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Research

# *One-sided Process Capability Assessment in the Presence of Measurement Errors*

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***In the manufacturing industry, many product characteristics are of one-sided specifications. The well-known process capability indices $C_{PU}$ and $C_{PL}$ are often used to measure process performance. Most capability research works have assumed no measurement errors. Unfortunately, such an assumption is not realistic even if the measurement is conducted using highly sophisticated advanced measuring instruments. Therefore, conclusions drawn regarding process capability are not reliable. In this paper, we consider the estimation and testing of $C_{PU}$ and $C_{PL}$ with the presence of measurement errors, to obtain adjusted lower confidence bounds and critical values for true process capability, which can be used to determine whether the factory processes meet the capability requirement when the measurement errors are unavoidable. Copyright © 2005 John Wiley & Sons, Ltd.***

KEY WORDS: confidence bound; critical value; gauge measurement errors; process capability indices; one-sided specification

## 1. INTRODUCTION

Process capability indices have been widely used in the manufacturing industry to provide quantitative measures on process potential and performance (see Borges and Ho[1], Chen and Hsu[2], Chen and Chen[3], Ding[4], Hoffman[5], Kotz and Johnson[6], Nahar *et al.*[7], Noorossana[8], Pearn and Lin[9], Perakis and Xekalaki[10], Spiring *et al.*[11], Wu and Pearn[12], Pearn and Wu[13], Zimmer *et al.*[14] and many others). In the manufacturing industry, many product characteristics are of one-sided specifications. The process capability indices $C_{PU}$ and $C_{PL}$ are often used to measure process performance (Kane[15]), and have been defined as

$$C_{PU} = \frac{USL - \mu}{3\sigma}, \quad C_{PL} = \frac{\mu - LSL}{3\sigma}$$

where $LSL$ is the lower specification limit, $USL$ is the upper specification limit, $\mu$ is the process mean and $\sigma$ is the process standard deviation. If the quality characteristic of the manufacturing process is normally distributed, the process yield $\rho\%$ can be expressed by $\rho\% = \Phi(3C_I)$, where $\Phi$ is the cumulative distribution function of the standard normal distribution, and $C_I = C_{PU}$ or $C_{PL}$. It is clear that the relationship between the index $C_I$ and process yield is one-to-one. Thus, the index $C_I$ provides an exact measure of process yield. Table I displays

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Table I. The corresponding process yield and NCPPM for $C_{\mathrm{I}}$

| $C_{\mathrm{I}}$ | Process yield $\rho\%$ | NCPPM |
|---|---|---|
| 1.00 | 0.998 650 1020 | 1350 |
| 1.33 | 0.999 966 9634 | 33 |
| 1.50 | 0.999 996 6023 | 3.4 |
| 1.67 | 0.999 999 7278 | 0.272 |
| 2.00 | 0.999 999 9990 | 0.001 |

some commonly used capability values of $C_{\mathrm{I}}$, the corresponding process yield and non-conformity units in parts per million (NCPPM).

In current practice, a process is called 'inadequate' if $C_{\mathrm{I}} < 1.00$, 'marginally capable' if $1.00 \leq C_{\mathrm{I}} < 1.33$, 'satisfactory' if $1.33 \leq C_{\mathrm{I}} < 1.50$, 'excellent' if $1.50 \leq C_{\mathrm{I}} < 2.00$ and 'super' if $2.00 \leq C_{\mathrm{I}}$. Montgomery[16] recommended some minimum quality requirements on $C_{\mathrm{I}}$. For existing processes, the capability must be no less than 1.25, and for new processes, the capability must be no less than 1.45. For existing processes on safety, strength, or critical parameters, the capability must be no less than 1.45, and for new processes on safety, strength, or critical parameters, the capability must be no less than 1.60. Using the index $C_{\mathrm{I}}$, the practitioners can evaluate their process capability and make decisions.

In practice, no measurement is free from errors even if the measurement is conducted using highly sophisticated advanced measuring instruments. Any variation in the measurement process has a direct impact on capability estimation and judgment about the true process capability. Clearly, conclusions about process capability based on the empirical index values are not reliable. To analyze the effects of measurement errors on true process capability, Mittag[17] and Levinson[18] discussed the behavior of theoretical process capability indices in the presence of measurement errors. Bordignon and Scagliarini[19] performed some statistical analysis in estimating $C_{\mathrm{P}}$ and $C_{\mathrm{P}K}$.

In this paper, we consider the one-sided process capability indices $C_{\mathrm{PU}}$ and $C_{\mathrm{PL}}$. We first develop the relationship between the true process capability and the empirical process capability. We then show that the empirical confidence bound of capability estimation severely underestimates the true capability. When performing capability testing, both the $\alpha$-risk and the power of the test decrease substantially with the presence of measurement errors. To estimate the capability accurately and improve the power with given $\alpha$-risk, adjusted confidence bounds and critical values are provided. An application example on TFT-LCDs (thin-film-transistor liquid crystal displays) is also presented.

## 2. EMPIRICAL PROCESS CAPABILITY

Suppose that $X \sim N(\mu, \sigma^2)$ is the relevant quality characteristic of a manufacturing process, and $M \sim N(0, \sigma_M^2)$ is a random variable describing the measurement errors. Assuming that $X$ and $M$ are mutually independent, instead of measuring the true variable $X$, the empirical data $Y \sim N(\mu_Y = \mu, \sigma_Y^2 = \sigma^2 + \sigma_M^2)$ is observed and measured. The empirical process capability indices $C_{\mathrm{PU}}^Y$ and $C_{\mathrm{PL}}^Y$ are obtained after substituting $\sigma_Y$ for $\sigma$. We first define the degree of error contamination $\tau$ (see Mittag[17]),

$$\tau = \frac{\sigma_M}{\sigma}$$

to obtain the following relationship between the empirical process capability index $C_{\mathrm{I}}^Y$ and the true process capability index $C_{\mathrm{I}}$:

$$\frac{C_{\mathrm{I}}^Y}{C_{\mathrm{I}}} = \frac{1}{\sqrt{1+\tau^2}}$$

Table II. Process capability with $\tau = 0(0.1)1.0$ for various $C_{\mathrm{I}}$

| | $\tau$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_{\mathrm{I}}$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.50 | 0.50 | 0.49 | 0.48 | 0.46 | 0.45 | 0.43 | 0.41 | 0.39 | 0.37 | 0.35 |
| 1.00 | 1.00 | 0.98 | 0.96 | 0.93 | 0.89 | 0.86 | 0.82 | 0.78 | 0.74 | 0.71 |
| 1.33 | 1.32 | 1.30 | 1.27 | 1.23 | 1.19 | 1.14 | 1.09 | 1.04 | 0.99 | 0.94 |
| 1.50 | 1.49 | 1.47 | 1.44 | 1.39 | 1.34 | 1.29 | 1.23 | 1.17 | 1.11 | 1.06 |
| 1.67 | 1.66 | 1.64 | 1.60 | 1.55 | 1.49 | 1.43 | 1.37 | 1.30 | 1.24 | 1.18 |
| 2.00 | 1.99 | 1.96 | 1.92 | 1.86 | 1.79 | 1.71 | 1.64 | 1.56 | 1.49 | 1.41 |
| 2.50 | 2.49 | 2.45 | 2.39 | 2.32 | 2.24 | 2.14 | 2.05 | 1.95 | 1.86 | 1.77 |

where $C_{\mathrm{PU}}^Y$ or $C_{\mathrm{PL}}^Y$ is denoted here as $C_{\mathrm{I}}^Y$. Since the variation of the empirical data we observe is greater than the variation of the original data (without measurement errors), the denominator of the index $C_{\mathrm{I}}$ becomes larger, and we would understate the true capability of the process if we calculate the process capability based on the empirical data from $Y$.

In Table II, we tabulate some empirical process capabilities with $\tau = 0(0.1)1.0$ for various true process capabilities $C_{\mathrm{I}} = 0.50$, 1.00, 1.33, 1.50, 1.67, 2.00 and 2.50. If $\tau = 1.0$, then for $C_{\mathrm{I}}^Y = 0.35$ the true process capability is $C_{\mathrm{I}} = 0.50$, and for $C_{\mathrm{I}}^Y = 1.77$ the true process capability $C_{\mathrm{I}} = 2.50$. The empirical process capability is more likely to diverge from the true capability when the measurement error increases. It is obvious that the gauge accuracy is less important if the required process capability is only marginally capable, and becomes more critical as the true capability requirement gets more stringent.

## 3. *ESTIMATING EMPIRICAL PROCESS CAPABILITY*

Since the process parameters $\mu$ and $\sigma$ are unknown, we therefore cannot evaluate the actual process capability. However, given sample data taken from the process, we could estimate process capability. Denoting by $\{X_i, i = 1, \ldots, n\}$ the random sample of size $n$ from the quality characteristics $X$, the natural estimators of $C_{\mathrm{PU}}$ and $C_{\mathrm{PL}}$ are

$$\hat{C}_{\mathrm{PU}} = \frac{USL - \bar{X}}{3S}, \quad \hat{C}_{\mathrm{PL}} = \frac{\bar{X} - LSL}{3S}$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $S = [\sum_{i=1}^{n}(X_i - \bar{X})/(n-1)]^{1/2}$ are conventional estimators of $\mu$ and $\sigma$. Chou and Owen[20] showed that under the normality assumption, the estimators $\hat{C}_{\mathrm{PU}}$ and $\hat{C}_{\mathrm{PL}}$ are distributed as $ct_{n-1}(\delta)$, where $c = (3\sqrt{n})^{-1}$, and $t_{n-1}(\delta)$ is a non-central $t$ distribution with $n-1$ degrees of freedom and non-centrality parameter $\delta = 3\sqrt{n}C_{\mathrm{PU}}$ and $\delta = 3\sqrt{n}C_{\mathrm{PL}}$, respectively. By adding the well-known correction factor,

$$b_{n-1} = \sqrt{\frac{2}{n-1}}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)^{-1}$$

to $\hat{C}_{\mathrm{PU}}$ and $\hat{C}_{\mathrm{PL}}$, such as $\tilde{C}_{\mathrm{PU}} = b_{n-1}\hat{C}_{\mathrm{PU}}$ and $\tilde{C}_{\mathrm{PL}} = b_{n-1}\hat{C}_{\mathrm{PL}}$, Pearn and Chen[21] showed that $\tilde{C}_{\mathrm{PU}}$ and $\tilde{C}_{\mathrm{PL}}$ are uniformly minimum variance unbiased estimators (UMVUEs) of $C_{\mathrm{PU}}$ and $C_{\mathrm{PL}}$. Thus, given a sample $\{Y_i, i = 1, \ldots, n\}$, the estimators of $C_{\mathrm{PU}}$ and $C_{\mathrm{PL}}$ are

$$\tilde{C}_{\mathrm{PU}}^Y = b_{n-1}\frac{USL - \bar{Y}}{3S_Y}, \quad \tilde{C}_{\mathrm{PL}}^Y = b_{n-1}\frac{\bar{Y} - LSL}{3S_Y}$$

Based on the same argument as used in Chou and Owen[20] and Pearn and Chen[21], the estimator $\tilde{C}_{\mathrm{I}}^Y$ ($\tilde{C}_{\mathrm{PU}}^Y$ or $\tilde{C}_{\mathrm{PL}}^Y$) is distributed as $dt_{n-1}(\delta^Y)$, where $d = b_{n-1}(3\sqrt{n})^{-1}$ and $t_{n-1}(\delta^Y)$ is a non-central $t$ distribution with

Table III. $\tau_0$ values for $n = 5(5)100$

| $n$ | $\tau_0$ | $n$ | $\tau_0$ | $n$ | $\tau_0$ | $n$ | $\tau_0$ |
|---|---|---|---|---|---|---|---|
| 5 | 1.439 | 30 | 0.279 | 55 | 0.199 | 80 | 0.163 |
| 10 | 0.587 | 35 | 0.255 | 60 | 0.189 | 85 | 0.157 |
| 15 | 0.431 | 40 | 0.237 | 65 | 0.181 | 90 | 0.153 |
| 20 | 0.356 | 45 | 0.222 | 70 | 0.174 | 95 | 0.149 |
| 25 | 0.310 | 50 | 0.209 | 75 | 0.168 | 100 | 0.145 |

$n-1$ degrees of freedom and non-centrality parameter $\delta^Y = 3\sqrt{n}C_{\mathrm{I}}/\sqrt{1+\tau^2}$. The mean, the variance and the mean squared error of the estimator $\tilde{C}_{\mathrm{I}}^Y$ are

$$E(\tilde{C}_{\mathrm{I}}^Y) = \frac{C_{\mathrm{I}}}{\sqrt{1+\tau^2}}$$

$$\mathrm{Var}(\tilde{C}_{\mathrm{I}}^Y) = \left\{\frac{\Gamma((n-1)/2)\Gamma((n-3)/2)}{[\Gamma((n-2)/2)]^2} - 1\right\}\frac{(C_{\mathrm{I}})^2}{1+\tau^2} + \frac{\Gamma((n-1)/2)\Gamma((n-3)/2)}{9n[\Gamma((n-2)/2)]^2}$$

$$\mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y) = \left(\frac{1}{\sqrt{1+\tau^2}} - 1\right)^2 (C_{\mathrm{I}})^2 + \left\{\frac{\Gamma((n-1)/2)\Gamma((n-3)/2)}{[\Gamma((n-2)/2)]^2} - 1\right\}\frac{(C_{\mathrm{I}})^2}{1+\tau^2} + \frac{\Gamma((n-1)/2)\Gamma((n-3)/2)}{9n[\Gamma((n-2)/2)]^2}$$

For $\tau > 0$, $\tilde{C}_{\mathrm{I}}^Y$ is a biased estimator of $C_{\mathrm{I}}$, and the bias $(1/\sqrt{1+\tau^2} - 1)C_{\mathrm{I}}$ decreases in $\tau$. Since $\Gamma((n-1)/2)\Gamma((n-3)/2)/[\Gamma((n-2)/2)]^2 - 1$ is positive, then $\mathrm{Var}(\tilde{C}_{\mathrm{I}}^Y) < \mathrm{Var}(\tilde{C}_{\mathrm{I}})$. To compare $\mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y)$ with $\mathrm{MSE}(\tilde{C}_{\mathrm{I}})$, we consider the function $f(C_{\mathrm{I}}, n, \tau) = \mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y)/\mathrm{MSE}(\tilde{C}_{\mathrm{I}})$. By some reduction, we have $f(C_{\mathrm{I}}, n, \tau) = 1$ if and only if

$$\tau = \frac{2\Gamma((n-2)/2)\sqrt{\Gamma((n-1)/2)\Gamma((n-3)/2) - [\Gamma((n-2)/2)]^2}}{2[\Gamma((n-2)/2)]^2 - \Gamma((n-1)/2)\Gamma((n-3)/2)}$$

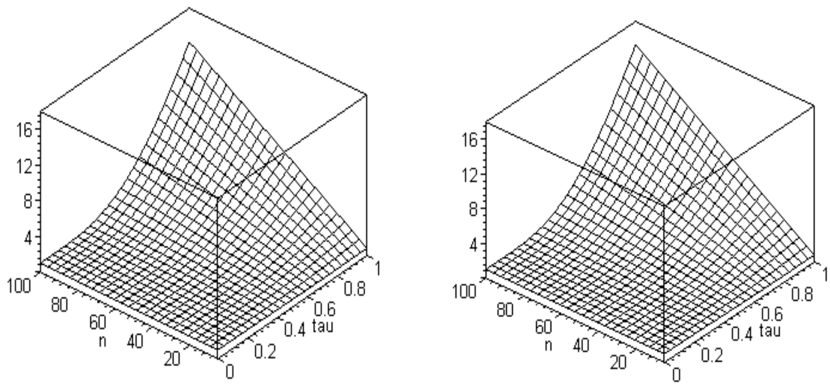
or $\tau = 0$. Denote the right-hand side of the above formula by $\tau_0$ and we have $f(C_{\mathrm{I}}, n, \tau) > 1$ if $\tau > \tau_0$ and $f(C_{\mathrm{I}}, n, \tau) < 1$ if $\tau < \tau_0$ exclusive of 0. This represents that $\mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y) > \mathrm{MSE}(\tilde{C}_{\mathrm{I}})$ if $\tau > \tau_0$, $\mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y) < \mathrm{MSE}(\tilde{C}_{\mathrm{I}})$ if $\tau < \tau_0$ exclusive of 0, and $\mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y) = \mathrm{MSE}(\tilde{C}_{\mathrm{I}})$ if $\tau = \tau_0$ or 0.

Table III lists the $\tau_0$ values for $n = 5(5)100$. Figures 1(a) and (b) display the surface plots of the ratios $\gamma = f(C_{\mathrm{I}}, n, \tau)$ with $n = 5(1)100$ and $\tau$ in [0, 1] for $C_{\mathrm{I}} = 1.00$, and 1.33. The value $\tau_0$ is greater than 0.5 for small $n$ ($n \le 10$), and greater than 0.2 for $n \le 50$. When $50 < n \le 100$, $\tau_0$ is between 0.7 and 0.2. For large $n$, $\gamma$ is greater than 1 for almost every value of $\tau$, and $\gamma$ increases if $\tau$ increases. The maximum values of $\gamma$ are 14.239, and 15.347, respectively, and the minimum values of $\gamma$ are 0.806 (1/1.241), and 0.797 (1/1.255), respectively. The maximum values of $\gamma$ occur at $n = 100$ and $\tau = 1$, and the minimum values of $\gamma$ occur at $n = 5$ and $\tau = 0.788$. The difference between $\mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y)$ and $\mathrm{MSE}(\tilde{C}_{\mathrm{I}})$ with $\gamma > 1$ is more significant than that with $\gamma < 1$.

## 4. EMPIRICAL LOWER CONFIDENCE BOUND

The lower confidence bounds present a measure on the minimum capability of the process based on the sample data. Let $k_1 = 3\tilde{C}_{\mathrm{PU}}/b_{n-1}$ and $k_2 = 3\tilde{C}_{\mathrm{PL}}/b_{n-1}$, and we have $USL = \bar{X} + k_1 S$ and $LSL = \bar{X} - k_2 S$.




(a) (b)

Figure 1. Surface plot of $\gamma$ with $n = 5(1)100$ and $\tau$ in [0, 1] for: (a) $C_{\mathrm{I}} = 1.00$; (b) $C_{\mathrm{I}} = 1.33$

A $100\theta\%$ lower confidence bound $C_{\mathrm{U}}$ for $C_{\mathrm{PU}}$ satisfies $P(C_{\mathrm{PU}} \geq C_{\mathrm{U}}) = \theta$. It can be written as

$$\begin{aligned} P(C_{\mathrm{PU}} \geq C_{\mathrm{U}}) &= P\left(\frac{USL - \mu}{3\sigma} \geq C_{\mathrm{U}}\right) \\ &= P\left(\frac{Z - 3\sqrt{n}C_{\mathrm{U}}}{S/\sigma} \geq -\frac{3\tilde{C}_{\mathrm{PU}}}{b_{n-1}}\sqrt{n}\right) = P(t_{n-1}(\delta_{\mathrm{U}} = -3\sqrt{n}C_{\mathrm{U}}) \geq t_1) = \theta \end{aligned}$$

Similarly, a $100\theta\%$ lower confidence bound $C_{\mathrm{L}}$ for $C_{\mathrm{PL}}$ satisfies $P(C_{\mathrm{PL}} \geq C_{\mathrm{L}}) = \theta$. It can be shown as $P(t_{n-1}(\delta_{\mathrm{L}} = 3\sqrt{n}C_{\mathrm{L}}) \leq t_2) = \theta$, where $Z$ is distributed as $N(0, 1)$, $t_1 = -k_1\sqrt{n}$ and $t_2 = k_2\sqrt{n}$. To find the exact $100\theta\%$ lower confidence bounds, Pearn and Shu[22] provided an algorithm and a *Matlab* program to solve the above equations. With measurement errors, we use $\tilde{C}_{\mathrm{I}}^Y$ to estimate $C_{\mathrm{I}}$ but not $\tilde{C}_{\mathrm{I}}$. Thus, $t_1^Y = -(3\tilde{C}_{\mathrm{PU}}^Y/b_{n-1})\sqrt{n}$ and $t_2^Y = (3\tilde{C}_{\mathrm{PL}}^Y/b_{n-1})\sqrt{n}$, instead of $t_1$ and $t_2$, are substituted into the equations to obtain the confidence bounds. Denote the bounds originated from $t_1^Y$ and $t_2^Y$ as $C_{\mathrm{U}}^Y$ and $C_{\mathrm{L}}^Y$. The confidence coefficient by the confidence bound $C_{\mathrm{U}}^Y$ (denoted by $\theta^Y$) we obtained is

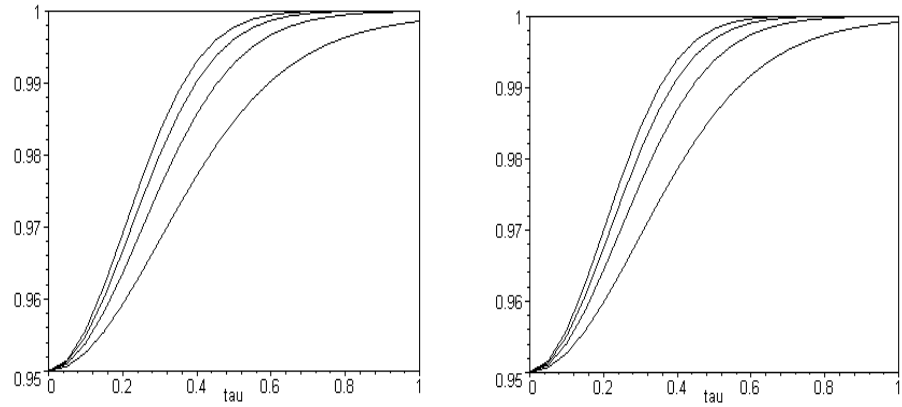
$$\begin{aligned} \theta^Y = P(C_{\mathrm{PU}} \geq C_{\mathrm{U}}^Y) &= P\left(\frac{USL - \mu_Y}{3\sigma_Y}\sqrt{1+\tau^2} \geq C_{\mathrm{U}}^Y\right) \\ &= P\left(\frac{\bar{Y} + k_1^Y S_Y - \mu_Y}{3\sigma_Y} \geq \frac{C_{\mathrm{U}}^Y}{\sqrt{1+\tau^2}}\right) = P\left(\frac{Z - 3\sqrt{n}C_{\mathrm{U}}^Y/\sqrt{1+\tau^2}}{S_Y/\sigma_Y} \geq -k_1^Y\sqrt{n}\right) \\ &= P\left(\frac{Z - 3\sqrt{n}C_{\mathrm{U}}^Y/\sqrt{1+\tau^2}}{S_Y/\sigma_Y} \geq -\frac{3\tilde{C}_{\mathrm{PU}}\sqrt{n}}{b_{n-1}}\right) = P\left(t_{n-1}\left(\delta_{\mathrm{U}}^Y = \frac{-3\sqrt{n}C_{\mathrm{U}}^Y}{\sqrt{1+\tau^2}}\right) \geq t_1^Y\right) \end{aligned}$$

where $k_1^Y = 3\tilde{C}_{\mathrm{PU}}^Y/b_{n-1_1}$, and $\theta^Y$ can be also obtained by the confidence bound $C_{\mathrm{L}}^Y$, expressed as

$$\theta^Y = P\left(t_{n-1}\left(\delta_{\mathrm{L}}^Y = \frac{3\sqrt{n}C_{\mathrm{L}}^Y}{\sqrt{1+\tau^2}}\right) \leq t_2^Y\right)$$

Figures 2(a) and (b) plot $\theta^Y$ versus $\tau$ with $n = 25(25)100$ and $\tilde{C}_{\mathrm{I}} = 1.00$, and 1.33, for 95% confidence intervals (since $E(\tilde{C}_{\mathrm{I}}^Y) = E(\tilde{C}_{\mathrm{I}}/\sqrt{1+\tau^2})$, we consider the cases with $\tilde{C}_{\mathrm{I}}^Y = \tilde{C}_{\mathrm{I}}/\sqrt{1+\tau^2}$). Since $\tilde{C}_{\mathrm{I}}^Y$ is smaller than $\tilde{C}_{\mathrm{I}}$ in the presence of measurement errors, and $C_{\mathrm{U}}^Y$ (or $C_{\mathrm{L}}^Y$) is smaller than $C_{\mathrm{U}}$ (or $C_{\mathrm{L}}$), it is necessary that $\theta^Y$ is always greater than $\theta$. Severely underestimating the true process capability may result in high production cost, losing the power of competition. For instance, suppose that a process has a 95% lower confidence bound, 1.256,




(a) (b)

Figure 2. Plots of $\theta^Y$ versus $\tau$ with $n = 25(25)100$ (from top to bottom) for 95% confidence intervals and: (a) $\tilde{C}_{\mathrm{I}} = 1.00$; (b) $\tilde{C}_{\mathrm{I}} = 1.33$

with $n = 50$, which has met the threshold of an 'excellent' process. However, the bound may be calculated as 1.073 with measurement errors $\tau = 0.6$. The coefficient increases to 0.998, but the process may be determined as a 'capable' process rather than a 'satisfactory' process.

## 5. CAPABILITY TESTING BASED ON EMPIRICAL DATA

We usually use statistical testing to determine whether our processes meet the capability requirement. The null hypothesis is $H_0 : C_{\mathrm{I}} \le c$ (process is not capable), and the alternative hypothesis is $H_0 : C_{\mathrm{I}} > c$ (process is capable) of testing, where $c$ is our required process capability. The critical value is used to determine whether the null hypothesis should be rejected. If the point estimator of the process capability is greater than the critical value, we reject the null hypothesis and conclude that the process is capable. Otherwise, we would believe that the process is incapable. Suppose that the nominal size of our statistical testing is $\alpha$ (type I error), the critical value $c_0$ can be determined by

$$\alpha = P(\tilde{C}_{\mathrm{I}} \ge c_0 \mid C_{\mathrm{I}} = c)$$
$$c_0 = \frac{b_{n-1}}{3\sqrt{n}} t_{n-1,\alpha}(\delta = 3\sqrt{n}c)$$

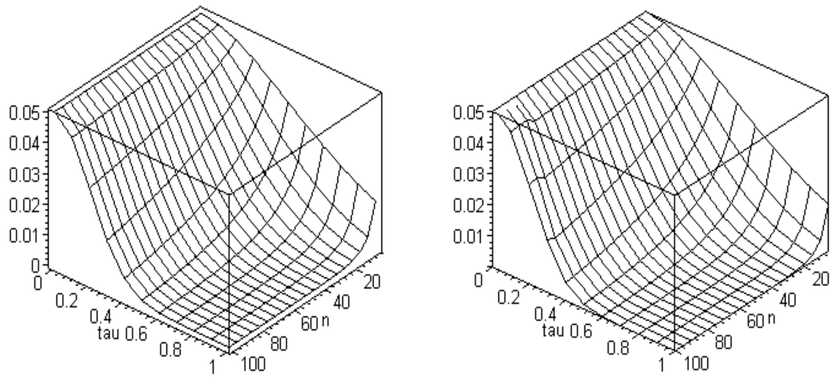
where $t_{n-1,\alpha}(\delta)$ is the upper $\alpha$th quantile of $t_{n-1}(\delta)$ distribution. The power of the test can be calculated as

$$\begin{aligned}\pi(C_{\mathrm{I}}) &= P(\tilde{C}_{\mathrm{I}} > c_0 \mid C_{\mathrm{I}}) = P(3\sqrt{n}\tilde{C}_{\mathrm{I}} > 3\sqrt{n}c_0 \mid C_{\mathrm{I}}) \\ &= P(t_{n-1}(\delta = 3\sqrt{n}C_{\mathrm{I}}) > t_{n-1,\alpha}(\delta = 3\sqrt{n}c))\end{aligned}$$

However, in the presence of measurement errors, the $\alpha$-risk (denoted by $\alpha^Y$) and the power (denoted by $\pi^Y$) are

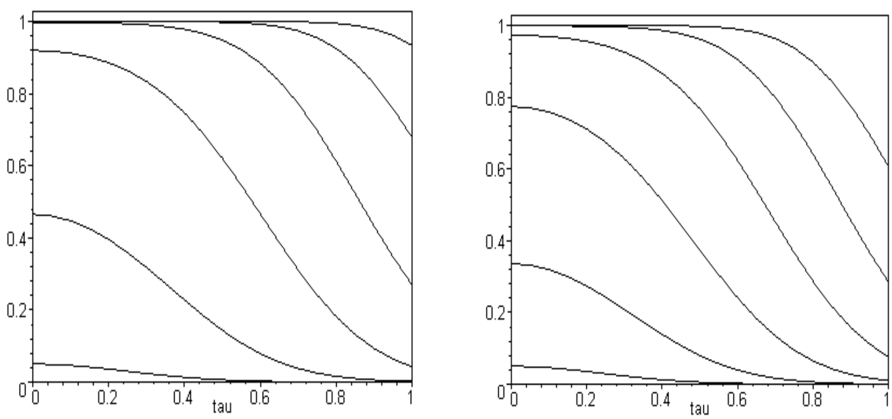
$$\begin{aligned}\alpha^Y &= P(\tilde{C}_{\mathrm{I}}^Y \ge c_0 \mid C_{\mathrm{I}} = c) = P(3\sqrt{n}\tilde{C}_{\mathrm{I}}^Y \ge 3\sqrt{n}c_0 \mid C_{\mathrm{I}} = c) \\ &= P\left(\frac{3\sqrt{n}}{b_{n-1}}\tilde{C}_{\mathrm{I}}^Y \ge \frac{3\sqrt{n}}{b_{n-1}}c_0 \,\middle|\, C_{\mathrm{I}} = c\right) = P\left(t_{n-1}(\delta^Y = 3\sqrt{n}C_{\mathrm{I}}^Y) \ge \frac{3\sqrt{n}}{b_{n-1}}c_0 \,\middle|\, C_{\mathrm{I}} = c\right) \\ &= P\left(t_{n-1}\left(\delta^Y = 3\sqrt{n}\frac{c}{\sqrt{1+\tau^2}}\right) \ge t_{n-1,\alpha}(\delta = 3\sqrt{n}c)\right)\end{aligned}$$




(a) (b)

Figure 3. Surface plot of $\alpha^Y$ with $n = 5(1)100$, $\tau \in [0, 1]$, $\alpha = 0.05$, for (a) $c = 1.00$; (b) $c = 1.33$




(a) (b)

Figure 4. Plots of $\pi^Y$ versus $\tau$, with $n = 50$, $\alpha = 0.05$, for (a) $c = 1.00$, $C_{\mathrm{I}} = 1.00(0.20)2.00$; (b) $c = 1.33$, $C_{\mathrm{I}} = 1.33(0.20)2.33$ (from bottom to top)

$$\begin{aligned}
\pi^Y(C_{\mathrm{I}}) &= P(\tilde{C}_{\mathrm{I}}^Y > c_0 \mid C_{\mathrm{I}}) = P(3\sqrt{n}\tilde{C}_{\mathrm{I}}^Y > 3\sqrt{n}c_0 \mid C_{\mathrm{I}}) \\
&= P\left(\frac{3\sqrt{n}}{b_{n-1}}\tilde{C}_{\mathrm{I}}^Y > \frac{3\sqrt{n}}{b_{n-1}}c_0 \,\middle|\, C_{\mathrm{I}}\right) = P\left(t_{n-1}(\delta^Y = 3\sqrt{n}C_{\mathrm{I}}^Y) > \frac{3\sqrt{n}}{b_{n-1}}c_0 \,\middle|\, C_{\mathrm{I}}\right) \\
&= P\left(t_{n-1}\left(\delta^Y = 3\sqrt{n}\frac{C_{\mathrm{I}}}{\sqrt{1+\tau^2}}\right) > t_{n-1,\alpha}(\delta = 3\sqrt{n}c)\right)
\end{aligned}$$

Earlier discussions indicate that we underestimate the true process capability using $\tilde{C}_{\mathrm{I}}^Y$ instead of $\tilde{C}_{\mathrm{I}}$. The probability that $\tilde{C}_{\mathrm{I}}^Y$ is greater than $c_0$ would be less than that of using $\tilde{C}_{\mathrm{I}}$. Thus, the $\alpha$-risk using $\tilde{C}_{\mathrm{I}}^Y$ to estimate $C_{\mathrm{I}}$ is less than the $\alpha$-risk if using $\tilde{C}_{\mathrm{I}}$ to estimate $C_{\mathrm{I}}$. The power, if using $\tilde{C}_{\mathrm{I}}^Y$ to estimate $C_{\mathrm{I}}$, is also less than the power if using $\tilde{C}_{\mathrm{I}}$. That is, we have $\alpha^Y \le \alpha$ and $\pi^Y \le \pi$. Figures 3(a) and (b) are the surface plots of $\alpha^Y$ with $n = 5(1)100$ and $\tau \in [0, 1]$ for $C_{\mathrm{I}} = 1.00, 1.33$ and $\alpha = 0.05$. Figures 4(a) and (b) are the plots of $\pi^Y$ versus $\tau$ with $n = 50$ and $\alpha = 0.05$ for $c = 1.00, 1.33$ and $C_{\mathrm{I}} = c(0.20)c + 1$. Note that for $\tau = 0$, $\alpha^Y = \alpha$ and $\pi^Y = \pi$ in those figures.

In Figures 3(a) and (b), $\alpha^Y$ decreases as $\tau$ or $n$ increases, and the decreasing rate is more significant with large $c$ values. We find that for large $\tau$ values $\alpha^Y$ is smaller than $1 \times 10^{-5}$. In Figures 4(a) and (b), $\pi^Y$ decreases as $\tau$ increases, but increases as $n$ increases. Decrement of $\pi^Y$ by $\tau$ is more significant for large $c$ values. Because of measurement errors, $\pi^Y$ may decrease significantly. For instance, in Figure 4(a) the $\pi^Y$ value ($c = 1.00, n = 50$) for $C_{\mathrm{I}} = 1.40$ is $\pi^Y = 0.920$ if there is no measurement error ($\tau = 0$). However, when $\tau = 1.0$, $\pi^Y$ decreases to 0.042 and the decrement of the power is 0.878.

## 6. MODIFIED LOWER CONFIDENCE BOUNDS AND CRITICAL VALUES

We have shown that the coefficients increase owing to underestimating the lower confidence bounds. We have also shown that both the $\alpha$-risk and the power of the test decrease in measurement error. The probability of passing non-conforming product units decreases, but the probability of correctly judging a capable process as incapable also decreases. Since the lower confidence bound of the process capability is severely underestimated, and the power becomes much weaker, the producers cannot firmly state that their processes meet the capability requirement even if their processes are sufficiently capable. Good product units would be incorrectly rejected in this case (rejected products are either scrapped or require rework). Unnecessary cost to the producers may accompany those incorrect decisions. Improving the gauge capability and training the operators by proper education are some ways to reduce the measurement errors. Nevertheless, measurement errors may be unavoidable in most manufacturing processes. Thus, in this section, we adjust the confidence bounds to give a more precise estimation of process capability, and revise critical values to improve the power for testing hypothesis.

Suppose that the desired confidence coefficient is $\theta$, the adjusted confidence interval of $C_{\mathrm{PU}}$ with confidence interval bound $C_{\mathrm{U}}^*$, and can be established as

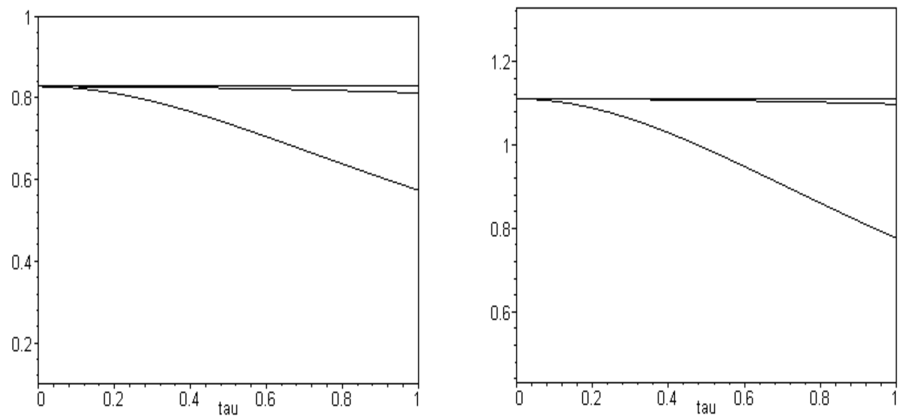
$$\begin{aligned}
\theta = P(C_{\mathrm{PU}} \geq C_{\mathrm{U}}^*) &= P\left(\frac{USL - \mu_Y}{3\sigma_Y}\sqrt{1+\tau^2} \geq C_{\mathrm{U}}^*\right) \\
&= P\left(\frac{\bar{Y} + k_1^Y S_Y - \mu_Y}{3\sigma_Y} \geq \frac{C_{\mathrm{U}}^*}{\sqrt{1+\tau^2}}\right) = P\left(\frac{Z - 3\sqrt{n}C_{\mathrm{U}}^*/\sqrt{1+\tau^2}}{S_Y/\sigma_Y} \geq -k_1^Y\sqrt{n}\right) \\
&= P\left(\frac{Z - 3\sqrt{n}C_{\mathrm{U}}^*/\sqrt{1+\tau^2}}{S_Y/\sigma_Y} \geq -\frac{3\tilde{C}_{\mathrm{PU}}\sqrt{n}}{b_{n-1}}\right) = P\left(t_{n-1}\left(\delta_{\mathrm{U}}^* = \frac{-3\sqrt{n}C_{\mathrm{U}}^*}{\sqrt{1+\tau^2}}\right) \geq t_1^Y\right)
\end{aligned}$$

Similarly, the adjusted confidence interval of $C_{\mathrm{PL}}$ with confidence interval bound $C_{\mathrm{L}}^*$, can be established as

$$\theta = P\left(t_{n-1}\left(\delta_{\mathrm{L}}^* = \frac{3\sqrt{n}C_{\mathrm{L}}^*}{\sqrt{1+\tau^2}}\right) \leq t_2^Y\right)$$

To find the exact $100\theta\%$ lower confidence bounds, an *S-plus* program has been developed to solve the equations. Figures 5(a) and (b) are comparisons among $C_{\mathrm{U}}$, $C_{\mathrm{U}}^Y$, and $C_{\mathrm{U}}^*$ for $\tilde{C}_{\mathrm{PU}} = 1.00$, 1.33 with $n = 50$, where $C_{\mathrm{U}}$ is the 95% lower confidence bound of $\tilde{C}_{\mathrm{PU}}$, $C_{\mathrm{U}}^Y$ is the 95% lower confidence bound of $\tilde{C}_{\mathrm{PU}}^Y$, and $C_{\mathrm{U}}^*$ is the adjusted 95% lower confidence bound for $\tilde{C}_{\mathrm{PU}}^Y$. Note that, in this case, the probability that the interval with the bound $C_{\mathrm{U}}^Y$ contains the actual $C_{\mathrm{PU}}$ value is greater than that of the interval with the bound $C_{\mathrm{U}}$ or $C_{\mathrm{U}}^*$, while the probability that the interval with the bound $C_{\mathrm{U}}$ or $C_{\mathrm{U}}^*$ contains the actual $C_{\mathrm{PU}}$ value is just 0.95. From Figures 5(a) and (b), we see that the lower confidence bounds remained underestimated, even if we adjust the formula to calculate the bounds. However, the magnitude of underestimation using adjusted confidence bounds is significantly reduced.

In order to improve the power of the test, we consider the revised critical values $c_0^*$ satisfied $c_0^* < c_0$. Thus, the probability that $\tilde{C}_{\mathrm{I}}^Y$ is greater than $c_0^*$ is greater than the probability that $\tilde{C}_{\mathrm{I}}^Y$ is greater than $c_0$.




(a) (b)

Figure 5. Plots of $C_{\mathrm{U}}$, $C_{\mathrm{U}}^{*}$ and $C_{\mathrm{U}}^{Y}$ (from top to bottom) versus $\tau$ with $n = 50$ and for: (a) $\tilde{C}_{\mathrm{PU}} = 1.00$; (b) $\tilde{C}_{\mathrm{PU}} = 1.33$

Both the $\alpha$-risk and the power increase when we use $c_0^*$ as a new critical value in the testing. Suppose that the $\alpha$-risk using the revised critical value $c_0^*$ is $\alpha^*$, the revised critical $c_0^*$ must satisfy

$$\begin{aligned}
\alpha^* &= P(\tilde{C}_{\mathrm{I}}^{Y} \geq c_0^* \mid C_{\mathrm{I}} = c) = P(3\sqrt{n}\tilde{C}_{\mathrm{I}}^{Y} \geq 3\sqrt{n}c_0^* \mid C_{\mathrm{I}} = c) \\
&= P\left(\frac{3\sqrt{n}}{b_{n-1}}\tilde{C}_{\mathrm{I}}^{Y} \geq \frac{3\sqrt{n}}{b_{n-1}}c_0^* \,\middle|\, C_{\mathrm{I}} = c\right) = P\left(t_{n-1}(\delta^Y = 3\sqrt{n}C_{\mathrm{I}}^{Y}) \geq \frac{3\sqrt{n}}{b_{n-1}}c_0^* \,\middle|\, C_{\mathrm{I}} = c\right) \\
&= P\left(t_{n-1}\left(\delta^Y = 3\sqrt{n}\frac{c}{\sqrt{1+\tau^2}}\right) \geq \frac{3\sqrt{n}}{b_{n-1}}c_0^*\right)
\end{aligned}$$

To ensure that the $\alpha$-risk is within the preset magnitude, we let $\alpha^* = \alpha$, thus $c_0^*$ can be obtained as

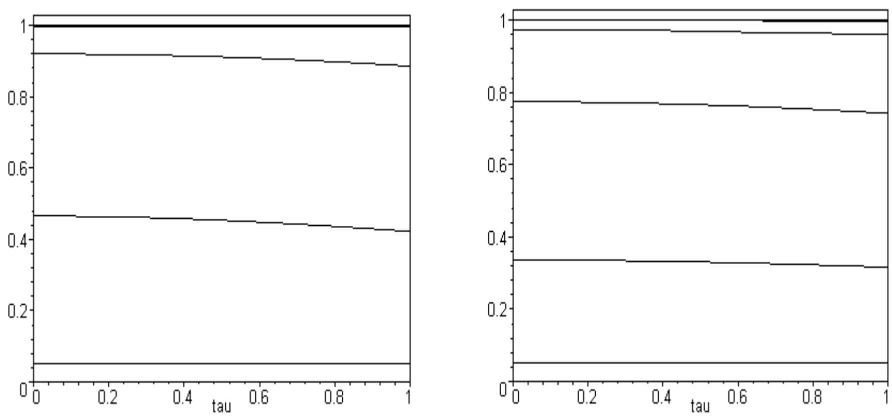
$$c_0^* = \frac{b_{n-1}}{3\sqrt{n}}t_{n-1,\alpha}\left(\delta^Y = 3\sqrt{n}\frac{c}{\sqrt{1+\tau^2}}\right)$$

and the power $\pi^*$ is

$$\begin{aligned}
\pi^*(C_{\mathrm{I}}) &= P(\tilde{C}_{\mathrm{I}}^{Y} > c_0^* \mid C_{\mathrm{I}}) = P(3\sqrt{n}\tilde{C}_{\mathrm{I}}^{Y} > 3\sqrt{n}c_0^* \mid C_{\mathrm{I}}) \\
&= P\left(\frac{3\sqrt{n}}{b_{n-1}}\tilde{C}_{\mathrm{I}}^{Y} > \frac{3\sqrt{n}}{b_{n-1}}c_0^* \,\middle|\, C_{\mathrm{I}}\right) = P\left(t_{n-1}(\delta^Y = 3\sqrt{n}C_{\mathrm{I}}^{Y}) > \frac{3\sqrt{n}}{b_{n-1}}c_0^* \,\middle|\, C_{\mathrm{I}}\right) \\
&= P\left(t_{n-1}\left(\delta^Y = 3\sqrt{n}\frac{C_{\mathrm{I}}}{\sqrt{1+\tau^2}}\right) > t_{n-1,\alpha}\left(\delta^Y = 3\sqrt{n}\frac{c}{\sqrt{1+\tau^2}}\right)\right)
\end{aligned}$$

Figures 6(a) and (b) plot $\pi^*$ versus $\tau$ with $n = 50$ and $\alpha = 0.05$ for $c = 1.00$, 1.33, and $C_{\mathrm{I}} = c(0.20)c + 1$. From those figures, we see that the powers corresponding to the adjusted critical values $c_0^*$ remain decreasing in measurement error, but the decrements originating from the new critical values $c_0^*$ are very small. We have improved a certain degree of power. For instance, if we compare the $\pi^Y$ values in Figure 4(a) ($c = 1.00$, $n = 50$) for $C_{\mathrm{I}} = 1.40$ with the $\pi^*$ values in Figure 6(a) ($c = 1.00$, $n = 50$) for $C_{\mathrm{I}} = 1.40$, we see that $\pi^Y = 0.042$ and $\pi^* = 0.885$ with $\tau = 1.0$. In this case, using the adjusted critical values $c_0^*$ the power is improved by 0.843. Tables IV–VII provide the revised critical values for some commonly used capability requirements. Using these tables, the practitioner may select the proper critical values for capability testing.




(a) (b)

Figure 6. Plots of $\pi^*$ versus $\tau$, with $n = 50$, $\alpha = 0.05$, for: (a) $c = 1.00$, $C_{\mathrm{I}} = 1.00(0.20)2.00$; (b) $c = 1.33$, $C_{\mathrm{I}} = 1.33(0.20)2.33$ (from bottom to top)

Table IV. Critical values for $C_{\mathrm{I}} = 1.00$, with $n = 10(10)100$, $\tau = 0.1(0.1)1.0$

| | | $\tau$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $1-\alpha$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 10 | 0.950 | 1.534 | 1.512 | 1.479 | 1.436 | 1.386 | 1.332 | 1.276 | 1.221 | 1.166 | 1.114 |
| | 0.975 | 1.707 | 1.684 | 1.647 | 1.599 | 1.544 | 1.484 | 1.423 | 1.361 | 1.301 | 1.243 |
| | 0.990 | 1.948 | 1.921 | 1.879 | 1.825 | 1.763 | 1.695 | 1.625 | 1.555 | 1.487 | 1.422 |
| 20 | 0.950 | 1.336 | 1.318 | 1.288 | 1.251 | 1.207 | 1.159 | 1.110 | 1.061 | 1.013 | 0.967 |
| | 0.975 | 1.429 | 1.409 | 1.378 | 1.338 | 1.291 | 1.241 | 1.189 | 1.137 | 1.086 | 1.037 |
| | 0.990 | 1.548 | 1.527 | 1.494 | 1.450 | 1.400 | 1.346 | 1.290 | 1.234 | 1.179 | 1.127 |
| 30 | 0.950 | 1.263 | 1.245 | 1.217 | 1.181 | 1.140 | 1.095 | 1.048 | 1.001 | 0.956 | 0.912 |
| | 0.975 | 1.330 | 1.312 | 1.283 | 1.245 | 1.201 | 1.154 | 1.105 | 1.056 | 1.009 | 0.963 |
| | 0.990 | 1.416 | 1.396 | 1.365 | 1.325 | 1.279 | 1.229 | 1.178 | 1.126 | 1.076 | 1.027 |
| 40 | 0.950 | 1.222 | 1.205 | 1.178 | 1.143 | 1.102 | 1.058 | 1.013 | 0.968 | 0.923 | 0.881 |
| | 0.975 | 1.277 | 1.259 | 1.231 | 1.194 | 1.152 | 1.107 | 1.060 | 1.013 | 0.967 | 0.922 |
| | 0.990 | 1.345 | 1.327 | 1.297 | 1.259 | 1.215 | 1.167 | 1.118 | 1.069 | 1.021 | 0.974 |
| 50 | 0.950 | 1.195 | 1.178 | 1.152 | 1.117 | 1.078 | 1.035 | 0.990 | 0.946 | 0.902 | 0.860 |
| | 0.975 | 1.242 | 1.225 | 1.197 | 1.162 | 1.121 | 1.076 | 1.030 | 0.984 | 0.939 | 0.896 |
| | 0.990 | 1.301 | 1.282 | 1.254 | 1.217 | 1.174 | 1.128 | 1.080 | 1.032 | 0.985 | 0.940 |
| 60 | 0.950 | 1.176 | 1.159 | 1.133 | 1.099 | 1.060 | 1.018 | 0.974 | 0.930 | 0.887 | 0.846 |
| | 0.975 | 1.218 | 1.200 | 1.173 | 1.139 | 1.098 | 1.055 | 1.009 | 0.964 | 0.920 | 0.878 |
| | 0.990 | 1.269 | 1.251 | 1.223 | 1.187 | 1.145 | 1.100 | 1.053 | 1.006 | 0.961 | 0.917 |
| 70 | 0.950 | 1.161 | 1.145 | 1.119 | 1.085 | 1.047 | 1.005 | 0.961 | 0.918 | 0.875 | 0.835 |
| | 0.975 | 1.199 | 1.182 | 1.155 | 1.121 | 1.081 | 1.038 | 0.994 | 0.949 | 0.905 | 0.863 |
| | 0.990 | 1.245 | 1.228 | 1.200 | 1.165 | 1.124 | 1.079 | 1.033 | 0.987 | 0.942 | 0.899 |
| 80 | 0.950 | 1.149 | 1.133 | 1.107 | 1.074 | 1.036 | 0.994 | 0.951 | 0.908 | 0.866 | 0.826 |
| | 0.975 | 1.184 | 1.167 | 1.141 | 1.107 | 1.068 | 1.025 | 0.981 | 0.937 | 0.894 | 0.852 |
| | 0.990 | 1.227 | 1.209 | 1.182 | 1.147 | 1.107 | 1.063 | 1.017 | 0.972 | 0.927 | 0.884 |
| 90 | 0.950 | 1.140 | 1.124 | 1.098 | 1.065 | 1.027 | 0.986 | 0.943 | 0.900 | 0.859 | 0.818 |
| | 0.975 | 1.172 | 1.156 | 1.129 | 1.096 | 1.057 | 1.015 | 0.971 | 0.927 | 0.884 | 0.843 |
| | 0.990 | 1.211 | 1.194 | 1.168 | 1.133 | 1.093 | 1.049 | 1.004 | 0.959 | 0.915 | 0.873 |
| 100 | 0.950 | 1.132 | 1.116 | 1.090 | 1.058 | 1.020 | 0.979 | 0.936 | 0.894 | 0.852 | 0.812 |
| | 0.975 | 1.162 | 1.146 | 1.120 | 1.086 | 1.048 | 1.006 | 0.962 | 0.919 | 0.876 | 0.835 |
| | 0.990 | 1.199 | 1.182 | 1.155 | 1.121 | 1.081 | 1.038 | 0.994 | 0.949 | 0.905 | 0.863 |

Table V. Critical values for $C_{\mathrm{I}} = 1.33$, with $n = 10(10)100$, $\tau = 0.1(0.1)1.0$

| | | $\tau$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $1-\alpha$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 10 | 0.950 | 2.018 | 1.990 | 1.945 | 1.887 | 1.820 | 1.748 | 1.673 | 1.598 | 1.524 | 1.453 |
| | 0.975 | 2.244 | 2.212 | 2.163 | 2.099 | 2.025 | 1.944 | 1.861 | 1.778 | 1.697 | 1.619 |
| | 0.990 | 2.557 | 2.521 | 2.464 | 2.392 | 2.308 | 2.216 | 2.122 | 2.028 | 1.936 | 1.847 |
| 20 | 0.950 | 1.762 | 1.737 | 1.698 | 1.647 | 1.588 | 1.525 | 1.459 | 1.393 | 1.328 | 1.266 |
| | 0.975 | 1.881 | 1.855 | 1.813 | 1.759 | 1.696 | 1.629 | 1.559 | 1.489 | 1.420 | 1.354 |
| | 0.990 | 2.036 | 2.008 | 1.962 | 1.904 | 1.837 | 1.764 | 1.688 | 1.613 | 1.539 | 1.468 |
| 30 | 0.950 | 1.667 | 1.643 | 1.606 | 1.558 | 1.502 | 1.441 | 1.379 | 1.316 | 1.255 | 1.196 |
| | 0.975 | 1.754 | 1.729 | 1.690 | 1.639 | 1.581 | 1.517 | 1.452 | 1.386 | 1.322 | 1.260 |
| | 0.990 | 1.864 | 1.838 | 1.796 | 1.743 | 1.681 | 1.614 | 1.544 | 1.475 | 1.407 | 1.342 |
| 40 | 0.950 | 1.614 | 1.591 | 1.555 | 1.508 | 1.454 | 1.395 | 1.334 | 1.274 | 1.214 | 1.157 |
| | 0.975 | 1.685 | 1.661 | 1.623 | 1.574 | 1.518 | 1.457 | 1.394 | 1.331 | 1.269 | 1.209 |
| | 0.990 | 1.773 | 1.748 | 1.708 | 1.658 | 1.598 | 1.534 | 1.468 | 1.402 | 1.337 | 1.275 |
| 50 | 0.950 | 1.579 | 1.557 | 1.521 | 1.476 | 1.423 | 1.365 | 1.305 | 1.246 | 1.187 | 1.131 |
| | 0.975 | 1.640 | 1.617 | 1.580 | 1.533 | 1.478 | 1.418 | 1.357 | 1.295 | 1.234 | 1.176 |
| | 0.990 | 1.715 | 1.691 | 1.653 | 1.603 | 1.546 | 1.484 | 1.420 | 1.355 | 1.292 | 1.232 |
| 60 | 0.950 | 1.555 | 1.532 | 1.497 | 1.452 | 1.400 | 1.343 | 1.285 | 1.226 | 1.168 | 1.113 |
| | 0.975 | 1.608 | 1.585 | 1.549 | 1.503 | 1.449 | 1.390 | 1.330 | 1.269 | 1.210 | 1.153 |
| | 0.990 | 1.675 | 1.651 | 1.613 | 1.565 | 1.509 | 1.448 | 1.385 | 1.323 | 1.261 | 1.202 |
| 70 | 0.950 | 1.536 | 1.514 | 1.479 | 1.435 | 1.383 | 1.327 | 1.269 | 1.211 | 1.154 | 1.099 |
| | 0.975 | 1.584 | 1.562 | 1.526 | 1.480 | 1.427 | 1.369 | 1.310 | 1.250 | 1.191 | 1.135 |
| | 0.990 | 1.644 | 1.621 | 1.584 | 1.536 | 1.481 | 1.422 | 1.360 | 1.298 | 1.237 | 1.179 |
| 80 | 0.950 | 1.521 | 1.499 | 1.465 | 1.421 | 1.369 | 1.314 | 1.256 | 1.198 | 1.142 | 1.088 |
| | 0.975 | 1.566 | 1.543 | 1.508 | 1.463 | 1.410 | 1.353 | 1.294 | 1.235 | 1.177 | 1.121 |
| | 0.990 | 1.620 | 1.597 | 1.561 | 1.514 | 1.459 | 1.401 | 1.340 | 1.279 | 1.219 | 1.161 |
| 90 | 0.950 | 1.509 | 1.487 | 1.453 | 1.409 | 1.358 | 1.303 | 1.246 | 1.189 | 1.133 | 1.079 |
| | 0.975 | 1.550 | 1.528 | 1.493 | 1.448 | 1.396 | 1.339 | 1.281 | 1.222 | 1.165 | 1.110 |
| | 0.990 | 1.601 | 1.578 | 1.542 | 1.495 | 1.442 | 1.384 | 1.323 | 1.263 | 1.204 | 1.147 |
| 100 | 0.950 | 1.498 | 1.477 | 1.443 | 1.399 | 1.349 | 1.294 | 1.237 | 1.180 | 1.125 | 1.071 |
| | 0.975 | 1.537 | 1.515 | 1.481 | 1.436 | 1.384 | 1.328 | 1.270 | 1.212 | 1.155 | 1.100 |
| | 0.990 | 1.584 | 1.562 | 1.526 | 1.480 | 1.427 | 1.369 | 1.310 | 1.250 | 1.191 | 1.135 |

## 7. APPLICATION EXAMPLE

TFT-LCDs (thin-film-transistor liquid crystal display) consist of a lower glass plate on which the TFT is formed, an upper glass plate on which the color filter is formed, and the injected liquid crystal between both glass plates (see Figure 7(a)). The TFT plays a critical role in transmitting and controlling electric signals, which determines the amount of voltage applied to the liquid crystal. The liquid crystal controls light permeability using different molecular structures that vary in accordance with the voltage. In this way, the desired color and image is displayed as it passes through the color filter (see Figure 7(b)). The TFT-LCD consumes less energy compared to a CRT (cathode-ray tube), is slimmer and weighs less. The TFT-LCD has emerged as the most widely used display solution, because of its high reliability, viewing quality and performance, compact size and environment-friendly features. Because of the heat resistance, non-conductance and simple processing steps, non-alkali thin-film glass is the major material of manufacturing TFT-LCD. While manufacturing non-alkali thin-film glass, flatness is one of the critical quality characteristics. If the flatness of glass is not in control, the TFT-LCD products may result in a certain degree of chromatic aberration.

Consider a supplier in manufacturing TFT-LCD products in Taiwan, the production specifications of flatness for a particular model of non-alkali thin-film glass are $USL = 25$ $\mu$m (0.0025 mm) and $T = 0$ $\mu$m. A total of 60 observations were collected which are displayed in Table VIII. To determine whether the process is 'satisfactory' ($C_{\mathrm{PU}} > 1.33$) with unavoidable measurement errors $\tau = 0.4$, we propose the following procedure. Step 1: determine the capability requirement $c$ (normally chosen as 1.00, 1.33, 1.50) and the $\alpha$-risk (normally set to 0.01, 0.025 or 0.05). Step 2: calculate the value of the point estimator $\tilde{C}_{\mathrm{I}}$ from the sample.

Table VI. Critical values for $C_{\mathrm{I}} = 1.50$, with $n = 10(10)100$, $\tau = 0.1(0.1)1.0$

| | | $\tau$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $1-\alpha$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 10 | 0.950 | 2.270 | 2.237 | 2.187 | 2.121 | 2.046 | 1.963 | 1.878 | 1.793 | 1.710 | 1.630 |
| | 0.975 | 2.522 | 2.487 | 2.430 | 2.358 | 2.274 | 2.183 | 2.089 | 1.995 | 1.903 | 1.814 |
| | 0.990 | 2.873 | 2.832 | 2.768 | 2.686 | 2.591 | 2.487 | 2.381 | 2.274 | 2.169 | 2.069 |
| 20 | 0.950 | 1.982 | 1.954 | 1.910 | 1.852 | 1.786 | 1.714 | 1.639 | 1.565 | 1.491 | 1.421 |
| | 0.975 | 2.116 | 2.086 | 2.038 | 1.977 | 1.907 | 1.830 | 1.751 | 1.671 | 1.594 | 1.519 |
| | 0.990 | 2.289 | 2.257 | 2.205 | 2.140 | 2.063 | 1.981 | 1.895 | 1.810 | 1.726 | 1.646 |
| 30 | 0.950 | 1.876 | 1.849 | 1.807 | 1.752 | 1.689 | 1.621 | 1.550 | 1.479 | 1.410 | 1.343 |
| | 0.975 | 1.973 | 1.945 | 1.901 | 1.844 | 1.777 | 1.706 | 1.632 | 1.557 | 1.485 | 1.415 |
| | 0.990 | 2.096 | 2.066 | 2.020 | 1.959 | 1.889 | 1.813 | 1.735 | 1.656 | 1.579 | 1.505 |
| 40 | 0.950 | 1.817 | 1.791 | 1.750 | 1.697 | 1.636 | 1.569 | 1.501 | 1.432 | 1.365 | 1.300 |
| | 0.975 | 1.896 | 1.869 | 1.826 | 1.771 | 1.707 | 1.638 | 1.567 | 1.495 | 1.425 | 1.358 |
| | 0.990 | 1.995 | 1.966 | 1.921 | 1.864 | 1.797 | 1.725 | 1.650 | 1.575 | 1.501 | 1.431 |
| 50 | 0.950 | 1.778 | 1.753 | 1.713 | 1.661 | 1.601 | 1.536 | 1.469 | 1.401 | 1.335 | 1.272 |
| | 0.975 | 1.846 | 1.820 | 1.778 | 1.724 | 1.662 | 1.595 | 1.525 | 1.456 | 1.387 | 1.322 |
| | 0.990 | 1.930 | 1.902 | 1.859 | 1.803 | 1.738 | 1.668 | 1.596 | 1.523 | 1.452 | 1.383 |
| 60 | 0.950 | 1.750 | 1.725 | 1.686 | 1.635 | 1.576 | 1.512 | 1.445 | 1.379 | 1.314 | 1.251 |
| | 0.975 | 1.811 | 1.785 | 1.744 | 1.691 | 1.630 | 1.564 | 1.496 | 1.427 | 1.360 | 1.295 |
| | 0.990 | 1.884 | 1.858 | 1.815 | 1.761 | 1.697 | 1.629 | 1.558 | 1.486 | 1.417 | 1.350 |
| 70 | 0.950 | 1.729 | 1.705 | 1.665 | 1.615 | 1.557 | 1.493 | 1.428 | 1.362 | 1.298 | 1.236 |
| | 0.975 | 1.784 | 1.758 | 1.718 | 1.666 | 1.606 | 1.541 | 1.473 | 1.406 | 1.339 | 1.276 |
| | 0.990 | 1.850 | 1.824 | 1.782 | 1.729 | 1.666 | 1.599 | 1.529 | 1.459 | 1.391 | 1.325 |
| 80 | 0.950 | 1.713 | 1.688 | 1.649 | 1.599 | 1.542 | 1.479 | 1.414 | 1.349 | 1.285 | 1.223 |
| | 0.975 | 1.763 | 1.737 | 1.698 | 1.646 | 1.587 | 1.522 | 1.456 | 1.389 | 1.323 | 1.260 |
| | 0.990 | 1.823 | 1.797 | 1.756 | 1.703 | 1.642 | 1.575 | 1.507 | 1.438 | 1.370 | 1.305 |
| 90 | 0.950 | 1.699 | 1.675 | 1.636 | 1.587 | 1.529 | 1.467 | 1.402 | 1.338 | 1.274 | 1.213 |
| | 0.975 | 1.745 | 1.720 | 1.681 | 1.630 | 1.571 | 1.507 | 1.441 | 1.375 | 1.310 | 1.248 |
| | 0.990 | 1.802 | 1.776 | 1.735 | 1.683 | 1.622 | 1.556 | 1.488 | 1.420 | 1.353 | 1.289 |
| 100 | 0.950 | 1.688 | 1.663 | 1.625 | 1.576 | 1.519 | 1.457 | 1.393 | 1.328 | 1.265 | 1.205 |
| | 0.975 | 1.731 | 1.706 | 1.667 | 1.617 | 1.558 | 1.495 | 1.429 | 1.363 | 1.299 | 1.237 |
| | 0.990 | 1.784 | 1.758 | 1.718 | 1.666 | 1.606 | 1.541 | 1.473 | 1.406 | 1.339 | 1.276 |

Step 3: check the appropriate table listed in Tables IV–VII and find the corresponding critical value $c_0^*$ based on $\alpha$, $\tau$ and $n$. Step 4: conclude that the process meets the capability requirement if $\tilde{C}_{\mathrm{I}}$ is greater than $c_0^*$. Otherwise, we do not have enough information to conclude that the process is capable.

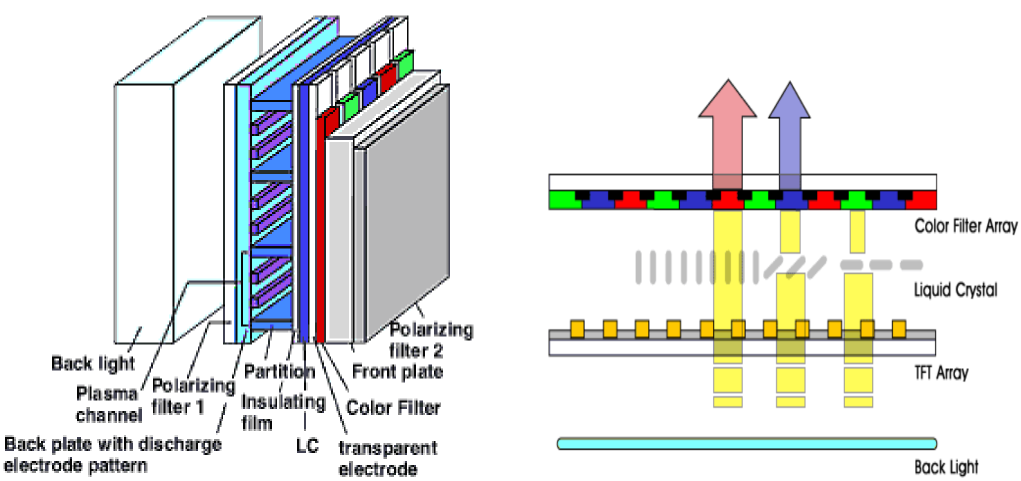
With the proposed procedure, we first determine that $c = 1.33$ and $\alpha = 0.05$. Based on the sample data of 60 observations, we obtain the sample mean $\bar{Y} = 11.93$, the sample standard deviation $S_Y = 2.85$ and the point estimator $\tilde{C}_{\mathrm{PU}}^Y = 1.511$. From Table VI, we find the critical value $c_0^* = 1.452$ based on $\alpha$, $\tau$ and $n$. Since $\tilde{C}_{\mathrm{PU}}^Y > c_0^*$, we conclude that the process is 'satisfactory'. Moreover, by inputting $\tilde{C}_{\mathrm{PU}}^Y$, $\tau$, $n$ and the desired confidence coefficient $\theta = 0.95$ into the computer program, we can obtain the 95% lower confidence bound of this process capability as 1.385.

## 8. *CONCLUSIONS*

In this paper, we investigated the estimation and testing the one-sided process capability index $C_{\mathrm{I}}$ with measurement errors. We considered the estimator $\tilde{C}_{\mathrm{I}}^Y$ rather than $\tilde{C}_{\mathrm{I}}$ for estimating $C_{\mathrm{I}}$, using the sample data contaminated by random measurement errors. The estimator $\tilde{C}_{\mathrm{I}}^Y$ underestimates the true process capability, and the bias decreases in $\tau$, with $\mathrm{Var}(\tilde{C}_{\mathrm{I}}^Y) < \mathrm{Var}(\tilde{C}_{\mathrm{I}})$, and $\mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y) > \mathrm{MSE}(\tilde{C}_{\mathrm{I}})$ if $\tau > \tau_0$, $\mathrm{MSE}(\tilde{C}_{\mathrm{I}}^Y) < \mathrm{MSE}(\tilde{C}_{\mathrm{I}})$ if $\tau < \tau_0$. In estimating the capability, the confidence bounds are severely underestimated in the presence of

Table VII. Critical values for $C_I = 2.00$, with $n = 10(10)100$, $\tau = 0.1(0.1)1.0$

| | | $\tau$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $1-\alpha$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 10 | 0.950 | 3.011 | 2.968 | 2.900 | 2.813 | 2.711 | 2.601 | 2.487 | 2.373 | 2.261 | 2.153 |
| | 0.975 | 3.345 | 3.297 | 3.221 | 3.124 | 3.012 | 2.889 | 2.763 | 2.636 | 2.513 | 2.393 |
| | 0.990 | 3.807 | 3.752 | 3.667 | 3.556 | 3.428 | 3.290 | 3.146 | 3.002 | 2.862 | 2.726 |
| 20 | 0.950 | 2.632 | 2.594 | 2.535 | 2.458 | 2.369 | 2.273 | 2.173 | 2.072 | 1.974 | 1.880 |
| | 0.975 | 2.808 | 2.767 | 2.704 | 2.622 | 2.527 | 2.425 | 2.318 | 2.212 | 2.107 | 2.007 |
| | 0.990 | 3.036 | 2.992 | 2.924 | 2.835 | 2.733 | 2.622 | 2.508 | 2.393 | 2.280 | 2.172 |
| 30 | 0.950 | 2.492 | 2.456 | 2.400 | 2.327 | 2.242 | 2.151 | 2.056 | 1.961 | 1.868 | 1.779 |
| | 0.975 | 2.620 | 2.582 | 2.523 | 2.446 | 2.358 | 2.262 | 2.162 | 2.063 | 1.965 | 1.871 |
| | 0.990 | 2.782 | 2.742 | 2.679 | 2.598 | 2.504 | 2.402 | 2.297 | 2.191 | 2.088 | 1.989 |
| 40 | 0.950 | 2.414 | 2.380 | 2.325 | 2.254 | 2.172 | 2.084 | 1.992 | 1.900 | 1.810 | 1.723 |
| | 0.975 | 2.518 | 2.482 | 2.425 | 2.351 | 2.266 | 2.174 | 2.078 | 1.982 | 1.888 | 1.798 |
| | 0.990 | 2.648 | 2.610 | 2.550 | 2.473 | 2.383 | 2.286 | 2.186 | 2.085 | 1.987 | 1.892 |
| 50 | 0.950 | 2.364 | 2.330 | 2.276 | 2.207 | 2.127 | 2.040 | 1.950 | 1.859 | 1.771 | 1.686 |
| | 0.975 | 2.453 | 2.418 | 2.362 | 2.290 | 2.207 | 2.117 | 2.024 | 1.930 | 1.839 | 1.751 |
| | 0.990 | 2.563 | 2.526 | 2.468 | 2.393 | 2.307 | 2.213 | 2.115 | 2.018 | 1.922 | 1.830 |
| 60 | 0.950 | 2.328 | 2.294 | 2.241 | 2.173 | 2.094 | 2.008 | 1.920 | 1.831 | 1.744 | 1.660 |
| | 0.975 | 2.406 | 2.372 | 2.317 | 2.247 | 2.165 | 2.077 | 1.985 | 1.893 | 1.803 | 1.717 |
| | 0.990 | 2.503 | 2.467 | 2.411 | 2.338 | 2.253 | 2.161 | 2.066 | 1.970 | 1.877 | 1.787 |
| 70 | 0.950 | 2.300 | 2.267 | 2.215 | 2.147 | 2.069 | 1.984 | 1.897 | 1.809 | 1.723 | 1.640 |
| | 0.975 | 2.371 | 2.337 | 2.283 | 2.214 | 2.133 | 2.046 | 1.956 | 1.865 | 1.777 | 1.691 |
| | 0.990 | 2.458 | 2.423 | 2.367 | 2.296 | 2.212 | 2.122 | 2.028 | 1.935 | 1.843 | 1.755 |
| 80 | 0.950 | 2.278 | 2.245 | 2.193 | 2.127 | 2.049 | 1.965 | 1.878 | 1.791 | 1.706 | 1.624 |
| | 0.975 | 2.343 | 2.310 | 2.257 | 2.188 | 2.108 | 2.022 | 1.933 | 1.843 | 1.756 | 1.671 |
| | 0.990 | 2.423 | 2.388 | 2.333 | 2.263 | 2.181 | 2.091 | 1.999 | 1.907 | 1.816 | 1.729 |
| 90 | 0.950 | 2.260 | 2.228 | 2.176 | 2.110 | 2.033 | 1.950 | 1.863 | 1.777 | 1.692 | 1.611 |
| | 0.975 | 2.321 | 2.287 | 2.235 | 2.167 | 2.088 | 2.003 | 1.914 | 1.825 | 1.738 | 1.655 |
| | 0.990 | 2.395 | 2.360 | 2.306 | 2.236 | 2.155 | 2.067 | 1.975 | 1.884 | 1.795 | 1.709 |
| 100 | 0.950 | 2.245 | 2.213 | 2.162 | 2.096 | 2.020 | 1.937 | 1.851 | 1.765 | 1.681 | 1.600 |
| | 0.975 | 2.302 | 2.269 | 2.217 | 2.149 | 2.071 | 1.986 | 1.898 | 1.810 | 1.724 | 1.641 |
| | 0.990 | 2.371 | 2.337 | 2.283 | 2.214 | 2.133 | 2.046 | 1.956 | 1.865 | 1.777 | 1.691 |




(a) (b)

Figure 7. (a) Structure of TFT-LCD; (b) light passing through the color filter

Table VIII. 60 observations for flatness (in $\mu$m)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 14.40 | 4.47 | 11.18 | 8.29 | 9.38 | 8.73 | 11.64 | 6.59 | 12.55 | 12.83 | 14.40 |
| 12.18 | 14.73 | 12.22 | 10.42 | 11.56 | 14.37 | 11.76 | 8.06 | 10.03 | 5.45 | 12.18 |
| 14.40 | 15.28 | 9.60 | 15.01 | 12.36 | 14.69 | 10.71 | 6.96 | 8.88 | 16.30 | 14.40 |
| 15.53 | 15.22 | 12.02 | 12.95 | 10.50 | 15.09 | 11.23 | 8.33 | 13.76 | 12.19 | 15.53 |
| 9.93 | 9.14 | 10.41 | 15.34 | 12.94 | 10.24 | 14.44 | 12.54 | 10.40 | 13.47 | 9.93 |
| 13.22 | 16.93 | 18.41 | 11.19 | 15.09 | 9.40 | 12.22 | 12.17 | 13.80 | 12.60 | 13.22 |

measurement errors. When we use statistical testing to determine if the process meets the capability requirement, we observe that the power of the test decreases in measurement errors. As the measurement errors are unavoidable in most manufacturing industry, to obtain a more accurate confidence bound and improve the power with appropriate $\alpha$-risk, we must adjust the confidence bounds and the critical values. For practical purpose, we tabulated some adjusted critical values for the engineers to use in their factory applications.

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