$f(E_0) = [81, 100]$, $f(E_1) = [1, 20]$, $f(E_2) = [21, 40]$, $f(E_3) = [41, 60]$,
$f(E_4) = [61, 80]$.

In order to find the α-labeling number of a given tree T with n edges, it is left to prove that there exists an (n, l)-crown H such that $n|l$ and H can be decomposed into isomorphic copies of T. First, we need a definition of bilabeling.

Let $G = (A, B)$ be a bipartite graph with q edges. We call a function $f : V(G) \to [1, m]$ a bilabeling of G if $f|_A$ and $f|_B$ are injective functions. We let the strength of f be $\max\{f(x) | x \in A \cup B\}$. A bilabeling of $G = (A, B)$ is a ρ-bilabeling if $\{f(y) - f(x) | x \in A, y \in B$ and $\{x, y\} \in E(G)\} = [0, q-1]$, where $q = |E(G)|$. Moreover, if a ρ-bilabeling of $G = (A, B)$ has strength $|E(G)|$, then the ρ-bilabeling is in fact the β-bilabeling of G or bigraceful labeling named by Ringel et al. [5]. Not surprisingly, they use this labeling to tackle the $K_{n,n}$ tree decomposition conjecture. In what follows, we shall use a ρ-bilabeling to obtain our main result. First, we claim that a tree does have a ρ-bilabeling with larger strength.

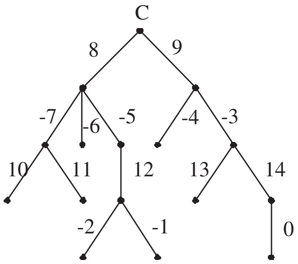


Fig. 1.

Proposition 2.2. *Let T be a tree with $n>0$ edges. Then T has a ρ-bilabeling with strength not greater than $\lceil r/2\rceil n$ where r is the radius of T.*

Proof. Let c be a vertex in the center of T. Then we construct a rooted tree T_c with c as the root by assigning direction to the edges of T. Since T is a bipartite graph let $T=(A,B)$ where $c\in A$. For convenience, we also let $V'=V(T)\backslash\{c\}$, $|A|=s$, and $|B|=t$. Then we have $n=s+t-1$. If $s=1$, then T is star, and it is easy to see that T has a bigraceful labeling. This completes the proof. Otherwise, we assume $s>1$. In T_c, each vertex v in V' has in-degree one, we can denote the unique arc adjacent to v by e_v. Hence, the arc set of T_c, $E(T_c)=\{e_v|\,v\in V'\}$ and we partition $E(T_c)$ into two sets $E_A=\{e_v\in E(T_c)|\,v\in A\}$ and $E_B=\{e_v\in E(T_c)|\,v\in B\}$. Let P_v be the unique path joint from the root c to v for each vertex $v\in V'$. Then on the path P_v, the edges occur in E_B and E_A alternately and this implies the following fact.

Fact 1. $|E(P_v)\cap E_A|=\lfloor|E(P_v)|/2\rfloor$ *and* $|E(P_v)\cap E_B|=\lceil|E(P_v)|/2\rceil$ *for each* $v\in V'$.

In order to construct a ρ-bilabeling, we first label the arcs of T_c. Since $|E(T_c)|=n$, we can define a bijective function g mapping $E(T_c)$ to $[-(s-2),0]\cup[s-1,n-1]$ by the following rules.

Rule 1: $g(E_A)=[-(s-2),0]$ and $g(E_B)=[s-1,n-1]$.

Rule 2: For each pair e_v and $e_{v'}$ in the same arc set, if $|E(P_v)|<|E(P_{v'})|$, then $g(e_v)<g(e_{v'})$.

Rule 3: For each pair e_v and $e_{v'}$ in the same arc set, let e_u and $e_{u'}$ be the previous arc of e_v and $e_{v'}$ in P_v and $P_{v'}$, respectively. Then $g(e_v)<g(e_{v'})$ provided that $g(e_u)<g(e_{u'})$; otherwise, $g(e_v)>g(e_{v'})$.

See Fig. 1 for an example.

By Rule 1 of the definition of g and Fact 1, we have the following:

Fact 2. $\sum_{e_u\in E(P_v)} g(e_u)>0$ *for each* $v\in V'$.

Now, we are ready to define the desired vertex labeling f. Let f be a function mapping $V(T)$ to a set of nonnegative integers which is defined as follows:

1. $f(c)=1$ and
2. $f(v)=\sum_{e_u\in E(P_v)} g(e_u)+1$ where $v\in V'$.

See Fig. 2 for example.

Since the length of each path starting from the center is no more than the radius in a tree, we have

$$\begin{aligned} f(v) &= \sum_{e_u\in E(p_v)} g(e_u)+1 \leqslant \sum_{e_u\in E(p_v)\cap E_B} g(e_u)+1 \\ &\leqslant |E(P_v)\cap E_B|(n-1)+1 \\ &= \lceil|E(P_v)|/2\rceil(n-1)+1 \leqslant \lceil r/2\rceil(n-1)+1 \leqslant \lceil r/2\rceil n \end{aligned}$$

for each $v\in V'$ (by Fact 1 and Rule 1). Hence the strength of f is at most $\lceil r/2\rceil n$. It is left to show that f is a ρ-bilabeling.

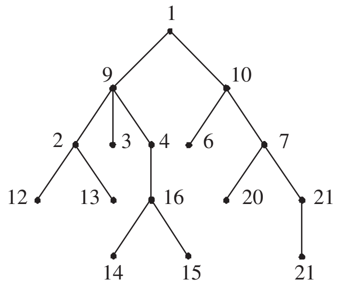


Fig. 2.

Claim 1. *f is a bilabeling*.

By Fact 2 and $f(c)=1$, we have $f(v)\geqslant 1$ for each $v\in V(T)$. This implies that f maps $V(T)$ into a set of positive integers. It is left to claim that both $f|_A$ and $f|_B$ are injective. Clearly, by the definition of f and Fact 2, $f(c)\neq f(v)$ for each $v\in A\backslash\{c\}$. For any pair of vertices v and v' both in $A\backslash\{v\}$ (or both in B), let

$$P_v = c - v_1 - v_2 - \cdots - v_l(=v) \quad \text{and}$$
$$P_{v'} = c - v'_1 - v'_2 - \cdots - v'_m(=v').$$

For convenience, we also let $e_{v_i}=e_i$ and $e_{v'_j}=e'_j$ for $i=1,2,\dots l$, $j=1,2,\dots,m$.

Case 1: $l=m$.

Suppose that $g(e_v)<g(e_{v'})$. Then $g(e_i)<g(e'_i)$ for $i=l-1,l-2,\dots,1$ by Rule 3. Hence

$$f(v)=\sum_{i=1}^{l} g(e_i)+1<\sum_{j=1}^{m} g(e'_j)+1=f(v').$$

Case 2: $l\neq m$.

Suppose that $l<m$. Then $|E(P_v)|<|E(P'_v)|$, and we have $g(e_v)<g(e_{v'})$ by Rule 2. Also, by Rule 3, we have $g(e_{l-i})<g(e'_{m-i})$ for $i=1,2,\dots,l-1$. Hence, by Fact 2, we have

$$\begin{aligned} f(v) &= \sum_{i=1}^{l} g(e_i)+1 < \sum_{j=m-l+1}^{m} g(e'_j)+1 \\ &< \sum_{e_w\in E(P_u)} g(e_w) + \sum_{j=m-l+1}^{m} g(e'_j)+1 \\ &= \sum_{j=1}^{m} g(e'_j)+1 = f(v'), \end{aligned}$$

where $u=v'_{m-l}$.

In any case, we have that both $f|_A$ and $f|_B$ are injective (See Fig. 3 for example). Therefore, we have the Claim.

Claim 2. $\{f(y)-f(x)|x\in A, y\in B$ *and* $\{x,y\}\in E(T)\}=[0,n-1]$.

For each arc $e_v=(u,v)\in E(T_c)$, that is, for each edge $\{u,v\}\in E(T)$, if $e_v\in E_A$, then $v\in A$ and $u\in B$. Hence, we have

$$f(u)-f(v)=\left[\sum_{w\in P_u} g(e_w)+1\right]-\left[\sum_{w\in P_v} g(e_w)+1\right]=-g(e_v).$$

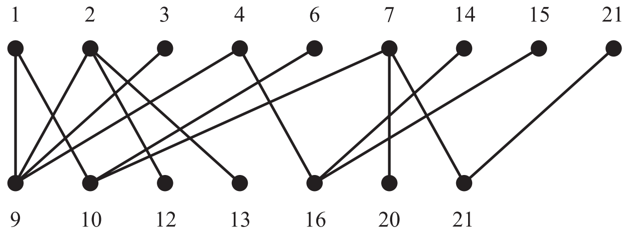


Fig. 3.

Otherwise, $u \in A$ and $v \in B$, and we have

$$f(v) - f(u) = \left[\sum_{w \in P_v} g(e_w) + 1\right] - \left[\sum_{w \in P_u} g(e_w) + 1\right] = g(e_v).$$

This implies that

$$\begin{aligned}&\{f(y) - f(x) | x \in A, y \in B \text{ and } \{x, y\} \in E(T)\} \\ &\quad = \{-g(e_v) | e_v \in E_A\} \cup \{g(e_v) | e_v \in E_B\} \\ &\quad = \{0, 1, 2, \ldots, s-2\} \cup \{s-1, s, s+1, \ldots, n-1\} \text{ (by Rule 1)} \\ &\quad = [0, n-1].\end{aligned}$$

Therefore, we have the proof. □

Proposition 2.3. *Let G be a bipartite graph with n edges. If G has a ρ-bilabeling of strength m, then an (n, l)-crown H can be decomposed into isomorphic copies of G for each $l \geqslant m$.*

Proof. Let $H = (U, V)$ be an (n, l)-crown, $l \geqslant m$. Therefore, U and V are l-set, let them be $\{u_0, u_1, \ldots, u_{l-1}\}$ and $\{v_0, v_1, \ldots, v_{l-1}\}$, respectively. By the definition of an (n, l)-crown $u_i v_j \in E(H)$ if and only if $j = i, i+1, \ldots, i+n-1 (\text{mod}\, l)$.

Now, let $G = (A, B)$ be a bipartite graph with n edges where $A = \{a_0, a_1, \ldots, a_{s-1}\}$ and $B = \{b_0, b_1, \ldots, b_{t-1}\}$. By the hypothesis, G has a ρ-bilabeling f. For convenience, let $f(a_i) = s_i$ and $f(b_j) = t_j$, respectively. It is not difficult to see the following bipartite graphs $G_1, G_2, \ldots,$ and G_l are isomorphic to G where $G_j = (A_j, B_j)$ is defined as follows:

(i) $A_j = \{u_{s_i + j (\text{mod}\, l)} | i \in \mathbb{Z}_s\}$ and $B_j = \{v_{t_{i'} + j (\text{mod}\, l)} | i' \in \mathbb{Z}_t\}$, and
(ii) $u_{s_i + j (\text{mod}\, l)}$ is adjacent to $v_{t_{i'} + j (\text{mod}\, l)}$ if and only if a_i and $b_{i'}$ are adjacent in G.

It is left to show that $G_1, G_2, \ldots,$ and G_l form a decomposition of H. Suppose not. Then, let $u_\alpha v_\beta \in E(G_{j_1}) \cap E(G_{j_2})$, $j_1 \neq j_2$. This implies that we have two distinct edges $x_1 y_1$ and $x_2 y_2$ in $E(G)$ such that $f(x_1) + j_1 \equiv f(x_2) + j_2 \equiv \alpha \ (\text{mod}\, l)$ and $f(y_1) + j_1 \equiv f(y_2) + j_2 \equiv \beta \ (\text{mod}\, l)$.

Hence, $f(y_1) - f(x_1) = f(y_2) - f(x_2)$. By the fact that f is a ρ-bilabeling, $f(y_1) - f(x_1) = f(y_2) - f(x_2)$ if and only if $x_1 y_1 = x_2 y_2$. This leads to a contradiction. Thus, $G_1, G_2, \ldots,$ and G_l decompose H. □

With the above propositions, we are able to obtain the following result.

Theorem 2.4. *Let r be the radius of a tree T with n edges. Then $T_\alpha \leqslant \lceil \frac{r}{2} \rceil n$.*

Proof. By Proposition 2.2, T has a ρ-bilabeling with strength $\lceil r/2 \rceil n$. Since an $(n, \lceil r/2 \rceil n)$-crown has an α-labeling (by Proposition 2.1) and an $(n, \lceil r/2 \rceil n)$-crown can be decomposed into $\lceil r/2 \rceil n$ isomorphic copies of T (by Proposition 2.3), we conclude that $T_\alpha \leqslant \lceil r/2 \rceil n$. □

Corollary 2.5. *For these trees T with constant radius and n edges, $T_\alpha \leqslant O(n)$.*

Corollary 2.6. $T_\alpha \leqslant O(n^2)$.

Proof. This is a direct consequence of the fact that the radius of a tree with n edges is at most $\lceil n/2 \rceil$. □

By combining Theorems 1.1 and 2.4, we also have a result which is a slight improvement of the research work obtained by Kézdy and Snevily [4].

Corollary 2.7. *Let T be a tree with n edges and radius r, then T decomposes* K_t *for some* $t \leqslant (r+1)n^2 + 1$.

3. Concluding remark

As mentioned in Section 1, Snevily conjectured that the α-labeling number of a tree with n edges is at most n. Note that there are trees which do not have α-labelings. For these trees T, $T_\alpha \geqslant 2$. We believe that $T_\alpha \leqslant n$ is quite possible. Hopefully, by finding a better ρ-bilabeling with smaller strength, we can prove the conjecture in the near future.

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