On the Spanning w-Wide Diameter of the Star Graph*

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Let u and v be any two distinct nodes of an undirected graph G, which is k-connected. A container $C(u,v)$ between u and v is a set of internally disjoint paths $\{P_1, P_2, \ldots, P_w\}$ between u and v where $1 \leq w \leq k$. The width of $C(u,v)$ is w and the length of $C(u,v)$ {written as $l[C(u,v)]$} is $\max\{l(P_i) \mid 1 \leq i \leq w\}$. A w-container $C(u,v)$ is a container with width w. The w-wide distance between u and v, $d_w(u,v)$, is $\min\{l(C(u,v)) \mid C(u,v)$ is a w-container$\}$. A w-container $C(u,v)$ of the graph G is a w^*-container if every node of G is incident with a path in $C(u,v)$. That means that the w-container $C(u,v)$ spans the whole graph. Let S_n be the n-dimensional star graph with $n \geq 5$. It is known that S_n is bipartite. In this article, we show that, for any pair of distinct nodes u and v in different partite sets of S_n, there exists an $(n-1)^*$-container $C(u,v)$ and the $(n-1)$-wide distance $d_{(n-1)}(u,v)$ is less than or equal to $\frac{n!}{n-2}+1$. In addition, we also show the existence of a 2^*-container $C(u,v)$ and the 2-wide distance $d_2(u,v)$ is bounded above by $\frac{n!}{2}+1$. **NETWORKS, Vol. 48(4), 235–249 2006**

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1. BASIC DEFINITIONS

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph, in which the nodes correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer networks. The n-cube is one of the most popular topologies [17]. The n-dimensional star network S_n was proposed in [1] as "an attractive alternative to the n-cube" topology for interconnecting processors in parallel computers. Since its introduction, the network S_n has received considerable attention. Akers and Krishnamurthy [1] showed that the star graphs are node transitive and edge transitive. Jwo et al. [15] showed that the star graphs are bipartite. Star graphs are able to embed grids [15]: trees [3, 5, 8], and hypercubes [22]. Cycle embeddings and path embeddings are studied in [10–13, 15, 18, 23]. The diameter and fault diameters of star graphs were computed in [1, 16, 24]. Some interesting properties of star graphs are studied in [7, 9, 19].

For graph definitions and notation we follow [4]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u,v) \mid (u,v)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set*. A graph G is *vertex transitive* if there is an isomorphism f from G into itself such that $f(u) = v$ for any two nodes u and v of G. A graph G is *edge transitive* if there is an isomorphism f from G into itself such that $f((u,v)) = (x,y)$ for any two edges (u,v) and (x,y) of G. For a node u in graph G, $N_G(u)$ denotes the *neighborhood* of u, which is the set $\{v \mid (u,v) \in E\}$. For any node u of V, we denote the *degree* of u by $\deg_G(u) = |N_G(u)|$.

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A graph G is *k-regular* if $\deg_G(u) = k$ for all nodes u in G. Two nodes u and v are *adjacent* if $(u, v) \in E$. A *path* P is a sequence of adjacent nodes, written as $\langle v_1, v_2, \ldots, v_k \rangle$, in which the nodes $v_1, v_2, \ldots, v_k$ are distinct except that possibly $v_1 = v_k$. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_2, v_1 \rangle$. Let $I(P) = V(P) - \{v_1, v_k\}$ be the set of the *internal nodes* of P. A set of paths $\{P_1, P_2, \ldots, P_k\}$ is *internally node-disjoint* (abbreviated as *disjoint*) if $I(P_i) \cap I(P_j) = \emptyset$ for any $i \neq j$. The *length* of a path Q, $l(Q)$, is the number of edges in Q. We also write the path $\langle v_1, v_2, \ldots, v_k \rangle$ as $\langle v_1, Q_1, v_i, v_{i+1}, \ldots, v_j, Q_2, v_t, \ldots, v_k \rangle$, where Q_1 is the path $\langle v_1, v_2, \ldots, v_i \rangle$ and Q_2 is the path $\langle v_j, v_{j+1}, \ldots, v_t \rangle$. Hence, it is possible to write a path as $\langle v_1, Q, v_1, v_2, \ldots, v_k \rangle$ if $l(Q) = 0$. We use $d(u, v)$ to denote the *distance* between u and v, that is, the length of a shortest path joining u and v. The *diameter* of a graph G, $D(G)$, is defined as $\max\{d(u, v) \mid u, v \in V\}$. A path is a *hamiltonian path* if it contains all nodes of G. A graph G is *hamiltonian connected* if for any two distinct nodes of G there is a hamiltonian path of G between them. A *cycle* is a path with at least three nodes such that the first node is the same as the last node. A *hamiltonian cycle* of G is a cycle that traverses every node of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle.

The *connectivity* of G, $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's Theorem [21] that there are k internally node-disjoint paths joining any two distinct nodes u and v when $k \leq \kappa(G)$. A *container* $C(u, v)$ between two distinct nodes u and v in G is a set of internally disjoint paths $\{P_1, P_2, \ldots, P_r\}$ between u and v. The *width* of $C(u, v)$ is r. A *w-container* is a container of width w. The *length* of $C(u, v) = \{P_1, \ldots, P_r\}$, $l(C(u, v))$, is $\max\{l(P_i) \mid 1 \leq i \leq r\}$. The *w-wide distance* between u and v, $d_w(u, v)$, is $\min\{l(C(u, v)) \mid C(u, v)$ is a w-container$\}$. The *w-diameter* of G, $D_w(G)$, is $\max\{d_w(u, v) \mid u, v \in V, u \neq v\}$. In particular, the *wide diameter* of G is $D_{\kappa(G)}(G)$. The wide diameter is used to measure the performance of multipath communication in networks [14].

In this article, we are interested in a specific type of container. We say that a w-container $C(u, v)$ is a *w^*-container* if every node of G is incident with a path in $C(u, v)$. A graph G is *w^*-connected* if there exists a w^*-container between any two distinct nodes u and v. Obviously, a graph G is 1^*-connected if and only if it is hamiltonian connected. Moreover, a graph G is 2^*-connected if it is hamiltonian. The study of w^*-connected graphs is motivated by the globally 3^*-connected graphs proposed by Albert et al. [2]. A globally 3^*-connected graph is a 3-regular 3^*-connected graph. Assume that a graph G is w^*-connected. Obviously, $w \leq \kappa(G)$. A graph G is *super spanning connected* if G is w^*-connected for any w with $1 \leq w \leq \kappa(G)$. In [19, 26], some families of graphs are proved to be super spanning connected.

Graph containers do exist in engineering designed information and telecommunication networks or in biological and neural systems ([1, 14] and its references). The study of w-container, w-wide distance, and their w^*-versions plays a pivotal role in design and implementation of parallel routing and efficient information transmission in large-scale networking systems. In biological informatics and neuroinformatics, the existence and structure of a w^*-container signifies the cascade effect in the signal transduction system and the reaction in a metabolic pathway.

A graph G is *bipartite* if its node set can be partitioned into two subsets V_1 and V_2 such that every edge joins nodes between V_1 and V_2. Let G be a bipartite graph with bipartition V_1 and V_2 such that $|V_1| \geq |V_2|$. Suppose that there exists a w^*-container $C(u, v) = \{P_1, P_2, \ldots, P_w\}$ in G joining u to v with $u, v \in V_1$. Obviously, the number of nodes in P_i is $2t_i + 1$ for some integer t_i. There are $t_i - 1$ nodes of P_i in V_1 other than u and v, and t_i nodes of P_i in V_2. As a consequence, $|V_1| = \sum_{i=1}^{w}(t_i - 1) + 2$ and $|V_2| = \sum_{i=1}^{w} t_i$. Therefore, any bipartite graph G with $\kappa(G) \geq 3$ is not w^*-connected for any w, $3 \leq w \leq \kappa(G)$.

Let G be a w^*-laceable bipartite graph with bipartite node sets V_1 and V_2 and $|V_1 \cup V_2| \geq 2$. From the above discussion, $|V_1| = |V_2|$. For this reason, a bipartite graph is *w^*-laceable* if there exists a w^*-container between any two nodes from different partite sets for some w, $1 \leq w \leq \kappa(G)$. A 1^*-laceable graph is also known as a *hamiltonian laceable graph* [25]. Moreover, a graph G is 2^*-laceable if and only if it is hamiltonian. All 1^*-laceable graphs except K_1 and K_2 are 2^*-laceable. A bipartite graph G is *super spanning laceable* if G is i^*-laceable for all $1 \leq i \leq \kappa(G)$. Recently, Chang et al. [6] proved that the n-dimensional hypercube Q_n is super spanning laceable for every positive integer n. It was proved in [19] that the n-dimensional star graph S_n is super spanning laceable if and only if $n \neq 3$.

We also define the *w^{*}-laceable distance* between any two nodes u and v from different partite sets, $d_w^{s_L}(u, v)$, as $\min\{l(C(u, v)) \mid C(u, v)$ is a w^*-container$\}$. The *w^{*_L}-diameter* of G, denoted by $D_w^{s_L}(G)$, is defined as $\max\{d_w^{s_L}(u, v) \mid u$ and v are nodes from different partite sets$\}$. In particular, the *spanning wide diameter* of G is $D_{\kappa(G)}^{s_L}(G)$. In this article, we evaluate the spanning wide diameter of S_n and the 2^{*_L}-diameter of S_n.

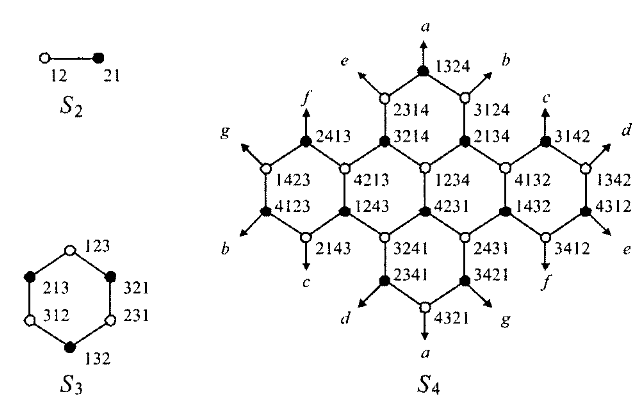

FIG. 1. The star graphs S_2, S_3, and S_4.

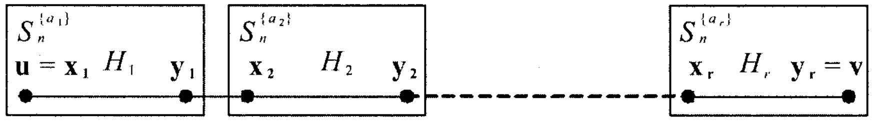

FIG. 2. Illustration for Lemma 3.

In Section 2, we give the definition of star graphs and introduce some basic properties of star graphs. In Section 3, we prove some hamiltonian path properties of star graphs. Then we discuss the spanning wide diameter of S_n in Section 4. In Section 5, we discuss the 2^{*_L}-spanning diameter of S_n.

2. STAR GRAPHS AND THEIR PROPERTIES

Assume that $n \geq 2$. We use $\langle n\rangle$ to denote the set $\{1, 2, \dots, n\}$, where n is a positive integer. A *permutation* on $\langle n\rangle$ is a sequence of n distinct elements $u_i \in \langle n\rangle$, $u_1u_2 \dots u_i \dots u_n$. An *inversion* of $u_1u_2 \dots u_i \dots u_n$ is a pair (i,j) such that $u_i < u_j$ and $i > j$. An *even permutation* is a permutation with an even number of inversions, and an *odd permutation* is a permutation with an odd number of inversions. The *n-dimensional star network*, denoted by S_n, is a graph with the node set $V(S_n) = \{u_1u_2 \dots u_n \mid u_i \in \langle n\rangle$ and $u_i \neq u_j$ for $i \neq j\}$. The edges are specified as follows: $u_1u_2 \dots u_i \dots u_n$ is adjacent to $v_1v_2 \dots v_i \dots v_n$ by an edge in dimension i with $2 \leq i \leq n$ if $v_j = u_j$ for $j \notin \{1, i\}$, $v_1 = u_i$ and $v_i = u_1$. By definition, S_n is an $(n-1)$-regular graph with $n!$ nodes. Moreover, it is node transitive and edge transitive [1]. The star graphs S_2, S_3, and S_4 are shown in Figure 1 for an illustration.

We use boldface to denote nodes in S_n. Hence, $\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}$ denotes a sequence of nodes in S_n. We use $\mathbf{e}$ to denote the element $12 \dots n$. It is known that S_n is a bipartite graph with one partite set containing those nodes corresponding to odd permutations and the other partite set containing those nodes corresponding to even permutations. We will use white nodes to represent those even permutation nodes and use black nodes to represent those odd permutation nodes. Let $\mathbf{u} = u_1u_2 \dots u_n$ be any node of the star graph S_n. We say that u_i is the i-th coordinate of $\mathbf{u}$, denoted by $(\mathbf{u})_i$, for $1 \leq i \leq n$. By the definition of S_n, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \leq i \leq n$. For this reason, we use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. Obviously, $((\mathbf{u})^i)^i = \mathbf{u}$. For $1 \leq i \leq n$, let $S_n^{\{i\}}$ denote the subgraph of S_n induced by those nodes $\mathbf{u}$ with $(\mathbf{u})_n = i$. Obviously, S_n can be decomposed into n subgraphs $S_n^{\{i\}}$, $1 \leq i \leq n$, and each $S_n^{\{i\}}$ is isomorphic to S_{n-1}. Thus, the star graph can be constructed recursively. Obviously, $\mathbf{u} \in S_n^{\{(\mathbf{u})_n\}}$ and $(\mathbf{u})^n \in S_n^{\{(\mathbf{u})_1\}}$. Let $I \subseteq \langle n\rangle$. We use S_n^I to denote the subgraph of S_n induced by $\cup_{i \in I} V\big(S_n^{\{i\}}\big)$. For $1 \leq i \neq j \leq n$, we use $E^{i,j}$ to denote the set of edges between $S_n^{\{i\}}$ and $S_n^{\{j\}}$. For $1 \leq i \neq j \leq n$, we use $S_n^{\{(i,j)\}}$ to denote the subgraph of S_n induced by those nodes $\mathbf{u}$ with $(\mathbf{u})_{n-1} = i$ and $(\mathbf{u})_n = j$. Obviously, $S_n^{\{(i,j)\}} \neq S_n^{\{(j,i)\}}$ and $S_n^{\{(i,j)\}}$ is isomorphic to S_{n-2}.

Lemma 1 ([23]). *Assume that $n \geq 4$. $|E^{i,j}| = (n-2)!$ for any $1 \leq i \neq j \leq n$. Moreover, there are $\frac{(n-2)!}{2}$ edges joining black nodes of $S_n^{\{i\}}$ to white nodes of $S_n^{\{j\}}$.*

Lemma 2 ([1]). *Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct nodes of S_n with $d(\mathbf{u}, \mathbf{v}) \leq 2$. Then $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Moreover, $\{((\mathbf{u})^i)_1 \mid 2 \leq i \leq n-1\} = \langle n\rangle - \{(\mathbf{u})_1, (\mathbf{u})_n\}$ if $n \geq 3$.*

3. HAMILTONIAN PATHS OF STAR GRAPHS

Theorem 1 ([11]). *S_n is hamiltonian laceable if and only if $n \neq 3$.*

Theorem 2 ([18]). *Suppose that $\mathbf{w}$ is any black node of S_n with $n \geq 4$. Then there is a hamiltonian path P of $S_n - \{\mathbf{w}\}$ between any two distinct white nodes $\mathbf{u}$ and $\mathbf{v}$.*

Lemma 3. *Let $n \geq 5$ and $I = \{a_1, a_2, \dots, a_r\}$ be a subset of $\langle n\rangle$ for some $r \in \langle n\rangle$. Assume that $\mathbf{u}$ is a white node in $S_n^{\{a_1\}}$ and $\mathbf{v}$ is a black node in $S_n^{\{a_r\}}$. Then there is a hamiltonian path P of S_n^I joining $\mathbf{u}$ to $\mathbf{v}$.*

Proof. Let $\mathbf{x_1} = \mathbf{u}$ and $\mathbf{y_r} = \mathbf{v}$. Obviously, $S_n^{\{a_i\}}$ is isomorphic to S_{n-1} for every $i \in \langle r\rangle$. By Theorem 1, this result holds on $r = 1$. Suppose that $r \geq 2$. By Lemma 1, there are $(n-2)!/2 \geq 3$ edges joining black nodes of $S_n^{\{a_i\}}$ to white nodes of $S_n^{\{a_{i+1}\}}$ for every $i \in \langle r-1\rangle$. For every $i \in \langle r-1\rangle$, we can choose a black node $\mathbf{y_i} \in S_n^{\{a_i\}}$ and a white node $\mathbf{x_{i+1}} \in S_n^{\{a_{i+1}\}}$ such that $(\mathbf{y_i}, \mathbf{x_{i+1}}) \in E^{a_i, a_{i+1}}$. By Theorem 1, there is a hamiltonian path H_i of $S_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every $i \in \langle r\rangle$. Then $\langle \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \dots, \mathbf{x_r}, H_r, \mathbf{y_r}\rangle$ forms the desired hamiltonian path of S_n^I joining $\mathbf{u}$ to $\mathbf{v}$. See Figure 2 for an illustration. ∎

Lemma 4. *Assume that $\mathbf{r}$ and $\mathbf{s}$ are any two adjacent nodes of S_n with $n \geq 4$. Then, for any white node $\mathbf{u}$ in $S_n - \{\mathbf{r}, \mathbf{s}\}$ and for any $i \in \langle n\rangle$, there exists a hamiltonian path P of $S_n - \{\mathbf{r}, \mathbf{s}\}$ joining $\mathbf{u}$ to some black node $\mathbf{v}$ of $S_n - \{\mathbf{r}, \mathbf{s}\}$ with $(\mathbf{v})_1 = i$.*

Proof. Because S_n is node transitive and edge transitive, we assume that $\mathbf{r} = \mathbf{e}$ and $\mathbf{s} = (\mathbf{e})^2$. Obviously, both $\mathbf{e}$ and $(\mathbf{e})^2$ are in $S_n^{\{n\}}$. We prove this lemma by induction on n. Suppose that $n = 4$. The required hamiltonian paths of $S_4 - \{1234, 2134\}$ are listed below.

⟨1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 4312, 2314, 3214, 4213, 2413, 1423, 3421, 2431, 4231, 3241, 1243⟩
⟨1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 4312, 2314, 3214, 4213, 1243, 3241, 4231, 2431, 3421, 1423, 2413⟩
⟨1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 2431, 4231, 3241, 1243, 4213, 3214, 2314, 4312, 3412, 2413, 1423, 3421⟩
⟨1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 2431, 3421, 1423, 2413, 3412, 4312, 2314, 3214, 4213, 1243, 3241, 4231⟩
⟨1423, 3421, 4321, 2341, 1342, 4312, 3412, 2413, 4213, 3214, 2314, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 2431, 4231, 3241, 1243⟩
⟨1423, 3421, 2431, 4231, 3241, 1243, 4213, 3214, 2314, 4312, 1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 2413⟩
⟨1423, 3421, 2431, 4231, 3241, 1243, 4213, 2413, 3412, 1432, 4132, 3142, 2143, 4123, 3124, 1324, 4321, 2341, 1342, 4312, 2314, 3214⟩
⟨1423, 3421, 2431, 4231, 3241, 1243, 2143, 3142, 4132, 1432, 3412, 2413, 4213, 3214, 2314, 4312, 1342, 2341, 4321, 1324, 3124, 4123⟩
⟨2143, 3142, 4132, 1432, 3412, 2413, 1423, 4123, 3124, 1324, 4321, 3421, 2431, 4231, 3241, 2341, 1342, 4312, 2314, 3214, 4213, 1243⟩
⟨2143, 3142, 4132, 1432, 2431, 4231, 3241, 1243, 4213, 3214, 2314, 1324, 3124, 4123, 1423, 3421, 4321, 2341, 1342, 4312, 3412, 2413⟩
⟨2143, 3142, 4132, 1432, 2431, 4231, 3241, 1243, 4213, 3214, 2314, 1324, 3124, 4123, 1423, 2413, 3412, 4312, 1342, 2341, 4321, 3421⟩
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⟨2314, 3214, 4213, 1243, 3241, 4231, 2431, 3421, 1423, 2413, 3412, 4312, 1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432⟩
⟨2314, 3214, 4213, 1243, 3241, 4231, 2431, 1432, 4132, 3142, 2143, 4123, 3124, 1324, 4321, 3421, 1423, 2413, 3412, 4312, 1342, 2341⟩
⟨2314, 4312, 1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 2413, 1423, 3421, 2431, 4231, 3241, 1243, 4213, 3214⟩
⟨2314, 3214, 4213, 1243, 3241, 4231, 2431, 3421, 1423, 2413, 3412, 1432, 4132, 3142, 2143, 4123, 3124, 1324, 4321, 2341, 1342, 4312⟩
⟨2431, 4231, 3241, 1243, 4213, 3214, 2314, 4312, 1342, 2341, 4321, 3421, 1423, 2413, 3412, 1432, 4132, 3142, 2143, 4123, 3124, 1324⟩
⟨2431, 4231, 3241, 1243, 4213, 3214, 2314, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 4312, 1342, 2341, 4321, 3421, 1423, 2413⟩
⟨2431, 4231, 3241, 1243, 4213, 3214, 2314, 4312, 1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 2413, 1423, 3421⟩
⟨2431, 3421, 4321, 1324, 3124, 4123, 1423, 2413, 3412, 1432, 4132, 3142, 2143, 1243, 4213, 3214, 2314, 4312, 1342, 2341, 3241, 4231⟩
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⟨3241, 4231, 2431, 1432, 4132, 3142, 1342, 2341, 4321, 3421, 1423, 2413, 3412, 4312, 2314, 3214, 4213, 1243, 2143, 4123, 3124, 1324⟩
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⟨4132, 3142, 2143, 4123, 3124, 1324, 4321, 2341, 1342, 4312, 2314, 3214, 4213, 1243, 3241, 4231, 2431, 3421, 1423, 2413, 3412, 1432⟩
⟨4132, 3142, 2143, 4123, 3124, 1324, 4321, 3421, 1423, 2413, 3412, 1432, 2431, 4231, 3241, 1243, 4213, 3214, 2314, 4312, 1342, 2341⟩
⟨4132, 3142, 2143, 4123, 3124, 1324, 4321, 2341, 1342, 4312, 2314, 3214, 4213, 1243, 3241, 4231, 2431, 1432, 3412, 2413, 1423, 3421⟩
⟨4132, 3142, 2143, 4123, 3124, 1324, 2314, 3214, 4213, 1243, 3241, 4231, 2431, 1432, 3412, 2413, 1423, 3421, 4321, 2341, 1342, 4312⟩
⟨4213, 3214, 2314, 4312, 1342, 2341, 4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 2413, 1423, 3421, 2431, 4231, 3241, 1243⟩
⟨4213, 3214, 2314, 4312, 1342, 2341, 4321, 1324, 3124, 4123, 1423, 3421, 2431, 4231, 3241, 1243, 2143, 3142, 4132, 1432, 3412, 2413⟩
⟨4213, 3214, 2314, 4312, 1342, 2341, 4321, 1324, 3124, 4123, 2143, 1243, 3241, 4231, 2431, 3421, 1423, 2413, 3412, 1432, 4132, 3142⟩
⟨4213, 3214, 2314, 4312, 1342, 2341, 3241, 1243, 2143, 3142, 4132, 1432, 3412, 2413, 1423, 4123, 3124, 1324, 4321, 3421, 2431, 4231⟩
⟨4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 2413, 1423, 3421, 2431, 4231, 3241, 2341, 1342, 4312, 2314, 3214, 4213, 1243⟩
⟨4321, 1324, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 2413, 1423, 3421, 2431, 4231, 3241, 1243, 4213, 3214, 2314, 4312, 1342, 2341⟩
⟨4321, 1324, 3124, 4123, 2143, 1243, 4213, 3214, 2314, 4312, 1342, 2341, 3241, 4231, 2431, 3421, 1423, 2413, 3412, 1432, 4132, 3142⟩
⟨4321, 1324, 3124, 4123, 2143, 1243, 3241, 2341, 1342, 3142, 4132, 1432, 3412, 4312, 2314, 3214, 4213, 2413, 1423, 3421, 2431, 4231⟩

Assume that this result holds in S_k for every $4 \le k \le n-1$. We have the following cases:

CASE 1. $\mathbf{u} \in S_n^{\{n\}}$. By induction, there is a hamiltonian path P of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to a black node $\mathbf{x}$ with $(\mathbf{x})_1 = n-1$. Note that $(\mathbf{x})^n$ is a white node of $S_n^{\{n-1\}}$. We choose a black node $\mathbf{v}$ in $S_n^{\langle n-2\rangle}$ with $(\mathbf{v})_1 = i$. By Lemma 3, there is a hamiltonian path Q of $S_n^{\langle n-1\rangle}$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, P, \mathbf{x}, (\mathbf{x})^n, Q, \mathbf{v}\rangle$ forms the desired hamiltonian path of $S_n - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to $\mathbf{v}$ with $(\mathbf{v})_1 = i$. See Figure 3(a) for an illustration.

CASE 2. $\mathbf{u} \in S_n^{\{k\}}$ for some $k \in \langle n-1\rangle$. By Lemma 1, there are $(n-2)!/2 \ge 3$ edges joining black nodes of $S_n^{\{k\}}$ to white nodes of $S_n^{\{n\}}$. We can choose a white node $\mathbf{x} \in S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ with $(\mathbf{x})_1 = k$. By Theorem 1, there is a hamiltonian path P

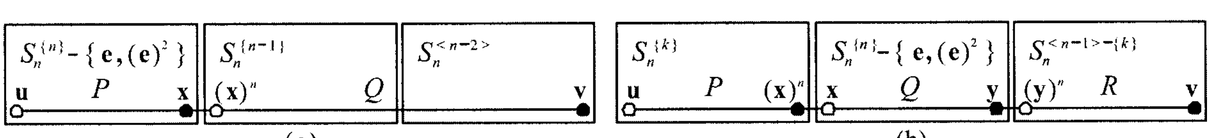

FIG. 3. Illustration for Lemma 4.

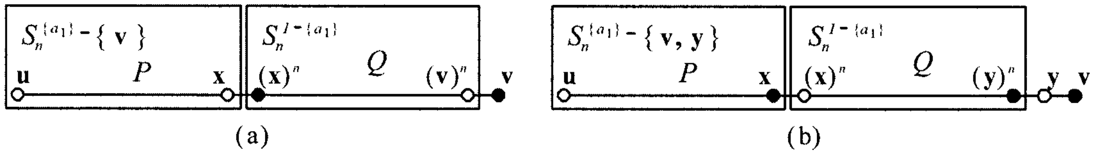

FIG. 4. Illustration for Theorem 3.

of $S_n^{\{k\}}$ joining $\mathbf{u}$ to the black node $(\mathbf{x})^n$. By induction, there is a hamiltonian path Q of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{x}$ to a black node $\mathbf{y}$ with $(\mathbf{y})_1 \in \langle n-1\rangle - \{k\}$. We choose a black node $\mathbf{v}$ in $S_n^{\langle n-1\rangle - \{k,(\mathbf{y})_1\}}$ with $(\mathbf{v})_1 = i$. By Lemma 3, there exists a hamiltonian path R of $S_n^{\langle n-1\rangle - \{k\}}$ joining the white node $(\mathbf{y})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, P, (\mathbf{x})^n, \mathbf{x}, Q, \mathbf{y}, (\mathbf{y})^n, R, \mathbf{v}\rangle$ forms the desired hamiltonian path of $S_n - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to $\mathbf{v}$ with $(\mathbf{v})_1 = i$. See Figure 3(b) for an illustration. ∎

Theorem 3. *Let $n \geq 5$ and $I = \{a_1, a_2, \ldots, a_r\}$ be a subset of $\langle n\rangle$ for some $r \in \langle n\rangle$. Then S_n^I is hamiltonian laceable.*

Proof. Let $\mathbf{u}$ be a white node and $\mathbf{v}$ be a black node of S_n^I. By Lemma 3, this theorem holds on either $r = 1$ or $r \geq 2$ and $(\mathbf{u})_n \neq (\mathbf{v})_n$. Thus, we assume that $r \geq 2$ and $(\mathbf{u})_n = (\mathbf{v})_n$. Without loss of generality, we assume that $(\mathbf{u})_n = (\mathbf{v})_n = a_1$.

CASE 1. $(\mathbf{v})^n \in S_n^I$. Without loss of generality, we assume that $(\mathbf{v})^n \in S_n^{\{a_r\}}$. By Theorem 2, there is a hamiltonian path P of $S_n^{\{a_1\}} - \{\mathbf{v}\}$ joining $\mathbf{u}$ to a white node $\mathbf{x}$ with $(\mathbf{x})_1 = a_2$. By Lemma 3, there is a hamiltonian path Q of $S_n^{I-\{a_1\}}$ joining the black node $(\mathbf{x})^n$ to the white node $(\mathbf{v})^n$. Then $\langle \mathbf{u}, P, \mathbf{x}, (\mathbf{x})^n, Q, (\mathbf{v})^n, \mathbf{v}\rangle$ forms the desired hamiltonian path of S_n^I joining $\mathbf{u}$ to $\mathbf{v}$. See Figure 4(a) for an illustration.

CASE 2. $(\mathbf{u})^n \notin S_n^I$ and $(\mathbf{v})^n \notin S_n^I$. We can choose a white node $\mathbf{y}$ with $\mathbf{y}$ being a neighbor of $\mathbf{v}$ in $S_n^{\{a_1\}}$ and $(\mathbf{y})_1 = a_r$. Obviously, $\mathbf{y} \neq \mathbf{u}$. By Lemma 4, there is a hamiltonian path P of $S_n^{\{a_1\}} - \{\mathbf{v}, \mathbf{y}\}$ joining $\mathbf{u}$ to a black node $\mathbf{x}$ with $(\mathbf{x})_1 = a_2$. By Lemma 3, there is a hamiltonian path Q of $S_n^{I-\{a_1\}}$ joining the white node $(\mathbf{x})^n$ to the black node $(\mathbf{y})^n$. Then $\langle \mathbf{u}, P, \mathbf{x}, (\mathbf{x})^n, Q, (\mathbf{y})^n, \mathbf{y}, \mathbf{v}\rangle$ is the desired hamiltonian path of S_n^I joining $\mathbf{u}$ to $\mathbf{v}$. See Figure 4(b) for an illustration. ∎

Theorem 4. *Assume that* $\mathbf{r}$ *and* $\mathbf{s}$ *are two adjacent nodes of S_n with $n \geq 5$. Then $S_n - \{\mathbf{r}, \mathbf{s}\}$ is hamiltonian laceable.*

Proof. Because S_n is node transitive and edge transitive, we assume that $\mathbf{r} = \mathbf{e}$ and $\mathbf{s} = (\mathbf{e})^2$. Obviously, both $\mathbf{e}$ and $(\mathbf{e})^2$ are in $S_n^{\{n\}}$. Let $\mathbf{u}$ be a white node and $\mathbf{v}$ be a black node of $S_n - \{\mathbf{e}, (\mathbf{e})^2\}$. We want to find a hamiltonian path of $S_n - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to $\mathbf{v}$.

CASE 1. $\mathbf{u}, \mathbf{v} \in S_n^{\{n\}}$. By Lemma 4, there is a hamiltonian path P of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to a black node $\mathbf{y}$ with $(\mathbf{y})_1 = 1$. We write P as $\langle \mathbf{u}, Q_1, \mathbf{x}, \mathbf{v}, Q_2, \mathbf{y}\rangle$. (Note that $l(Q_1) = 0$ if $\mathbf{u} = \mathbf{x}$ and $l(Q_2) = 0$ if $\mathbf{v} = \mathbf{y}$.) By Theorem 3, there is a hamiltonian path R of $S_n^{\langle n-1\rangle}$ joining the black node $(\mathbf{x})^n$ to the white node $(\mathbf{y})^n$. Then $\langle \mathbf{u}, Q_1, \mathbf{x}, (\mathbf{x})^n, R, (\mathbf{y})^n, \mathbf{y}, (Q_2)^{-1}, \mathbf{v}\rangle$ is the desired hamiltonian path of $S_n - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to $\mathbf{v}$. See Figure 5(a) for an illustration.

CASE 2. $\mathbf{u}, \mathbf{v} \in S_n^{\{k\}}$ for some $k \in \langle n-1\rangle$. By Theorem 1, there is a hamiltonian path P of $S_n^{\{k\}}$ joining $\mathbf{u}$ to $\mathbf{v}$. By Lemma 1, there are $(n-2)!/2 \geq 3$ edges joining white nodes of $S_n^{\{k\}}$ to black nodes of $S_n^{\{n\}}$. We can choose a white node $\mathbf{x}$ of $S_n^{\{k\}}$, with $(\mathbf{x})^n$ being a black node of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$. We write P as $\langle \mathbf{u}, Q_1, \mathbf{x}, \mathbf{y}, Q_2, \mathbf{v}\rangle$. (Note that $l(Q_1) = 0$ if $\mathbf{u} = \mathbf{x}$ and $l(Q_2) = 0$ if $\mathbf{v} = \mathbf{y}$.) Because $d(\mathbf{x}, \mathbf{y}) = 1$, by Lemma 2, $(\mathbf{y})_1 \in \langle n-1\rangle - \{k\}$. By Lemma 4, there is a hamiltonian path R of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $(\mathbf{x})^n$ to a white node $\mathbf{z}$ with $(\mathbf{z})_1 \in \langle n-1\rangle - \{k\}$. By Theorem 3, there is a hamiltonian path T of $S_n^{\langle n-1\rangle - \{k\}}$ joining the black node $(\mathbf{z})^n$ to the white node $(\mathbf{y})^n$. Then $\langle \mathbf{u}, Q_1, \mathbf{x}, (\mathbf{x})^n, R, \mathbf{z}, (\mathbf{z})^n, T, (\mathbf{y})^n, \mathbf{y}, Q_2, \mathbf{v}\rangle$ is the desired hamiltonian path of $S_n - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to $\mathbf{v}$. See Figure 5(b) for an illustration.

CASE 3. $\mathbf{u} \in S_n^{\{n\}}$ and $\mathbf{v} \in S_n^{\{k\}}$ for some $k \in \langle n-1\rangle$. By Lemma 4, there is a hamiltonian path P of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to a black node $\mathbf{x}$ with $(\mathbf{x})_1 \in \langle n-1\rangle$. By Theorem 3, there is a hamiltonian path Q of $S_n^{\langle n-1\rangle}$ joining the white node $(\mathbf{x})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, P, \mathbf{x}, (\mathbf{x})^n, Q, \mathbf{v}\rangle$ is the desired hamiltonian path of $S_n - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to $\mathbf{v}$. See Figure 5(c) for an illustration.

CASE 4. $\mathbf{u} \in S_n^{\{k\}}$ and $\mathbf{v} \in S_n^{\{l\}}$ with k, l, and n being distinct. By Lemma 1, there are $(n-2)!/2 \geq 3$ edges joining black nodes of $S_n^{\{k\}}$ to white nodes of $S_n^{\{n\}}$. We choose a black node $\mathbf{x}$ of $S_n^{\{k\}}$ with $(\mathbf{x})^n$ being a white node of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$. By Theorem 1, there is a hamiltonian path P of $S_n^{\{k\}}$ joining $\mathbf{u}$ to $\mathbf{x}$. By Lemma 4, there is a hamiltonian path Q of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $(\mathbf{x})^n$ to a black node $\mathbf{y}$ with $(\mathbf{y})_1 \in \langle n-1\rangle - \{k\}$. By Theorem 3, there is a hamiltonian path R of $S_n^{\langle n-1\rangle - \{k\}}$ joining

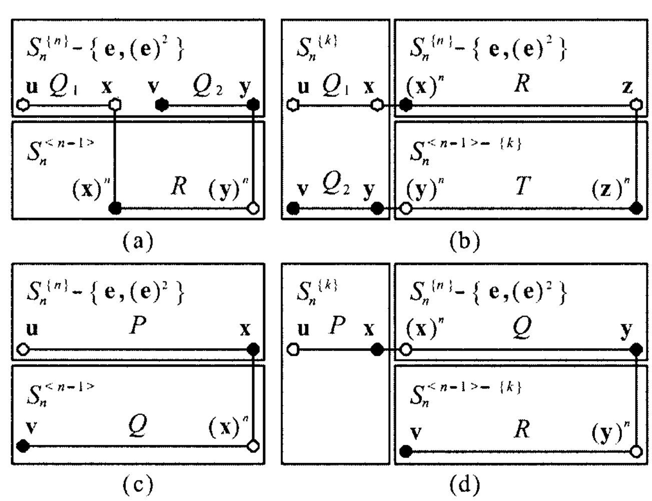

FIG. 5. Illustration for Theorem 4.

the white node $(\mathbf{y})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, P, \mathbf{x}, (\mathbf{x})^n, Q, \mathbf{y}, (\mathbf{y})^n, R, \mathbf{v} \rangle$ is the desired hamiltonian path of $S_n - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{u}$ to $\mathbf{v}$. See Figure 5(d) for an illustration. ∎

Lemma 5. *Assume that $n \geq 5$. Suppose that $\mathbf{p}$ and $\mathbf{q}$ are two different white nodes of S_n, and $\mathbf{r}$ and $\mathbf{s}$ are two different black nodes of S_n. Then there exist two disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{p}$ to $\mathbf{r}$, (2) P_2 joins $\mathbf{q}$ to $\mathbf{s}$, and (3) $P_1 \cup P_2$ spans S_n.*

Proof. Because S_n is edge transitive, we assume that $\mathbf{p} \in S_n^{\{n\}}$ and $\mathbf{q} \in S_n^{\{n-1\}}$. Suppose that $\mathbf{r} \in S_n^{\{i\}}$ and $\mathbf{s} \in S_n^{\{j\}}$.

CASE 1. $i, j \in \langle n-2 \rangle$ with $i \neq j$. By Theorem 3, there is a hamiltonian path P_1 of $S_n^{\{i,n\}}$ joining $\mathbf{p}$ to $\mathbf{r}$. Again, there is a hamiltonian path P_2 of $S_n^{\langle n-1 \rangle - \{i\}}$ joining $\mathbf{q}$ to $\mathbf{s}$. Then P_1 and P_2 are the desired paths.

CASE 2. $i, j \in \langle n-2 \rangle$ with $i = j$. We can choose a white node $\mathbf{x}$ with $\mathbf{x}$ being a neighbor of $\mathbf{s}$ in $S_n^{\{i\}}$ and $(\mathbf{x})_1 \in \langle n-1 \rangle - \{i\}$. By Lemma 4, there is a hamiltonian path Q of $S_n^{\{i\}} - \{\mathbf{s}, \mathbf{x}\}$ joining $\mathbf{r}$ to a white node $\mathbf{y}$ with $(\mathbf{y})_1 = n$. By Theorem 3, there is a hamiltonian path P of $S_n^{\{n\}}$ joining $\mathbf{p}$ to the black node $(\mathbf{y})^n$. Moreover, there is a hamiltonian path R of $S_n^{\langle n-1 \rangle - \{i\}}$ joining $\mathbf{q}$ to the black node $(\mathbf{x})^n$. Then $P_1 = \langle \mathbf{p}, P, (\mathbf{y})^n, \mathbf{y}, Q^{-1}, \mathbf{r} \rangle$ and $P_2 = \langle \mathbf{q}, R, (\mathbf{x})^n, \mathbf{x}, \mathbf{s} \rangle$ are the desired paths.

CASE 3. Either ($i = n$ and $j \in \langle n-1 \rangle$), or ($i \in \langle n \rangle - \{n-1\}$ and $j = n-1$). By symmetry, we assume that $i = n$ and $j \in \langle n-1 \rangle$. By Theorem 3, there is a hamiltonian path P_1 of $S_n^{\{n\}}$ joining $\mathbf{p}$ to $\mathbf{r}$. Moreover, there is a hamiltonian path P_2 of $S_n^{\langle n-1 \rangle}$ joining $\mathbf{q}$ to $\mathbf{s}$. Then P_1 and P_2 are the desired paths.

CASE 4. Either ($i = n-1$ and $j \in \langle n-2 \rangle$), or ($i \in \langle n-2 \rangle$ and $j = n$). By symmetry, we assume that $i = n-1$ and $j \in \langle n-2 \rangle$. By Lemma 1, there exist $(n-2)!/2 \geq 3$ edges joining white nodes of $S_n^{\{n-1\}}$ to black nodes of $S_n^{\{n\}}$. We can choose a white node $\mathbf{x}$ in $S_n^{\{n-1\}} - \{\mathbf{q}\}$ with $(\mathbf{x})_1 = n$. By Theorem 3, there is a hamiltonian path R of $S_n^{\{n-1\}}$ joining $\mathbf{q}$ to $\mathbf{r}$. We write R as $\langle \mathbf{q}, R_1, \mathbf{y}, \mathbf{x}, R_2, \mathbf{r} \rangle$. By Theorem 3, there is a hamiltonian path P of $S_n^{\{n\}}$ joining $\mathbf{p}$ to the black node $(\mathbf{x})^n$. Because $d(\mathbf{x}, \mathbf{y}) = 1$, by Lemma 2, $(\mathbf{y})^n \in S_n^{\langle n-2 \rangle}$. By Theorem 3, there exists a hamiltonian path Q of $S_n^{\langle n-2 \rangle}$ joining the white node $(\mathbf{y})^n$ to $\mathbf{s}$. Then $P_1 = \langle \mathbf{p}, P, (\mathbf{x})^n, \mathbf{x}, R_2, \mathbf{r} \rangle$ and $P_2 = \langle q, R_1, \mathbf{y}, (\mathbf{y})^n, Q, \mathbf{s} \rangle$ are the desired paths.

CASE 5. $i = n-1$ and $j = n$. By Theorem 3, there is a hamiltonian path Q of $S_n^{\{n\}}$ joining $\mathbf{p}$ to $\mathbf{s}$. Again, there is a hamiltonian path R of $S_n^{\{n-1\}}$ joining $\mathbf{q}$ to $\mathbf{r}$. We choose a white node $\mathbf{x} \in S_n^{\{n\}}$ with $(\mathbf{x})_1 = n-1$. We write Q as $\langle \mathbf{p}, Q_1, \mathbf{x}, \mathbf{y}, Q_2, \mathbf{s} \rangle$ and write R as $\langle \mathbf{q}, R_1, \mathbf{w}, (\mathbf{x})^n, R_2, \mathbf{r} \rangle$. Obviously, $\mathbf{y}$ is a black node and $\mathbf{w}$ is a white node. Because $d(\mathbf{x}, \mathbf{y}) = 1$, by Lemma 2, $(\mathbf{y})_1 \in \langle n-2 \rangle$. Because $d((\mathbf{x})^n, \mathbf{w}) = 1$, by Lemma 2, $(\mathbf{w})_1 \in \langle n-2 \rangle$. By Theorem 3, there exists a hamiltonian path W of $S_n^{\langle n-2 \rangle}$ joining the black node $(\mathbf{w})^n$ to the white node $(\mathbf{y})^n$. Then $P_1 = \langle \mathbf{p}, Q_1, \mathbf{x}, (\mathbf{x})^n, R_2, \mathbf{r} \rangle$ and $P_2 = \langle \mathbf{q}, R_1, \mathbf{w}, (\mathbf{w})^n, W, (\mathbf{y})^n, \mathbf{y}, Q_2, \mathbf{s} \rangle$ are the desired paths.

CASE 6. Either $i = j = n$ or $i = j = n-1$. By symmetry, we assume that $i = j = n$. By Theorem 3, there is a hamiltonian path P of $S_n^{\{n\}}$ joining $\mathbf{p}$ to $\mathbf{s}$. We can write P as $\langle \mathbf{p}, R_1, \mathbf{r}, \mathbf{x}, R_2, \mathbf{s} \rangle$. By Theorem 3, there is a hamiltonian path Q of $S_n^{\langle n-1 \rangle}$ joining $\mathbf{q}$ to the black node $(\mathbf{x})^n$. Then $P_1 = \langle \mathbf{p}, R_1, \mathbf{r} \rangle$ and $P_2 = \langle \mathbf{q}, Q, (\mathbf{x})^n, \mathbf{x}, R_2, \mathbf{s} \rangle$ are the desired paths. ∎

4. THE $(n-1)^{*L}$-DIAMETER OF S_n

Let $\mathbf{u}$ be a node of S_n with $n \geq 4$ and let m be any integer with $3 \leq m \leq n$. We set $F_m(\mathbf{u}) = \{(\mathbf{u})^i \mid 3 \leq i \leq m\} \cup \{((\mathbf{u})^i)^{i-1} \mid 3 \leq i \leq m\}$.

Lemma 6. *Assume that $\mathbf{u}$ is a white node of S_n and $j \in \langle n \rangle$ with $n \geq 4$. Then there is a hamiltonian path P of $S_n - F_n(\mathbf{u})$ joining $\mathbf{u}$ to some black node $\mathbf{v}$ with $(\mathbf{v})_1 = j$.*

Proof. We prove this lemma by induction on n. Because S_n is node transitive, we assume that $\mathbf{u} = \mathbf{e}$. Suppose that $n = 4$. The required hamiltonian paths of $S_4 - F_4(\mathbf{e})$ are listed below:

$j = 1$	⟨1234, 2134, 3124, 4123, 1423, 3421, 2431, 1432, 4132, 3142, 2143, 1243, 4213, 2413, 3412, 4312, 1342, 2341, 4321, 1324⟩
$j = 2$	⟨1234, 2134, 4132, 3142, 1342, 4312, 3412, 1432, 2431, 3421, 1423, 2413, 4213, 1243, 2143, 4123, 3124, 1324, 4321, 2341⟩
$j = 3$	⟨1234, 2134, 4132, 1432, 2431, 3421, 1423, 4123, 3124, 1324, 4321, 2341, 1342, 4312, 3412, 2413, 4213, 1243, 2143, 3142⟩
$j = 4$	⟨1234, 2134, 3124, 1324, 4321, 2341, 1342, 3142, 4132, 1432, 2431, 3421, 1423, 4123, 2143, 1243, 4213, 2413, 3412, 4312⟩

Assume that this statement holds on any S_k for every $4 \leq k \leq n-1$. We have $F_n(\mathbf{e}) = F_{n-1}(\mathbf{e}) \cup \{(\mathbf{e})^n, ((\mathbf{e})^n)^{n-1}\}$. By induction, there is a hamiltonian path P of $S_n^{\{n\}} - F_{n-1}(\mathbf{e})$ joining $\mathbf{e}$ to a black node $\mathbf{x}$ with $(\mathbf{x})_1 = 1$. By Lemma 4, there is a hamiltonian path Q of $S_n^{\{1\}} - \{(\mathbf{e})^n, ((\mathbf{e})^n)^{n-1}\}$ joining the white node $(\mathbf{x})^n$ to a black node $\mathbf{y}$ with $(\mathbf{y})_1 = 2$. We can choose a black node $\mathbf{z}$ of $S_n^{\langle n-1 \rangle - \{1\}}$ with $(\mathbf{z})_1 = j$. By Theorem 3, there exists a hamiltonian path R of $S_n^{\langle n-1 \rangle - \{1\}}$ joining the white node $(\mathbf{y})^n$ to $\mathbf{z}$. Then $\langle \mathbf{e}, P, \mathbf{x}, (\mathbf{x})^n, Q, \mathbf{y}, (\mathbf{y})^n, R, \mathbf{z} \rangle$ is a desired hamiltonian path. ∎

Lemma 7. *Let $\mathbf{u} = u_1u_2u_3u_4$ be any white node of S_4. There exist three paths P_1, P_2, and P_3 such that (1) P_1 joins $\mathbf{u}$ to the black node $u_2u_4u_1u_3$ with $l(P_1) = 7$, (2) P_2 joins $\mathbf{u}$ to the white node $u_3u_4u_1u_2$ with $l(P_2) = 8$, (3) P_3 joins $\mathbf{u}$ to the white node $u_4u_1u_3u_2$ with $l(P_3) = 8$, and (4) $P_1 \cup P_2 \cup P_3$ spans S_4.*

Proof. Because S_4 is node transitive, we assume that $\mathbf{u} = 1234$. Then we set

$$P_1 = \langle 1234, 3214, 4213, 1243, 2143, 4123, 1423, 2413 \rangle,$$

$$P_2 = \langle 1234, 4231, 3241, 2341, 4321, 3421, 2431, 1432, 3412 \rangle, \text{ and}$$

$$P_3 = \langle 1234, 2134, 3124, 1324, 2314, 4312, 1342, 3142, 4132 \rangle.$$

Obviously, P_1, P_2, and P_3 are the desired paths. ∎

Lemma 8. *Let $\mathbf{u} = u_1u_2u_3u_4$ be any white node of S_4. Let $i_1i_2i_3$ be a permutation of u_2, u_3, and u_4. There exist four paths P_1, P_2, P_3, and P_4 of S_4 such that (1) P_1 joins $\mathbf{u}$ to a white node $\mathbf{w}$ with $(\mathbf{w})_1 = i_1$ and $l(P_1) = 2$, (2) P_2 joins $\mathbf{u}$ to a white node $\mathbf{x}$ with $(\mathbf{x})_1 = i_2$ and $l(P_2) = 2$, (3) P_3 joins $\mathbf{u}$ to a black node $\mathbf{y}$ with $(\mathbf{y})_1 = i_3$ and $l(P_3) = 19$, (4) P_4 joins $\mathbf{u}$ to a black node $\mathbf{z}$ with $\mathbf{z} \neq \mathbf{y}$, $(\mathbf{z})_1 = i_3$, and $l(P_4) = 19$, (5) $P_1 \cup P_2 \cup P_3$ spans S_4, (6) $P_1 \cup P_2 \cup P_4$ spans S_4, (7) $V(P_1) \cap V(P_2) \cap V(P_3) = \{\mathbf{u}\}$, and (8) $V(P_1) \cap V(P_2) \cap V(P_4) = \{\mathbf{u}\}$.*

Proof. Because S_4 is node transitive, we assume that $\mathbf{u} = 1234$. Because $\mathbf{u} = 1234$, we have $\{i_1, i_2\} \subset \{2, 3, 4\}$ and $i_3 \in \{2, 3, 4\} - \{i_1, i_2\}$. Without loss of generality, we suppose that $i_1 < i_2$. The required four paths are listed below.

$i_1 = 2$	$P_1 = \langle 1234, 4231, 2431 \rangle$
$i_2 = 3$	$P_2 = \langle 1234, 2134, 3124 \rangle$
$i_3 = 4$	$P_3 = \langle 1234, 3214, 2314, 1324, 4321, 3421, 1423, 4123, 2143, 3142, 4132, 1432, 3412, 2413, 4213, 1243, 3241, 2341, 1342, 4312 \rangle$
	$P_4 = \langle 1234, 3214, 2314, 1324, 4321, 3421, 1423, 2413, 4213, 1243, 3241, 2341, 1342, 4312, 3412, 1432, 4132, 3142, 2143, 4123 \rangle$
$i_1 = 2$	$P_1 = \langle 1234, 4231, 2431 \rangle$
$i_2 = 4$	$P_2 = \langle 1234, 3214, 4213 \rangle$
$i_3 = 3$	$P_3 = \langle 1234, 2134, 3124, 4123, 2143, 1243, 3214, 2314, 1342, 4312, 2314, 1324, 4321, 3421, 1423, 2413, 3412, 1432, 4132, 3142 \rangle$
	$P_4 = \langle 1234, 2134, 3142, 1324, 2314, 4312, 1342, 3142, 4132, 1432, 3412, 2413, 1423, 4123, 2143, 1243, 3241, 2341, 4321, 3421 \rangle$
$i_1 = 3$	$P_1 = \langle 1234, 2134, 3124 \rangle$
$i_2 = 4$	$P_2 = \langle 1234, 3214, 4213 \rangle$
$i_3 = 2$	$P_3 = \langle 1234, 4231, 3241, 1243, 2143, 4123, 1423, 2413, 3412, 4312, 2314, 1324, 4321, 3421, 2431, 1432, 4132, 3142, 1342, 2341 \rangle$
	$P_4 = \langle 1234, 4231, 3241, 1243, 2143, 4123, 1423, 3421, 2431, 1432, 4132, 3142, 1342, 2341, 4321, 1324, 2314, 4312, 3412, 2413 \rangle$

Thus, this statement is proved. ∎

Lemma 9. *Assume that $n \geq 5$ and $i_1i_2 \ldots i_{n-1}$ is an $(n-1)$-permutation on $\langle n \rangle$. Let $\mathbf{u}$ be any white node of S_n. Then there exist $(n-1)$ paths $P_1, P_2, \ldots, P_{n-1}$ of S_n such that (1) P_1 joins $\mathbf{u}$ to a black node $\mathbf{y_1}$ with $(\mathbf{y_1})_1 = i_1$ and $l(P_1) = n(n-2)! - 1$, (2) P_j joins $\mathbf{u}$ to a white node $\mathbf{y_j}$ with $(\mathbf{y_j})_1 = i_j$ and $l(P_j) = n(n-2)!$ for every $2 \leq j \leq n-1$, (3) $\cup_{j=1}^{n-1} P_j$ spans S_n, and (4) $\cap_{j=1}^{n-1} V(P_j) = \{\mathbf{u}\}$.*

Proof. The proof of this lemma is rather tedious. The authors strongly suggest the reader skim over the proof first and comprehend the details later.

Because S_n is node transitive, we assume that $\mathbf{u} = \mathbf{e}$. Without loss of generality, we suppose that $i_2 < i_3 < \cdots < i_{n-1}$.

CASE 1. $n = 5$. Hence, $n(n-2)! = 30$. We have $i_2 \neq 4$, $i_3 \geq 2$, and $i_4 \geq 3$. We set $\mathbf{x_1} = (\mathbf{e})^5$ and $\mathbf{x_i} = ((\mathbf{x_{i-1}})^i)^5$ for every $2 \leq i \leq 4$, and $\mathbf{x_5} = ((\mathbf{x_4})^3)^5$. Note that $\mathbf{x_i}$ is a black node in $S_n^{\{i\}}$ for every $i \in \langle 4 \rangle$ and $\mathbf{x_5}$ is a black node in $S_n^{\{1\}}$. Obviously, $\mathbf{x_1} \neq \mathbf{x_5}$. We set $H = \langle \mathbf{e}, \mathbf{x_1}, (\mathbf{x_1})^2, \mathbf{x_2}, (\mathbf{x_2})^3, \mathbf{x_3}, (\mathbf{x_3})^4, \mathbf{x_4}, (\mathbf{x_4})^3, \mathbf{x_5} \rangle$.

CASE 1.1. $i_1 = 3$. We have $i_2 \neq 4$, $i_3 \neq 3$, and $i_4 \neq 1$. Let $\mathbf{u_1} = 24135$, $\mathbf{u_2} = 41325$, and $\mathbf{u_3} = 34125$. We set

$$W_1 = \langle \mathbf{e} = 12345, 32145, 42135, 12435, 21435, 41235, 14235, 24135 = \mathbf{u_1} \rangle,$$

$$W_2 = \langle \mathbf{e} = 12345, 21345, 31245, 13245, 23145, 43125, 13425, 31425, 41325 = \mathbf{u_2} \rangle, \text{ and}$$

$$W_3 = \langle \mathbf{e} = 12345, 42315, 32415, 23415, 43215, 34215, 24315, 14325, 34125 = \mathbf{u_3} \rangle.$$

Obviously, $W_1 \cup W_2 \cup W_3$ spans $S_5^{\{5\}}$ and $V(W_i) \cap V(W_j) = \{\mathbf{e}\}$ for every $i,j \in \langle 3 \rangle$ with $i \neq j$. By Lemma 4, there exists a hamiltonian path Q_1 of $S_5^{\{2\}} - \{\mathbf{x_2}, (\mathbf{x_2})^3\}$ joining the white node $(\mathbf{u_1})^5$ to a black node $\mathbf{y_1}$ with $(\mathbf{y_1})_1 = i_1$. Again, there exists a hamiltonian path Q_2 of $S_5^{\{4\}} - \{\mathbf{x_4}, (\mathbf{x_4})^3\}$ joining the black node $(\mathbf{u_2})^5$ to a white node $\mathbf{y_2}$ with $(\mathbf{y_2})_1 = i_2$. Moreover, there exists a hamiltonian path Q_3 of $S_5^{\{3\}} - \{\mathbf{x_3}, (\mathbf{x_3})^4\}$ joining the black node $(\mathbf{u_3})^5$ to a white node $\mathbf{y_3}$ with $(\mathbf{y_3})_1 = i_3$. Similarly, there exists a hamiltonian path Q_4 of $S_5^{\{1\}} - \{\mathbf{x_1}, (\mathbf{x_1})^2\}$ joining the black node $\mathbf{x_5}$ to a white node $\mathbf{y_4}$ with $(\mathbf{y_4})_1 = i_4$. We set

$$P_1 = \langle \mathbf{e}, W_1, \mathbf{u_1}, (\mathbf{u_1})^5, Q_1, \mathbf{y_1} \rangle,$$

$$P_2 = \langle \mathbf{e}, W_2, \mathbf{u_2}, (\mathbf{u_2})^5, Q_2, \mathbf{y_2} \rangle,$$

$$P_3 = \langle \mathbf{e}, W_3, \mathbf{u_3}, (\mathbf{u_3})^5, Q_3, \mathbf{y_3} \rangle, \text{ and}$$

$$P_4 = \langle \mathbf{e}, H, \mathbf{x_5}, Q_4, \mathbf{y_4} \rangle.$$

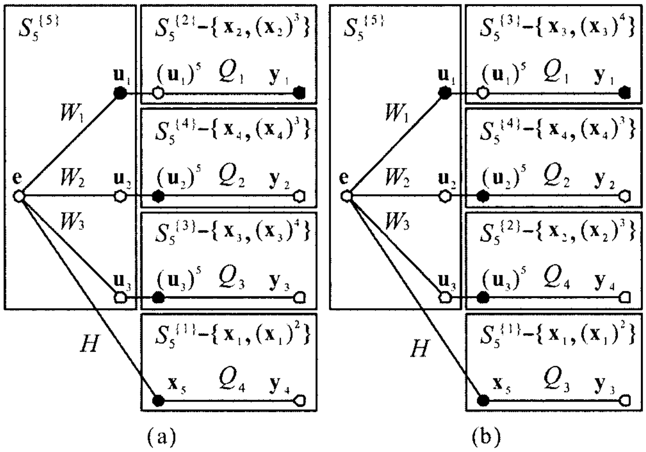

FIG. 6. Illustration of case 1.

Obviously, $l(P_1) = 29$ and $l(P_i) = 30$ for every $2 \le i \le 4$. Apparently, P_1, P_2, P_3, and P_4 are the desired paths. See Figure 6(a) for an illustration.

CASE 1.2. $i_1 \neq 3$. We have $i_2 \neq 4$, $i_3 \neq 1$, and $i_4 \neq 2$. Let $\mathbf{u_1} = 31425$, $\mathbf{u_2} = 42135$, and $\mathbf{u_3} = 21435$. We set

$$W_1 = \langle \mathbf{e} = 12345, 21345, 41325, 14325, 34125, 43125, 13425, 31425 = \mathbf{u_1} \rangle,$$

$$W_2 = \langle \mathbf{e} = 12345, 32145, 23145, 13245, 31245, 41235, 14235, 24135, 42135 = \mathbf{u_2} \rangle, \text{ and}$$

$$W_3 = \langle \mathbf{e} = 12345, 42315, 24315, 34215, 43215, 23415, 32415, 12435, 21435 = \mathbf{u_3} \rangle.$$

Obviously, $W_1 \cup W_2 \cup W_3$ spans $S_5^{\{5\}}$ and $V(W_i) \cap V(W_j) = \{\mathbf{e}\}$ for every $i, j \in \langle 3 \rangle$ with $i \neq j$. By Lemma 4, there exists a hamiltonian path Q_1 of $S_5^{\{3\}} - \{\mathbf{x_3}, (\mathbf{x_3})^4\}$ joining the white node $(\mathbf{u_1})^5$ to a black node $\mathbf{y_1}$ with $(\mathbf{y_1})_1 = i_1$. Again, there exists a hamiltonian path Q_2 of $S_5^{\{4\}} - \{\mathbf{x_4}, (\mathbf{x_4})^3\}$ joining the black node $(\mathbf{u_2})^5$ to a white node $\mathbf{y_2}$ with $(\mathbf{y_2})_1 = i_2$. Moreover, there exists a hamiltonian path Q_3 of $S_5^{\{1\}} - \{\mathbf{x_1}, (\mathbf{x_1})^2\}$ joining the black node $\mathbf{x_5}$ to a white node $\mathbf{y_3}$ with $(\mathbf{y_3})_1 = i_3$. Similarly, there exists a hamiltonian path Q_4 of $S_5^{\{2\}} - \{\mathbf{x_2}, (\mathbf{x_2})^3\}$ joining the black node $(\mathbf{u_3})^5$ to a white node $\mathbf{y_4}$ with $(\mathbf{y_4})_1 = i_4$. We set

$$P_1 = \langle \mathbf{e}, W_1, \mathbf{u_1}, (\mathbf{u_1})^5, Q_1, \mathbf{y_1} \rangle,$$

$$P_2 = \langle \mathbf{e}, W_2, \mathbf{u_2}, (\mathbf{u_2})^5, Q_2, \mathbf{y_2} \rangle,$$

$$P_3 = \langle \mathbf{e}, H, \mathbf{x_5}, Q_3, \mathbf{y_3} \rangle, \text{ and}$$

$$P_4 = \langle \mathbf{e}, W_3, \mathbf{u_3}, (\mathbf{u_3})^5, Q_4, \mathbf{y_4} \rangle.$$

Obviously, $l(P_1) = 29$ and $l(P_i) = 30$ for every $2 \le i \le 4$. Apparently, P_1, P_2, P_3, and P_4 are the desired paths. See Figure 6(b) for an illustration.

CASE 2. $n \ge 6$. Because $n - 1 \ge 5$, we have $i_k \neq k + 2$ for every $2 \le k \le n - 4$, $i_{n-3} \neq 1$, $i_{n-2} \neq 2$, and $i_{n-1} \neq 3$. We set $\mathbf{u_j} = (\mathbf{e})^{j+2}$ and $\mathbf{v_j} = ((\mathbf{e})^{j+2})^{j+1}$ for every $j \in \langle n - 4 \rangle$. Thus, $\mathbf{u_j}$ is a black node in $S_n^{\{(n-1,n)\}}$ and $\mathbf{v_j}$ is a white node in $S_n^{\{(n-1,n)\}}$ for every $j \in \langle n - 4 \rangle$. Note that $F_{n-2}(\mathbf{e}) = \{\mathbf{u_j} \mid j \in \langle n - 4 \rangle\} \cup \{\mathbf{v_j} \mid j \in \langle n - 4 \rangle\}$.

By Lemma 6, there is a hamiltonian path P of $S_n^{\{(n-1,n)\}} - F_{n-2}(\mathbf{e})$ joining $\mathbf{e}$ to a black node $\mathbf{x_1}$ with $(\mathbf{x_1})_1 = 2$. We recursively set $\mathbf{x_j}$ as the unique neighbor of $(\mathbf{x_{j-1}})^{n-1}$ in $S_n^{\{(j,n)\}}$ with $(\mathbf{x_j})_1 = j + 1$ for every $2 \le j \le n - 4$, and we set $\mathbf{x_{n-3}}$ as the unique neighbor of $(\mathbf{x_{n-4}})^{n-1}$ in $S_n^{\{(n-3,n)\}}$ with $(\mathbf{x_{n-3}})_1 = n - 1$. It is easy to see that $\mathbf{x_j}$ is a black node for $1 \le j \le n - 3$ and $\{(\mathbf{x_j})^{n-1}, \mathbf{x_{j+1}}\} \subset S_n^{\{(j+1,n)\}}$ for $1 \le j \le n - 4$. We construct P_j for every $1 \le j \le n - 1$ as follows:

1. $j \in \langle n - 4 \rangle - \{1\}$. By Lemma 4, there is a hamiltonian path T_j of $S_n^{\{(j+1,n)\}} - \{(\mathbf{x_j})^{n-1}, \mathbf{x_{j+1}}\}$ joining the black node $(\mathbf{v_j})^{n-1}$ to a white node $\mathbf{z_j}$ with $(\mathbf{z_j})_1 = j + 2$. Again, there is a hamiltonian path T'_j of $S_n^{\{j+2\}}$ joining the black node $(\mathbf{z_j})^n$ to a white node $\mathbf{y_j}$ with $(\mathbf{y_j})_1 = i_j$. Then we set P_j as $\langle \mathbf{e}, \mathbf{u_j}, \mathbf{v_j}, (\mathbf{v_j})^{n-1}, T_j, \mathbf{z_j}, (\mathbf{z_j})^n, T'_j, \mathbf{y_j} \rangle$. Obviously, $l(P_j) = n(n-2)!$.
2. $j = n - 3$. We choose a white node $\mathbf{y_{n-3}}$ in $S_n^{\{1\}}$ with $(\mathbf{y_{n-3}})_1 = i_{n-3}$. Note that there are $((n-3)!/2)$ edges joining some black nodes of $S_n^{\{(n-2,n)\}}$ to some white nodes of $S_n^{\{1\}}$ and there are $((n-3)!/2)$ edges joining some white nodes of $S_n^{\{(n-2,n)\}}$ to some black nodes of $S_n^{\{1\}}$. We choose a white node $\mathbf{r}$ in $S_n^{\{1\}}$ with $(\mathbf{r})^n$ being a black node in $S_n^{\{(n-2,n)\}}$ and choose a black node $\mathbf{s}$ in $S_n^{\{1\}}$ with $(\mathbf{s})^n$ being a white node in $S_n^{\{(n-2,n)\}}$. By Lemma 5, there exist two disjoint paths H_1 and H_2 of $S_n^{\{1\}}$ such that (1) H_1 joins $(\mathbf{e})^n$ to $\mathbf{r}$, (2) H_2 joins $\mathbf{s}$ to $\mathbf{y_{n-3}}$, and (3) $H_1 \cup H_2$ spans $S_n^{\{1\}}$. By Theorem 3, there is a hamiltonian path H of $S_n^{\{(n-2,n)\}}$ joining the black node $(\mathbf{r})^n$ to the white node $(\mathbf{s})^n$. We set P_{n-3} as $\langle \mathbf{e}, (\mathbf{e})^n, H_1, \mathbf{r}, (\mathbf{r})^n, H, (\mathbf{s})^n, \mathbf{s}, H_2, \mathbf{y_{n-3}} \rangle$. Obviously, $l(P_{n-3}) = n(n-2)!$.
3. $j = n - 1$. By Lemma 4, there is a hamiltonian path Q_1 of $S_n^{\{(2,n)\}} - \{(\mathbf{x_1})^{n-1}, \mathbf{x_2}\}$ joining the black node $(\mathbf{v_1})^{n-1}$ to a white node $\mathbf{q}$ with $(\mathbf{q})_1 = 3$. Again, there is a hamiltonian path Q_2 of $S_n^{\{3\}}$ joining the black node $(\mathbf{q})^n$ to a white node $\mathbf{y_{n-1}}$ with $(\mathbf{y_{n-1}})_1 = i_{n-1}$. We set P_{n-1} as $\langle \mathbf{e}, \mathbf{u_1}, \mathbf{v_1}, (\mathbf{v_1})^{n-1}, Q_1, \mathbf{q}, (\mathbf{q})^n, Q_2, \mathbf{y_{n-1}} \rangle$. Obviously, $l(P_{n-1}) = n(n-2)!$.
4. We construct P_1 and P_{n-2} dependent on whether $i_1 = n - 1$ or not. We set L as $\langle \mathbf{x_1}, (\mathbf{x_1})^{n-1}, \mathbf{x_2}, (\mathbf{x_2})^{n-1}, \ldots, \mathbf{x_{n-4}}, (\mathbf{x_{n-4}})^{n-1}, \mathbf{x_{n-3}}, (\mathbf{x_{n-3}})^n \rangle$. By Theorem 1, there is a hamiltonian path W of $S_n^{\{(1,n)\}}$ joining the black node $(\mathbf{e})^{n-1}$ to a white node $\mathbf{p}$ with $(\mathbf{p})_1 = 2$.

Suppose that $i_1 \neq n - 1$. By Theorem 1, there is a hamiltonian path R of $S_n^{\{n-1\}}$ joining the white node $(\mathbf{x_{n-3}})^n$ to a black node $\mathbf{y_1}$ with $(\mathbf{y_1})_1 = i_1$. Again, there exists a hamiltonian path Z of $S_{n-1}^{\{2\}}$ joining the black node $(\mathbf{p})^n$ to a white node $\mathbf{y_{n-2}}$ with $(\mathbf{y_{n-2}})_1 = i_{n-2}$. We set P_1 as $\langle \mathbf{e}, P, \mathbf{x_1}, L, (\mathbf{x_{n-3}})^n, R, \mathbf{y_1} \rangle$ and P_{n-2} as $\langle \mathbf{e}, (\mathbf{e})^{n-1}, W, \mathbf{p}, (\mathbf{p})^n, Z, \mathbf{y_{n-2}} \rangle$. Obviously, $l(P_1) = n(n-2)! - 1$ and $l(P_{n-2}) = n(n-2)!$. Apparently, $P_1, P_2, \ldots, P_{n-1}$ are the desired paths. See Figure 7(a) for an illustration for the case $n = 7$.

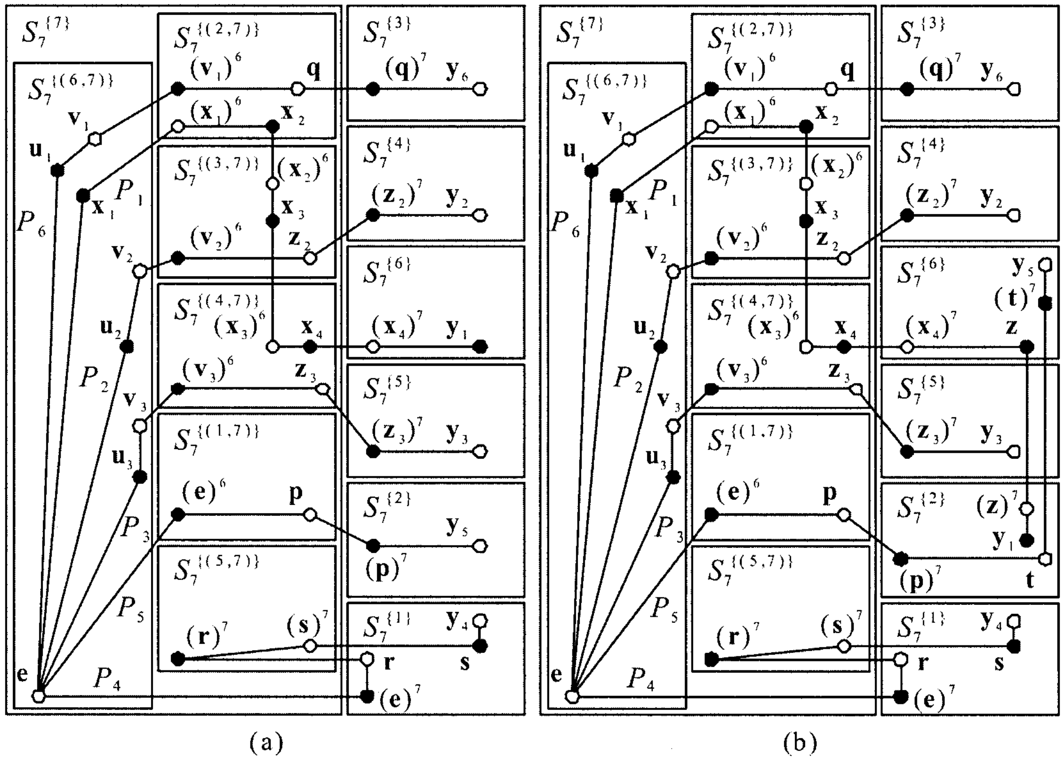

FIG. 7. Illustration of case 3 with $n = 7$.

Suppose that $i_1 = n - 1$. Note that $i_{n-2} \neq n - 1$. Because $(\mathbf{x_{n-3}})^n$ is a white node in $S_n^{\{n-1\}}$ with $((\mathbf{x_{n-3}})^n)_1 = n$ and $((\mathbf{x_{n-3}})^n)_n = n - 1$, there is a black node $\mathbf{z}$ in $S_n^{\{n-1\}}$ such that $\mathbf{z}$ is the unique neighbor of $(\mathbf{x_{n-3}})^n$ with $(\mathbf{z})_1 = 2$. Because $((\mathbf{x_{n-3}})^n)_{n-1} = n - 3$, we have $(\mathbf{z})_{n-1} = n - 3$. Note that $(\mathbf{z})^n$ is a white node in $S_n^{\{2\}}$ with $((\mathbf{z})^n)_{n-1} = n - 3$. Because $(\mathbf{p})^n$ is a black node in $S_n^{\{2\}}$ with $((\mathbf{p})^n)_1 = n$ and $((\mathbf{p})^n)_n = 2$, there is a white node $\mathbf{t}$ in $S_n^{\{2\}}$ such that $\mathbf{t}$ is the unique neighbor of $(\mathbf{p})^n$ with $(\mathbf{t})_1 = n - 1$. Because $(\mathbf{p})_{n-1} = 1$ and $(\mathbf{p})_n = n$, we have $((\mathbf{p})^n)_{n-1} = 1$ and $((\mathbf{p})^n)_1 = n$. Because $((\mathbf{p})^n)_{n-1} = 1$, we have $(\mathbf{t})_{n-1} = 1$. Because $((\mathbf{z})^n)_{n-1} = n - 3$ and $(\mathbf{t})_{n-1} = 1$, we have $(\mathbf{z})^n \neq \mathbf{t}$. By Theorem 4, there is a hamiltonian path W_1 of $S_n^{\{2\}} - \{(\mathbf{p})^n, \mathbf{t}\}$ joining $(\mathbf{z})^n$ to a black node $\mathbf{y_1}^n$ with $(\mathbf{y_1})_1 = i_1$. Again, there is a hamiltonian path W_2 of $S_n^{\{n-1\}} - \{(\mathbf{x_{n-3}})^n, \mathbf{z}\}$ joining $(\mathbf{t})^n$ to a white node $\mathbf{y_{n-2}}$ with $(\mathbf{y_{n-2}})_1 = i_{n-2}$. We set P_1 as $\langle \mathbf{e}, P, \mathbf{x_1}, L, (\mathbf{x_{n-3}})^n, \mathbf{z}, (\mathbf{z})^n, W_1, \mathbf{y_1} \rangle$ and P_{n-2} as $\langle \mathbf{e}, (\mathbf{e})^{n-1}, W, \mathbf{p}, (\mathbf{p})^n, \mathbf{t}, (\mathbf{t})^n, W_2, \mathbf{y_{n-2}} \rangle$. Obviously, $l(P_1) = n(n-2)! - 1$ and $l(P_{n-2}) = n(n-2)!$. Apparently, $P_1, P_2, \ldots, P_{n-1}$ are the desired paths. See Figure 7(b) for an illustration for the case $n = 7$. ■

TABLE 1. All Hamiltonian Cycles in S_4.

⟨1234, 4231, 3241, 1243, 4213, 3214, 2314, 1324, 4321, 2341, 1342, 4312, 3412, 2413, 1423, 3421, 2431, 1432, 4132, 3142, 2143, 4123, 3124, 2134, 1234⟩
⟨1234, 4231, 3241, 1243, 4213, 2413, 3412, 1432, 2431, 3421, 1423, 4123, 2143, 3142, 4132, 2134, 3124, 1324, 4321, 2341, 1342, 4312, 2314, 3214, 1234⟩
⟨1234, 4231, 3241, 1243, 2143, 4123, 3124, 2134, 4132, 3142, 1342, 2341, 4321, 1324, 2314, 4312, 3412, 1432, 2431, 3421, 1423, 2413, 4213, 3214, 1234⟩
⟨1234, 4231, 3241, 2341, 1342, 4312, 2314, 1324, 4321, 3421, 2431, 1432, 3412, 2413, 1423, 4123, 3124, 2134, 4132, 3142, 2143, 1243, 4213, 3214, 1234⟩
⟨1234, 4231, 3241, 2341, 1342, 3142, 2143, 1243, 4213, 3214, 2314, 4312, 3412, 2413, 1423, 4123, 3124, 1324, 4321, 3421, 2431, 1432, 4132, 2134, 1234⟩
⟨1234, 4231, 3241, 2341, 4321, 3421, 2431, 1432, 4132, 3142, 1342, 4312, 3412, 2413, 1423, 4123, 2143, 1243, 4213, 3214, 2314, 1324, 3124, 2134, 1234⟩
⟨1234, 4231, 2431, 1432, 3412, 4312, 2314, 3214, 4213, 2413, 1423, 3421, 4321, 1324, 3124, 4123, 2143, 1243, 3241, 2341, 1342, 3142, 4132, 2134, 1234⟩
⟨1234, 4231, 2431, 1432, 4132, 2134, 3124, 1324, 4321, 3421, 1423, 4123, 2143, 3142, 1342, 2341, 3241, 1243, 4213, 2413, 3412, 4312, 2314, 3214, 1234⟩
⟨1234, 4231, 2431, 1432, 4132, 3142, 2143, 1243, 3241, 2341, 1342, 4312, 3412, 2413, 4213, 3214, 2314, 1324, 4321, 3421, 1423, 4123, 3124, 2134, 1234⟩
⟨1234, 4231, 2431, 3421, 1423, 2413, 3412, 1432, 4132, 2134, 3124, 4123, 2143, 3142, 1342, 4312, 2314, 1324, 4321, 2341, 3241, 1243, 4213, 3214, 1234⟩
⟨1234, 4231, 2431, 3421, 1423, 4123, 3124, 1324, 4321, 2341, 3241, 1243, 2143, 3142, 1342, 4312, 2314, 3214, 4213, 2413, 3412, 1432, 4132, 2134, 1234⟩
⟨1234, 4231, 2431, 3421, 4321, 2341, 3241, 1243, 4213, 2413, 1423, 4123, 2143, 3142, 1342, 4312, 3412, 1432, 4132, 2134, 3124, 1324, 2314, 3214, 1234⟩
⟨1234, 3214, 4213, 1243, 3241, 4231, 2431, 1432, 3412, 2413, 1423, 3421, 4321, 2341, 1342, 4312, 2314, 1324, 3124, 4123, 2143, 3142, 4132, 2134, 1234⟩
⟨1234, 3214, 4213, 2413, 3412, 4312, 2314, 1324, 3124, 4123, 1423, 3421, 4321, 2341, 1342, 3142, 2143, 1243, 3241, 4231, 2431, 1432, 4132, 2134, 1234⟩
⟨1234, 3214, 4213, 2413, 1423, 4123, 2143, 1243, 3241, 4231, 2431, 3421, 4321, 2341, 1342, 3142, 4132, 1432, 3412, 4312, 2314, 1324, 3124, 2134, 1234⟩
⟨1234, 3214, 2314, 4312, 1342, 3142, 4132, 1432, 3412, 2413, 4213, 1243, 2143, 4123, 1423, 3421, 2431, 4231, 3241, 2341, 4321, 1324, 3124, 2134, 1234⟩
⟨1234, 3214, 2314, 1324, 4321, 3421, 2431, 4231, 3241, 2341, 1342, 4312, 3412, 1432, 4132, 3142, 2143, 1243, 4213, 2413, 1423, 4123, 3124, 2134, 1234⟩
⟨1234, 3214, 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431, 1432, 3412, 4312, 1342, 3142, 4132, 2134, 1234⟩

Using depth first search, we list all hamiltonian cycles in S_4 in Table 1.

Lemma 10. $D_3^{S_L}(S_4) = 15$.

Proof. Let **u** be any white node in S_4, and let **v** be any black node in S_4. Because S_4 is node transitive, we assume that $\mathbf{u} = 1234$. Suppose that $d(\mathbf{u}, \mathbf{v}) = 1$. Because S_4 is edge transitive, we assume that $\mathbf{v} = 2134$. Let $\{P_1, P_2, P_3\}$ be a 3*-container joining **u** to **v**. Because S_4 is 3-regular, one of three paths, say P_3, is $\langle \mathbf{u}, \mathbf{v} \rangle$. Thus, $P_1 \cup P_2^{-1}$ forms a hamiltonian cycle of S_4 not using the edge $(\mathbf{u}, \mathbf{v})$. From Table 1, we obtain $d_3^{S_L}(\mathbf{u}, \mathbf{v}) = 15$. Thus, $D_3^{S_L}(S_4) \geq 15$. Suppose that $d(\mathbf{u}, \mathbf{v}) \neq 1$. Then $\mathbf{v} \in \{1324, 1243, 1432, 2413, 2341, 3142, 4123, 4312, 3421\}$. We find the following set of 3*-containers of S_4 between $\mathbf{u} = 1234$ and **v**:

$C\langle 1234, 1324\rangle$	$P_1 = \langle 1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324\rangle$
	$P_2 = \langle 1234, 2134, 4132, 3142, 1342, 2341, 4321, 1324\rangle$
	$P_3 = \langle 1234, 3214, 4213, 2413, 1423, 3421, 2431, 1432, 3412, 4312, 2314, 1324\rangle$
$C(1234, 1243)$	$P_1 = \langle 1234, 2134, 4132, 1432, 3412, 2413, 4213, 1243\rangle$
	$P_2 = \langle 1234, 3214, 2314, 4312, 1342, 3142, 2143, 1243\rangle$
	$P_3 = \langle 1234, 4231, 2431, 3421, 1423, 4123, 3124, 1324, 4321, 2341, 3241, 1243\rangle$
$C(1234, 1432)$	$P_1 = \langle 1234, 3214, 2314, 1324, 4321, 3421, 2431, 1432\rangle$
	$P_2 = \langle 1234, 4231, 3241, 2341, 1342, 4312, 3412, 1432\rangle$
	$P_3 = \langle 1234, 2134, 3124, 4123, 1423, 2413, 4213, 1243, 2143, 3142, 4132, 1432\rangle$
$C(1234, 2413)$	$P_1 = \langle 1234, 3214, 4213, 2413\rangle$
	$P_2 = \langle 1234, 4231, 2431, 3421, 4321, 2341, 3241, 1243, 2143, 4123, 1423, 2413\rangle$
	$P_3 = \langle 1234, 2134, 3124, 1324, 2314, 4312, 1342, 3142, 4132, 1432, 3412, 2413\rangle$
$C(1234, 2341)$	$P_1 = \langle 1234, 4231, 3241, 2341\rangle$
	$P_2 = \langle 1234, 3214, 2314, 4312, 3412, 2413, 4213, 1243, 2143, 3142, 1342, 2341\rangle$
	$P_3 = \langle 1234, 2134, 4132, 1432, 2431, 3421, 1423, 4123, 3124, 1324, 4321, 2341\rangle$
$C(1234, 3142)$	$P_1 = \langle 1234, 2134, 4132, 3142\rangle$
	$P_2 = \langle 1234, 4231, 3241, 2341, 4321, 3421, 2431, 1432, 3412, 4312, 1342, 3142\rangle$
	$P_3 = \langle 1234, 3241, 2341, 4321, 3421, 1423, 4123, 2143, 1243, 4213, 2413, 3412\rangle$
$C(1234, 4123)$	$P_1 = \langle 1234, 2134, 3124, 4123\rangle$
	$P_2 = \langle 1234, 3214, 4213, 2413, 3412, 4312, 2314, 1324, 4321, 3421, 1423, 4123\rangle$
	$P_3 = \langle 1234, 4231, 2431, 1432, 4132, 3142, 1342, 2341, 3241, 1243, 2143, 4213\rangle$
$C(1234, 4312)$	$P_1 = \langle 1234, 3214, 2314, 4312\rangle$
	$P_2 = \langle 1234, 2134, 4132, 3142, 2143, 4123, 3124, 1324, 4321, 2341, 1342, 4312\rangle$
	$P_3 = \langle 1234, 4231, 3241, 1243, 4213, 2413, 1423, 3421, 2431, 1432, 3412, 4312\rangle$
$C(1234, 3421)$	$P_1 = \langle 1234, 4231, 2431, 3421\rangle$
	$P_2 = \langle 1234, 2134, 3124, 4123, 2143, 3142, 4132, 1432, 3412, 2413, 1423, 3421\rangle$
	$P_3 = \langle 1234, 3214, 4213, 1243, 3241, 2341, 1342, 4312, 2314, 1324, 4321, 3421\rangle$

From this table, $d_3^{S_L}(\mathbf{u}, \mathbf{v}) \leq 15$ if $d(\mathbf{u}, \mathbf{v}) \neq 1$. Hence, $D_3^{S_L}(S_4) = 15$. ∎

Lemma 11. $D_{n-1}^{S_L}(S_n) \geq \frac{n!}{n-2} + 1 = (n-1)! + 2(n-2)! + 2(n-3)! + 1$ *if* $n \geq 5$.

Proof. Let **u** and **v** be two adjacent nodes of S_n. Obviously, **u** and **v** are in different partite sets. Let $\{P_1, P_2, \ldots, P_{n-1}\}$ be any $(n-1)^*$-container of S_n joining **u** to **v**. Obviously, one of these paths is $\langle \mathbf{u}, \mathbf{v} \rangle$. Thus, $\max\{l(P_i) \mid 1 \leq i \leq n-1\} \geq \lceil \frac{n!-2}{n-2} \rceil + 1 = \lceil \frac{n!}{n-2} - \frac{2}{n-2} \rceil + 1 = \frac{n!}{n-2} + 1$. Hence, $d_{n-1}^{S_L}(\mathbf{u}, \mathbf{v}) \geq \frac{n!}{n-2} + 1$ and $D_{n-1}^{S_L}(S_n) \geq \frac{n!}{n-2} + 1$. ∎

Lemma 12. $D_4^{S_L}(S_5) \leq 41$.

Proof. Let **u** be any white node and **v** be any black node of S_5. Obviously, $d(\mathbf{u}, \mathbf{v})$ is odd.

Case 1. $d(\mathbf{u}, \mathbf{v}) = 1$. Because S_5 is node transitive and edge transitive, we may assume that $\mathbf{u} = \mathbf{e} = 12345$ and $\mathbf{v} = (\mathbf{e})^5 = 52341$. By Lemma 7, there exist three paths P_1, P_2, and P_3 of $S_5^{\{5\}}$ such that (1) P_1 joins 12345 to the black node 24135 with $l(P_1) = 7$, (2) P_2 joins 12345 to the white node 34125 with $l(P_2) = 8$, (3) P_3 joins 12345 to the white node 41325 with $l(P_3) = 8$, and (4) $P_1 \cup P_2 \cup P_3$ spans $S_5^{\{5\}}$. Similarly, there exist three paths Q_1, Q_2, and Q_3 of $S_5^{\{1\}}$ such that (1) Q_1 joins 52341 to the white node 24531 with $l(Q_1) = 7$, (2) Q_2 joins 52341 to the black node 34521 with $l(Q_2) = 8$, (3) Q_3 joins 52341 to the black node 45321 with $l(Q_3) = 8$, and (4) $Q_1 \cup Q_2 \cup Q_3$ spans $S_5^{\{1\}}$. By Theorem 1, there is a hamiltonian path R_1 of $S_5^{\{2\}}$

joining the white node 54132 to the black node 14532, there is a hamiltonian path R_2 of $S_5^{\{3\}}$ joining the black node 54123 to the white node 14523, and there is a hamiltonian path R_3 of $S_5^{\{4\}}$ joining the black node 51324 to the white node 15324. Then we set

$$T_1 = \langle \mathbf{e} = 12345, P_1, 24135, 54132, R_1, 14532, 24531, (Q_1)^{-1}, 52341 = (\mathbf{e})^5\rangle,$$

$$T_2 = \langle \mathbf{e} = 12345, P_2, 34125, 54123, R_2, 14523, 34521, (Q_2)^{-1}, 52341 = (\mathbf{e})^5\rangle,$$

$$T_3 = \langle \mathbf{e} = 12345, P_3, 41325, 51324, R_3, 15324, 45321, (Q_3)^{-1}, 52341 = (\mathbf{e})^5\rangle, \text{ and}$$

$$T_4 = \langle \mathbf{e} = 12345, 52341 = (\mathbf{e})^5\rangle.$$

Obviously, $\{T_1, T_2, T_3, T_4\}$ is a 4*-container of S_5 between $\mathbf{e}$ and $(\mathbf{e})^5$. Moreover, $l(T_1) = 39$, $l(T_2) = l(T_3) = 41$, and $l(T_4) = 1$. Thus, $d_4^{S_L}(\mathbf{e}, (\mathbf{e})^5) \le 41$.

CASE 2. $d(\mathbf{u}, \mathbf{v}) \ge 3$. Because $d(\mathbf{u}, \mathbf{v}) \ge 3$, there is $i \in \{2, 3, 4, 5\}$ such that $(\mathbf{u})_i \ne (\mathbf{v})_i$ and $\{(\mathbf{u})_i, (\mathbf{v})_i\} \cap \{(\mathbf{u})_1, (\mathbf{v})_1\} = \emptyset$. Without loss of generality, we assume that $(\mathbf{u})_5 \ne (\mathbf{v})_5$ and $\{(\mathbf{u})_5, (\mathbf{v})_5\} \cap \{(\mathbf{u})_1, (\mathbf{v})_1\} = \emptyset$. Moreover, we assume that $(\mathbf{u})_5 = 5$, $(\mathbf{v})_5 = 4$, $(\mathbf{u})_1 = 1$, and $(\mathbf{v})_1 \ne 5$. Because $(\mathbf{u})_1 = 1$ and $(\mathbf{u})_5 = 5$, we have $\{(\mathbf{u})_2, (\mathbf{u})_3, (\mathbf{u})_4\} = \{2, 3, 4\}$.

SUBCASE 2.1. $(\mathbf{v})_1 = 1$. We have $\{(\mathbf{v})_2, (\mathbf{v})_3, (\mathbf{v})_4\} = \{2, 3, 5\}$. By Lemma 8, there exist four paths P_1, P_2, P_3, and P_4 of $S_5^{\{5\}}$ such that (1) P_1 joins $\mathbf{u}$ to a white node $\mathbf{w}$ with $(\mathbf{w})_1 = 2$ and $l(P_1) = 2$, (2) P_2 joins $\mathbf{u}$ to a white node $\mathbf{x}$ with $(\mathbf{x})_1 = 3$ and $l(P_1) = 2$, (3) P_3 joins $\mathbf{u}$ to a black node $\mathbf{y}$ with $(\mathbf{y})_1 = 4$ and $l(P_3) = 19$, (4) P_4 joins $\mathbf{u}$ to a black node $\mathbf{z} \ne \mathbf{y}$ with $(\mathbf{z})_1 = 4$ and $l(P_4) = 19$, (5) $P_1 \cup P_2 \cup P_3$ spans $S_5^{\{5\}}$, (6) $P_1 \cup P_2 \cup P_4$ spans $S_5^{\{5\}}$, (7) $V(P_1) \cap V(P_2) \cap V(P_3) = \{\mathbf{u}\}$, and (8) $V(P_1) \cap V(P_2) \cap V(P_4) = \{\mathbf{u}\}$.

Similarly, there exist four paths Q_1, Q_2, Q_3, and Q_4 of $S_5^{\{4\}}$ such that (1) Q_1 joins $\mathbf{v}$ to a black node $\mathbf{p}$ with $(\mathbf{p})_1 = 2$ and $l(Q_1) = 2$, (2) Q_2 joins $\mathbf{v}$ to a black node $\mathbf{q}$ with $(\mathbf{q})_1 = 3$ and $l(Q_2) = 2$, (3) Q_3 joins $\mathbf{v}$ to a white node $\mathbf{r}$ with $(\mathbf{r})_1 = 5$ and $l(Q_3) = 19$, (4) Q_4 joins $\mathbf{v}$ to a white node $\mathbf{s} \ne \mathbf{r}$ with $(\mathbf{s})_1 = 5$ and $l(Q_4) = 19$, (5) $Q_1 \cup Q_2 \cup Q_3$ spans $S_5^{\{4\}}$, (6) $Q_1 \cup Q_2 \cup Q_4$ spans $S_5^{\{4\}}$, (7) $V(Q_1) \cap V(Q_2) \cap V(Q_3) = \{\mathbf{v}\}$, and (8) $V(Q_1) \cap V(Q_2) \cap V(Q_4) = \{\mathbf{v}\}$.

By Lemma 1, there are exactly three edges joining some black nodes of $S_5^{\{5\}}$ to some white nodes of $S_5^{\{4\}}$. By the pigeon-hole principle, at least one node in $\{\mathbf{y}, \mathbf{z}\}$ is adjacent to a node in $\{\mathbf{r}, \mathbf{s}\}$. Without loss of generality, we assume that $\mathbf{y}$ is adjacent to $\mathbf{r}$. Let T_1 be the hamiltonian path of $S_5^{\{1\}}$ joining the black node $(\mathbf{u})^5$ to the white node $(\mathbf{v})^5$, T_2 be the hamiltonian path of $S_5^{\{2\}}$ joining the black node $(\mathbf{w})^5$ to the white node $(\mathbf{p})^5$, and T_3 be the hamiltonian path of $S_5^{\{3\}}$ joining the black node $(\mathbf{x})^5$ to the white node $(\mathbf{q})^5$. We set

$$H_1 = \langle \mathbf{u}, (\mathbf{u})^5, T_1, (\mathbf{v})^5, \mathbf{v}\rangle,$$

$$H_2 = \langle \mathbf{u}, P_1, \mathbf{w}, (\mathbf{w})^5, T_2, (\mathbf{p})^5, \mathbf{p}, Q_1^{-1}, \mathbf{v}\rangle,$$

$$H_3 = \langle \mathbf{u}, P_2, \mathbf{x}, (\mathbf{x})^5, T_3, (\mathbf{q})^5, \mathbf{q}, Q_2^{-1}, \mathbf{v}\rangle, \text{ and}$$

$$H_4 = \langle \mathbf{u}, P_3, \mathbf{y}, \mathbf{r}, Q_3^{-1}, \mathbf{v}\rangle.$$

Obviously, $\{H_1, H_2, H_3, H_4\}$ is a 4*-container of S_5 between $\mathbf{u}$ and $\mathbf{v}$. Moreover, $l(H_1) = 25$, $l(H_2) = l(H_3) = 29$, and $l(H_4) = 39$. Thus, $d_4^{S_L}(\mathbf{u}, \mathbf{v}) \le 41$.

SUBCASE 2.2. $(\mathbf{v})_1 = a \in \{2, 3\}$. We have $\{(\mathbf{v})_2, (\mathbf{v})_3, (\mathbf{v})_4\} = \{1, 2, 3, 5\} - \{a\}$. Let b be the only element in $\{2, 3\} - \{a\}$. By Lemma 8, there exist four paths P_1, P_2, P_3, and P_4 of $S_5^{\{5\}}$ such that (1) P_1 joins $\mathbf{u}$ to a white node $\mathbf{w}$ with $(\mathbf{w})_1 = a$ and $l(P_1) = 2$, (2) P_2 joins $\mathbf{u}$ to a white node $\mathbf{x}$ with $(\mathbf{x})_1 = b$ and $l(P_2) = 2$, (3) P_3 joins $\mathbf{u}$ to a black node $\mathbf{y}$ with $(\mathbf{y})_1 = 4$ and $l(P_3) = 19$, (4) P_4 joins $\mathbf{u}$ to a black node $\mathbf{z} \ne \mathbf{y}$ with $(\mathbf{z})_1 = 4$ and $l(P_4) = 19$, (5) $P_1 \cup P_2 \cup P_3$ spans $S_5^{\{5\}}$, (6) $P_1 \cup P_2 \cup P_4$ spans $S_5^{\{5\}}$, (7) $V(P_1) \cap V(P_2) \cap V(P_3) = \{\mathbf{u}\}$, and (8) $V(P_1) \cap V(P_2) \cap V(P_4) = \{\mathbf{u}\}$.

Again, there exist four paths Q_1, Q_2, Q_3, and Q_4 of $S_5^{\{4\}}$ such that (1) Q_1 joins $\mathbf{v}$ to a black node $\mathbf{p}$ with $(\mathbf{p})_1 = 1$ and $l(Q_1) = 2$, (2) Q_2 joins $\mathbf{v}$ to a black node $\mathbf{q}$ with $(\mathbf{q})_1 = b$ and $l(Q_2) = 2$, (3) Q_3 joins $\mathbf{v}$ to a white node $\mathbf{r}$ with $(\mathbf{r})_1 = 5$ and $l(Q_3) = 19$, (4) Q_4 joins $\mathbf{v}$ to a white node $\mathbf{s} \ne \mathbf{r}$ with $(\mathbf{s})_1 = 5$ and $l(Q_4) = 19$, (5) $Q_1 \cup Q_2 \cup Q_3$ spans $S_5^{\{4\}}$, (6) $Q_1 \cup Q_2 \cup Q_4$ spans $S_5^{\{4\}}$, (7) $V(Q_1) \cap V(Q_2) \cap V(Q_3) = \{\mathbf{v}\}$, and (8) $V(Q_1) \cap V(Q_2) \cap V(Q_4) = \{\mathbf{v}\}$.

By Lemma 1, there are exactly three edges joining some black nodes of $S_5^{\{5\}}$ to some white nodes of $S_5^{\{4\}}$. By the pigeon-hole principle, at least one node in $\{\mathbf{y}, \mathbf{z}\}$ is adjacent to a node in $\{\mathbf{r}, \mathbf{s}\}$. Without loss of generality, we assume that $\mathbf{y}$ is adjacent to $\mathbf{r}$. Let T_1 be the hamiltonian path of $S_5^{\{1\}}$ joining the black node $(\mathbf{u})^5$ to the white node $(\mathbf{p})^5$, T_2 be the hamiltonian path of $S_5^{\{a\}}$ joining the black node $(\mathbf{w})^5$ to the white node $(\mathbf{v})^5$, and T_3 be the hamiltonian path of $S_5^{\{b\}}$ joining the black node $(\mathbf{x})^5$ to the white node $(\mathbf{q})^5$. We set

$$H_1 = \langle \mathbf{u}, (\mathbf{u})^5, T_1, (\mathbf{p})^5, \mathbf{p}, Q_1^{-1}, \mathbf{v}\rangle,$$

$$H_2 = \langle \mathbf{u}, P_1, \mathbf{w}, (\mathbf{w})^5, T_2, (\mathbf{v})^5, \mathbf{v}\rangle,$$

$$H_3 = \langle \mathbf{u}, P_2, \mathbf{x}, (\mathbf{x})^5, T_3, (\mathbf{q})^5, \mathbf{q}, Q_2^{-1}, \mathbf{v}\rangle, \text{ and}$$

$$H_4 = \langle \mathbf{u}, P_3, \mathbf{y}, \mathbf{r}, Q_3^{-1}, \mathbf{v}\rangle.$$

Obviously, $\{H_1, H_2, H_3, H_4\}$ is a 4*-container of S_5 between $\mathbf{u}$ and $\mathbf{v}$. Moreover, $l(H_1) = l(H_2) = 27$, $l(H_3) = 29$, and $l(H_4) = 39$. Thus, $d_4^{S_L}(\mathbf{u}, \mathbf{v}) \le 41$. ∎

Lemma 13. *$d_{n-1}^{s_L}(\mathbf{u}, \mathbf{v}) \le (n-1)! + 2(n-2)! + 2(n-3)! + 1 = \frac{n!}{n-2} + 1$ for every $n \ge 6$.*

Proof. Let $\mathbf{u}$ be any white node and $\mathbf{v}$ be any black node of S_n. Obviously, $d(\mathbf{u}, \mathbf{v})$ is odd.

CASE 1. $d(\mathbf{u}, \mathbf{v}) = 1$. Because the star graph is node transitive and edge transitive, we may assume that $\mathbf{u} = \mathbf{e}$ and $\mathbf{v} = (\mathbf{e})^n$.

By Lemma 9, there exist $(n-2)$ paths $P_1, P_2, \ldots, P_{n-2}$ of $S_n^{\{n\}}$ such that (1) P_1 joins $\mathbf{e}$ to a black node $\mathbf{x_1}$ with $(\mathbf{x_1})_1 = 2$ and $l(P_1) = (n-1)(n-3)! - 1$, (2) P_i joins $\mathbf{e}$ to a white node $\mathbf{x_i}$ with $(\mathbf{x_i})_1 = i+1$ and $l(P_i) = (n-1)(n-3)!$ for $2 \le i \le n-2$, (3) $\cup_{i=1}^{n-2} P_i$ spans $S_n^{\{n\}}$, and (4) $\cap_{i=1}^{n-2} V(P_i) = \{\mathbf{e}\}$. Again, there exist $n-2$ paths $Q_1, Q_2, \ldots, Q_{n-2}$ of $S_n^{\{1\}}$ such that (1) Q_1 joins $(\mathbf{e})^n$ to a white node $\mathbf{y_1}$ with $(\mathbf{y_1})_1 = 2$ and $l(Q_1) = (n-1)(n-3)! - 1$, (2) Q_i joins $(\mathbf{e})^n$ to a black node $\mathbf{y_i}$ with $(\mathbf{y_i})_1 = i+1$ and $l(Q_i) = (n-1)(n-3)!$ for $2 \le i \le n-2$, (3) $\cup_{i=1}^{n-2} Q_i$ spans $S_n^{\{1\}}$, and (4) $\cap_{i=1}^{n-2} V(Q_i) = \{\mathbf{v}\}$.

By Theorem 1, there is a hamiltonian path R_1 of $S_n^{\{2\}}$ joining the white node $(\mathbf{x_1})^n$ to the black node $(\mathbf{y_1})^n$. Again, there is a hamiltonian path R_i of $S_n^{\{i+1\}}$ joining the black node $(\mathbf{x_i})^n$ to the black node $(\mathbf{y_i})^n$ for every $2 \le i \le n-2$.

We set $H_i = \langle \mathbf{e}, P_i, \mathbf{x_i}, (\mathbf{x_i})^n, R_i, (\mathbf{y_i})^n, \mathbf{y_i}, Q_i^{-1}, (\mathbf{e})^n \rangle$ for every $1 \le i \le n-2$ and $H_{n-1} = \langle \mathbf{e}, (\mathbf{e})^n \rangle$. Then $\{H_1, H_2, \ldots, H_{n-1}\}$ is an $(n-1)^*$-container between $\mathbf{e}$ and $(\mathbf{e})^n$. Obviously, $l(H_1) = (n-1)! + 2(n-2)! + 2(n-3)! - 1$, $l(H_i) = (n-1)! + 2(n-2)! + 2(n-3)! + 1$ for $2 \le i \le n-2$, and $l(H_{n-1}) = 1$. Hence, $d_{n-1}^{s_L}(\mathbf{e}, (\mathbf{e})^n) \le (n-1)! + 2(n-2)! + 2(n-3)! + 1$.

CASE 2. $d(\mathbf{u}, \mathbf{v}) \ge 3$. Because $d(\mathbf{u}, \mathbf{v}) \ge 3$, there is $i \in \langle n \rangle - \{1\}$ such that $(\mathbf{u})_i \ne (\mathbf{v})_i$ and $\{(\mathbf{u})_i, (\mathbf{v})_i\} \cap \{(\mathbf{u})_1, (\mathbf{v})_1\} = \emptyset$. Without loss of generality, we assume that $(\mathbf{u})_n \ne (\mathbf{v})_n$ and $\{(\mathbf{u})_n, (\mathbf{v})_n\} \cap \{(\mathbf{u})_1, (\mathbf{v})_1\} = \emptyset$. Moreover, we assume that $(\mathbf{u})_n = n$, $(\mathbf{v})_n = n-1$, $(\mathbf{u})_1 = 1$, and $(\mathbf{v})_1 \ne 5$.

SUBCASE 2.1. $(\mathbf{v})_1 = 1$. By Lemma 9, there are $(n-2)$ paths $P_1, P_2, \ldots, P_{n-2}$ of $S_n^{\{n\}}$ such that (1) P_1 joins $\mathbf{u}$ to a black node $\mathbf{x_1}$ with $(\mathbf{x_1})_1 = 1$ and $l(P_1) = (n-1)(n-3)! - 1$, (2) P_i joins $\mathbf{u}$ to a white node $\mathbf{x_i}$ with $(\mathbf{x_i})_1 = i$ and $l(P_i) = (n-1)(n-3)!$ for $2 \le i \le n-2$, (3) $\cup_{i=1}^{n-2} P_i$ spans $S_n^{\{n\}}$, and (4) $\cap_{i=1}^{n-2} V(P_i) = \{\mathbf{u}\}$. Again, there are $(n-2)$ paths $Q_1, Q_2, \ldots, Q_{n-2}$ of $S_n^{\{n-1\}}$ such that (1) Q_1 joins $\mathbf{v}$ to a white node $\mathbf{y_1}$ with $(\mathbf{y_1})_1 = 1$ and $l(Q_1) = (n-1)(n-3)! - 1$, (2) Q_i joins $\mathbf{v}$ to a black node $\mathbf{y_i}$ with $(\mathbf{y_i})_1 = i$ and $l(Q_i) = (n-1)(n-3)!$ for $2 \le i \le n-2$, (3) $\cup_{i=1}^{n-2} Q_i$ spans $S_n^{\{n-1\}}$, and (4) $\cap_{i=1}^{n-2} V(Q_i) = \{\mathbf{v}\}$.

By Lemma 5, there are two disjoint paths H_1 and H_2 of $S_n^{\{1\}}$ such that (1) H_1 joins the white node $(\mathbf{x_1})^n$ to the black node $(\mathbf{y_1})^n$, (2) H_2 joins the black node $(\mathbf{u})^n$ to the white node $(\mathbf{v})^n$, and (3) $H_1 \cup H_2$ spans $S_n^{\{1\}}$. By Theorem 1, there is a hamiltonian path R_i of $S_n^{\{i\}}$ joining the black node $(\mathbf{x_i})^n$ to the white node $(\mathbf{y_i})^n$ for every $2 \le i \le n-2$. We set

$$T_1 = \langle \mathbf{u}, P_1, \mathbf{x_1}, (\mathbf{x_1})^n, H_1, (\mathbf{y_1})^n, \mathbf{y_1}, Q_1^{-1}, \mathbf{v} \rangle,$$

$$T_i = \langle \mathbf{u}, P_i, \mathbf{x_i}, (\mathbf{x_i})^n, R_i, (\mathbf{y_i})^n, \mathbf{y_i}, Q_i^{-1}, \mathbf{v} \rangle \quad \text{for } 2 \le i \le n-2, \text{ and}$$

$$T_{n-1} = \langle \mathbf{u}, (\mathbf{u})^n, H_2, (\mathbf{v})^n, \mathbf{v} \rangle.$$

Obviously, $\{T_1, T_2, \ldots, T_{n-1}\}$ is an $(n-1)^*$-container of S_n between $\mathbf{u}$ and $\mathbf{v}$. Moreover, $l(T_i) \le (n-1)! + 2(n-2)! + 2(n-3)! + 1$. Thus, $d_{n-1}^{s_L}(\mathbf{u}, \mathbf{v}) \le (n-1)! + 2(n-2)! + 2(n-3)! + 1$.

SUBCASE 2.2. $(\mathbf{v})_1 = t \in \langle n-2 \rangle - \{1\}$. By Lemma 9, there are $(n-2)$ paths $P_1, P_2, \ldots, P_{n-2}$ of $S_n^{\{n\}}$ such that (1) P_1 joins $\mathbf{u}$ to a black node $\mathbf{x_1}$ with $(\mathbf{x_1})_1 = 1$ and $l(P_1) = (n-1)(n-3)! - 1$, (2) P_i joins $\mathbf{u}$ to a white node $\mathbf{x_i}$ with $(\mathbf{x_i})_1 = i$ and $l(P_i) = (n-1)(n-3)!$ for $2 \le i \le n-2$, (3) $\cup_{i=1}^{n-2} P_i$ spans $S_n^{\{n\}}$, and (4) $\cap_{i=1}^{n-2} V(P_i) = \{\mathbf{u}\}$. Again, there are $(n-2)$ paths $Q_1, Q_2, \ldots, Q_{n-2}$ of $S_n^{\{n-1\}}$ such that (1) Q_1 joins $\mathbf{v}$ to a white node $\mathbf{y_1}$ with $(\mathbf{y_1})_1 = 1$ and $l(Q_1) = (n-1)(n-3)! - 1$, (2) Q_i joins $\mathbf{v}$ to a black node $\mathbf{y_i}$ with $(\mathbf{y_i})_1 = i$ and $l(Q_i) = (n-1)(n-3)!$ for $2 \le i \le n-2$, (3) $\cup_{i=1}^{n-2} Q_i$ spans $S_n^{\{n-1\}}$, and (4) $\cap_{i=1}^{n-2} V(Q_i) = \{\mathbf{v}\}$.

Because $(\mathbf{v})^n$ is a white node in $S_n^{\{t\}}$ with $((\mathbf{v})^n)_1 = (\mathbf{v})_n = n-1$ and $((\mathbf{v})^n)_n = (\mathbf{v})_1 = t \ne 1$, we can choose a black node $\mathbf{w}$ in $N_{S_n^{\{t\}}}((\mathbf{v})^n)$ with $(\mathbf{w})_1 = 1$. By Lemma 5, there exist two disjoint paths H_1 and H_2 of $S_n^{\{1\}}$ such that (1) H_1 joins the white node $(\mathbf{x_1})^n$ to the black node $(\mathbf{y_1})^n$, (2) H_2 joins the black node $(\mathbf{u})^n$ to the white node $(\mathbf{w})^n$, and (3) $H_1 \cup H_2$ spans $S_n^{\{1\}}$.

By Theorem 4, there exists a hamiltonian path R_t of $S_n^{\{t\}} - \{(\mathbf{v})^n, \mathbf{w}\}$ joining the black node $(\mathbf{x_t})^n$ to the white node $(\mathbf{y_t})^n$. By Theorem 1, there exists a hamiltonian path R_i of $S_n^{\{i\}}$ joining the black node $(\mathbf{x_i})^n$ to the white node $(\mathbf{y_i})^n$ for every $2 \le i \le n-2$ with $i \ne t$. We set

$$T_1 = \langle \mathbf{u}, P_1, \mathbf{x_1}, (\mathbf{x_1})^n, H_1, (\mathbf{y_1})^n, \mathbf{y_1}, Q_1^{-1}, \mathbf{v} \rangle,$$

$$T_i = \langle \mathbf{u}, P_i, \mathbf{x_i}, (\mathbf{x_i})^n, R_i, (\mathbf{y_i})^n, \mathbf{y_i}, Q_i^{-1}, \mathbf{v} \rangle \quad \text{for } 2 \le i \le n-2, \text{ and}$$

$$T_{n-1} = \langle \mathbf{u}, (\mathbf{u})^n, H_2, (\mathbf{w})^n, \mathbf{w}, (\mathbf{v})^n, \mathbf{v} \rangle.$$

Obviously, $\{T_1, T_2, \ldots, T_{n-1}\}$ is an $(n-1)^*$-container of S_n between $\mathbf{u}$ and $\mathbf{v}$. Moreover, $l(T_i) \le (n-1)! + 2(n-2)! + 2(n-3)! + 1$. Thus, $d_{n-1}^{s_L}(\mathbf{u}, \mathbf{v}) \le (n-1)! + 2(n-2)! + 2(n-3)! + 1$. ∎

Theorem 5.

$$D_{n-1}^{s_L}(S_n) = \begin{cases} 1 & \text{if } n = 2, \\ 5 & \text{if } n = 3, \\ 15 & \text{if } n = 4, \text{ and} \\ (n-1)! + 2(n-2)! \\ \quad +2(n-3)! + 1 & \text{if } n \ge 5. \end{cases}$$

Proof. It is easy to check that $D_1^{s_L}(S_2) = 1$ and $D_2^{s_L}(S_3) = 5$. By Lemma 10, $D_3^{s_L}(S_4) = 15$. By Lemmas 11, 12, and 13, we have $D_{n-1}^{s_L}(S_n) = (n-1)! + 2(n-2)! + 2(n-3)! + 1$ if $n \ge 5$. Hence, this statement is proved. ∎

5. THE 2^{*L}-DIAMETER S_n

Lemma 14. $D_2^{s_L}(S_4) = 15$.

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct nodes of S_4. Because S_4 is node transitive, we assume that $\mathbf{u} = 1234$. Let $\{P_1, P_2\}$ be a 2^*-container joining $\mathbf{u}$ to $\mathbf{v}$. Thus, $P_1 \cup P_2^{-1}$ is a hamiltonian cycle of S_4. In Table 1, we list all the possible hamiltonian cycles of S_4. From Table 1, we know $d_2^{s_L}(1234, \mathbf{v}) = 13$ if $\mathbf{v} \in \{1423, 3142, 2413, 1243, 3421, 1432, 2341, 4123, 4312\}$ and $d_2^{s_L}(1234, \mathbf{v}) = 15$ if $\mathbf{v} \in \{2134, 3214, 4231\}$. Hence, $D_2^{s_L}(S_4) = 15$. ∎

Lemma 15. *Assume that a and b are any two distinct elements of $\langle 4 \rangle$ and $\mathbf{u}$ is any white node of S_4. There exist two paths P_1 and P_2 of S_4 such that (1) P_1 joins $\mathbf{u}$ to a black node $\mathbf{x}$ with $(\mathbf{x})_1 = a$ and $l(P_1) = 5$, (2) P_2 joins $\mathbf{u}$ to a white node $\mathbf{y}$ with $(\mathbf{y})_1 = b$ and $l(P_2) = 18$, and (3) $P_1 \cup P_2$ spans S_4.*

Proof. Because S_4 is node transitive, we may assume that $\mathbf{u} = 1234$. The required two paths are listed below.

$P_1 = \langle 1234, 3214, 4213, 2413, 3412, 1432 \rangle$	$P_2 = \langle 1234, 4231, 2431, 3421, 1423, 4123, 3124, 2134, 4132, 3142, 1342, 4312, 2314, 1324, 4321, 2341, 3241, 1243, 2143 \rangle$
$P_1 = \langle 1234, 3214, 4213, 2413, 3412, 1432 \rangle$	$P_2 = \langle 1234, 4231, 2431, 3421, 1423, 4123, 2143, 1243, 3241, 2341, 4321, 1324, 2314, 4312, 1342, 3142, 4132, 2134, 3124 \rangle$
$P_1 = \langle 1234, 3214, 4213, 2413, 3412, 1432 \rangle$	$P_2 = \langle 1234, 4231, 2431, 3421, 1423, 4123, 3124, 2134, 4132, 3142, 2143, 1243, 3241, 2341, 1342, 4312, 2314, 1324, 4321 \rangle$
$P_1 = \langle 1234, 3214, 2314, 4312, 1342, 2341 \rangle$	$P_2 = \langle 1234, 2134, 3124, 1324, 4321, 3421, 2431, 4231, 3241, 1243, 4213, 2413, 3412, 1432, 4132, 3142, 2143, 4123, 1423 \rangle$
$P_1 = \langle 1234, 2134, 4132, 3142, 1342, 2341 \rangle$	$P_2 = \langle 1234, 3214, 4213, 2413, 1423, 3421, 4321, 1324, 2314, 4312, 3412, 1432, 2431, 4231, 3241, 1243, 2143, 4123, 3124 \rangle$
$P_1 = \langle 1234, 3214, 2314, 4312, 1342, 2341 \rangle$	$P_2 = \langle 1234, 2134, 4132, 3142, 2143, 4123, 3124, 1324, 4321, 3421, 1423, 2413, 3412, 1432, 2431, 4231, 3241, 1243, 4213 \rangle$
$P_1 = \langle 1234, 2134, 3124, 1324, 2314, 3214 \rangle$	$P_2 = \langle 1234, 4231, 3241, 2341, 4321, 3421, 2431, 1432, 4132, 3142, 1342, 4312, 3412, 2413, 4213, 1243, 2143, 4123, 1423 \rangle$
$P_1 = \langle 1234, 3214, 4213, 2413, 1423, 3421 \rangle$	$P_2 = \langle 1234, 4231, 3241, 1243, 2143, 4123, 3124, 2134, 4132, 3142, 1342, 2341, 4321, 1324, 2314, 4312, 3412, 1432, 2431 \rangle$
$P_1 = \langle 1234, 3214, 4213, 2413, 1423, 3421 \rangle$	$P_2 = \langle 1234, 2134, 3124, 4123, 2143, 1243, 3241, 4231, 2431, 1432, 3412, 4312, 2314, 1324, 4321, 2341, 1342, 3142, 4132 \rangle$
$P_1 = \langle 1234, 3214, 4213, 1243, 3241, 4231 \rangle$	$P_2 = \langle 1234, 2134, 4132, 3142, 2143, 4123, 3124, 1324, 2314, 4312, 1342, 2341, 4321, 3421, 2431, 1432, 3412, 2413, 1423 \rangle$
$P_1 = \langle 1234, 3214, 4213, 1243, 3241, 4231 \rangle$	$P_2 = \langle 1234, 2134, 4132, 3142, 2143, 4123, 3124, 1324, 2314, 4312, 1342, 2341, 4321, 3421, 1423, 2413, 3412, 1432, 2431 \rangle$
$P_1 = \langle 1234, 3214, 4213, 1243, 3241, 4231 \rangle$	$P_2 = \langle 1234, 2134, 4132, 3142, 2143, 4123, 1423, 2413, 3412, 1432, 2431, 3421, 4321, 2341, 1342, 4312, 2314, 1324, 3124 \rangle$

Hence, this statement is proved. ∎

Theorem 6.

$$D_2^{s_L}(S_n) = \begin{cases} 5 & \text{if } n = 3, \\ 15 & \text{if } n = 4, \text{and} \\ \frac{n!}{2} + 1 & \text{if } n \geq 5. \end{cases}$$

Proof. It is easy to check that $D_2^{s_L}(S_3) = 5$. By Lemma 14, we have that $D_2^{s_L}(S_4) = 15$. Thus, we assume that $n \geq 5$. Let $\mathbf{u}$ be a white node and $\mathbf{v}$ be a black node of S_n. Let P_1 and P_2 be any 2*-container of S_n joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, $\max\{l(P_1), l(P_2)\} \geq \frac{n!}{2} + 1$. Hence, $d_2^{s_L}(\mathbf{u}, \mathbf{v}) \geq \frac{n!}{2} + 1$ and $D_2^{s_L}(S_n) \geq \frac{n!}{2} + 1$. Hence, we only need to show that $d_2^{s_L}(\mathbf{u}, \mathbf{v}) \leq \frac{n!}{2} + 1$. Because S_n is edge transitive, we assume that $\mathbf{u} \in S_n^{\{n\}}$ and $\mathbf{v} \in S_n^{\{n-1\}}$.

CASE 1. $n = 5$. By Lemma 15, there exist two paths H_1 and H_2 of $S_5^{\{5\}}$ such that (1) H_1 joins $\mathbf{u}$ to a black node $\mathbf{x}$ with $(\mathbf{x})_1 = 1$ and $l(H_1) = 5$, (2) H_2 joins $\mathbf{u}$ to a white node $\mathbf{y}$ with $(\mathbf{y})_1 = 3$ and $l(H_2) = 18$, and (3) $H_1 \cup H_2$ spans $S_5^{\{5\}}$. Again, there exist two paths T_1 and T_2 of $S_5^{\{4\}}$ such that (1) T_1 joins $\mathbf{v}$ to a white node $\mathbf{p}$ with $(\mathbf{p})_1 = 2$ and $l(T_1) = 5$, (2) T_2 joins $\mathbf{v}$ to a black node $\mathbf{q}$ with $(\mathbf{q})_1 = 3$ and $l(T_2) = 18$, and (3) $T_1 \cup T_2$ spans $S_5^{\{4\}}$. By Lemma 3, there is a hamiltonian path R of $S_5^{\{1,2\}}$ joining the white node $(\mathbf{x})^5$ to the black node $(\mathbf{p})^5$. Again, there is a hamiltonian path Z of $S_5^{\{3\}}$ joining the black node $(\mathbf{y})^5$ to the white node $(\mathbf{q})^5$. We set

$$L_1 = \langle \mathbf{u}, H_1, \mathbf{x}, (\mathbf{x})^5, R, (\mathbf{p})^5, \mathbf{p}, T_1^{-1}, \mathbf{v} \rangle \text{ and}$$
$$L_2 = \langle \mathbf{u}, H_2, \mathbf{y}, (\mathbf{y})^5, Z, (\mathbf{q})^5, \mathbf{q}, T_2^{-1}, \mathbf{v} \rangle.$$

Obviously, $\{L_1, L_2\}$ is a 2*-container. Moreover, $l(L_1) = 59$ and $l(L_2) = 61$. Hence, $d_2^{s_L}(\mathbf{u}, \mathbf{v}) \leq \frac{n!}{2} + 1$. See Figure 8(a) for an illustration.

CASE 2. $n \geq 6$ is even. Let $\mathbf{x}$ be a neighbor of $\mathbf{u}$ in $S_n^{\{n\}}$ with $(\mathbf{x})_1 \in \langle n-2 \rangle$. Let $\mathbf{y}$ be a neighbor of $\mathbf{v}$ in $S_n^{\{n-1\}}$. Let $\mathbf{z}$ be a neighbor of $\mathbf{y}$ in $S_n^{\{n-1\}}$ with $(\mathbf{z})_1 \in \langle n-2 \rangle - \{(\mathbf{v})_1, (\mathbf{y})_1, (\mathbf{x})_1\}$. Let $a_1 a_2 \ldots a_{n-2}$ be a permutation of $\langle n-2 \rangle$ such that $a_1 = (\mathbf{x})_1$ and $a_{n-2} = (\mathbf{z})_1$. Let $H = \{a_1, a_2, \ldots, a_{\frac{n-2}{2}}\}$. By Theorem 2, there is a hamiltonian path P of $S_n^{\{n\}} - \{\mathbf{x}\}$ joining $\mathbf{u}$ to a white node $\mathbf{p}$ with $(\mathbf{p})_1 = a_{n-2}$. By Theorem 4, there is a hamiltonian path Q of $S_n^{\{n-1\}} - \{\mathbf{y}, \mathbf{z}\}$ joining a white node $\mathbf{q}$ with $(\mathbf{q})_1 = a_1$ to $\mathbf{v}$. By Theorem 3, there is a hamiltonian path R of S_n^H joining the white node $(\mathbf{x})^n$ to the black

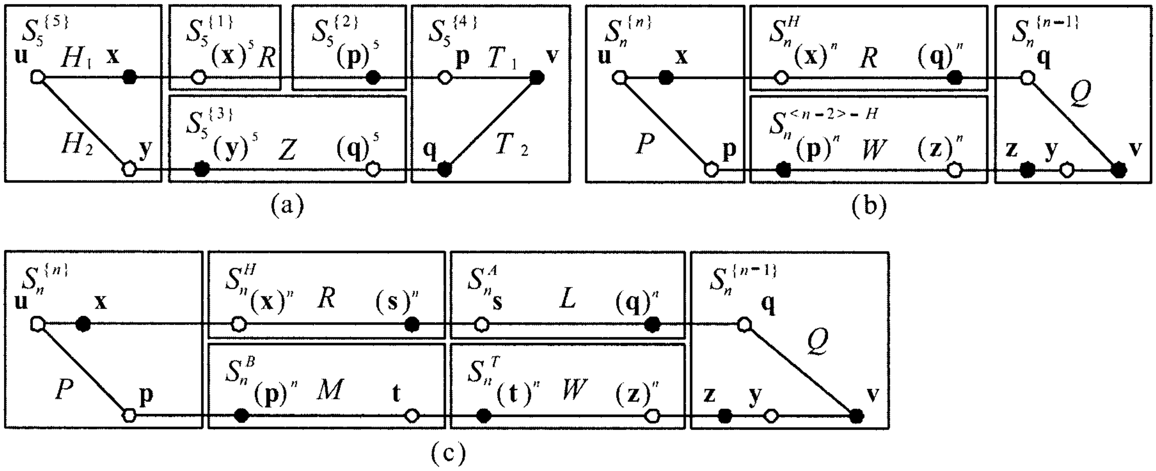

FIG. 8. Illustration for Theorem 6.

node $(\mathbf{q})^n$. Again, there is a hamiltonian path W of $S_n^{\langle n-2\rangle - H}$ joining the black node $(\mathbf{p})^n$ to the white node $(\mathbf{z})^n$. We set

$$L_1 = \langle \mathbf{u}, \mathbf{x}, (\mathbf{x})^n, R, (\mathbf{q})^n, \mathbf{q}, Q, \mathbf{v}\rangle \text{ and}$$
$$L_2 = \langle \mathbf{u}, P, \mathbf{p}, (\mathbf{p})^n, W, (\mathbf{z})^n, \mathbf{z}, \mathbf{y}, \mathbf{v}\rangle.$$

Obviously, $\{L_1, L_2\}$ is a 2^*-container of S_n between $\mathbf{u}$ and $\mathbf{v}$. Because $l(L_1) = \frac{n!}{2} - 1$ and $l(L_2) = \frac{n!}{2} + 1$, we have $d_2^{S_L}(\mathbf{u}, \mathbf{v}) \leq \frac{n!}{2} + 1$. See Figure 8(b) for an illustration.

CASE 3. $n \geq 7$ is odd. Let $\mathbf{x}$ be a neighbor of $\mathbf{u}$ in $S_n^{\{n\}}$ with $(\mathbf{x})_1 \in \langle n-2\rangle$. Let $\mathbf{y}$ be a neighbor of $\mathbf{v}$ in $S_n^{\{n-1\}}$. Let $\mathbf{z}$ be a neighbor of $\mathbf{y}$ in $S_n^{\{n-1\}}$ with $(\mathbf{z})_1 \in \langle n-2\rangle - \{(\mathbf{v})_1, (\mathbf{y})_1, (\mathbf{x})_1\}$. Let $a_1a_2\ldots a_{n-2}$ be a permutation of $\langle n-2\rangle$ such that $a_1 = (\mathbf{x})_1$ and $a_{n-3} = (\mathbf{z})_1$. Let $H = \{a_1, a_2, \ldots, a_{\frac{n-3}{2}}\}$ and $T = \{a_{\frac{n-3}{2}+1}, a_{\frac{n-3}{2}+2}, \ldots, a_{n-3}\}$. We set $A = \{(i, a_{n-2}) \mid i \in H \cup \{n-1\}\}$ and $B = \{(i, a_{n-2}) \mid i \in T \cup \{n\}\}$. Let S_n^A denote the subgraph of S_n induced by $\cup_{i \in H \cup \{n-1\}} S_n^{\{(i, a_{n-2})\}}$, and let S_n^B denote the subgraph of S_n induced by $\cup_{i \in T \cup \{n\}} S_n^{\{(i, a_{n-2})\}}$. By Theorem 2, there is a hamiltonian path P of $S_n^{\{n\}} - \{\mathbf{x}\}$ joining $\mathbf{u}$ to a white node $\mathbf{p}$ with $(\mathbf{p})_1 = a_{n-2}$ and $(\mathbf{p})_{n-2} = a_{n-3}$. By Theorem 4, there is a hamiltonian path Q of $S_n^{\{n-1\}} - \{\mathbf{y}, \mathbf{z}\}$ joining a white node $\mathbf{q}$ with $(\mathbf{q})_1 = a_{n-2}$ and $(\mathbf{q})_{n-1} = a_1$ to $\mathbf{v}$. By Theorem 3, there is a hamiltonian path L of S_n^A joining a white node $\mathbf{s}$ with $(\mathbf{s})_1 = a_1$ to the black node $(\mathbf{q})^n$. Again, there is a hamiltonian path M of S_n^B joining the black node $(\mathbf{p})^n$ to a white node $\mathbf{t}$ with $(\mathbf{t})_1 = a_{n-3}$. By Lemma 3, there is a hamiltonian path R of S_n^H joining the white node $(\mathbf{x})^n$ to the black node $(\mathbf{s})^n$. Again, there is a hamiltonian path W of S_n^T joining the black node $(\mathbf{t})^n$ to the white node $(\mathbf{z})^n$. We set

$$L_1 = \langle \mathbf{u}, \mathbf{x}, (\mathbf{x})^n, R, (\mathbf{s})^n, \mathbf{s}, L, (\mathbf{q})^n, \mathbf{q}, Q, \mathbf{v}\rangle \text{ and}$$
$$L_2 = \langle \mathbf{u}, P, \mathbf{p}, (\mathbf{p})^n, M, \mathbf{t}, (\mathbf{t})^n, W, (\mathbf{z})^n, \mathbf{z}, \mathbf{y}, \mathbf{v}\rangle.$$

Obviously, $\{L_1, L_2\}$ is a 2^*-container of S_n between $\mathbf{u}$ and $\mathbf{v}$. Because $l(L_1) = \frac{n!}{2} - 1$ and $l(L_2) = \frac{n!}{2} + 1$, we have $d_2^{S_L}(\mathbf{u}, \mathbf{v}) \leq \frac{n!}{2} + 1$. See Figure 8(c) for an illustration. ∎

6. CONCLUSION

In this study, we prove that $D_{n-1}^{S_L}(S_n) = (n-1)! + 2(n-2)! + 2(n-3)! + 1 = \frac{n!}{n-2} + 1$ and $D_2^{S_L}(S_n) = \frac{n!}{2} + 1$ for $n \geq 5$. Actually, we prove that $d_2^{S_L}(\mathbf{u}, \mathbf{v}) = \frac{n!}{2} + 1$ for any two vertices $\mathbf{u}$ and $\mathbf{v}$ from different bipartite sets of S_n.

Recently, we have proved that S_n is super laceable [19]. Hence, we can study $D_k^{S_L}(S_n)$ for $1 \leq k \leq n-1$. We conjecture that $D_k^{S_L}(S_n) = \frac{n!}{k} + 1$ for $n \geq 5$ and $3 \leq k \leq n-2$. Furthermore, we believe that $d_k^{S_L}(\mathbf{u}, \mathbf{v}) = \frac{n!}{k} + 1$ for any two nodes $\mathbf{u}$ and $\mathbf{v}$ from different bipartite sets of S_n for $n \geq 5$ and $3 \leq k \leq n-2$.

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