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Integral Equations and Operator Theory

Polar Decompositions of $C_0(N)$ Contractions

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Abstract. Let A be a bounded linear operator on a complex separable Hilbert space H. We show that A is a $C_0(N)$ contraction if and only if $A = U(I - \sum_{j=1}^{d} r_j(x_j \otimes x_j))$, where U is a singular unitary operator with multiplicity $d \leq N$, $0 < r_1, \ldots, r_d \leq 1$ and $x_1, \ldots, x_d$ are orthonormal vectors satisfying $\bigvee\{U^k x_j : k \geq 0, 1 \leq j \leq d\} = H$. For a $C_0(N)$ contraction, this gives a complete characterization of its polar decompositions with unitary factors.

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1. Introduction

As a generalization of the polar coordinate of a complex number $z = re^{i\theta}$, the polar decompositions of a bounded linear operator on a complex Hilbert space reveal important features of the operator. In this paper, we characterize the polar decompositions of $C_0(N)$ contractions.

Recall that if A is an operator on a Hilbert space, then A can be decomposed as $A = V(A^*A)^{1/2}$, where V is a partial isometry (V isometric on $(\ker V)^{\perp}$) and $(A^*A)^{1/2}$ denotes the positive square root of A^*A. Each of these decompositions is called a *polar decomposition* of A. In particular, if $\ker A$ and $\ker A^*$ have equal dimensions, then V can be taken to be a unitary operator. For properties of polar decompositions, the reader may consult [7, Chapter 16].

A contraction A ($\|A\| \leq 1$) is of *class* C_0 if it is completely nonunitary and satisfies $\phi(A) = 0$ for some nonzero function ϕ in the Hardy space H^∞ on the unit disc. Recall that a contraction is *completely nonunitary* (*c.n.u.*) if it has no nontrivial reducing subspace on which it is unitary. It is known that the *defect indices* $d_A \equiv \operatorname{rank}(I - A^*A)^{1/2}$ and $d_{A^*} \equiv \operatorname{rank}(I - AA^*)^{1/2}$ of a C_0 contraction A are equal to each other. A is of *class* $C_0(N)$ (N a positive integer) if it is a C_0 contraction with $d_A = d_{A^*} \leq N$. An equivalent condition for a contraction to be of class $C_0(N)$ can be given in terms of the asymptotic behavior of its powers. Recall that a contraction A is of *class* $C_{0\cdot}$ (resp., $C_{\cdot 0}$) if $A^n x \to 0$ (resp., ${A^*}^n x \to 0$) for every vector x; it is of *class* $C_{1\cdot}$ (resp., $C_{\cdot 1}$) if $A^n x \not\to 0$ (resp., ${A^*}^n x \not\to 0$) for every $x \neq 0$. We also define the $C_{\alpha\beta}$ class, $\alpha, \beta = 0, 1$, as the intersection of the classes $C_{\alpha\cdot}$ and $C_{\cdot\beta}$. It is known that a contraction A is of class $C_0(N)$ if and only if A is in C_{00} and $d_A = d_{A^*} \leq N$. Such operators originate from the Sz.-Nagy-Foiaş dilation theory of contractions and were studied intensively in the 1960s and '70s. The standard references are [10] and [1].

2. $C_0(N)$ contraction

Our main theorem on the polar decompositions of $C_0(N)$ contractions is the following:

Theorem 2.1. *A is a $C_0(N)$ contraction on H if and only if*

$$A = U(I - \sum_{j=1}^{d} r_j (x_j \otimes x_j)),$$

where U is a singular unitary operator with multiplicity $d \leq N$, $0 < r_1, \ldots, r_d \leq 1$ and $x_1, \ldots, x_d$ are orthonormal vectors satisfying $\bigvee\{U^k x_j : k \geq 0, 1 \leq j \leq d\} = H$. In this case, $I - \sum_{j=1}^{d} r_j(x_j \otimes x_j) = (A^*A)^{1/2}$, $\bigvee\{x_1, \ldots, x_d\} = \operatorname{ran}(I - A^*A)$ and $d = d_A$.

Recall that a unitary operator is said to be *singular* if its spectral measure is mutually singular with respect to the Lebesgue measure on the unit circle. The *multiplicity* μ_A of an operator A on H is the minimum cardinality of any subset of vectors $\{x_\lambda\}_{\lambda \in \Lambda}$ in H such that $\bigvee\{A^k x_\lambda : k \geq 0, \lambda \in \Lambda\} = H$. For any nonzero vector x, $x \otimes x$ denotes the rank-one operator $(x \otimes x)y = \langle y, x \rangle x$ for y in H, where $\langle \cdot, \cdot \rangle$ is the inner product in H. For any operator A, we use $\operatorname{ran} A$ to denote its range, and $\sigma(A)$ (resp., $\sigma_p(A)$) its spectrum (resp., point spectrum). $\mathbb{D}$ denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane.

We prove this theorem via a series of lemmas.

Lemma 2.2. *If A is a C_0 contraction, then* $\dim\ker A = \dim\ker A^*$ *and hence $A = U(A^*A)^{1/2}$ for some unitary operator U.*

Proof. Note that A (resp., A^*) is similar to a direct sum $A_1 \oplus A_2$ (resp., $A_1' \oplus A_2'$), where A_1 (resp., A_1') is a nilpotent operator on a finite-dimensional space and A_2 (resp., A_2') is invertible (cf. [10, Proposition III.7.1]). Thus $A_1 \oplus A_2$ is similar to ${A_1'}^* \oplus {A_2'}^*$, from which we infer the similarity of A_1 and ${A_1'}^*$ and therefore that of A_1 and A_1'. Hence

$$\dim\ker A = \dim\ker A_1 = \dim\ker A_1' = \dim\ker A^*. \qquad \square$$

Now we show that every unitary factor in a polar decomposition of a $C_0(N)$ contraction behaves as asserted in Theorem 2.1.

Lemma 2.3. *If $A = U(A^*A)^{1/2}$ with U unitary is a $C_0(N)$ contraction, then U is singular and $\mu_U \leq d_A$.*

Proof. Since $d_A = \operatorname{rank}(I - A^*A)$ is finite, we have $A^*A = I + F_1$ for some operator F_1 with $-I \leq F_1 \leq 0$ and $\operatorname{rank} F_1 = d_A$. Then $(A^*A)^{1/2} = I + F_2$, where $-I \leq F_2 \leq 0$ and $\operatorname{rank} F_2 = d_A$. Therefore, $A = U(A^*A)^{1/2} = U + UF_2$. That U is singular unitary now follows from [13, Proposition 3.7]. On the other hand, [13, Theorem 4.1] implies that

$$\mu_U \leq \operatorname{rank} UF_2 = \operatorname{rank} F_2 = d_A,$$

completing the proof. □

An example shows that $\mu_U < d_A$ is possible here.

Example 2.4. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$. Then A is a $C_0(2)$ contraction with $d_A = \mu_A = 2$. A polar decomposition of A is given by $A = U(A^*A)^{1/2}$ with

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \text{and} \quad (A^*A)^{1/2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the eigenvalues of U, ± 1 and $\pm i$, are all distinct, we infer that $\mu_U = 1$.

We are now ready to prove the necessity of the assertion in Theorem 2.1.

Lemma 2.5. *If A is a $C_0(N)$ contraction on H, then $A = U(I - \sum_{j=1}^{d} r_j(x_j \otimes x_j))$, where U is singular unitary with $\mu_U \leq d \equiv d_A, 0 < r_1, \ldots, r_d \leq 1$ and $x_1, \ldots, x_d$ are orthonormal vectors satisfying $\bigvee\{U^k x_j : k \geq 0, 1 \leq j \leq d\} = H$.*

Proof. By Lemma 2.2 and the proof of Lemma 2.3, A has a polar decomposition $U(I - F)$, where U is singular unitary with $\mu_U \leq d_A \equiv d$ and $0 \leq F \leq I$ with $\operatorname{rank} F = d$. Let $r_j, j = 1, \ldots, d$, be the nonzero eigenvalues of F with the corresponding orthonormal eigenvectors $x_j, j = 1, \ldots, d$. Then $F = \sum_{j=1}^{d} r_j(x_j \otimes x_j)$. Let K denote the subspace $\bigvee\{U^k x_j : k \geq 0, 1 \leq j \leq d\}$. We now check that $K = H$. Since U is singular unitary by Lemma 2.3, the invariant subspace K of U actually reduces U (cf. [11, Lemma 3]). For any vector y in $K^\perp$, we have

$$Ay = U(I - F)y = Uy - U(\sum_{j=1}^{d} r_j \langle y, x_j \rangle x_j) = Uy \in K^\perp$$

and

$$A^*y = (I - F)U^*y = U^*y - \sum_{j=1}^{d} r_j \langle U^*y, x_j \rangle x_j = U^*y \in K^\perp.$$

This shows that $K^\perp$ reduces A and $A|K^\perp = U|K^\perp$ is unitary. Since A has no unitary part, we must have $K^\perp = \{0\}$ and thus $K = H$ completing the proof. □

We now prepare for the proof of the sufficiency in Theorem 2.1.

Lemma 2.6. *Let U be a singular unitary operator with finite multiplicity on H. If $x_1, \ldots, x_d$ $(1 \leq d < \infty)$ are orthonormal vectors satisfying $\bigvee\{U^k x_j : k \geq 0, 1 \leq j \leq d\} = H$ and $L = (\bigvee\{x_1, \ldots, x_d\})^\perp$, then the compression $P_L U|L$ of U to L (P_L is the orthogonal projection from H onto L) is a $C_0(d)$ contraction.*

Proof. Let $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ with respect to the decomposition $H = L \oplus L^\perp$. Then $U_1 = P_L U|L$. Let $V = U_1 \oplus 0$ on $H = L \oplus L^\perp$. Since V is a finite-rank perturbation of U, the invertibility of U implies that V is Fredholm with $\dim\ker V = \dim\ker V^* < \infty$. Since $\dim\ker V = \dim\ker U_1 + d$ and $\dim\ker V^* = \dim\ker U_1^* + d$, we obtain $\dim\ker U_1 = \dim\ker U_1^* < \infty$. It follows from the equality $d_{U_1} + \dim\ker U_1^* = d_{U_1^*} + \dim\ker U_1$ [6, Lemma 4] that $d_{U_1} = d_{U_1^*}$. We also have

$$d_{U_1} = \operatorname{rank}(I - U_1^* U_1) = \operatorname{rank} U_3^* U_3 \leq d < \infty.$$

We next show that U_1 is c.n.u. Indeed, if $U_1 = U_1' \oplus U_2'$ on $L = L_1 \oplus L_2$, where U_1' is unitary and U_2' is c.n.u., then

$$U = \begin{bmatrix} U_1' & 0 & 0 \\ 0 & U_2' & * \\ 0 & * & U_4 \end{bmatrix} \quad \text{on} \quad H = L_1 \oplus L_2 \oplus L^\perp,$$

in which case $H = \bigvee\{U^k x_j : k \geq 0, 1 \leq j \leq d\} \subseteq L_2 \oplus L^\perp$ implies that $L_1 = \{0\}$. Thus U_1 is c.n.u. as asserted.

Finally, to show that U_1 is of class $C_0(d)$, we triangulate U_1 as

$$U_1 = \begin{bmatrix} U_{01} & & & * \\ & U_{11} & & \\ & & U_{00} & \\ 0 & & & U_{10} \end{bmatrix},$$

where U_{ij} is of class C_{ij}, $i, j = 0, 1$ (cf. [12, Lemma 3.2]). Since U_{01} and U_{10} are contractions with unequal defect indices and U_{11} is a c.n.u. C_{11} contraction, they cannot have a singular unitary dilation (cf. [13, Proposition 3.5]). As the singular unitary U is their dilation, we conclude that U_{01}, U_{11} and U_{10} do not appear in the above triangulation. Thus $U_1 = U_{00}$ is of class C_{00}. Since the defect indices of U_1 are at most d, we obtain that U_1 is of class $C_0(d)$ as asserted. □

Lemma 2.7. *Under the assumptions and notations of Lemma 2.6, the operator UP_L is a $C_0(N)$ contraction with defect indices equal to d.*

Proof. Let $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ on $H = L \oplus L^\perp$. Then $A' \equiv UP_L = \begin{bmatrix} U_1 & 0 \\ U_3 & 0 \end{bmatrix}$. We have

$$\begin{aligned} d_{A'} &= \operatorname{rank}(I - A'^* A') = \operatorname{rank} \begin{bmatrix} I - U_1^* U_1 - U_3^* U_3 & 0 \\ 0 & I \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = d < \infty. \end{aligned}$$

On the other hand, by Lemma 2.6, for any $y = y_1 \oplus y_2$ in $H = L \oplus L^\perp$ both

$$\|A'^n y\| = \| \begin{bmatrix} U_1 & 0 \\ U_3 & 0 \end{bmatrix}^n \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \| = \| \begin{bmatrix} U_1^n y_1 \\ U_3 U_1^{n-1} y_1 \end{bmatrix} \|$$

and

$$\|A'^{*n} y\| = \| \begin{bmatrix} U_1^* & U_3^* \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \| = \| \begin{bmatrix} U_1^{*n} y_1 + U_1^{*n-1} U_3^* y_2 \\ 0 \end{bmatrix} \|$$

converge to zero as n approaches infinity. This shows that A' is of class C_{00} and hence of class $C_0(N)$ with $d_{A'} = d_{A'^*} = d$ (cf. [10, Theorem VI.5.2]). □

We can now prove the sufficiency in Theorem 2.1.

Lemma 2.8. *Let U be a singular unitary operator with multiplicity $d\,(1 \le d < \infty)$ on H. If $x_1, \ldots, x_d$ are orthonormal vectors with $\bigvee\{U^k x_j : k \ge 0, 1 \le j \le d\} = H$ and $r_1, \ldots, r_d$ are scalars satisfying $0 < r_j \le 1$ for all j, then $A = U(I - \sum_{j=1}^d r_j (x_j \otimes x_j))$ is a $C_0(N)$ contraction with $d_A = d_{A^*} = d$.*

Proof. Since $I - \sum_{j=1}^d r_j (x_j \otimes x_j)$ can be represented as

$$\begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix}, \quad \text{where } S = \begin{bmatrix} 1 - r_1 & & 0 \\ & \ddots & \\ 0 & & 1 - r_d \end{bmatrix},$$

we may just as well assume that it is already of this form. Then

$$d_A = \operatorname{rank}(I - A^*A) = \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & I - S^2 \end{bmatrix} = d$$

and

$$d_{A^*} = \operatorname{rank}(I - AA^*) = \operatorname{rank} U(I - A^*A)U^* = d_A.$$

We next check that A is c.n.u. Assume otherwise that A has a unitary part on the subspace M of H. Then for any nonzero vector y in M and any $n \ge 0$, we have $\|A^{n+1} y\| = \|y\|$ and hence $A^n y$ is in $\ker(I - A^*A)$. Since $\ker(I - A^*A) = L \equiv (\bigvee\{x_1, \ldots, x_d\})^\perp$, we obtain $A^n y \in L$ for all $n \ge 0$. Let $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ on $H = L \oplus L^\perp$. Then

$$A^n y = A(A^{n-1} y) = \begin{bmatrix} U_1 & U_2 S \\ U_3 & U_4 S \end{bmatrix} \begin{bmatrix} A^{n-1} y \\ 0 \end{bmatrix} = \begin{bmatrix} U_1 & 0 \\ U_3 & 0 \end{bmatrix} \begin{bmatrix} A^{n-1} y \\ 0 \end{bmatrix}.$$

Repeating this process $n - 1$ times yields $A^n y = \begin{bmatrix} U_1 & 0 \\ U_3 & 0 \end{bmatrix}^n y$. Hence

$$\| \begin{bmatrix} U_1 & 0 \\ U_3 & 0 \end{bmatrix}^n y \| = \|A^n y\| = \|y\|,$$

which cannot converge to zero as n approaches infinity. This contradicts Lemma 2.7. Thus A must be c.n.u.

Finally, to check that A is of class $C_0(N)$, we use an argument similar to the one in the proof of Lemma 2.6. Let

$$A = \begin{bmatrix} A_{01} & & & * \\ & A_{11} & & \\ & & A_{00} & \\ 0 & & & A_{10} \end{bmatrix},$$

where A_{ij} is of class C_{ij}, $i, j = 0, 1$. Since $d_A = d_{A^*} = d$, A can be dilated to a unitary operator

$$W = \begin{bmatrix} A & * \\ * & * \end{bmatrix} \quad \text{on} \quad H \oplus H'$$

with $\dim H' = d$ (cf. [13, Proposition 3.6]). Hence A_{01}, A_{11} and A_{10} all dilate to W. Since A_{01} and A_{10} have unequal defect indices and A_{11} is a c.n.u. C_{11} contraction, [13, Proposition 3.5] says that W cannot be singular unitary. But W is a finite-rank perturbation of the singular unitary operator $U \oplus I$ on $H \oplus H'$. The Kato-Rosenblum theorem [8, 9] implies that W is also singular unitary. This contradiction yields that A_{01}, A_{11} and A_{10} cannot appear in the above triangulation of A and therefore $A = A_{00}$ is a $C_0(N)$ contraction as required. □

Theorem 2.1 now follows from Lemmas 2.5 and 2.8.

Corollary 2.9. *For any singular unitary operator U with $\mu_U < \infty$, there is a sequence of $C_0(N)$ contractions $\{A_n\}$ such that $d_{A_n} = \operatorname{rank}(A_n - U) = \mu_U$ for all n and $A_n \to U$ in norm as $n \to \infty$.*

Proof. Assume that U acts on H. Let $A_n = U(I - \sum_{j=1}^{d} r_j^{(n)}(x_j \otimes x_j))$ be as in Lemma 2.8 with $d = \mu_U, 0 < r_1^{(n)}, \ldots, r_d^{(n)} \leq 1$ for all n, $r_j^{(n)} \to 0$ as $n \to \infty$ for all j and orthonormal vectors $x_1, \ldots, x_d$ satisfying $\bigvee\{U^k x_j : k \geq 0, 1 \leq j \leq d\} = H$. That the A_n's are $C_0(N)$ contractions with $d_{A_n} = \mu_U$ for all n is by Lemma 2.8. That $\operatorname{rank}(A_n - U) = \mu_U$ and $A_n \to U$ in norm as $n \to \infty$ are immediate. □

3. Compressions of the shift

In this section, we restrict ourselves to compressions of the shift. Such operators are exactly the $C_0(1)$ contractions or, equivalently, those which are unitarily equivalent to $S(\phi)$ for some (nonconstant) inner function ϕ, $S(\phi)$ being the operator on $H^2 \ominus \phi H^2$ defined by $f \mapsto P(zf(z))$, where P denotes the (orthogonal) projection from H^2 onto $H^2 \ominus \phi H^2$. They are called the *compressions of the shift* because each can be dilated to the (simple) unilateral shift S, $(Sf)(z) = zf(z)$ for f in H^2. Properties of $S(\phi)$ are discussed in detail in [1, Section 3.1]; its unitary perturbation is considered in [3].

Theorem 2.1, when recast in this case, takes the following form.

Proposition 3.1. *A is a compression of the shift if and only if $A = U(I - r(x \otimes x))$, where U is a cyclic singular unitary operator, $0 < r \leq 1$ and x is a unit cyclic vector for U.*

An operator B on H is *cyclic* if $\mu_B = 1$; a vector x for which $\bigvee\{B^k x : k \geq 0\} = H$ is called a *cyclic vector* for B.

Proposition 3.1 is essentially proved in [3, Theorem 9.2]. However, its proof there (at least for one direction) depends on [3, Lemma 9.1], which, unfortunately, is not true in general. We now briefly recount the construction therein and give a counterexample to this lemma. For any singular measure μ on $\partial\mathbb{D}$ and any c in $\mathbb{D}$, the operator $T_{\mu,c}$ is defined by

$$T_{\mu,c} f = e^{it}\left(f - \frac{\langle f, e^{-it}\rangle e^{-it}}{\|e^{-it}\|^2}\right) + c\frac{\langle f, e^{-it}\rangle}{\|e^{-it}\|^2}$$

on $L^2(\mu)$. If U denotes the unitary operator $(Uf)(e^{it}) = e^{it} f(e^{it})$ on $L^2(\mu)$ and $x = e^{-it}/\|e^{-it}\|$ in $L^2(\mu)$, then x is a unit cyclic vector for U and $T_{\mu,c}$ is exactly the operator $U(I - (1-c)(x \otimes x))$. [3, Lemma 9.1] claims that two such operators $T_{\mu,c}$ and $T_{\mu',c}$ are unitarily equivalent if and only if μ and μ' are equal. The next example shows that this is not necessarily the case. It was worked out jointly with H.-L. Gau.

Example 3.2. Let μ (resp., μ') be the probability measure supported on the two points ± 1 (resp., $\pm i$), each with mass $1/2$. Then $L^2(\mu)$ (resp., $L^2(\mu')$) can be identified with the two-dimensional space $\mathbb{C}^2$ under the correspondence $f \mapsto (f(1)/\sqrt{2}, f(-1)/\sqrt{2})$ (resp., $g \mapsto (g(i)/\sqrt{2}, g(-i)/\sqrt{2})$). For $c = 0$, the operators $A \equiv T_{\mu,0}$ and $A' \equiv T_{\mu',0}$ are given by

$$A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} a \\ b \end{bmatrix} - \frac{1}{2}\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\rangle \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \frac{1}{2}(a+b)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$A'\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\left(\begin{bmatrix} a \\ b \end{bmatrix} - \frac{1}{2}\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} -i \\ i \end{bmatrix}\right\rangle \begin{bmatrix} -i \\ i \end{bmatrix}\right) = \frac{1}{2}(a+b)i\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

on $\mathbb{C}^2$. In particular, A and A' are both unitarily equivalent to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. However, it is obvious that neither μ nor μ' is even absolutely continuous with respect to the other.

The next proposition relates the (point) spectra of the unitary factors in the polar decompositions of a compression of the shift.

Proposition 3.3. *Let A be a compression of the shift on H with the polar decompositions $A = U_1(A^*A)^{1/2} = U_2(A^*A)^{1/2}$, where U_1 and U_2 are unitary. Then either $\sigma_p(U_1) = \sigma_p(U_2)$ or $\sigma_p(U_1) \cap \sigma_p(U_2) = \emptyset$. In the former case, U_1 equals U_2 while in the latter, $\sigma(U_1) \cap \sigma(U_2)$ equals $\sigma_e(B)$, the essential spectrum of $B \equiv P_K A|K$, the compression of A to $K \equiv \ker(I - A^*A)$.*

Proof. This depends on results from [3]. For $j = 1, 2$, let $U_j = \begin{bmatrix} U_{j1} & U_{j2} \\ U_{j3} & U_{j4} \end{bmatrix}$ on $H = K \oplus K^\perp$. Since $A = U_1 = U_2$ on K, we have $B = U_{11} = U_{21}$ and $A' \equiv \begin{bmatrix} U_{11} & 0 \\ U_{13} & 0 \end{bmatrix} = \begin{bmatrix} U_{21} & 0 \\ U_{23} & 0 \end{bmatrix}$. As were shown in Lemmas 2.6 and 2.7, B and A' are compressions of the shift. If ϕ is the minimal function of B, then $z\phi(z)$ is the minimal function of A'. Since U_j, $j = 1, 2$, is a rank-one perturbation of A', it equals one of the U_w's ($|w| = 1$) defined in [3, (1)] (cf. [3, Remark 2.3]). Say, $U_j = U_{w_j}$, where $|w_j| = 1, j = 1, 2$. By [3, Corollary 3.3], the spectrum $\sigma(U_j)$ equals $\{\lambda \in \partial\mathbb{D} : \phi$ not analytically continuable at $\lambda\} \cup \{\lambda \in \partial\mathbb{D} : \phi$ analytically continuable at λ and $\lambda\phi(\lambda) = w_j\}$. Hence $\sigma(U_j) = \sigma_e(B) \cup \sigma_p(U_j)$. If $w_1 = w_2$, then $U_1 = U_2$. On the other hand, if $w_1 \neq w_2$, then obviously $\sigma_p(U_1)$ and $\sigma_p(U_2)$ are disjoint and thus $\sigma(U_1) \cap \sigma(U_2) = \sigma_e(B)$ as asserted. □

For compressions of the shift on a finite-dimensional space, more can be said. Such operators are characterized as those contractions A with $\sigma(A) \subseteq \mathbb{D}$ and rank $(I - A^*A) = 1$. We call an n-by-n matrix with these properties an $\mathcal{S}_n$-*matrix*. The next proposition, due to Bryan Cain [2], gives different expressions of the polar decompositions of $\mathcal{S}_n$-matrices.

Proposition 3.4. *The following are equivalent for an n-by-n matrix A :*

(a) *A is an $\mathcal{S}_n$-matrix*;

(b) *$A = U(I_n - rxx^*)$, where U is a unitary matrix with distinct eigenvalues, $0 < r \leq 1$ and x is a unit cyclic vector for U*;

(c) *A is unitarily equivalent to $U'(I_n - rx'x'^*)$, where U' is a diagonal unitary matrix with distinct eigenvalues, $0 < r \leq 1$ and x' is a unit vector with all components nonzero*;

(d) *$A = U(I_n - rP)$, where U is a unitary matrix, $0 < r \leq 1$ and P is a rank-one (orthogonal) projection whose kernel contains no eigenvector of U and whose range contains a cyclic vector of U*;

(e) *A is unitarily equivalent to VD, where V is a unitary matrix such that all its eigenvectors have a nonzero first component and it has $[1\ 0\ \ldots\ 0]^T$ as a cyclic vector, and D is the diagonal matrix* diag $(s, 1, \ldots, 1)$ *with $0 \leq s < 1$.*

Proof. (a)⇔(b): This follows from Theorem 2.1.

(b)⇒(c): Let $A = U(I_n - rxx^*)$ be as in (b). If W is a unitary matrix such that $U' \equiv W^*UW$ is diagonal, then U' has distinct eigenvalues and $W^*AW = U'(I_n - rx'x'^*)$, where $x' = W^*x$ is a cyclic vector for U'. The cyclicity of x' implies that all its components are nonzero.

(c)⇒(a): If A is unitarily equivalent to $A' \equiv U(I_n - rxx^*)$ as in (c), then, since A' is an $\mathcal{S}_n$-matrix by the equivalence of (a) and (b), the same is true for A.

(b)⇒(d): Let $A = U(I_n - rxx^*)$ be as in (b). Then $P \equiv xx^*$ is a rank-one (orthogonal) projection. If y is an eigenvector of U ($Uy = \lambda y$ for some λ, $|\lambda| = 1$) which is also in ker P, then

$$Ay = U(I_n - rP)y = Uy = \lambda y.$$

This says that λ is an eigenvalue of A, which contradicts $\sigma(A) \subseteq \mathbb{D}$ since A is an $\mathcal{S}_n$-matrix by (a). On the other hand, ran P contains the cyclic vector x of U. Hence P has the asserted properties in (d).

(d)⇒(e): If $A = U(I_n - rP)$ as in (d), then, letting W be a unitary matrix such that $W^*PW = \text{diag}\,(1, 0, \ldots, 0)$, we have $W^*AW = (W^*UW)\text{diag}\,(1-r, 1, \ldots, 1)$. Hence A is unitarily equivalent to $U'D$, where $U' = W^*UW$ and $D = \text{diag}\,(1 - r, 1, \ldots, 1)$. If $y = [0\ y_2\ \ldots\ y_n]^T$ is an eigenvector of U', then Wy is an eigenvector of U and is also in $\ker P$. This contradicts (d). Moreover, since any nonzero vector x in $\text{ran}\, P$ is cyclic for U, we have that $W^*x = [c\ 0\ \ldots\ 0]^T$, where $|c| = \|x\|$, is cyclic for U'. Since $c \neq 0$, (e) holds.

(e)⇒(a): Assume that A is unitarily equivalent to $A' \equiv VD$ as in (e). Then

$$(A'^* A')^{1/2} = D = I_n - rxx^*,$$

where $r = 1 - s$ and $x = [1\ 0\ \ldots\ 0]^T$. We check that the eigenvalues of V are all distinct. Indeed, if V has an eigenvalue λ with multiplicity at least 2, then, letting $K = \ker\,(V - \lambda I_n)$ and $M = 0 \oplus \mathbb{C}^{n-1}$, we have

$$\begin{aligned} \dim\,(K \cap M) &= \dim K + \dim M - \dim\,(K \vee M) \\ &\geq 2 + (n-1) - n \\ &= 1. \end{aligned}$$

This implies that V has an eigenvector with first component zero, which contradicts our assumption in (e). Thus the eigenvalues of V are all distinct. We infer from the equivalence of (a) and (b) that A' is an $\mathcal{S}_n$-matrix. Hence the same is true for A. □

Now we show how the characteristic polynomial of an $\mathcal{S}_n$-matrix A can be expressed in terms of r and the entries of U' and x' in Proposition 3.4 (c).

Proposition 3.5. *Let A be an $\mathcal{S}_n$-matrix with polar decomposition $U'(I_n - rx'x'^*)$ as in Proposition* 3.4 (c). *If U' has eigenvalues $u_1, \ldots, u_n$ and $x' = [x_1\ \ldots\ x_n]^T$, then the characteristic polynomial of A is given by*

$$\sum_{j=1}^{n} |x_j|^2 (z - u_1) \cdots (z - (1-r)u_j) \cdots (z - u_n).$$

Proof. We may assume that $A = U(I_n - rxx^*)$. If $y = -rUx = [-ru_1x_1\ \ldots\ -ru_nx_n]^T$, then $A = U + yx^*$. By a result of Anderson [5, Lemma 2], the characteristic polynomial of A is given by

$$(z - u_1) \cdots (z - u_n) + \sum_{j=1}^{n} ru_jx_j\overline{x_j}(z - u_1) \cdots \widehat{(z - u_j)} \cdots (z - u_n),$$

where the "∧" over $z - u_j$ denotes that this term is absent from the product. Since

$$(z - u_1) \cdots (z - u_n) = \sum_{j=1}^{n} |x_j|^2 (z - u_1) \cdots (z - u_n),$$

the above polynomial can be simplified to

$$\sum_{j=1}^{n} |x_j|^2 (z-u_1)\cdots\widehat{(z-u_j)}\cdots(z-u_n)(z-u_j+ru_j),$$

which is the same as the asserted form. □

Finally, for $\mathcal{S}_n$-matrices Proposition 3.3 can be further refined to the following.

Proposition 3.6. *Let A be an $\mathcal{S}_n$-matrix with the polar decompositions $A = U_1(I_n - r_1x_1x_1^*) = U_2(I_n - r_2x_2x_2^*)$ as in Proposition* 3.4 (b). *Then*

(a) $r_1 = r_2$,
(b) $x_1 = \lambda x_2$ *for some* λ, $|\lambda| = 1$, *and*
(c) *either the eigenvalues of U_1 and U_2 coincide or they are disjoint. In the former case, U_1 and U_2 are equal; in the latter, their eigenvalues alternate around $\partial\mathbb{D}$.*

Proof. Since $I_n - r_1x_1x_1^* = (A^*A)^{1/2} = I_n - r_2x_2x_2^*$, (a) and (b) follow easily. Proposition 3.3 settles all the assertions in (c) except the eigenvalue alternation. Since U_1 and U_2 are n-by-n unitary dilations of the $\mathcal{S}_{n-1}$-matrix $B \equiv P_K A|K$, $K = \ker(I_n - A^*A)$, by Lemma 2.6, this assertion is an easy consequence of the fact that the n-gons formed by the eigenvalues of U_1 and U_2 circumscribe the numerical range of B (cf. [4, Theorem 2.1]). □

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