Extending the discussion on coverage intervals and statistical coverage intervals

2006 Metrologia 43 L43

(http://iopscience.iop.org/0026-1394/43/6/N03)

INSTITUTE OF PHYSICS PUBLISHING METROLOGIA

Metrologia **43** (2006) L43–L44 doi:10.1088/0026-1394/43/6/N03

SHORT COMMUNICATION

Extending the discussion on coverage intervals and statistical coverage intervals

Lin-An Chen and Hui-Nien Hung

Institute of Statistics, National Chiao Tung University, Hsinchu, Taiwan

Received 17 July 2006
Published 20 October 2006
Online at stacks.iop.org/Met/43/L43

Abstract
Willink (2004 *Metrologia* **41** L5–6) is concerned that, in the society of metrology, there is potential for confusion between coverage interval and statistical coverage interval and he makes a precise interpretaion of these two terms. We further clarify that the confidence of a coverage interval is actually a statistical coverage interval.

Quite often a scientist is less interested in estimating parameters and more concerned about gaining a notion about where individual observations or measurements might fall. There are two attempts to determine bounds from this notion. The coverage interval (also called the reference interval in clinical chemistry) refers to population-based reference values obtained from a well-defined (normal or healthy people) group of reference individuals. This is an interval with two confidence limits that covers the individual values in the population in some probabilistic sense. One method of establishing a bound on single values in the population is to determine a confidence interval for a fixed proportion of the measurements. A common problem in clinical chemistry is to determine the coverage interval for a particular test. This reference interval represents a region of the distribution of normal or healthy people. Once the reference interval is determined, any patient with a suspected disease may have the test and the result of the test can be compared with the reference interval. A result outside the reference interval may then be taken as confirmation of the disease.

The statistical coverage intervals are statistical intervals that contain (or cover) at least a proportion p of a population with a stated confidence $1-\alpha$. In recent years, the International Organization for Standardization (ISO 3534-2 1993, ISO 16269-6 2005) and several expert scientists have advocated calculation of the coverage interval and the statistical coverage interval. Due to the potential for confusion between the terms 'coverage interval' and 'statistical coverage interval', Willink (2004) provided a clear interpretation of the roles that these two terms play in the literature on metrology. The interpretation has been taken further by Perruchet (2004) in describing the differences between the terms 'confidence interval', 'coverage interval' and 'statistical coverage interval'. In this paper, we want to clarify the relation between 'coverage interval' and 'statistical coverage interval'.

Suppose that a quantity (called a random variable in statistics) X has a distribution with probability density function $f(x)$, and generally this function involves an unknown parameter θ. A $100p\%$ coverage interval for this quantity is any interval (a, b) such that $p = P(X \in (a, b))$. On the other hand, Wilks (1941) introduces a p-content statistical coverage interval with confidence $1-\alpha$ as any random interval (T_1, T_2) that satisfies

$$P\{P_X[(T_1, T_2)] \geqslant p\} \geqslant 1-\alpha. \qquad (1)$$

There is a vast literature (see, for example, Wald (1943), Paulson (1943), Guttman (1970) and, for a recent review, Patel (1986)) that introduces techniques in constructing a p-content statistical coverage interval with confidence $1-\alpha$. However, this variety of techniques generally involves approximation or simulation for the construction of a statistical coverage interval. Thus, the connection between a statistical coverage interval and a coverage interval has been unclear.

Goodman and Madansky (1962) implicitly applied the concept that a $100(1-\alpha)\%$ confidence interval (T_1, T_2) of a $100p\%$ coverage interval (a, b) in the sense that

$$P\{T_1 \leqslant a < b \leqslant T_2\} = 1-\alpha$$

is a p-content statistical coverage interval with confidence $1-\alpha$. Here we formally prove that any statistical coverage interval is a confidence interval of a coverage interval with some confidence. Suppose that (a, b) is a $100p\%$ coverage interval and we have a sample $X_1, \ldots, X_n$ from a distribution with distribution F_X. Let (T_1, T_2) be a $100(1-\alpha)\%$ confidence interval of (a, b). The following statements will help us clarify some relations between the coverage interval and the statistical coverage interval:

$$\begin{aligned} P\{P_X[(T_1, T_2)] \geqslant p\} \\ = P\{F_X(T_2) - F_X(T_1) \geqslant p\} \\ = P\{F_X(T_2) - F_X(T_1) \geqslant F_X(b) - F_X(a)\} \\ \geqslant P\{F_X(T_1) \leqslant F_X(a) < F_X(b) \leqslant F_X(T_2)\} \\ \geqslant P\{T_1 \leqslant a < b \leqslant T_2\} \end{aligned} \quad (2)$$

as F_X is non-decreasing. □

Points of interest include the following:

(a) If we choose a random interval (T_1, T_2) that is a $100(1-\alpha)\%$ confidence interval of (a, b), this indicates that $P\{P_X[(T_1, T_2)] \geqslant p\} \geqslant 1-\alpha$ such that (T_1, T_2) is a p-content statistical coverage interval at confidence $1-\alpha$.

(b) Suppose that (T_1, T_2) is only a random interval. Then it is still a p-content statistical coverage interval at confidence $1-\alpha = P\{T_1 \leqslant a < b \leqslant T_2\}$. The fact that every random interval is also a statistical coverage interval is noteworthy.

This connection contributes the construction of a statistical coverage interval through the use of a coverage interval. Choosing any a p-content coverage interval, then any random interval which is a $100(1-\alpha)\%$ confidence interval of it is a p-content statistical coverage interval at confidence coefficient $1-\alpha$.

Perhaps the most significant observation is that in (*a*). Suppose we have a quantity X that has a normal distribution $N(\mu, \sigma^2)$ where σ is a known constant. With a sample $X_1, ..., X_n$, let $\hat{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Consider the p-content coverage interval

$$(\mu - z_{\frac{1+p}{2}}\sigma < \mu + z_{\frac{1+p}{2}}\sigma). \quad (3)$$

The following shows that

$$\left(\hat{X} - z_{\frac{1+p}{2}}\sigma - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \hat{X} + z_{\frac{1+p}{2}}\sigma + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) \quad (4)$$

is a $100(1-\alpha)\%$ confidence interval of the coverage interval in (3):

$$\begin{aligned} P\bigg(\hat{X} - z_{\frac{1+p}{2}}\sigma - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leqslant \mu - z_{\frac{1+p}{2}}\sigma \\ < \mu + z_{\frac{1+p}{2}}\sigma \leqslant \hat{X} + z_{\frac{1+p}{2}}\sigma + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\bigg) \\ = P\bigg(-z_{\frac{1+p}{2}}\sigma - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leqslant \hat{X} - \mu - z_{\frac{1+p}{2}}\sigma \\ < \hat{X} - \mu + z_{\frac{1+p}{2}}\sigma \leqslant z_{\frac{1+p}{2}}\sigma + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\bigg) \\ = P\left(-z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leqslant \hat{X} - \mu \leqslant z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) \\ = P\left(-z_{1-\frac{\alpha}{2}} \leqslant \frac{\hat{X} - \mu}{\sigma/\sqrt{n}} \leqslant z_{1-\frac{\alpha}{2}}\right) \\ = 1 - \alpha. \end{aligned}$$

The random interval of (4) is a γ-content statistical coverage interval at confidence $1-\alpha$ because it is a $100(1-\alpha)\%$ confidence interval of a p coverage interval.

We further assume that both parameters μ and σ are unknown. Wald and Wolfowitz (1946) first introduced the normal tolerance interval of the form

$$(\bar{X} - kS, \bar{X} + kS). \quad (5)$$

As noted in Guttman (1970), it is exceedingly complicated to derive k to meet the requirement (1) for preassigned p and $1-\alpha$. This leads to the approximation techniques by Wald and Wolfowitz (1946), Weissberg and Beatty (1960) and Odeh and Owen (1980). Let $T = t(r, c)$ represent a random variable with a non-central t distribution with r degrees of freedom and a non-centrality parameter c. We also let $t_\alpha(r, c)$ satisfy $\alpha = P(T \geqslant t_\alpha(r, c))$. We may show that

$$\left(\bar{X} - t_{1-\frac{\alpha}{2}}\left(n-1, \sqrt{n}z_{\frac{1+p}{2}}\right)\frac{S}{\sqrt{n}}, \bar{X} + t_{1-\frac{\alpha}{2}}\left(n-1, \sqrt{n}z_{\frac{1+p}{2}}\right)\frac{S}{\sqrt{n}}\right)$$

is a $100(1-\alpha)\%$ confidence interval for the coverage interval of (3) and then it is also a p-content tolerance interval at confidence $1-\alpha$. Moreover, from (2), an interval of (5) for any $k > 0$ is also a p-content tolerance interval at some confidence.

Acknowledgments

We are grateful to the editor and one referee for comments which greatly improved the presentation of this paper. This research work was partially supported by the National Science Council of Taiwan Grant No NSC 95-2118-M-009-007, NSC 94-2118-M-009-003 and MOE ATU Programme.

References

Goodman L A and Madansky A 1962 Parameter-free and nonparametric tolerance limits: the exponential case *Technometrics* **4** 75–95

Guttman I 1970 *Statistical Tolerance Regions: Classical and Bayesian* (London: Griffin)

ISO 3534-2 1993 *Statistics—Vocabulary and Symbols—Part 2: Statistical Quality Control*

ISO 16269-6 2005 *Statistical Interpretation of Data—Part 6: Determination of Statistical Tolerance Intervals*

Odeh R E and Owen D B 1980 *Tables for Normal Tolerance Limits, Sampling Plans, and Screening* (New York: Marcel Dekker)

Patel J K 1986 Tolerance limits— a review *Commun. Stat.—Theory Methods* **15** 2719–62

Paulson E 1943 A note on tolerance limits *Ann. Math. Stat.* **14** 90–3

Perruchet C 2004 Some differences between the applied statistical approach for measurement uncertainty theory and the traditional approach in metrology and testing *Advanced Mathematical and Computational Tools in Metrolgy VI* ed P Ciarlini and F Pavese (Singapore: World Scientific)

Wald A 1943 An extension of Wilks' method for setting tolerance limits *Ann. Math. Stat.* **14** 45–55

Wald A and Wolfowitz J 1946 Tolerance limits for a normal distribution *Ann. Math. Stat.* **17** 208–15

Weissberg A and Beatty G H 1960 Tables of tolerance-limits factors for normal distribution *Technometrics* **4** 483–500

Wilks S S 1941 Determination of sample sizes for setting tolerance limits *Ann. Math. Stat.* **12** 91–6

Willink R 2004 Coverage intervals and statistical coverage intervals *Metrologia* **41** L5–6