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Perturbed block circulant matrices and their application to the wavelet method of chaotic control

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Controlling chaos via wavelet transform was proposed by Wei *et al.* [Phys. Rev. Lett. **89**, 284103.1–284103.4 (2002)]. It was reported there that by modifying a tiny fraction of the wavelet subspace of a coupling matrix, the transverse stability of the synchronous manifold of a coupled chaotic system could be dramatically enhanced. The stability of chaotic synchronization is actually controlled by the second largest eigenvalue $\lambda_2(\alpha,\beta)$ of the (wavelet) transformed coupling matrix $C(\alpha,\beta)$ for each α and β. Here β is a mixed boundary constant and α is a scalar factor. In particular, $\beta=1$ (0) gives the nearest neighbor coupling with periodic (Neumann) boundary conditions. In this paper, we obtain two main results. First, the reduced eigenvalue problem for $C(\alpha,0)$ is completely solved. Some partial results for the reduced eigenvalue problem of $C(\alpha,\beta)$ are also obtained. Second, we are then able to understand behavior of $\lambda_2(\alpha,0)$ and $\lambda_2(\alpha,1)$ for any wavelet dimension $j\in\mathbb{N}$ and block dimension $n\in\mathbb{N}$. Our results complete and strengthen the work of Shieh *et al.* [J. Math. Phys. **47**, 082701.1–082701.10 (2006)] and Juang and Li [J. Math. Phys. **47**, 072704.1–072704.16 (2006)]. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2400828]

I. INTRODUCTION

Of concern here is the eigencurve problem for a class of "perturbed" block circulant matrices.

$$C(\alpha,\beta)\mathbf{b}=\lambda(\alpha,\beta)\mathbf{b}. \tag{1.1a}$$

Here $C(\alpha,\beta)$ is an $n\times n$ block matrix of the following form:

$$C(\alpha,\beta)=\begin{pmatrix} C_1(\alpha,\beta) & C_2(\alpha,1) & 0 & \cdots & 0 & C_2^T(\alpha,\beta) \\ C_2^T(\alpha,1) & C_1(\alpha,1) & C_2(\alpha,1) & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & C_2^T(\alpha,1) & C_1(\alpha,1) & C_2(\alpha,1) \\ C_2(\alpha,\beta) & 0 & \cdots & 0 & C_2^T(\alpha,1) & \hat{I}C_1(\alpha,\beta)\hat{I} \end{pmatrix}_{n\times n}. \tag{1.1b}$$

Here

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$$C_1(\alpha,\beta)=\begin{pmatrix} -1-\beta & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{2^j\times 2^j} - \frac{\alpha(1+\beta)}{2^{2j}}ee^T =: A_1(\beta,2^j) - \frac{\alpha(1+\beta)}{2^{2j}}ee^T, \tag{1.1c}$$

where $e=(1,1,\ldots,1)^T$, j is a positive integer, $\alpha>0$ is a (wavelet) scalar factor, and $\beta\in\mathbb{R}$ represents a mixed boundary constant. Moreover,

$$C_2(\alpha,\beta)=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \beta & 0 & \cdots & 0 \end{pmatrix} + \frac{\alpha\beta}{2^{2j}}ee^T =: A_2(\beta,2^j) + \frac{\alpha\beta}{2^{2j}}ee^T, \tag{1.1d}$$

$$\hat{I}=\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & & \cdot & \cdot & \cdot & \vdots \\ \vdots & \cdot & \cdot & \cdot & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}. \tag{1.1e}$$

The dimension of $C(\alpha,\beta)$ is $n2^j\times n2^j$. From here on, we shall call n and j the block and the wavelet dimensions of $C(\alpha,\beta)$, respectively. $C(\alpha,\beta)$ is a block circulant matrix (see, e.g., Ref. 1) only if $\beta=1$. It is well known, see, e.g., Theorem 5.6.4 of Ref. 1, that for each α the eigenvalues of $C(\alpha,1)$ consist of eigenvalues of a certain linear combinations of its block matrices. Such results are called the reduced eigenvalue problem for $C(\alpha,1)$.

This problem arises in the wavelet method for a chaotic control.[7] It is found there that the modification of a tiny fraction of wavelet subspaces of a coupling matrix could lead to a dramatic change in chaos synchronizing properties. We begin with describing their work. Let there be N nodes (oscillators). Assume $\mathbf{u}_i$ is the m-dimensional vector of dynamical variables of the ith node. Let the isolated (uncoupling) dynamics be $\dot{\mathbf{u}}_i=f(\mathbf{u}_i)$ for each node. Used in the coupling, $h:\mathbb{R}^m\to\mathbb{R}^m$ is an arbitrary function of each node's variables. Thus, the dynamics of the ith node is

$$\dot{\mathbf{u}}_i = f(\mathbf{u}_i) + \epsilon\sum_{j=1}^{N} a_{ij}h(\mathbf{u}_j), \quad i=1,2,\ldots,N, \tag{1.2a}$$

where ϵ is a coupling strength. The sum $\Sigma_{j=1}^N a_{ij}=0$. Let $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_N)^T$, $F(\mathbf{u})=(f(\mathbf{u}_1),f(\mathbf{u}_2),\ldots,f(\mathbf{u}_N))^T$, $H(\mathbf{u})=(h(\mathbf{u}_1),h(\mathbf{u}_2),\ldots,h(\mathbf{u}_N))^T$, and $A=(a_{ij})$. We may write Eq. (1.1a) as

$$\dot{\mathbf{u}} = F(\mathbf{u}) + \epsilon A\times H(\mathbf{u}). \tag{1.2b}$$

Here $\times$ is the direct product of two matrices B and C defined as follows. Let $B=(b_{ij})_{k_1\times k_2}$ be a $k_1\times k_2$ matrix and $C=(C_{ij})_{k_2\times k_3}$ be a $k_2\times k_3$ block matrix. Then

$$B \times C = \left(\sum_{l=1}^{k_2} b_{il} C_{lj} \right)_{k_1 \times k_3}.$$

Many coupling schemes are covered by Eq. (1.2b). For example, if the Lorenz system is used and the coupling is through its three components x, y, and z, then the function h is just the matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.3}$$

The choice of A will provide the connectivity of nodes. For instance, the nearest neighbor coupling with periodic, Neumann boundary conditions and mixed boundary conditions are, respectively, given as $A = A_1(1,N) + A_2(1,N) + A_2^T(1,N) =: A_P$, $A = A_1(0,N) + A_2(1,N)\hat{I} =: A_N$ and $A = A_1(\beta,N) + A_2(\beta,N) + A_2^T(\beta,N) + (1-\beta)A_2(1,N)\hat{I} =: A_M$, where those A_i's, $i=1,2$, are defined in Eqs. (1.1c) and (1.1d).

Mathematically speaking,[5] the second largest eigenvalue λ_2 of A is dominant in controlling the stability of chaotic synchronization, and the critical strength ϵ_c for synchronization can be determined in terms of λ_2,

$$\epsilon_c = \frac{L_{\max}}{-\lambda_2}. \tag{1.4}$$

The eigenvalues of $A = A_P$ are given by $\lambda_i = -4\sin^2[\pi(i-1)/N]$, $i=1,2,\ldots,N$. In general, a larger number of nodes give a smaller nonzero eigenvalue λ_2 in magnitude and, hence, a larger ϵ_c. In controlling a given system, it is desirable to reduce the critical coupling strength ϵ_c. The wavelet method in Ref. 7 will, in essence, transform A into $C(\alpha,\beta)$. Consequently, it is of great interest to study the second eigencurve of $C(\alpha,\beta)$ for each β. By the second largest eigencurve $\lambda_2(\alpha,\beta)$ of $C(\alpha,\beta)$ for fixed β, we mean that for given $\alpha>0$, $\lambda_2(\alpha,\beta)$ is the second largest eigenvalue of $C(\alpha,\beta)$. We remark that 0 is the largest eigenvalue of $C(\alpha,\beta)$ for any $\alpha>0$ and $\beta \in \mathbb{R}$. This is to say that for fixed β, $\lambda_2(\alpha,\beta)=0$ is the first eigencurve of $C(\alpha,\beta)$. A numerical simulation[7] of a coupled system of $N=512$ Lorenz oscillators shows that with $h=I_3$ and $A=A_P$, the critical coupling strength ϵ_c decreases linearly with respect to the increase of α up to a critical value α_c. The smallest ϵ_c is about 6, which is about 10^3 times smaller than the original critical coupling strength, indicating the efficiency of the proposed approach.

The mathematical verification of such phenomena is first achieved by Shieh *et al.*[6] Specifically, they solved the second eigencurve problem of $C(\alpha,1)$ with n being a multiple of 4 and j being any positive integer. Subsequently, in Ref. 4 the second eigencurve problem for $C(\alpha,0)$ and $C(\alpha,1)$ with n being any positive integer and $j=1$ are solved without touching on the reduced eigenvalue problem. In this paper, we obtain two main results. First, the reduced eigenvalue problem for $C(\alpha,0)$ is completely solved. Some partial results for the reduced eigenvalue problem of $C(\alpha,\beta)$ are also obtained. Second, we are then able to understand the behavior of $\lambda_2(\alpha,0)$ and $\lambda_2(\alpha,1)$ for any j and $n \in \mathbb{N}$.

II. REDUCED EIGENVALUE PROBLEMS

Writing the eigenvalue problem $C(\alpha,\beta)\mathbf{b} = \lambda\mathbf{b}$, where $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)^T$ and $\mathbf{b}_i \in \mathbb{C}^{2^j}$, in block component form, we get

$$C_2^T(\alpha,1)\mathbf{b}_{i-1} + C_1(\alpha,1)\mathbf{b}_i + C_2(\alpha,1)\mathbf{b}_{i+1} = \lambda\mathbf{b}_i, \quad 1 \leqslant i \leqslant n. \tag{2.1a}$$

Mixed boundary conditions would yield that

$$C_2^T(\alpha,1)\mathbf{b}_0 + C_1(\alpha,1)\mathbf{b}_1 + C_2(\alpha,1)\mathbf{b}_2 = \lambda\mathbf{b}_1 = C_1(\alpha,\beta)\mathbf{b}_1 + C_2(\alpha,1)\mathbf{b}_2 + C_2^T(\alpha,\beta)\mathbf{b}_n$$

and

$$C_2^T(\alpha,1)\mathbf{b}_{n-1}+C_1(\alpha,1)\mathbf{b}_n+C_2(\alpha,1)\mathbf{b}_{n+1}=\lambda\mathbf{b}_n=C_2(\alpha,\beta)\mathbf{b}_1+C_2^T(\alpha,1)\mathbf{b}_{n-1}+\hat{I}C_1(\alpha,\beta)\hat{I}\mathbf{b}_n,$$

or, equivalently,

$$\begin{aligned} C_2^T(\alpha,1)\mathbf{b}_0&=(C_1(\alpha,\beta)-C_1(\alpha,1))\mathbf{b}_1+C_2^T(\alpha,\beta)\mathbf{b}_n\\ &=\left[\begin{pmatrix}1-\beta & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}+\frac{\alpha(1-\beta)}{2^{2j}}ee^T\right]\mathbf{b}_1+\left[\begin{pmatrix}0 & \cdots & 0 & \beta\\ 0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 0\end{pmatrix}+\frac{\alpha\beta}{2^{2j}}ee^T\right]\mathbf{b}_n\\ &=(1-\beta)C_2^T(\alpha,1)\hat{I}\mathbf{b}_1+\beta C_2(\alpha,1)\mathbf{b}_n \end{aligned}\tag{2.1b}$$

and

$$C_2(\alpha,1)\mathbf{b}_{n+1}=(\hat{I}C_1(\alpha,\beta)\hat{I}-C_1(\alpha,1))\mathbf{b}_n+C_2(\alpha,\beta)\mathbf{b}_1=(1-\beta)C_2^T(\alpha,1)\hat{I}\mathbf{b}_n+\beta C_2(\alpha,1)\mathbf{b}_1.\tag{2.1c}$$

To study the block difference equation [Eq. (2.1)], we set

$$\mathbf{b}_j=\delta^j v,\tag{2.2}$$

where $\boldsymbol{v}\in\mathbb{C}^{2^j}$ and $\delta\in\mathbb{C}$.

Substituting Eq. (2.2) into Eq. (2.1a), we have

$$[C_2^T(\alpha,1)+\delta(C_1(\alpha,1)-\lambda I)+\delta^2C_2(\alpha,1)]\mathbf{v}=0.\tag{2.3}$$

To have a nontrivial solution $\boldsymbol{v}$ satisfying Eq. (2.3), we need to have

$$\det[C_2^T(\alpha,1)+\delta(C_1(\alpha,1)-\lambda I)+\delta^2C_2(\alpha,1)]=0.\tag{2.4}$$

Definition 2.1: Equation (2.4) is to be called the characteristic equation of the block difference equation [Eq. (2.1a)]. Let $\delta_k=\delta_k(\lambda)\neq 0$ and $\boldsymbol{v}_k=\boldsymbol{v}_k(\lambda)\neq 0$ be complex numbers and vectors, respectively, satisfying Eq. (2.3). Here $k=1,2,\ldots,m$ and $m\leqslant 2^j$. Assume that there exists a $\lambda\in\mathbb{C}$, such that $\mathbf{b}_j=\Sigma_{k=1}^m c_k\delta_k^j(\lambda)\boldsymbol{v}_k(\lambda)$, $j=0,1,\ldots,n+1$, satisfy Eqs. (2.1b) and (2.1c), where $c_k\in\mathbb{C}$. If, in addition, $\mathbf{b}_j$, $j=1,2,\ldots,n$, are not all zero vectors, then such $\delta_k(\lambda)$ is called a characteristic value of Eq. (2.1a), (2.1b), and (2.1c) or (1.1a) with respect to λ and $\boldsymbol{v}_k(\lambda)$ its corresponding characteristic vector.

Remark 2.1: Clearly, for each α and β, λ in Definition 2.1 is an eigenvalue of $C(\alpha,\beta)$.

Should no ambiguity arises, we will write $C_2^T(\alpha,1)=C_2^T$, $C_1(\alpha,1)=C_1$, and $C_2(\alpha,1)=C_2$. Likewise, we will write $A_2(\beta,2^j)=A_2(\beta)$ and $A_1(\beta,2^j)=A_1(\beta)$.

Proposition 2.1: *Let $\rho(\lambda)=\{\delta_i(\lambda):\delta_i(\lambda)$ is a root of Eq. (2.4)$\}$, and let $\bar{\rho}(\lambda)=\{1/\delta_i(\lambda):\delta_i(\lambda)$ is a root of Eq. (2.4)$\}$. Then $\rho(\lambda)=\bar{\rho}(\lambda)$. Let δ_i and δ_k be in $\rho(\lambda)$. We further assume that δ_i and $\boldsymbol{v}_i=(v_{i1},\ldots,v_{i2^j})^T$ satisfy Eq. (2.3). Suppose $\delta_i\cdot\delta_k=1$. Then δ_k and $\boldsymbol{v}_k=(v_{i2^j},v_{i2^j-1},\ldots,v_{i2},v_{i1})^T=:\boldsymbol{v}_i^s$ also satisfy Eq. (2.3). Conversely, if $\delta_i\cdot\delta_k\neq 1$, then $\boldsymbol{v}_k\neq\boldsymbol{v}_i^s$.*

Proof: To prove $\rho(\lambda)=\bar{\rho}(\lambda)$, we see that

$$\begin{aligned}\det[C_2^T+\delta(C_1-\lambda I)+\delta^2C_2]&=\delta^2\det\left[\frac{1}{\delta^2}C_2^T+\frac{1}{\delta}(C_1-\lambda I)+C_2\right]\\ &=\delta^2\det\left[\frac{1}{\delta^2}C_2^T+\frac{1}{\delta}(C_1-\lambda I)+C_2\right]^T\\ &=\delta^2\det\left[C_2^T+\frac{1}{\delta}(C_1-\lambda I)+\frac{1}{\delta^2}C_2\right].\end{aligned}$$

Thus, if δ is a root of Eq. (2.4), then so is $1/\delta$. To see the last assertion of the proposition, we write Eq. (2.3) with $\delta=\delta_i$ and $\boldsymbol{v}=\boldsymbol{v}_i$ in component form.

$$\sum_{m=1}^{2^j} [(C_2^T)_{lm}v_{im} + \delta_i(\bar{C}_1)_{lm}v_{im} + \delta_i^2(C_2)_{lm}v_{im}] = 0, \quad l = 1,2, \ldots ,2^j. \tag{2.5}$$

Here $\bar{C}_1 = C_1 - \lambda I$. Now the right hand side of Eq. (2.5) becomes

$$\left(\frac{1}{\delta_k}\right)^2 \left\{ \sum_{m=1}^{2^j} [(C_2)_{l(2^j+1-m)}v_{i(2^j+1-m)} + \delta_k(\bar{C}_1)_{l(2^j+1-m)}v_{i(2^j+1-m)} + \delta_k^2(C_2^T)_{l(2^j+1-m)}v_{i(2^j+1-m)}] \right\}$$
$$= \left(\frac{1}{\delta_k}\right)^2 \left\{ \sum_{m=1}^{2^j} [(C_2^T)_{(2^j+1-l)m}v_{i(2^j+1-m)} + \delta_k(\bar{C}_1)_{(2^j+1-l)m}v_{i(2^j+1-m)} + \delta_k^2(C_2)_{(2^j+1-l)m})v_{i(2^j+1-m)}] \right\},$$
$$l = 1,2, \ldots ,2^j. \tag{2.6}$$

We have used the fact that

$$(A)_{(2^j+1-l)m} = (A^T)_{l(2^j+1-m)}, \tag{2.7}$$

where $A=C_2^T$ or $\bar{C}_1$ or C_2 to justify the equality in Eq. (2.6). However, Eq. (2.7) follows from Eqs. (1.1c) and (1.1d). Letting $v_{i(2^j+1-m)}=v_{km}$, we have that the pair $(\delta_k, \boldsymbol{v}_k)$ satisfies Eq. (2.3). Suppose $\boldsymbol{v}_k=\boldsymbol{v}_i^s$, we see, similarly, that the pair $(1/\delta_i, \boldsymbol{v}_k)$ also satisfies Eq. (2.3). Thus $1/\delta_i=\delta_k$. □

Remark 2.2: Equation (2.4) is a palindromic equation. That is, for each λ, δ and δ^{-1} are both the roots of Eq. (2.4). However, the eigenvalue problem discussed here is not a palindromic eigenvalue problem.[3]

Definition 2.2: We shall call $\boldsymbol{v}^s$ and $-\boldsymbol{v}^s$, the symmetric vector and antisymmetric vector of $\boldsymbol{v}$, respectively. A vector $\boldsymbol{v}$ is symmetric (antisymmetric) if $\boldsymbol{v}=\boldsymbol{v}^s$ ($\boldsymbol{v}=-\boldsymbol{v}^s$).

Theorem 2.1: *Let $\delta_k=e^{(\pi k/n)i}$, k is an integer and $i=\sqrt{-1}$, then δ_{2k}, $k=0,1,\ldots,n-1$, are characteristic values of Eq.* (2.1a), (2.1b), and (2.1c) *with $\beta=1$. For each α, if $\lambda \in \mathbb{C}$ satisfies*

$$\det[C_2^T + \delta_{2k}(C_1 - \lambda I) + \delta_{2k}^2 C_2] = 0,$$

for some $k \in \mathbb{Z}$, $0 \leqslant k \leqslant n-1$, then λ is an eigenvalue of $C(\alpha,1)$.

Proof: Let λ be as assumed. Then there exists a $\boldsymbol{v} \in \mathbb{C}^{2^j}$, $\boldsymbol{v} \neq \mathbf{0}$ such that

$$[C_2^T + \delta_{2k}(C_1 - \lambda I) + \delta_{2k}^2 C_2]\mathbf{v} = \mathbf{0}.$$

Let $\mathbf{b}_j=\delta_{2k}^j\boldsymbol{v}$, $0 \leqslant j \leqslant n+1$. Then such $\mathbf{b}_j$'s satisfy Eqs. (2.1a), (2.1b), and (2.1c). We just proved the assertion of the theorem. □

Corollary 2.1: *Set*

$$\Gamma_k = C_1 + \delta_{2n-k}C_2^T + \delta_k C_2. \tag{2.8}$$

Then the eigenvalues of $C(\alpha,1)$, for each α, consist of eigenvalues of Γ_k, $k=0,2,4,\ldots,2(n-1)$. That is, $\rho(C(\alpha,1))=\cup_{k=0}^{n-1}\rho(\Gamma_{2k})$. Here $\rho(A)=$ *the spectrum of the matrix A.*

Remark 2.3: $C(\alpha,1)$ is a block circulant matrix. The assertion of Corollary 2.1 is not new (see, e.g., Theorem 5.6.4 of Ref. 1). Here we merely gave a different proof.

To study the eigenvalue of $C(\alpha,0)$ for each α, we begin with considering the eigenvalues and eigenvectors of $C_2^T+C_1+C_2$ and $C_2^T-C_1+C_2$.

Proposition 2.2: *Let $T_1(C)$ $(T_2(C))$ be the set of linearly independent eigenvectors of the matrix C that are symmetric (antisymmetric). Then $|T_1(C_2^T+C_1+C_2)|=|T_2(C_2^T+C_1+C_2)|=|T_1(C_2^T-C_1+C_2)|=|T_2(C_2^T-C_1+C_2)|=2^{j-1}$. Here $|A|$ denote the cardinality of the set A.*

Proof: We will only illustrate the case for $C_2^T-C_1+C_2=:C$. We first observe that $|T_1(C)|$ is less than or equal to 2^{j-1}. So is $|T_2(C)|$. We also remark that the cardinality of the set of all linearly independent eigenvectors of C is 2^j. If $0<|T_1(C)|<2^{j-1}$, there must exist an eigenvector $\boldsymbol{v}$ for which $\boldsymbol{v}\neq\boldsymbol{v}^s$, $\boldsymbol{v}\neq-\boldsymbol{v}^s$, and $\boldsymbol{v}\notin\text{span}\{T_1(C),T_2(C)\}$, the span of the vectors in $T_1(C)$ and $T_2(C)$. It then follows from Proposition 2.1 that $\boldsymbol{v}+\boldsymbol{v}^s$, a symmetric vector, is in the span$\{T_1(C)\}$. Moreover, $\boldsymbol{v}-\boldsymbol{v}^s$ is in span$\{T_2(C)\}$. Hence $\boldsymbol{v}\in\text{span}\{T_1(C),T_2(C)\}$, a contradiction. Hence, $|T_1(C)|=2^{j-1}$. Similarly, we conclude that $|T_2(C)|=2^{j-1}$. □

Theorem 2.2: *Let $\delta_k=\mathrm{e}^{(\pi k/n)i}$, where k is an integer and $i=\sqrt{-1}$. For each α, if $\lambda\in\mathbb{C}$ satisfies*

$$\det[C_2^T+\delta_k(C_1-\lambda I)+\delta_k^2C_2]=0,$$

for some $k\in\mathbb{Z}$, $1\leqslant k\leqslant n-1$, then λ is an eigenvalue of $C(\alpha,0)$. Let λ be the eigenvalue of $C_2^T+C_1+C_2$ $(-C_2^T+C_1-C_2)$ for which its associated eigenvector $\boldsymbol{v}$ satisfies $\hat{\boldsymbol{I}}\boldsymbol{v}=\boldsymbol{v}$ $(\hat{\boldsymbol{I}}\boldsymbol{v}=-\boldsymbol{v})$, then λ is also an eigenvalue of $C(\alpha,0)$.

Proof: For any $1\leqslant k\leqslant n-1$, let δ_k be as assumed. Let λ_k and ν_k be a number and a nonzero vector, respectively, satisfying

$$[C_2^T+\delta_k(C_1-\lambda_kI)+\delta_k^2C_2]\boldsymbol{v}_k=\mathbf{0}. \tag{2.9}$$

Using Proposition 2.1, we see that λ_k satisfies

$$\det[C_2^T+\delta_{2n-k}(C_1-\lambda_kI)+\delta_{2n-k}^2C_2]=0. \tag{2.10}$$

Let $\boldsymbol{v}_{2n-k}$ be a nonzero vector satisfying $[C_2^T+\delta_{2n-k}(C_1-\lambda_kI)+\delta_{2n-k}^2C_2]\boldsymbol{v}_{2n-k}=\mathbf{0}$. Letting

$$\mathbf{b}_i=\delta_k^i\boldsymbol{v}_k+\delta_k\delta_{2n-k}^i\boldsymbol{v}_{2n-k},\quad i=0,1,\ldots,n+1,$$

we conclude, via Eqs. (2.9) and (2.10), that $\mathbf{b}_i$ satisfy Eq. (2.1a) with $\lambda=\lambda_k$. Moreover,

$$\hat{\boldsymbol{I}}\mathbf{b}_1=\delta_k\hat{\boldsymbol{I}}\boldsymbol{v}_k+\hat{\boldsymbol{I}}\boldsymbol{v}_{2n-k}=\delta_k\boldsymbol{v}_{2n-k}+\boldsymbol{v}_k=\mathbf{b}_0.$$

We have used Proposition 2.1 to justify the second equality above. Similarly, $\mathbf{b}_{n+1}=\hat{\boldsymbol{I}}\mathbf{b}_n$. To see $\lambda=\lambda_k$, $1\leqslant k\leqslant n-1$, is indeed an eigenvalue of $C(\alpha,0)$ for each α, it remains to show that $\mathbf{b}_i\neq\mathbf{0}$ for some i. Using Proposition 2.1, we have that there exists an m, $1\leqslant m\leqslant 2^j$ such that $v_{km}=v_{(2n-k)(2^j-m+1)}\neq 0$. We first show that $\mathbf{b}_0\neq\mathbf{0}$. Let m be the index for which $v_{km}\neq 0$. Suppose $\mathbf{b}_0=\mathbf{0}$. Then

$$v_{km}+\delta_kv_{(2n-k)m}=0$$

and

$$v_{k(2^j-m+1)}+\delta_kv_{(2n-k)(2^j-m+1)}=v_{(2n-k)m}+\delta_kv_{km}=0.$$

And so, $v_{km}=\delta_k^2v_{km}$, a contradiction. Let λ and $\boldsymbol{v}$ be as assumed in the last assertion of theorem. Letting $\mathbf{b}_i=\boldsymbol{v}$ $(\mathbf{b}_i=(-1)^i\boldsymbol{v})$, we conclude that λ is an eigenvalue of $C(\alpha,0)$ with corresponding eigenvector $(\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_n)^T$. Thus, λ_k is an eigenvalue of $C(\alpha,0)$ for each α. □

Corollary 2.2: *Let $\delta_k=\mathrm{e}^{(\pi k/n)i}$, where k is an integer and $i=\sqrt{-1}$. Then, for each α, $\rho(C(\alpha,0))=\cup_{k=1}^{n-1}\rho(\Gamma_k)\cup\rho^S(\Gamma_0)\cup\rho^{AS}(\Gamma_n)$, where $\rho^S(A)$ $(\rho^{AS}(A))$ the set of eigenvalues of A for which their corresponding eigenvectors are symmetric (antisymmetric).*

We next consider the eigenvalues of $C(\alpha,\beta)$.

Theorem 2.3: *Let $\delta_k=\mathrm{e}^{(\pi k/n)i}$, where k is an integer and $i=\sqrt{-1}$. Then, for each α,*

$$\rho(C(\alpha,\beta))\supset\begin{cases}\displaystyle\bigcup_{k=1}^{[n/2]}\rho(\Gamma_{2k})\cup\rho^S(\Gamma_0), & n\text{ is odd}\\ \displaystyle\bigcup_{k=1}^{(n/2)-1}\rho(\Gamma_{2k})\cup\rho^S(\Gamma_0)\cup\rho^{AS}(\Gamma_n), & n\text{ is even}.\end{cases}$$

Here $[n/2]$ is the greatest integer that is less than or equal to $n/2$.

Proof: We illustrate only the case that n is even. Assume that k is such that $1 \leqslant k \leqslant n/2-1$. Let $\mathbf{b}_i = \delta_{2k}^i \boldsymbol{v}_{2k} + \delta_{2k}\delta_{2n-2k}^i \boldsymbol{v}_{2n-2k}$, we see clearly that such $\mathbf{b}_i$, $i=0,1,n,n+1$, satisfy both Neumann and periodic boundary conditions, respectively. And so

$$\mathbf{b}_0 = (1-\beta)\mathbf{b}_0 + \beta \mathbf{b}_0 = (1-\beta)\hat{I}\mathbf{b}_1 + \beta \mathbf{b}_n$$

and

$$\mathbf{b}_{n+1} = (1-\beta)\mathbf{b}_{n+1} + \beta \mathbf{b}_{n+1} = (1-\beta)\hat{I}\mathbf{b}_n + \beta \mathbf{b}_1.$$

Here, δ_{2k}, $1 \leqslant k \leqslant (n/2)-1$, are characteristic values of Eq. (2.1a), (2.1b), and (2.1c). Thus, if $\lambda \in \rho(\Gamma_{2k})$, then λ is an eigenvalue of $C(\alpha,\beta)$. The assertions for Γ_0 and Γ_n can be done similarly.□

Remark 2.4: If n is an even number, for each α and β, half of the eigenvalues of $C(\alpha,\beta)$ are independent of the choice of β. The other characteristic values of Eq. (2.1) seem to depend on β. It is of interest to find them.

III. THE SECOND EIGENCURVE OF $C(\alpha,0)$ AND $C(\alpha,1)$

We begin with considering the eigencurves of Γ_k, as given in Eq. (2.8). Clearly,

$$\Gamma_k = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \delta_{2n-k} \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ \delta_k & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{m\times m} - \frac{\alpha(2-2\cos(\pi k/n))}{m} ee^T =: D_1(k) - \alpha(k) ee^T, \tag{3.1}$$

where $m=2^j$. We next find a unitary matrix to diagonalize $D_1(k)$.

Remark 3.1: Let $(\lambda(k),\mathbf{v}(k))$ be the eigenpair of $D_1(k)$. If $e^T\boldsymbol{v}(k)=0$, then $\lambda(k)$ is also an eigenvalue of Γ_k.

Proposition 3.1: *Let*

$$\theta_{l,k} = \frac{2l\pi}{m} + \frac{k\pi}{nm}, \quad l=0,1,\ldots,m-1, \tag{3.2a}$$

$$\boldsymbol{p}_l(k) = (\mathrm{e}^{i\theta_{l,k}}, \mathrm{e}^{i2\theta_{l,k}}, \cdots, \mathrm{e}^{im\theta_{l,k}})^T, \tag{3.2b}$$

and

$$P(k) = \left(\frac{p_0(k)}{\sqrt{m}}, \ldots, \frac{p_{m-1}(k)}{\sqrt{m}}\right). \tag{3.2c}$$

(i) *Then $P(k)$ is a unitary matrix and $P^H(k)D_1(k)P(k)=\mathrm{diag}(\lambda_{0,k}\cdots\lambda_{m-1,k})$, where P^H is the conjugate transpose of P, and*

$$\lambda_{l,k} = 2\cos\theta_{l,k} - 2, \quad l=0,1,\ldots,m-. \tag{3.2d}$$

(ii) *Moreover, for $0 \leqslant k \leqslant 2n$, the eigenvalues of $D_1(k)$ are distinct if and only if $k \neq 0$, n,* or *$2n$.*

Proof: Let $\mathbf{b}=(b_1,\ldots,b_m)^T$. Writing the eigenvalue problem $D_1(k)\mathbf{b}=\lambda\mathbf{b}$ in component form, we get

$$b_{j-1} - (2+\lambda)b_j + b_{j+1} = 0, \quad j=2,3,\ldots,m-1, \tag{3.3a}$$

$$-(2+\lambda)b_1 + b_2 + \delta_{2n-k}b_m = 0, \tag{3.3b}$$

$$\delta_k b_1 + b_{m-1} - (2+\lambda) b_m = 0. \tag{3.3c}$$

Set $b_j = \delta^j$, where δ satisfies the characteristic equation $1-(2+\lambda)\delta+\delta^2=0$ of the system $D_1(k)\mathbf{b}=\lambda\mathbf{b}$. Then the boundary conditions (3.3b) and (3.3c) are reduced to

$$\delta^n = \delta_k. \tag{3.4}$$

Thus, the solutions $e^{i\theta_{l,k}}$, $l=0,1,\ldots,m-1$, of Eq. (3.4) are the candidates for the characteristic values of Eq. (3.3). Substituting $e^{i\theta_{l,k}}$ into Eq. (3.3a) and solving for λ, we see that $\lambda=\lambda_{l,k}$ are the candidates for the eigenvalues of $D_1(k)$. Clearly, $(\lambda,\mathbf{b})=(\lambda_{l,k},\boldsymbol{p}_l(k))$ satisfies $D_1(k)\mathbf{b}=\lambda\mathbf{b}$ and $\mathbf{b}=\boldsymbol{p}_l(k)\neq 0$. Thus, $\lambda=\lambda_{l,k}$ are, indeed, the eigenvalues of $D_1(k)$. To complete the proof of the proposition, it suffices to show that $P(k)$ is unitary. To this end, we need to compute $\boldsymbol{p}_l^H(k)\cdot\boldsymbol{p}_{l'}(k)$. Clearly, $\boldsymbol{p}_l^H(k)\cdot\boldsymbol{p}_l(k)=m$. Now, let $l\neq l'$, we have that

$$\boldsymbol{p}_l^H(k)\cdot\boldsymbol{p}_{l'}(k)=\sum_{j=1}^{m} e^{ij(\theta_{l,k}-\theta_{l',k})}=\sum_{j=1}^{m} e^{ij([2(l-l')/m]\pi)}=\frac{r(1-r^m)}{1-r}=0,$$

where $r=e^{i([2(l-l')/m]\pi)}$. Hence, $P(k)$ is unitary. The last assertion of the proposition is obvious. □

To prove the main results in this section, we also need the following proposition. Some assertions of the proposition are from Theorem 8.6.2 of Ref. 2.

Proposition 3.2: *Suppose* $D=\mathrm{diag}(d_1,\ldots,d_m)\in\mathbb{R}^{m\times m}$ *and that the diagonal entries satisfy* $d_1>\cdots>d_m$. *Let* $\gamma\neq 0$ *and* $\boldsymbol{z}=(z_1,\ldots,z_m)^T\in\mathbb{R}^n$. *Assume that* $(\lambda_i(\gamma),\boldsymbol{v}_i(\gamma))$ *are the eigenpairs of* $D+\gamma\boldsymbol{z}\boldsymbol{z}^T$ *with* $\lambda_1(\gamma)\geqslant\lambda(\gamma)\geqslant\cdots\geqslant\lambda_m(\gamma)$. *(i) Let* $A=\{k:1\leqslant k\leqslant m, z_k=0\}$, $A^c=\{1,\ldots,m\}-A$. *If* $k\in A$, *then* $d_k=\lambda_k$. *(ii) Assume* $\alpha>0$. *Then the following interlacing relations hold* $\lambda_1(\gamma)\geqslant d_1\geqslant\lambda_2(\gamma)\geqslant d_2\geqslant\cdots\geqslant\lambda_m(\gamma)\geqslant d_m$. *Moreover, the strict inequality holds for these indices* $i\in A^c$. *(iii) Let* $i\in A^c$, $\lambda_i(\gamma)$ *are strictly increasing in* γ *and* $\lim_{\alpha\to\infty}\lambda_i(\gamma)=\bar{\lambda}_i$ *for all* i, *where* $\bar{\lambda}_i$ *are the roots of* $g(\lambda)=\sum_{k\in A^c} z_i^2/(d_k-\lambda)$ *with* $\bar{\lambda}_i\in(d_i,d_{i-1})$. *In the case that* $1\in A^c$, $d_0=\infty$.

Proof: The proof of interlacing relations in (*ii*) and the assertion in (*i*) can be found in Theorem 8.6.2 of Ref. 2. We only prove the remaining assertions of the proposition. Rearranging $\boldsymbol{z}$ so that $\boldsymbol{z}^T=(0,0,\ldots,0,z_{i_1},\ldots,z_{i_k}):=(0,\ldots,0,\underset{\sim}{\boldsymbol{z}}^T)$, where $i_1<i_2<\cdots<i_k$ and $i_j\in A^c$, $j=1,\ldots,k$. The diagonal matrix D is rearranged accordingly. Let $D=\mathrm{diag}(D_1,D_2)$, where $D_2=\mathrm{diag}(d_{i_1},\ldots,d_{i_k})$. Following Theorem 8.6.2 of Ref. 2, we see that $\lambda_{i_j}(\gamma)$ are the roots of the scalar equation $f_\gamma(\lambda)$, where

$$f_\gamma(\lambda_{i_j}(\gamma))=1+\gamma\sum_{j=1}^{k}\frac{z_j^2}{d_{i_j}-\lambda_{i_j}(\gamma)}=0. \tag{3.5}$$

Differentiating the equation above with respect to γ, we get

$$\sum_{j=1}^{k}\frac{z_{i_j}^2}{d_{i_j}-\lambda_{i_j}(\gamma)}+\left(\gamma\sum_{j=1}^{k}\frac{z_{i_j}^2}{(d_{i_j}-\lambda_{i_k}(\gamma))^2}\right)\frac{d\lambda_{i_j}(\gamma)}{d\gamma}=0.$$

Thus,

$$\frac{d\lambda_{i_j}(\gamma)}{d\gamma}=\frac{1}{\gamma^2}\sum_{j=1}^{k}\frac{z_{i_j}^2}{(d_{i_j}-\lambda_{i_j}(\gamma))^2}>0.$$

Clearly, for each i_j, the limit of $\lambda_{i_j}(\gamma)$ as $\gamma\to\infty$ exists, say, $\bar{\lambda}_{i_j}$. Since, for $d_{i_j}<\lambda<d_{i_j-1}$,

$$\sum_{j=1}^{k}\frac{z_{i_j}^2}{d_{i_j}-\lambda_{i_j}(\gamma)}=\frac{1}{\gamma}.$$

Taking the limit as $\alpha\to\infty$ on both sides of the equation above, we get

$$\sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \bar{\lambda}_{i_j}} = 0. \tag{3.6}$$

as desired. □

We are now in the position to state the following theorems.

Theorem 3.1: *Let n and $m=2^j$ be given positive integers. For each k, $k=1,2,\ldots,n-1$, and α, we denote by $\lambda_{l,k}(\alpha)$, $l=0,1,\ldots,2^j-1$, the eigenvalues of Γ_k. For $k=1,2,\ldots,n-1$, we let $(\lambda_{l,k},u_{l,k})$, $l=0,1,\ldots,2^j-1$, be the eigenpairs of $D_1(k)$, as defined in* Eq. (3.1). *Then the following hold true.*

(i) *$\lambda_{l,k}(\alpha)$ is strictly decreasing in α, $l=0,1,\ldots,2^j-1$ and $k=1,2,\ldots,n-1$.*

(ii) *There exist $\lambda_{l,k}^*$ such that $\lim_{\alpha\to\infty}\lambda_{l,k}(\alpha)=\lambda_{l,k}^*$. Moreover, $g_k(\lambda_{l,k}^*)=0$, where*

$$g_k(\lambda) = \sum_{l=1}^{m} \frac{1}{(\lambda_{l-1,k})(\lambda_{l-1,k}+\lambda)}. \tag{3.7}$$

Proof: The first assertion of the theorem follows from Proposition 3.2 (iii). Let k be as assumed. Set, for $l=0,1,\ldots,m-1$,

$$z_{l+1} = \boldsymbol{p}_l^H(k)e = \sum_{j=1}^{m} e^{ij\theta_{l,k}} = \frac{e^{-\theta_{l,k}}(1-e^{-im\theta_{l,k}})}{1-e^{-\theta_{l,k}}} = \frac{e^{-\theta_{l,k}}(1-e^{-ik(\pi/n)})}{1-e^{-\theta_{l,k}}}.$$

Then

$$\bar{z}_{l+1}z_{l+1} = \frac{2-2\cos m\theta_{l,k}}{2-2\cos\theta_{l,k}} = \frac{2\cos(k\pi/n)-2}{\lambda_{l,k}} \neq 0. \tag{3.8}$$

Let $P(k)$ be as given in Eq. (3.2c). Then

$$-P^H(k)\cdot\Gamma_k\cdot P(k) = \mathrm{diag}(-\lambda_{0,k},\ldots,-\lambda_{m-1,k}) + \alpha(k)P_l^H(k)e(P_l^H(k)e)^H.$$

Note that if k is as assumed, it follows from Proposition 3.1(ii) that $\lambda_{l,k}$, $l=0,\ldots,m-1$, are distinct. Thus, we are in the position to apply Proposition 3.2. Specifically, by noting $A^c=\phi$, we see that $\lambda_{0,k}^*$ satisfies $g(\lambda)=0$, where

$$g(\lambda) = \sum_{l=1}^{m} \frac{1}{(\lambda_{l-1,k})(\lambda_{l-1,k}+\lambda)}.$$

We have used Eqs. (3.2d), (3.6), and (3.8), to find $g(\lambda)$. □

We next give an upper bound for $\lambda_{0,k}^*$, $k=1,2,\ldots,n-1$.

Theorem 3.2: *The following inequalities hold true:*

$$\lambda_{0,k}^* < \lambda_{0,n}, \quad k=1,2,\ldots,n-1. \tag{3.9}$$

Proof: To complete the proof of Eq. (3.9), it suffices to show that $g_k(-\lambda_{0,n})<0$. Now,

$$\begin{aligned} &g_k(-\lambda_{0,n}) \\ &= \sum_{l=1}^{m} \frac{1}{[2\cos([2(l-1)\pi/m]+(k\pi/nm))-2][2\cos([2(l-1)\pi/m]+(k\pi/nm))-2\cos(\pi/m)]} \\ &=: h(m,n,k) = h(2^j,n,k). \end{aligned} \tag{3.10}$$

We shall prove that $h(2^j,n,k)<0$ by the induction on j. For $j=1$, $h(2,n,k)=\frac{1}{2}[[1/\cos^2(k\pi/2n) -1]]<0$, $k=1,2,\ldots,n-1$. Assume $h(2^j,n,k)<0$. Here, $n\in\mathbb{N}$ and $k=1,2,\ldots,n-1$. We first note that

$$\cos\left(\frac{2(2^j+i-1)\pi}{2^{j+1}}+\frac{k\pi}{2^{j+1}n}\right)=-\cos\left(\frac{2(i-1)\pi}{2^{j+1}}+\frac{k\pi}{2^{j+1}n}\right)=:-\cos\theta_{i-1,k,j+1},\quad i=1,2,\ldots,2^j. \tag{3.11}$$

Moreover, upon using Eq. (3.11), we get that

$$\begin{aligned}
&\frac{1}{(\cos\theta_{i-1,k,j+1}-1)(\cos\theta_{i-1,k,j+1}-\cos\theta_{0,n,j+1})}+\frac{1}{(\cos\theta_{2^j+i-1,k,j+1}-1)(\cos\theta_{2^j+i-1,k,j+1}-\cos\theta_{0,n,j+1})}\\
&=\frac{1}{(\cos\theta_{i-1,k,j+1}-1)(\cos\theta_{i-1,k,j+1}-\cos\theta_{0,n,j+1})}+\frac{1}{(\cos\theta_{i-1,k,j+1}+1)(\cos\theta_{i-1,k,j+1}+\cos\theta_{0,n,j+1})}\\
&=\frac{2\cos^2\theta_{i-1,k,j+1}+2\cos\theta_{0,n,j+1}}{(\cos^2\theta_{i-1,k,j+1}-1)(\cos^2\theta_{i-1,k,j+1}-\cos^2\theta_{0,n,j+1})}\\
&=\frac{8(\cos^2\theta_{i-1,k,j+1}+\cos\theta_{0,n,j+1})}{(\cos 2\theta_{i-1,k,j+1}-1)(\cos 2\theta_{i-1,k,j+1}-\cos 2\theta_{0,n,j+1})}=\frac{2(\cos^2\theta_{i-1,k,j+1}+\cos\theta_{0,n,j+1})}{(\cos\theta_{i-1,k,j}-1)(\cos\theta_{i-1,k,j}-\cos\theta_{0,n,j})}.
\end{aligned} \tag{3.12}$$

We are now in a position to compute $h(2^{j+1},n,k)$. Using Eq. (3.12), we get that

$$\begin{aligned}
h(2^{j+1},n,k)&=\sum_{l=1}^{2^{j+1}}\frac{1}{4(\cos\theta_{l-1,k,j+1}-1)(\cos\theta_{l-1,k,j+1}-\cos\theta_{0,n,j+1})}\\
&=\sum_{l=1}^{2^j}\frac{2(\cos^2\theta_{l-1,k,j+1}+\cos\theta_{0,n,j+1})}{(\cos\theta_{l-1,k,j}-1)(\cos\theta_{i-1,k,j}-\cos\theta_{0,n,j})}\leqslant 8(\cos^2\theta_{0,k,j+1}+\cos\theta_{0,n,j+1})h(2^j,n,k).
\end{aligned} \tag{3.13}$$

We have used the facts that $\cos^2\theta_{0,k,j+1}>\cos^2\theta_{i-1,k,j+1}$, $i=2,\ldots,2^j$, and that the first term $(i=1)$ of the summation in Eq. (3.13) is negative while all the others are positive to justify the inequality in Eq. (3.13). It then follows from Eq. (3.13) that $h(2^{j+1},n,k)<0$. We just complete the proof of the theorem. □

Theorem 3.3: *Let n and j be the block and wavelet dimensions of $C(\alpha,1)$, respectively. Assume n and j are any positive integers. Let $\lambda_2(\alpha)$ be the second eigencurve of $C(\alpha,1)$. Then the following hold.*

(i) *$\lambda_2(\alpha)$ is a nonincreasing function of α.*
(ii) *If n is an even number, then $\lambda_2(\alpha)=\lambda_{0,n}$ whenever $\alpha\geqslant\alpha^*$ for some $\alpha^*>0$.*
(iii) *If n is an odd number, then $\lambda_2(\alpha)<\lambda_{0,n}$ whenever $\alpha\geqslant\bar{\alpha}$ for some $\bar{\alpha}>0$.*

Proof: We first remark that in the case of $\beta=1$, the set of the indices k's in Eq. (3.1) is $\{0,2,4,\ldots,2(n-1)\}:=I_n$. Suppose n is an even number. Then $n\in I_n$. Thus, $\delta_n=-1$, $\theta_{0,n}=\pi/m$, and $\boldsymbol{p}_0(n)=(e^{i(\pi/m)},e^{i(2\pi/m)},\ldots,e^{i\pi})^T$. Applying Proposition 3.1, we see that $\boldsymbol{p}_0(n)-\boldsymbol{p}_0^s(n)$, an antisymmetric vector, is also an eigenvector of $D_1(n)$. And so $e^T(\boldsymbol{p}_0(n)-\boldsymbol{p}_0^s(n))=0$. It then follows from Remark 3.1 that $\lambda_{0,n}$ is an eigenvalue of $\Gamma_n=D_1(n)-\rho(n)ee^T$ for all α. The first and second assertions of the theorem now follow from Theorems 3.1 and 3.2. Let n be an odd number. Then $\delta_i\cdot\delta_i\neq 1$ for any $i\in I_n$. Thus, if the pair $(\delta_i,\boldsymbol{v}_i)$ satisfy Eq. (2.3), then $\boldsymbol{v}_i\neq-\boldsymbol{v}_i^s$. Otherwise, the pair $(\delta_i,\boldsymbol{v}_i-(-\boldsymbol{v}_i)^s)=(\delta_i,\boldsymbol{v}_i+\boldsymbol{v}_i^s)$ also satisfy Eq. (2.3). This is a contradiction to the last assertion in Proposition 2.1. Thus, $\boldsymbol{v}_i^H\cdot e\neq 0$ for any $i\in I_n$. We then conclude, via Proposition 3.2 (iii) and Theorem 3.2, that the last assertion of the theorem holds. □

Remark 3.2: (i) Let the number of uncoupled (chaotic) oscillators be $N=2^jn$. If n is an odd number, then the wavelet method for controlling the coupling chaotic oscillators work even better in the sense that the critical coupling strength ϵ can be made even smaller. (ii) For n being a multiple of 4 and $j \in \mathbb{N}$, the assertions in Theorem 3.3 were first proved in Ref. 6 by a different method.

Theorem 3.4: *Let n and j be the block and wavelet dimensions of* $C(\alpha,0)$, *respectively. Assume n and j are any positive integers. Let* $\lambda_2(\alpha)$ *be the second eigencurve of* $C(\alpha,0)$. *Then for any n, there exists a* $\tilde{\alpha}$ *such that* $\lambda_2(\alpha)=\lambda_{0,n}$ *whenever* $\alpha \geqslant \tilde{\alpha}$.

Remark 3.3: For $n \in \mathbb{N}$ and $j=1$, the explicit formulas for the eigenvalues of $C(\alpha,0)$ were obtained in. Ref. 4 Such results are possible due to the fact that the dimension of the matrices in Eq. (2.4) is 2×2.

Davis, P. J., *Circulant Matrices* (Wiley, New York, 1979).

Golub, G. H. and Van Loan, C. F., *Matrix Computation* (The Hjons Hopkins University Press, Baltimore, 1989).

Hilliges, A., Mehl, C., and Mehrmann, V., Proceedings of the Fourth European Congress on Computational Methods in Applied Sciences and Engineering (EC-COMAS), Jyväskylä, Finland, 2004, pp. 24.7-28.7.

Juang, J. and Li, C.-L., “Eigenvalue problem and their application to the wavelet method of chaotic control,” J. Math. Phys. **47**, 072704.1–072704.16 (2006).

Pecora, L. M. and Carroll, T. L., “Master stability functions for synchronized coupled systems,” Phys. Rev. Lett. **80**, 2109–2112 (1998).

Shieh, S. F., Wang, Y. Q., Wei, G. W., and Lai, C.-H., “Mathematical analysis of the wavelet method of chaos control,” J. Math. Phys. **47**, 082701.1–082701.10 (2006).

Wei, G. W., Zhan, M., and Lai, C.-H., “Tailoring wavelets for chaos control,” Phys. Rev. Lett. **89**, 284103.1–284103.4 (2002).