Journal of Applied Statistics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/cjas20

Measuring Process Performance Based on Expected Loss with Asymmetric Tolerances

W. L. Pearn [a] , Y. C. Chang [b] & Chien-Wei Wu [c]

[a] Department of Industrial Engineering & Management , National Chiao Tung University , Taiwan

[b] Department of Industrial Engineering & Management , Ching Yun University , Taiwan

[c] Department of Industrial Engineering and Systems Management , Feng Chia University , Taiwan

Published online: 11 Jan 2007.

To cite this article: W. L. Pearn , Y. C. Chang & Chien-Wei Wu (2006) Measuring Process Performance Based on Expected Loss with Asymmetric Tolerances, Journal of Applied Statistics, 33:10, 1105-1120, DOI: 10.1080/02664760600746871

To link to this article: http://dx.doi.org/10.1080/02664760600746871

Journal of Applied Statistics
Vol. 33, No. 10, 1105–1120, December 2006

Routledge Taylor & Francis Group

Measuring Process Performance Based on Expected Loss with Asymmetric Tolerances

W.L. PEARN,* Y.C. CHANG** & CHIEN-WEI WU†

*Department of Industrial Engineering & Management, National Chiao Tung University, Taiwan, **Department of Industrial Engineering & Management, Ching Yun University, Taiwan, †Department of Industrial Engineering and Systems Management, Feng Chia University, Taiwan

ABSTRACT *By approaching capability from the point of view of process loss similar to C_{pm}, Johnson (1992) provided the expected relative loss L_e to consider the proximity of the target value. Putting the loss in relative terms, a user needs only to specify the target and the distance from the target at which the product would have zero worth to quantify the process loss. Tsui (1997) expressed the index L_e as $L_e = L_{ot} + L_{pe}$, which provides an uncontaminated separation between information concerning the process relative off-target loss (L_{ot}) and the process relative inconsistency loss (L_{pe}). Unfortunately, the index L_e inconsistently measures process capability in many cases, particularly for processes with asymmetric tolerances, and thus reflects process potential and performance inaccurately. In this paper, we consider a generalization, which we refer to as L''_e, to deal with processes with asymmetric tolerances. The generalization is shown to be superior to the original index L_e. In the cases of symmetric tolerances, the new generalization of process loss indices L''_e, L''_{ot} and L''_{pe} reduces to the original index L_e, L_{ot}, and L_{pe}, respectively. We investigate the statistical properties of a natural estimator of L''_e L''_{ot} and L''_{pe} when the underlying process is normally distributed. We obtained the rth moment, expected value, and the variance of the natural estimator $\hat{L}''_e$, $\hat{L}''_{ot}$, and $\hat{L}''_{pe}$. We also analyzed the bias and the mean squared error in each case. The new generalization L''_e measures process loss more accurately than the original index L_e.*

KEY WORDS: Asymmetric tolerances, bias, mean squared error, process capability indices, process loss indices

Introduction

In the last two decades, numerous process capability indices have been proposed to provide a unitless measure on whether a process is capable of reproducing items meeting the quality requirement preset by the product designer. Those indices are effective tools for process capability analysis and quality assurance. The formulas of those indices are easy to understand and straightforward to apply. Kane (1986) considered two basic indices C_p and C_{pk}, and investigated some properties of their estimators. Boyles (1991)

Correspondence Address: Y. C. Chang, Department of Industrial Engineering & Management, Ching Yun University, Taiwan. E-mail: ycchang@cyu.edu.tw

0266-4763 Print/1360-0532 Online/06/101105–16
DOI: 10.1080/02664760600746871

noted that C_{pk} is a yield-based index. Unfortunately, the designs of C_p and C_{pk} are independent of the target value T, which can fail to account for process targeting (the ability to cluster around the target).

For this reason, Chan *et al.* (1988) developed the index C_{pm}, which takes the process targeting into consideration. We note that the index C_{pm} is not originally designed to provide an exact measure on the number of non-conforming items. But, C_{pm} includes the process departure $(\mu - T)^2$ (rather than 6σ alone) in the denominator of the definition to reflect the degree of process targeting. Actually, the denominator of the index C_{pm} is the expected quadratic loss, which is closely related to process departure. For on target processes, the value of C_{pm} index reaches its maximum, implying that the process capability runs under the desired condition. We know that C_{pm} is a larger-the-better index and hence, small values of C_{pm} may be contributed by high expected loss resulting in poorer process capability. Therefore, the index C_{pm} is considered to be more sensitive than C_p and C_{pk} in reflecting process targeting. In the literature, another well-known larger-the-better index C_a introduced by Pearn *et al.* (1998) describes process capability in terms of process location only and provides a quantified measure of the amount that a process is off-target. Pearn *et al.* (1992) investigated the index C_{pmk}, which takes into account the process yield as well as the process loss. Those five well-known indices have been defined explicitly as:

$$C_p = \frac{USL - LSL}{6\sigma}, \ C_a = 1 - \frac{|\mu - T|}{d} \tag{1}$$

$$C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\}, \ C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} \tag{2}$$

$$C_{pmk} = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\} \tag{3}$$

where μ is the process mean, σ is the process standard deviation, USL is the upper specification limit, LSL is the lower specification limit, T is the target value and $d = (USL - LSL)/2$ is half the length of the specification interval.

Loss Function

The quadratic loss function is considered to distinguish the products that fall inside the specification limits by increasing the penalty as the departure from the target increases. To provide information on the variation about the target value, several possibilities have been tried. Hsiang & Taguchi (1985) first introduced the loss function approach to quality improvement with focuses on the reduction of variation around the target value. This concept pays attention to the product designer's original intent; that is, critical values at target lead to maximum product performance. In the development of this concept, Hsiang & Taguchi noted that any value x of a particular product's critical characteristic X incurs some monetary loss, which is denoted by $L(x)$, to the customer and/or society as it moves away from the target value. This loss function is defined as

$$L(x) = k(x - T)^2 \tag{4}$$

where k is some positive constant. Therefore, no loss is incurred when the characteristic is 'perfect' (i.e. $x = T$) and $L(x) = 0$, and increasing losses are incurred as the measured value moves away from the target. While the reasons for using a continuous loss function such as the loss function (4) are understood, obtaining precise estimates for the parameter k turns out to be uneasy.

Loss Index

The quadratic loss function itself does not provide any relationship between the specification limits and the unknown parameter k. To address these issues, Johnson (1992) developed the relative expected loss L_e for symmetric tolerances cases, which provides unitless measures on process performance for industrial applications. Using L_e for measuring process performance, estimation for parameter k becomes unnecessary. Tsui (1997) rewrote L_e as $L_e = L_{ot} + L_{pe}$, providing an uncontaminated separation between information concerning the relative off-target squared (L_{ot}) and the potential relative expected loss (L_{pe}). The index L_e is defined as the ratio of the expected quadratic loss and the square of half specification width:

$$L_e = \int_{-\infty}^{\infty} \left[\frac{(x-T)^2}{d^2}\right] \mathrm{d}F(x) = \left(\frac{\mu - T}{d}\right)^2 + \left(\frac{\sigma}{d}\right)^2 \qquad (5)$$

where once again μ is the process mean, σ is the process standard deviation, $d = (USL - LSL)/2$ is the half specification width, USL and LSL are the upper and the lower specification limits, T is the target value, and $F(x)$ is the cumulative distribution function of the measured characteristic. If we define $L_{ot} = [(\mu - T)/d]^2$ and $L_{pe} = (\sigma/d)^2$, then L_e can be expressed as $L_e = L_{ot} + L_{pe}$. We note that L_{ot} measures the relative process departure, which has been referred to as the process relative off-target loss index. On the other hand, L_{pe} measures process variation relative to the specification tolerance, which has been referred to as the process relative inconsistency loss index. The distributional and some statistical properties of estimators of these process loss indices (L_e, L_{ot}, L_{pe}) have been investigated in Pearn *et al.* (2004a).

We note that the mathematical relationship $L_e = (3C_{pm})^{-2}$, $L_{ot} = (1 - C_a)^2$, and $L_{pe} = (3C_p)^{-2}$ can be established. The greatest advantage of using L_e over C_{pm} is that the estimator of the former has better statistical properties than that of the latter, as the former does not involve a reciprocal transformation of process mean and variance.

Most research in quality assurance literature has a focus on cases in which the manufacturing tolerance is symmetric. A process is said to have a symmetric tolerance if the target value T is set to be the midpoint of the specification interval [LSL, USL], i.e. $T = m = (USL + LSL)/2$. Investigations on symmetric cases can be found in Kane (1986), Chan *et al.* (1988), Choi & Owen (1990), Boyles (1991), Pearn *et al.* (1992, 2004b, 2005), Vännman (1995), Vännman & Kotz (1995), and Spiring (1997). Although cases with symmetric tolerances are common in practical situations, cases with asymmetric tolerances also may occur in the manufacturing industry.

From the customer's point of view, asymmetric tolerances reflect that deviations from the target are less tolerable in one direction than in the other (see Boyles, 1994, Vännman (1997), and Wu & Tang (1998)). Usually they are not related to the shape of the supplier's process distribution. However, asymmetric tolerances can also arise in situations where the tolerances are symmetric to begin with, but the process distribution is skewed or follows a non-normal distribution. Dealing with this, the data have been transformed to achieve approximate normality, as shown by Chou *et al.* (1998) who have used Johnson curves to transform non-normal process data. Excluding Boyles (1994), Vännman

(1997), Pearn *et al*. (1998, 1999), and Chen *et al*. (1999), unfortunately, there has been comparatively little research published on cases with asymmetric tolerances.

In the asymmetric tolerances situation, using L_e would be risky and probably the results obtained misleading. Consider the following example with asymmetric tolerance (LSL, T, USL), where $T = (3USL + LSL)/4$ and $\sigma = d/3$. For processes A and B with $\mu_A = T - d/2 = m$ (the midpoint of the specification interval) and $\mu_B = T + d/2 = USL$. Both processes have the index value $L_e = 13/36$ and equal degree of clustering around the target, that is, $|\mu - T| = d/2$ for both processes A and B. However, the expected proportions non-conforming are approximately 0.27% for process A and 50% for process B. Obviously, L_e inconsistently measures process capability in this case, and is inappropriate for those with asymmetric tolerances. This problem calls for a need to generalize the index L_e to cover situations with asymmetric tolerances so that appropriate use of the process loss index can be continued.

A Generalization L''_e

In this section, we consider a new generalization of L_e to handle processes with asymmetric tolerances. We refer to this generalization as L''_e, which may be defined as follows:

$$L''_e = \left(\frac{A}{d^*}\right)^2 + \left(\frac{\sigma}{d^*}\right)^2 \tag{6}$$

where $A = \max\{(\mu - T) \cdot d/D_u, (T - \mu) \cdot d/D_l\}$, $D_u = USL - T$, $D_l = T - LSL$, $d^* = \min\{D_u, D_l\}$. We denoted $(A/d^*)^2$ by L''_{ot}, $(\sigma/d^*)^2$ by L''_{pe} and hence $L''_e = L''_{ot} + L''_{pe}$. Obviously, if the tolerances are symmetric ($T = m$), then $A = |\mu - T|$, $D_u = D_l = d$, and $d^* = d = (USL - LSL)/2$. Accordingly, the new generalization defined in equation (6) reduces to the original index L_e as in equation (5).

In developing the new generalization, we have replaced the term $|\mu - T|$ in L_e by A. This ensures that the new index obtains the minimal value at $\mu = T$ regardless of whether the tolerances are symmetric or asymmetric. By substituting the half specification width d by d^*, L''_e is sensitive to target value T and obtains a larger value when T is far from $m = (USL + LSL)/2$. For processes with asymmetric tolerances, the corresponding loss function is also asymmetric in T. We take into account the asymmetry of the loss function by adding the factors (d/D_u) and $(-d/D_l)$ to $(\mu - T)$ according to whether μ is greater or less than T. The factors (d/D_u) and $(-d/D_l)$ ensure that if processes A and B with $\mu_A > T$ and $\mu_B < T$ satisfy $(\mu_A - T)/D_u = (T - \mu_B)/D_l$, then the index values given to A and B are the same. In addition, it is easy to verify that if the process is on target, then $L''_e = L''_{pe} = (\sigma/d^*)^2$ is the minimum value.

Comparisons of L''_e and L_e

To examine some basic difference between L''_e and L_e, in the following, the generalization L''_e is compared with the original index L_e. We consider the following example with manufacturing specifications $LSL = T - 1.50d$, $USL = T + 0.50d$. Table 1 displays the values of L_e, L_{ot}, L_{pe}, L''_e, L''_{ot}, and L''_{pe} for various values of μ, with fixed $\sigma = d/4$. And these index values of L''_e, L''_{ot}, L_e, L_{ot}, L''_{pe} and L_{pe} versus μ are plotted in Figure 1 (from bottom to top in plot). We note that L_e and L_{ot} have the minimum value at the target. But their values at the upper specification limit (say, when the expected proportion non-conforming is 50%) are equal to those at the midpoint m. See Table 1, the values of L_e and L_{ot} are 0.313 and 0.250,

Table 1. A comparison among $L_e, L_{ot}, L_{pe}, L''_e, L''_{ot}$, and L''_{pe} for various values of μ with fixed $\sigma = d/4$

μ	L_e	L_{ot}	L_{pe}	L''_e	L''_{ot}	L''_{pe}
LSL	2.313	2.250	0.063	4.063	4.000	0.25
$\tilde{T} - 1.45d$	2.165	2.103	0.063	3.800	3.738	0.25
$\tilde{T} - 1.40d$	2.023	1.960	0.063	3.547	3.484	0.25
$\tilde{T} - 1.35d$	1.885	1.823	0.063	3.303	3.240	0.25
$\tilde{T} - 1.30d$	1.753	1.690	0.063	3.067	3.004	0.25
$\tilde{T} - 1.25d$	1.625	1.563	0.063	2.840	2.778	0.25
$\tilde{T} - 1.20d$	1.503	1.440	0.063	2.623	2.560	0.25
$\tilde{T} - 1.15d$	1.385	1.323	0.063	2.414	2.351	0.25
$\tilde{T} - 1.10d$	1.273	1.210	0.063	2.214	2.151	0.25
$\tilde{T} - 1.05d$	1.165	1.103	0.063	2.023	1.960	0.25
$\tilde{T} - 1.00d$	1.063	1.000	0.063	1.840	1.778	0.25
$\tilde{T} - 0.95d$	0.965	0.903	0.063	1.667	1.604	0.25
$\tilde{T} - 0.90d$	0.872	0.810	0.063	1.503	1.440	0.25
$\tilde{T} - 0.85d$	0.785	0.722	0.063	1.347	1.284	0.25
$\tilde{T} - 0.80d$	0.702	0.640	0.063	1.200	1.138	0.25
$\tilde{T} - 0.75d$	0.625	0.562	0.063	1.063	1.000	0.25
$\tilde{T} - 0.70d$	0.552	0.490	0.063	0.934	0.871	0.25
$\tilde{T} - 0.65d$	0.485	0.422	0.063	0.814	0.751	0.25
$\tilde{T} - 0.60d$	0.422	0.360	0.063	0.702	0.640	0.25
$\tilde{T} - 0.55d$	0.365	0.302	0.063	0.600	0.538	0.25
$\tilde{T} - 0.50d$	0.313	0.250	0.063	0.507	0.444	0.25
$\tilde{T} - 0.45d$	0.265	0.203	0.063	0.423	0.360	0.25
$\tilde{T} - 0.40d$	0.223	0.160	0.063	0.347	0.284	0.25
$\tilde{T} - 0.35d$	0.185	0.123	0.063	0.280	0.218	0.25
$\tilde{T} - 0.30d$	0.153	0.090	0.063	0.223	0.160	0.25
$\tilde{T} - 0.25d$	0.125	0.063	0.063	0.174	0.111	0.25
$\tilde{T} - 0.20d$	0.103	0.040	0.063	0.134	0.071	0.25
$\tilde{T} - 0.15d$	0.085	0.023	0.063	0.103	0.040	0.25
$\tilde{T} - 0.10d$	0.073	0.010	0.063	0.080	0.018	0.25
$\tilde{T} - 0.05d$	0.065	0.003	0.063	0.067	0.004	0.25
T	0.063	0.000	0.063	0.063	0.000	0.25
$T + 0.05d$	0.065	0.003	0.063	0.103	0.040	0.25
$T + 1.0d$	0.073	0.010	0.063	0.223	0.160	0.25
$T + 1.5d$	0.085	0.023	0.063	0.423	0.360	0.25
$T + 2.0d$	0.103	0.040	0.063	0.703	0.640	0.25
$T + 0.25d$	0.125	0.063	0.063	1.063	1.000	0.25
$T + 0.30d$	0.153	0.090	0.063	1.503	1.440	0.25
$T + 0.35d$	0.185	0.123	0.063	2.023	1.960	0.25
$T + 0.40d$	0.223	0.160	0.063	2.623	2.560	0.25
$T + 0.45d$	0.265	0.203	0.063	3.303	3.240	0.25
USL	0.313	0.250	0.063	4.063	4.000	0.25

respectively, either for $\mu = USL = T + 0.5d$ or $\mu = m = T - 0.5d$. These indices, being symmetric about the target value, do not take into account the location of the process mean.

On the other hand, the new index L''_e we proposed takes into account the location of the process mean for asymmetric tolerances. Thus, given two processes A and B with $\mu_A > T$ and $\mu_B < T$ satisfying $(\mu_A - T) = (T - \mu_B)$ and $D_l > D_u$, B has significantly higher yield

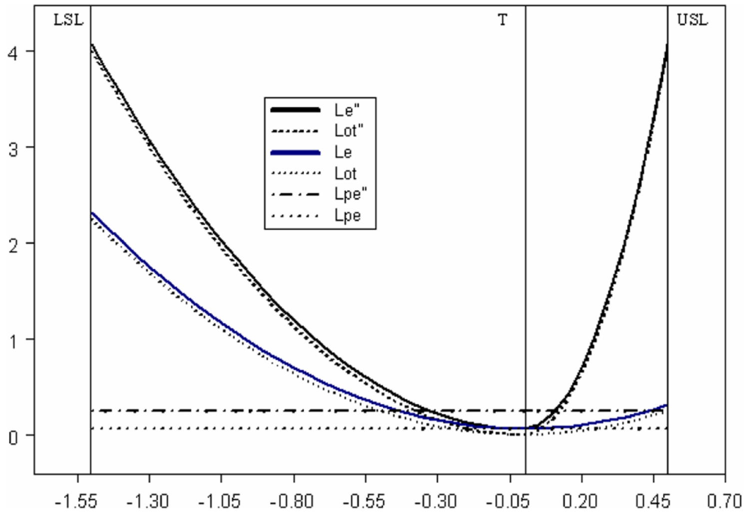

Figure 1. Plots of L''_e, L''_{ot}, L_e, L_{ot}, L''_{pe} and L_{pe} versus μ (top to bottom in plot)

that A, so the index value of the new generalization L''_e of A is greater than the index value of B. We note that L''_e is of the smaller-the-better type as one may expect, since process loss is smaller the better. An illustrative example is $L''_e = 4.063$ for $\mu_A = T + 0.5d$ and $L''_e = 0.507$ for $\mu_B = T - 0.5d$ in Table 1. These two process means have equal departure from the target, but B has significantly higher yield than A, so intuitively A should score higher than B. Therefore, we conclude that the proposed new generalization L''_e is superior to the original index L_e.

Estimation of the Process Loss Indices

We consider the case when the characteristic of the underlying process is normally distributed. Let $X_1, X_2, \ldots, X_n$ be a random sample drawing from a normal distribution with mean μ and variance σ^2 measuring the characteristic under investigation.

Estimation of L''_e

To estimate the new generalization of loss index L''_e, we consider the natural estimator which can be defined as follows:

$$\hat{L}''_e = \left(\frac{\hat{A}}{d^*}\right)^2 + \left(\frac{S_n}{d^*}\right)^2 \tag{7}$$

where $\hat{A} = \max\{(\bar{X} - T) \cdot d/D_u, (T - \bar{X}) \cdot d/D_l\}$, the mean μ is estimated by the sample mean, $\bar{X} = \sum_{i=1}^{n} X_i/n$, and the variance σ^2 by $S_n^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/n$, the maximum likelihood estimator. For the case where the production tolerance is symmetric, $\hat{A}$ may be simplified as $|\bar{X} - T|$. Therefore, the estimator $\hat{L}''_e$ reduces to $\hat{L}_e = (n^{-1}d^{-2}) \sum_{i=1}^{n} (X_i - T)^2$, the natural estimator of L_e discuss by Johnson (1992). Consequently, we may view the estimator $\hat{L}''_e$ as a direct extension of $\hat{L}_e$. Now we focus on some statistical properties of this natural estimator $\hat{L}''_e$.

Proposition 1

Let $X_1, X_2, \ldots, X_n$ be a random sample form $N(\mu, \sigma^2)$, $Y = \max^2\{d_u Z, -d_l Z\}$, where $Z = \sqrt{n}(\bar{X} - T)/\sigma$ is distributed as $N(\delta, 1)$ and $\delta = \sqrt{n}(\mu - T)/\sigma$. Then Y has a weighted non-central chi-square distribution with one degree of freedom (d.f.) and non-centrality parameter δ. The probability density function of Y is:

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j)\Gamma\left(\frac{1+j}{2}\right)\left[(d_u^{-2}) f_{Y_j}(y_u) + (-1)^j (d_l^{-2}) f_{Y_j}(y_l)\right] \tag{8}$$

where $P_j = (\sqrt{2}\delta)^j/(j!)$, $d_u = d/D_u, d_l = d/D_l$, $y_u = (y/d_u^2)$, $y_l = (y/d_l^2)$, $\lambda = \delta^2$, $\delta = \sqrt{n}(\mu - T)/\sigma$, and Y_j is distributed as χ^2_{1+j}. For the case when $d_u = d_l = 1$, this formula reduces to the probability density function of a non-central chi-square distribution with one d.f. and non-centrality parameter δ.

Proof

Based on the notation of Proposition 1, the cumulative distribution function of Y is:

$$F_Y(y) = \int_{-\sqrt{y}/d_l}^{\sqrt{y}/d_u} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(z-\delta)^2}{2}\right] \mathrm{d}z \tag{9}$$

Then

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \left[\frac{(d_u^{-2})}{2\sqrt{y_u}} e^{-y_u/2} e^{\delta\sqrt{y_u}} + \frac{(d_l^{-2})}{2\sqrt{y_l}} e^{-y_l/2} e^{\delta\sqrt{y_l}}\right] \tag{10}$$

Expanding e^y in power series, we obtain

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j)\Gamma\left(\frac{1+j}{2}\right)\left[(d_u^{-2}) f_{Yj}(y_u) + (-1)^j (d_l^{-2}) f_{Y_j}(y_l)\right] \tag{11}$$

This completes the proof.

Proposition 2

The rth moment about zero of $\hat{L}_e''$ is:

$$E(\hat{L}_e'')^r = \left(\frac{\sigma^2}{nd^{*2}}\right)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \cdot 2^r \cdot \Gamma\left(\frac{n+j}{2} + r\right) \cdot \left\{\sum_{i=0}^{r} \binom{r}{i} \frac{\Gamma([(1+j)/2] + i)}{\Gamma([(n+j)/2] + i)} \cdot \left[(d_u^2 - 1)^i + (-1)^j (d_l^2 - 1)^i\right]\right\} \tag{12}$$

Proof

For the sake of deriving the rth moment of $\hat{L}_e''$, the following notation is introduced:

1. $B = \sigma^2/(nd^{*2})$,
2. $K = (nS_n^2)/(\sigma^2)$,
3. $Y = \max^2\{d_u Z, -d_l Z\}$.

Assume that the process is normally distributed with mean μ and variance σ^2, then K is distributed as χ^2_{n-1}, Y is distributed as a weighted non-central chi-square distribution with one d.f. and non-centrality parameter δ (see Proposition 1). In the notation the estimator $\hat{L}''_e$ can be represented as $\hat{L}''_e = B(Y+K)$. Thus, the rth moment of $\hat{L}''_e$ is $E(\hat{L}''_e)^r = (B^r)E(Y+K)^r$. Since Y is distributed as a weighted non-central chi-square distribution with one d.f. and non-centrality parameter δ, we have

$$E(\hat{L}''_e)^r = (B^r)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\Gamma\left(\frac{1+j}{2}\right)\{E[K+(d_u^2)Y_j]^r + (-1)^j E[K+(d_l^2)Y_j]^r\} \quad (13)$$

where Y_j is distributed as χ^2_{1+j}. Let $H_j = Y_j/(K+Y_j)$ and $W_j = K+Y_j$. Under the assumption of normality, H_j and W_j are independent random variables (see, for instance, Johnson & Kotz, 1970), and H_j is distributed according to $\beta((1+j)/2, (n-1)/2)$. Furthermore, W_j has a chi-square distribution with $(n+j)$ degrees of freedom. Therefore

$$E(K+vY_j)^r = E(W_j)^r E(1+(v-1)H_j)^r \quad (14)$$

$$E(W_j)^r = \frac{2^r\Gamma((n+j)/2+r)}{\Gamma((n+j)/2)} \quad (15)$$

and

$$E(1+(v-1)H_j)^r = \sum_{i=0}^{r}\binom{r}{i}(v-1)^i\frac{\Gamma([(1+j)/2]+i)\Gamma((n+1)/2)}{\Gamma([(n+j)/2]+i)\Gamma((1+j)/2)} \quad (16)$$

Combining the results, we can obtain the rth moment of $\hat{L}''_e$ as stated in Proposition 2. This completes the proof.

From the derivations given in Propositions 1–2, we have the rth moment of $\hat{L}''_e$ as:

$$E(\hat{L}''_e)^r = \left(\frac{\sigma^2}{nd^{*2}}\right)^r\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)2^r\Gamma\left(\frac{n+j}{2}+r\right)$$
$$\left\{\sum_{i=0}^{r}\binom{r}{i}\frac{\Gamma([(1+j)/2]+i)}{\Gamma([(n+j)/2]+i)}\cdot\left[(d_u^2-1)^i + (-1)^j(d_l^2-1)^i\right]\right\} \quad (17)$$

where $P_j = (\sqrt{2}\delta)^j/(j!)$, $d_u = d/D_u$, $d_l = d/D_l$, $y_u = (y/d_u^2)$, $y_l = (y/d_l^2)$, $\lambda = \delta^2$, $\delta = \sqrt{n}(\mu - T)/\sigma$. In particular, the expected value and the variance of $\hat{L}''_e$ can be obtained as follows:

$$E(\hat{L}_e'') = \left(\frac{(n-1)\sigma^2}{nd^{*2}}\right) + \left(\frac{\sigma^2}{nd^{*2}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \times \sum_{j=0}^{\infty}(P_j)\cdot\Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot\left[d_u^2+(-1)^j d_l^2\right] \tag{18}$$

$$Var(\hat{L}_e'') = \left(\frac{\sigma^4}{n^2d^{*4}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \times \sum_{j=0}^{\infty}(P_j)\cdot\Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot(3+j)\cdot\left[d_u^4+(-1)^j d_l^4\right] - \left\{\left(\frac{\sigma^2}{nd^{*2}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\cdot\Gamma\left(\frac{1+j}{2}\right)\cdot(1+j)\cdot\left[d_u^2+(-1)^j d_l^2\right]\right\}^2 + \left(\frac{2(n-1)\sigma^4}{n^2d^{*4}}\right) \tag{19}$$

We note that the estimator $\hat{L}_e''$ is biased. The bias of $\hat{L}_e''$ may be computed as $Bias(\hat{L}_e'') = E(\hat{L}_e'') - L_e''$, and the mean squared error, which is more relevant to the analysis of process quality, is $MSE(\hat{L}_e'') = Var(\hat{L}_e'') + [Bias(\hat{L}_e'')]^2$. To explore the behavior of the estimator $\hat{L}_e''$, the bias and the mean squared error were calculated using computer software for various values of $a = (\mu - T)/\sigma$, $b = \sigma/d^*$, d_u, d_l, and sample size n. For example, Table 2 displays the bias and the MSE of $\hat{L}_e''$ for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)100$.

The results in Table 2 indicate that as $|a|$ increases, the bias and the mean squared error also increase. Further, as the sample size increases, the bias and the mean squared error decrease. The bias of $\hat{L}_e''$ versus n are plotted in Figure 2 with $a = -1.0, 0, 1.0$ (from bottom to top in the plot). Figure 3 plots the MSE of $\hat{L}_e''$ versus n with $a = 0, -1.0, 1.0$ (from bottom to top in the plot).

Table 2. The Bias and MSE of $\hat{L}_e''$ for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)100$

	$a=-1.0$		$a=-0.5$		$a=0$		$a=0.5$		$a=1.0$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	−0.0305	0.3835	−0.0289	0.2384	0.0128	0.2113	0.0546	0.4772	0.0562	1.2086
20	−0.0153	0.1941	−0.0152	0.1217	0.0064	0.1028	0.0280	0.2302	0.0281	0.5963
30	−0.0102	0.1299	−0.0102	0.0817	0.0043	0.0679	0.0187	0.1516	0.0187	0.3957
40	−0.0076	0.0976	−0.0076	0.0615	0.0032	0.0507	0.0141	0.1130	0.0141	0.2961
50	−0.0061	0.0782	−0.0061	0.0493	0.0026	0.0405	0.0112	0.0901	0.0112	0.2366
60	−0.0051	0.0652	−0.0051	0.0411	0.0021	0.0336	0.0094	0.0749	0.0094	0.1970
70	−0.0044	0.0559	−0.0044	0.0353	0.0018	0.0288	0.0080	0.0641	0.0080	0.1687
80	−0.0038	0.0490	−0.0038	0.0309	0.0016	0.0252	0.0070	0.0560	0.0070	0.1476
90	−0.0034	0.0435	−0.0034	0.0275	0.0014	0.0224	0.0062	0.0497	0.0062	0.1311
100	−0.0031	0.0392	−0.0031	0.0247	0.0013	0.0201	0.0056	0.0447	0.0056	0.1180

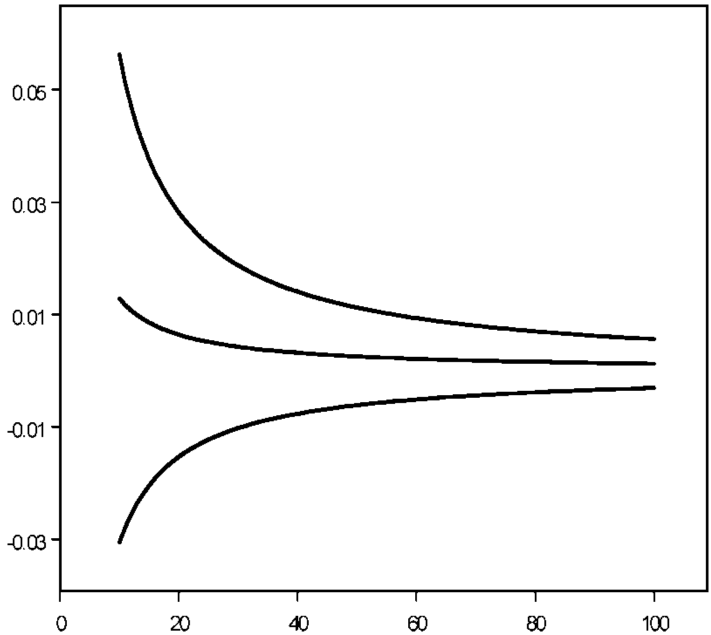

Figure 2. Plots of bias of $\hat{L}_e''$ versus n with $a = -1.0$, 0, 1.0 (bottom to top in plot)

Table 3 displays the relative error and relative bias of $\hat{L}_e''$, defined as $[MSE_R(\hat{L}_e'')]^{1/2} = \{E[(\hat{L}_e'' - L_e'')/L_e'']^2\}^{1/2}$ and $Bias_R(\hat{L}_e'') = [E(\hat{L}_e'') - L_e'']/L_e''$, respectively, for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)100$. The square root of the relative mean squared error is a direct measurement, which presents the expected relative error of the estimation from the true L_e''. For example, with $n = 100$, $a = 0.5$ we have

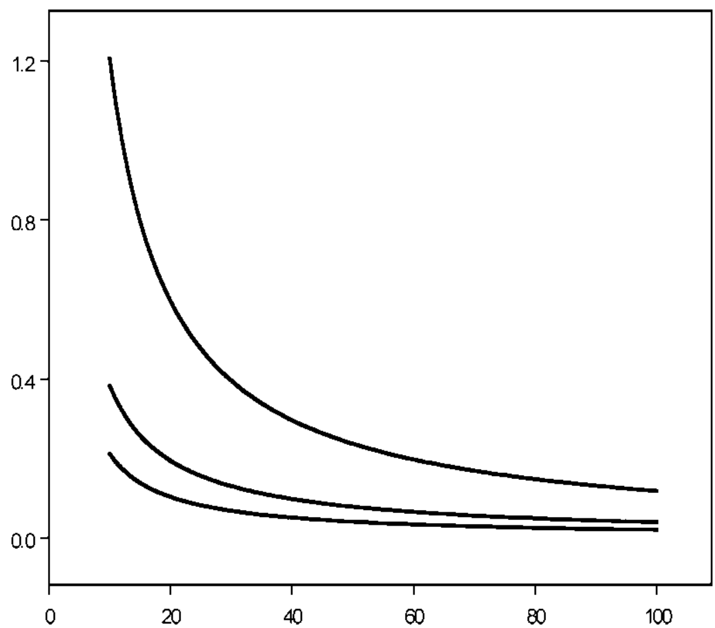

Figure 3. Plots of MSE of $\hat{L}_e''$ versus n with $a = 0$, -1.0, 1.0 (bottom to top in plot)

Table 3. The Bias_R and $\sqrt{\text{MSE}_R}$ of $\hat{L}_e''$ for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)100$

	$a=-1.0$		$a=-0.5$		$a=0$		$a=0.5$		$a=1.0$	
n	Bias_R	$\sqrt{\text{MSE}_R}$	Bias_R	$\sqrt{\text{MSE}_R}$	Bias_R	$\sqrt{\text{MSE}_R}$	Bias_R	$\sqrt{\text{MSE}_R}$	Bias_R	$\sqrt{\text{MSE}_R}$
10	−0.0180	0.3655	−0.0247	0.4160	0.0128	0.4597	0.0393	0.4968	0.0219	0.4290
20	−0.0090	0.2600	−0.0129	0.2973	0.0064	0.3207	0.0201	0.3450	0.0110	0.3013
30	−0.0060	0.2127	−0.0087	0.2435	0.0043	0.2606	0.0135	0.2800	0.0073	0.2455
40	−0.0045	0.1844	−0.0065	0.2112	0.0032	0.2252	0.0101	0.2418	0.0055	0.2124
50	−0.0036	0.1650	−0.0052	0.1891	0.0026	0.2011	0.0081	0.2159	0.0044	0.1898
60	−0.0030	0.1507	−0.0043	0.1728	0.0021	0.1834	0.0067	0.1968	0.0037	0.1732
70	−0.0026	0.1396	−0.0037	0.1600	0.0018	0.1697	0.0058	0.1821	0.0031	0.1603
80	−0.0023	0.1306	−0.0033	0.1497	0.0016	0.1587	0.0051	0.1702	0.0027	0.1499
90	−0.0020	0.1231	−0.0029	0.1412	0.0014	0.1495	0.0045	0.1604	0.0024	0.1413
100	−0.0018	0.1168	−0.0026	0.1340	0.0013	0.1418	0.0040	0.1521	0.0022	0.1340

$[MSE_R(\hat{L}_e'')]^{1/2} = 0.1521$. Thus, for $n = 100$, $a = 0.5$ we expect that the average error of $\hat{L}_e''$ would be no greater than 15.21% of the true L_e''. On the other hand, the relative bias $Bias_R(\hat{L}_e'')$ is investigated to analyze the accuracy of the natural estimator $\hat{L}_e''$. For example, with $n = 100$, $a = 0.5$ we has $Bias_R(\hat{L}_e'') = 0.0040$, that is, 0.4% relative bias for the true L_e''.

From the case where the production tolerance is symmetric, since $d_u = d_l = 1$, $\hat{L}_e''$ is an unbiased estimator of L_e'', or equivalently, $Bias(\hat{L}_e'') = 0$. The unbiased estimator depends only on the complete, sufficient statistic $(\bar{X}, S_n^2)$ for (μ, σ^2), by the Lehmann-Scheffé Theorem we know that $\hat{L}_e''$ is a uniformly minimum variance unbiased estimator (UMVUE) of L_e''. In addition, we have the rth moment of $\hat{L}_e''$ for symmetric tolerance as

$$E(\hat{L}_e'')^r = E(\hat{L}_e)^r = \left(\frac{\sigma^2}{nd^2}\right)^r \sum_{j=0}^{\infty} \left(\frac{e^{-\lambda/2}(\lambda/2)^j}{j!}\right)\left(\frac{2^r\Gamma((n/2)+j+r)}{\Gamma((n/2)+j)}\right) \tag{20}$$

Estimation of L_{ot}''

To estimate the new off-target loss index $L_{ot}'' = (A)^2/(d^*)^2$, we consider the natural estimator $\hat{L}_{ot}'' = (\hat{A})^2/(d^*)^2$. The rth moment about zero for $\hat{L}_{ot}''$ is:

$$E(\hat{L}_{ot}'')^r = \left(\frac{\sigma^2}{nd^{*2}}\right)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) 2^r \Gamma\left(\frac{1+j}{2}+r\right)\left[d_u^{2r} + (-1)^j d_l^{2r}\right] \tag{21}$$

In particular, the expected value and the variance of $\hat{L}_{ot}''$ can be obtained as follows:

$$E(\hat{L}_{ot}'') = \left(\frac{\sigma^2}{nd^{*2}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \cdot \Gamma\left(\frac{1+j}{2}\right) \cdot (1+j) \cdot \left[d_u^2 + (-1)^j d_l^2\right] \tag{22}$$

$$Var(\hat{L}_{ot}'') = \left(\frac{\sigma^4}{n^2 d^{*4}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \cdot \Gamma\left(\frac{1+j}{2}\right) \cdot (1+j) \cdot (3+j) \cdot \left[d_u^4 + (-1)^j d_l^4\right]$$
$$- \left\{\left(\frac{\sigma^2}{nd^{*2}}\right)\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \cdot \Gamma\left(\frac{1+j}{2}\right) \cdot (1+j) \cdot \left[d_u^2 + (-1)^j d_l^2\right]\right\}^2 \tag{23}$$

We note that the estimator $\hat{L}''_{ot}$ is biased. The bias of $\hat{L}''_{ot}$ may be computed as $Bias(\hat{L}''_{ot}) = E(\hat{L}''_{ot}) - L''_{ot}$, and the mean squared error, which is more relevant to the analysis of process quality, is $MSE(\hat{L}''_{ot}) = Var(\hat{L}''_{ot}) + [Bias(\hat{L}''_{ot})]^2$. Table 4 displays the bias and the MSE of $\hat{L}''_{ot}$ for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)50$. The results in Table 4 indicate that as $|a|$ increases, the mean squared error also increases. Further, as the sample size increases, the bias and the mean squared error decrease. The bias of $\hat{L}''_{ot}$ versus n are plotted in Figure 4 with $a = -1.0$, 0, 1.0 (from bottom to top in the plot). And Figure 5 plots the MSE of $\hat{L}''_{ot}$ versus n with $a = 0$, -1.0, 1.0 (from bottom to top in the plot).

Table 5 displays the relative error and relative bias of $\hat{L}''_{ot}$, defined as $[MSE_R(\hat{L}''_{ot})]^{1/2} = \{E[(\hat{L}''_{ot} - L''_{ot})/L''_{ot}]^2\}^{1/2}$ and $Bias_R(\hat{L}''_{ot}) = [E(\hat{L}''_{ot}) - L''_{ot}]/L''_{ot}$, respectively, for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)100$. The square root of the relative mean squared error is a direct measurement, which presents the expected relative error of the estimation from the true L''_{ot}. For example, with $n = 100$, $a = 0.5$ we have $[MSE_R(\hat{L}''_{ot})]^{1/2} = 0.4060$. Thus, for $n = 100$, $a = 0.5$ we expect that the average error of L''_{ot} would be no greater than 40.6% of the true L''_{ot}. On the other hand, the relative bias $Bias_R(\hat{L}''_{ot})$ is investigated to analyze the accuracy of the natural estimator L''_{ot}. For example, with $n = 100$, $a = 0.5$ we have $Bias_R(\hat{L}''_{ot}) = 0.0400$, that is, 4% relative bias for the true L''_{ot}.

For the case when the production tolerance is symmetric, $\hat{A}$ may be simplified as $|\bar{X} - T|$ and the estimator $\hat{L}''_{ot}$ reduces to $\hat{L}_{ot} = (\bar{X} - T)^2/(d)^2$, which is the maximum likelihood estimator (MLE) of L_{ot}. This is because that $\bar{X}$ is the MLE of μ, then by the invariance property of MLE the result follows. Thus, we have the rth moment of $\hat{L}''_{ot}$ for symmetric tolerance as

$$E(\hat{L}''_{ot})^r = E(\hat{L}_{ot})^r = \left(\frac{\sigma^2}{nd^2}\right)^r \sum_{j=0}^{\infty} \left(\frac{e^{-\lambda/2}(\lambda/2)^j}{j!}\right)\left(\frac{2^r\Gamma((1/2)+j+r)}{\Gamma((1/2)+j)}\right) \tag{24}$$

Estimation of L''_{pe}

The index L''_{pe} reflects the process inconsistency loss, and its natural estimator can be defined as $\hat{L}''_{pe} = S^2_{n-1}/d^{*2}$, where $S^2_{n-1} = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. This estimator is

Table 4. The Bias and MSE of $\hat{L}''_{ot}$ for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)100$

	$a = -1.0$		$a = -0.5$		$a = 0$		$a = 0.5$		$a = 1.0$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.0695	0.2074	0.0711	0.0626	0.1128	0.0439	0.1546	0.3181	0.1562	1.0498
20	0.0347	0.1001	0.0348	0.0277	0.0564	0.0110	0.0780	0.1405	0.0781	0.5066
30	0.0231	0.0659	0.0232	0.0177	0.0376	0.0049	0.0521	0.0895	0.0521	0.3337
40	0.0174	0.0491	0.0174	0.0130	0.0282	0.0027	0.0391	0.0656	0.0391	0.2487
50	0.0139	0.0392	0.0139	0.0102	0.0226	0.0018	0.0312	0.0518	0.0312	0.1982
60	0.0116	0.0326	0.0116	0.0084	0.0188	0.0012	0.0260	0.0427	0.0260	0.1648
70	0.0099	0.0279	0.0099	0.0072	0.0161	0.0009	0.0223	0.0364	0.0223	0.1410
80	0.0087	0.0243	0.0087	0.0063	0.0141	0.0007	0.0195	0.0317	0.0195	0.1232
90	0.0077	0.0216	0.0077	0.0055	0.0125	0.0005	0.0174	0.0280	0.0174	0.1094
100	0.0069	0.0194	0.0069	0.0050	0.0113	0.0004	0.0156	0.0251	0.0156	0.0984

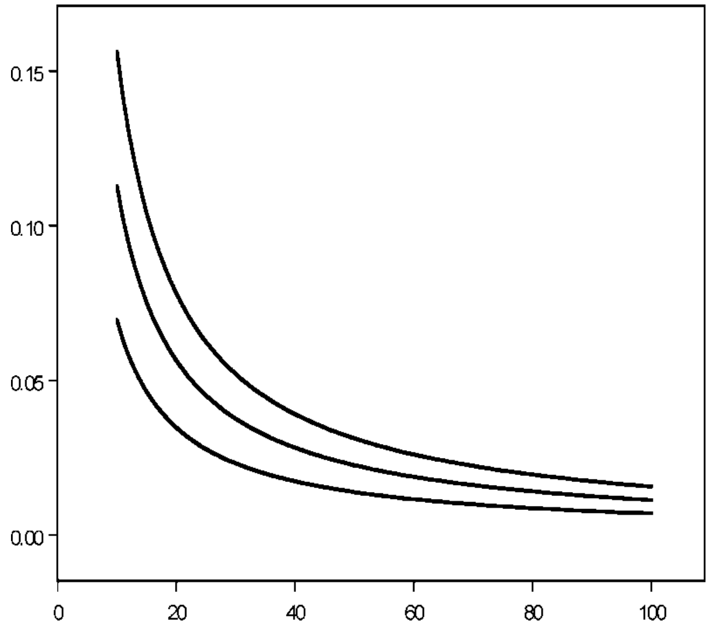

Figure 4. Plots of bias of $\hat{L}''_{ot}$ versus n with $a = -1.0$, 0, 1.0 (bottom to top in plot)

unbiased and depends only on the complete, sufficient statistic S_{n-1}^2 for σ^2. By the Lehmann–Scheffé Theorem we know that $\hat{L}''_{pe}$ is a uniformly minimum variance unbiased estimator (UMVUE) of L''_{pe}. On the assumption of normality, $\hat{L}''_{pe}$ is distributed as $\sigma^2/[(n-1)d^{*2}]$ times a chi-square variable with $(n-1)$ degrees of freedom. The rth

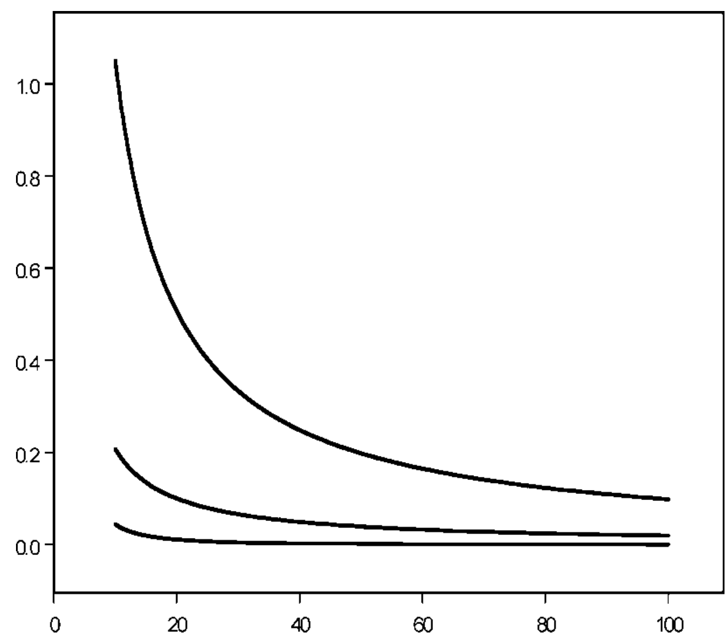

Figure 5. Plots of MSE of $\hat{L}''_{ot}$ versus n with $a = 0$, -1.0, 1.0 (bottom to top in plot)

Table 5. The Bias_R and $\sqrt{\text{MSE}_R}$ of $\hat{L}''_{ot}$ for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)100$

	$a = -1.0$		$a = -0.5$		$a = 0$		$a = 0.5$		$a = 1.0$	
n	Bias_R	$\sqrt{\text{MSE}_R}$	Bias_R	$\sqrt{\text{MSE}_R}$	Bias_R	$\sqrt{\text{MSE}_R}$	Bias_R	$\sqrt{\text{MSE}_R}$	Bias_R	$\sqrt{\text{MSE}_R}$
10	0.1000	0.6557	0.4093	1.4414	—	—	0.3959	1.4440	0.1000	0.6558
20	0.0500	0.4555	0.2007	0.9587	—	—	0.1997	0.9594	0.0500	0.4555
30	0.0333	0.3697	0.1334	0.7658	—	—	0.1333	0.7660	0.0333	0.3697
40	0.0250	0.3192	0.1000	0.6557	—	—	0.1000	0.6558	0.0250	0.3192
50	0.0200	0.2850	0.0800	0.5824	—	—	0.0800	0.5824	0.0200	0.2850
60	0.0167	0.2598	0.0667	0.5291	—	—	0.0667	0.5292	0.0167	0.2598
70	0.0143	0.2403	0.0571	0.4882	—	—	0.0571	0.4882	0.0143	0.2403
80	0.0125	0.2247	0.0500	0.4555	—	—	0.0500	0.4555	0.0125	0.2247
90	0.0111	0.2117	0.0444	0.4286	—	—	0.0444	0.4286	0.0111	0.2117
100	0.0100	0.2009	0.0400	0.4060	—	—	0.0400	0.4060	0.0100	0.2017

moment about zero for $\hat{L}''_{pe}$ is:

$$E(\hat{L}''_{pe})^r = \left(\frac{\sigma^2}{(n-1)d^{*2}}\right)^r \cdot \left(\frac{2^r\Gamma((n-1)/2+r)}{\Gamma((n-1)/2)}\right) \tag{25}$$

In particular, the expected value and the variance of $\hat{L}''_{pe}$ can be obtained as follows:

$$E(\hat{L}''_{pe}) = L''_{pe} \tag{26}$$

and

$$Var(\hat{L}''_{pe}) = \left(\frac{2\sigma^4}{(n-1)d^{*4}}\right) \tag{27}$$

For the case when the production tolerance is symmetric, d^* may be simplified as d and the estimator $\hat{L}''_{pe}$ reduces to $\hat{L}_{pe} = S^2_{n-1}/d^2$, which is a uniformly minimum variance unbiased estimator (UMVUE) of L_{pe}. The rth moment of $\hat{L}''_{ot}$ for symmetric tolerance becomes

$$E(\hat{L}''_{pe})^r = E(\hat{L}_{pe})^r = \left(\frac{\sigma^2}{(n-1)d^2}\right)^r \cdot \left(\frac{2^r\Gamma((n-1)/2+r)}{\Gamma((n-1)/2)}\right) \tag{28}$$

An Application Example

We consider a case study for the purpose of illustration. Consider the following example involving a factory manufacturing high density Light Emitting Diodes (LEDs). Application of LEDs is expanding rapidly since high intensity LEDs of a wide range of colors have been recently developed and become available, which enabled application of LEDs in a wide variety of areas such as instrument cluster lighting, color displays, traffic signals, roadway signs (barricade lights), airport signaling and lighting, automotive backlighting in dashboards and switches, telecommunication indicators and backlighting

Table 6. The 30 consecutive days $\hat{L}''_e$

0.644	0.817	0.942	0.691	0.754	0.458
0.485	0.610	0.707	0.577	0.732	0.512
0.683	0.764	0.870	0.653	0.574	0.623
0.551	0.690	0.582	0.744	0.658	0.491
0.725	0.673	0.455	0.649	0.971	0.521

in telephones and fax backlighting for audio and video equipment, backlighting in office equipment, indoor and outdoor message boards, flat backlight for LCDs, switches and symbols, illumination purposes, alternatives to incandescent lamps, etc.

LEDs are peculiar light sources very different from lamps in terms of physical size, flux level, spectrum, and spatial intensity distribution. And LED technology provides a number of benefits over incandescent bulbs. With a focus on the critical characteristic, the luminous intensity of LED sources, we examine a particular LED product model. The upper and the lower specification limits of luminous intensity are set to $USL = 100$ mcd, $LSL = 50$ mcd, and the target value is set to $T = 80$ mcd. We note that it is an asymmetric tolerance case.

Now we consider a particular type of LED manufacturing process. Historical data based on routine process monitoring shows that the process is under statistical control and the process distribution is justified and is shown to be fairly close to the normal distribution. A sample data collection procedure is implemented in the factory on a daily basis to monitor/control process quality. The factory production resource and schedule allows the data collection plan be implemented with a sample size $n \leq 40$. A simple approach to determine the true value (rather than an upper confidence bound) of L''_e is to perform the sampling on a routine basis consecutively for a number of, say 30, days. The calculated values of single-day $\hat{L}''_e$ for 30 consecutive days are displayed in Table 6. The average $\hat{L}''_e$ value for the 30 days is obtained as $E(\hat{L}''_e) = 0.660$. Checking Table 3, the values of $Bias_R(\hat{L}''_e)$ is between -0.0065 and 0.0101. Therefore, the true value of L''_e can be determined as $0.66/(1\text{-}0.65\%) = 0.6643$. The error of the approximation becomes negligibly small over time.

Conclusion

Johnson (1992) introduced the relative expected loss $L_e = L_{ot} + L_{pe}$, which provides an uncontaminated separation between information concerning the relative off-target loss (L_{ot}) and the relative inconsistency loss (L_{pe}). The definitions of L_{ot} and L_{pe} are the square of the ratio of the deviation of mean from the target and the half specification width, and the ratio of the process variance and the square of the half specification width, respectively. Both of them have clear interpretations on process loss. In this paper, we considered a new generalization L''_e, a modification of the process loss index L_e, to handle processes with asymmetric tolerances. The new generalization L''_e not only takes the proximity of the target value into consideration, but also takes into account the asymmetry of the specification limits. We also investigated the statistical properties of the natural estimator of process loss indices L''_e, L''_{ot}, and L''_{pe} assuming that the process is normally distributed. We obtained the rth moment, expected value, and the variance of the natural estimator $\hat{L}''_e$, $\hat{L}''_{ot}$, and $\hat{L}''_{pe}$, respectively. We also analyzed the bias and the MSE. The new generalization L''_e measures process loss more accurately than the original index L_e. Therefore, the new generalization L_e'' should be recommended for in-plant applications.

References

Boyles, R. A. (1991) The Taguchi capability index, *Journal of Quality Technology*, 23, pp. 17–26.

Boyles, R. A. (1994) Process capability with asymmetric tolerances, *Communications in Statistics-Simulation and Computation*, 23, pp. 615–643.

Chan, L. K., Cheng, S. W. & Spiring, F. A. (1988) A new measure of process capability: C_{pm}, *Journal of Quality Technology*, 20(3), pp. 162–175.

Chen, K. S., Pearn, W. L. & Lin, P. C. (1999) A new generalization of C_{pm} for processes with asymmetric tolerances, *International Journal of Reliability, Quality and Safety Engineering*, 6, pp. 383–398.

Choi, B. C. & Owen, D. B. (1990) A study of a new capability index, *Communications in Statistics: Theory and Methods*, 19, pp. 1231–1245.

Chou, Y. M., Polansky, A. M. & Mason, R. L. (1998) Transforming non-normal data to normality in statistical process control, *Journal of Quality Technology*, 30, pp. 133–141.

Hsiang, T. C. & Taguchi, G. (1985) A tutorial on quality control and assurance – the Taguchi methods, *ASA Annual Meeting*, Las Vegas, Nevada.

Johnson, T. (1992) The relationship of C_{pm} to squared error loss, *Journal of Quality Technology*, 24, pp. 211–215.

Johnson, N. L. & Kotz, S. (1970) *Distributions in Statistics. Continuous Univariate Distributions* -1 (New York: Wiley).

Kane, V. E. (1986) Process capability indices, *Journal of Quality Technology*, 18(1), pp. 41–52.

Pearn, W. L. & Chen, K. S. (1998) New generalization of process capability index C_{pk}, *Journal of Applied Statistics*, 25, pp. 801–810.

Pearn, W. L., Kotz, S. & Johnson, N. L. (1992) Distributional and inferential properties of process capability indices, *Journal of Quality Technology*, 24, pp. 216–231.

Pearn, W. L., Lin, G. H. & Chen, K. S. (1998) Distributional and inferential properties of the process accuracy and process precision indices, *Communications in Statistics: Theory and Methods*, 27(4), pp. 985–1000.

Pearn, W. L., Lin, P. C. & Chen, K. S. (1999) On the generalizations of the capability index C_{pmk} for asymmetric tolerances, *Far East Journal of Theoretical Statistics*, 3, pp. 49–66.

Pearn, W. L., Chang, Y. C. & Wu, C. W. (2004a) Differential and inferential properties of the process loss indices, *Journal of Applied Statistics*, 31(9), pp. 1115–1135.

Pearn, W. L., Chang, Y. C. & Wu, C. W. (2004b) Quality-yield measure for production processes with very low fraction defective, *International Journal of Production Research*, 42(23), pp. 4909–4925.

Pearn, W. L., Chang, Y. C. & Wu, C. W. (2005) Bootstrap approach for estimating process quality yield with application to light emitting diodes, *International Journal of Advanced Manufacturing Technology*, 25(5–6), pp. 560–570.

Spiring, F. A. (1997) A unifying approach to process capability indices, *Journal of Quality Technology*, 29, pp. 49–58.

Tsui, K. L. (1997) Interpretation of process capability indices and some alternatives, *Quality Engineering*, 9, pp. 587–596.

Vännman, K. (1995) A unified approach to capability indices, *Statistica Sinica*, 5, pp. 805–820.

Vännman, K. (1997) A general class of capability indices in the case of asymmetric tolerances, *Communications in Statistics: Theory and Methods*, 26, pp. 2049–2072.

Vännman, K. & Kotz, S. (1995) A superstructure of capability indices distributional properties and implications, *Scandinavian Journal of Statistics*, 22, pp. 477–491.

Wu, C. C. & Tang, G. R. (1998) Tolerance design for products with asymmetric quality losses, *International Journal of Production Research*, 36, pp. 2529–2541.