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Reliable Control of Nonlinear Systems via Variable Structure Scheme

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***Abstract*—This study proposes a class of variable structure stabilizing laws which make the closed-loop system be capable of tolerating the abnormal operation of actuators within a pre-specified subset of actuators. The ranges of acceptable change in control gain magnitude that preserves system's stability are estimated for the whole set of actuators. These ranges are shown to be able to be made larger than those obtained by linear quadratic regulator (LQR) reliable design (Veillette, 1995, and Liang *et al.*, 2000) by the choice of control parameters. Besides, this approach doesnot need the solution of Hamilton-Jacobi (HJ) equation or inequality, which is essential for optimal approaches such as LQR and H^∞ reliable designs. As a matter of fact, this approach can also relax the computational burden for solving the HJ equation or inequality.**

***Index Terms*—Hamilton–Jacobi (HJ) equation, nonlinear systems, reliable control, variable structure control (VSC).**

Manuscript received April 16, 2004; revised November 8, 2004, July 28, 2005, and December 30, 2005. Recommended by Associate Editor D. Nesic. This work was supported by the National Science Council, Taiwan, R.O.C., under Grants NSC 92-2213-E-009-124 and NSC 93-2218-E-009-037.

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Color versions of Figs. 1–3 are available online at http://ieeexplore.ieee.org.
Digital Object Identifier 10.1109/TAC.2006.880802

I. Introduction

The study of reliable control has recently attracted lots of attention (see, e.g., [1]–[3], [5], [8]–[10], and [12]–[15]). The objective of this study is to design an appropriate controller such that the closed-loop system can tolerate the abnormal operation of some specific control components and retain an overall system stability with acceptable system performance. An abnormal operation may include degradation, amplification and partial outage. From the approach viewpoint, in general, reliable control systems can be classified as active [1]–[3], [5] and passive [8]–[10], [12]–[15]. In this note, we consider only the passive issues. In an active reliable control system, faults are to be detected and identified by a fault detection and diagnosis (FDD) mechanism. Then the controllers are reconfigured according to the online detection results in real time. On the other hand, the passive approach exploits system's inherent redundancies to design a fixed controller so that the closed-loop system can achieve an acceptable performance not only during normal operation but also under various components fail without the need of FDD and controllers' reconfiguration. Although the performances of the active reliable control which uses controllers' reconfiguration are generally superior to those of passive one under various faulty situations, the active approach needs a reliable FDD but the passive one doesnot. This is important when the available reaction time is short after the occurrence of faults.

Several approaches for passive reliable control have been proposed, for example, linear matrix inequality (LMI)-based approach [10], algebraic Riccati equation (ARE)-based approach [12], [13], coprime factorization approach [14] and Hamilton–Jacobi (HJ)-based approach [9], [15]. Although the HJ-based approach is mainly for nonlinear systems, its reliable controllers need a solution of an HJ equation or inequality, which is known not easy to obtain. A power series method [6] may alleviate the difficulty of solving the HJ equation or inequality through computer calculation. However, the obtained solution is only an approximate one and, when system is complicated, the computational load grows fast as the order of the approximated solution increases. Due to these potential drawbacks of the HJ-based approach, this note investigates the reliability issues from the variable structure control (VSC) viewpoint, which is known to have the advantages of fast response and low sensitivity to model uncertainties and/or external disturbances (see, e.g., [4], [7], and [11]). In this note, we propose a VSC design that is shown to be able to tolerate the abnormal operation of actuators within a prespecified subset of actuators. The regions of acceptable change in control gain magnitude that preserves system's stability are also estimated. These regions are shown to be able to be made larger than those obtained in [9] and [13] by suitable choice of control parameters. Besides, the VSC approach needs not the solution of HJ equation or inequality. Thus, this approach can also alleviate the computational burden for solving the HJ equation or inequality.

This note is organized as follows. The reliable control problem and the main goal of the note are given in Section II. This is followed by designing the VSC controllers and analyzing their reliability. An example is also given in this section to demonstrate the use and the benefits of the design. Finally, Section IV gives concluding remarks.

II. Problem Statement

Consider a class of nonlinear control systems as described by

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}) + G_{\Omega_1}(\mathbf{x})\mathbf{u}_\Omega \tag{1}$$

and

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}) + G_{\Omega_2}(\mathbf{x})\mathbf{u}_\Omega + G_{\Omega'}(\mathbf{x})\mathbf{u}_{\Omega'} + \mathbf{d}. \tag{2}$$

Here, $\mathbf{x}_1 \in \mathbb{R}^{n_1}, \mathbf{x}_2 \in \mathbb{R}^{n_2}$, and $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T)^T$ denotes the system states, $\mathbf{u}_\Omega \in \mathbb{R}^{m_1}$ and $\mathbf{u}_{\Omega'} \in \mathbb{R}^{n_2}$ are the control inputs, $\mathbf{d} = (d_1, \ldots, d_{n_2})^T$ denotes possible model uncertainties and/or external disturbances, and $(\cdot)^T$ denotes transpose of a matrix or a vector. $\mathbf{f}_1(\mathbf{x}) \in \mathbb{R}^{n_1}, \mathbf{f}_2(\mathbf{x}) \in \mathbb{R}^{n_2}, G_{\Omega_1}(\mathbf{x}) \in \mathbb{R}^{n_1 \times m_1}, G_{\Omega_2}(\mathbf{x}) \in \mathbb{R}^{n_2 \times m_1}$, and $G_{\Omega'}(\mathbf{x}) \in \mathbb{R}^{n_2 \times n_2}$ are smooth functions. For the interest of study, we assume that $\mathbf{f}_1(\mathbf{0}) = \mathbf{0}$ and $\mathbf{f}_2(\mathbf{0}) = \mathbf{0}$. Note that we have divided the control inputs into two disjoint groups Ω and Ω' within which the abnormal operation of actuators in the set Ω must be tolerated. We also note that system (1)–(2) might come from a general nonlinear affine system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u}$ through a diffeomorphic transformation. When $n_1 = n_2, \mathbf{f}_1(\mathbf{x}) = \mathbf{x}_2$ and $G_{\Omega_1}(\mathbf{x}) = \mathbf{0}$, system (1)–(2) reduces to an important class of second order dynamical systems.

If all the actuators in the set Ω fail to operate, (1)–(2) becomes

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}) \tag{3}$$

and

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}) + G_{\Omega'}(\mathbf{x})\mathbf{u}_{\Omega'} + \mathbf{d}. \tag{4}$$

In practical applications, the number of susceptible actuators in Ω may be selected to be as many as possible. In addition, we assume that system (3)–(4) is in regular form. That is, $G_{\Omega'}(\mathbf{x})$ is a nonsingular matrix, as described in Assumption 1. This assumption is necessary for the existence of the equivalent control (see, e.g., [4]).

Assumption 1: The origin of system (3)–(4) is locally asymptotically stabilizable and $G_{\Omega'}(\mathbf{x})$ is a nonsingular matrix.

In addition to Assumption 1, we also impose the next two assumptions.

Assumption 2: Suppose there exists a function $\mathbf{x}_2 = \phi(\mathbf{x}_1)$ such that the reduced-order system $\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \phi(\mathbf{x}_1))$ has an asymptotically stable (AS) equilibrium point at the origin $\mathbf{x}_1 = \mathbf{0}$.

Assumption 3: There exist functions $\rho_i(\mathbf{x}, t) \geq 0, i = 1, \ldots, n_2$, such that $|d_i| \leq \rho_i(\mathbf{x}, t)$.

The objective of this study is then to organize $\mathbf{u}_{\Omega'}$ and $\mathbf{u}_\Omega$ so that the origin of the closed-loop system is AS even when the actuators in the set Ω experience abnormal operation. The susceptible actuators in this design are used to improve system performance when they are available.

III. Main Results

To achieve the objective of the note as stated previously, in this section we first employ the VSC technique to design the reliable controllers. This is followed by analyzing the overall reliability of the designed system. Finally, we present an illustrative example to demonstrate the benefits of the design.

A. Design of Reliable Controllers

The idea is first to organize a VSC law $\mathbf{u}_{\Omega'}$ as if $\mathbf{u}_\Omega$ is unavailable. Then the remaining controls $\mathbf{u}_\Omega$ are designed to promote the overall system performances. Suppose now that all the actuators in Ω are unavailable. Then, by Assumption 2, we choose the sliding surface to be

$$\mathbf{s} = \mathbf{x}_2 - \phi(\mathbf{x}_1) = 0. \tag{5}$$

It follows from (3)–(4) that

$$\dot{\mathbf{s}} = \mathbf{f}_2(\mathbf{x}) + G_{\Omega'}(\mathbf{x})\mathbf{u}_{\Omega'} - \frac{\partial \phi}{\partial \mathbf{x}_1} \cdot \mathbf{f}_1(\mathbf{x}) + \mathbf{d}. \tag{6}$$

Following the VSC design procedure [11], the VSC law for actuators in Ω' is designed as

$$\mathbf{u}_{\Omega'}^* = G_{\Omega'}^{-1}(\mathbf{x}) \left\{ \frac{\partial \phi}{\partial \mathbf{x}_1} \cdot \mathbf{f}_1(\mathbf{x}) - \mathbf{f}_2(\mathbf{x}) - \Lambda_{\Omega'} \cdot \operatorname{sgn}(\mathbf{s}) \right\} \tag{7}$$

where $\Lambda_{\Omega'} = \operatorname{diag}(\eta_1, \ldots, \eta_{n_2})$ with $\eta_i > \rho_i(\mathbf{x}, t) + r_i$ and $r_i > 0$ for all $i = 1, \ldots, n_2, \operatorname{sgn}(\cdot)$ denotes the sign function and $\operatorname{sgn}(\mathbf{s}) := (\operatorname{sgn}(s_1), \ldots, \operatorname{sgn}(s_{n_2}))^T$. By direct calculation

$$\mathbf{s}^T \dot{\mathbf{s}} \leq -\sum_{i=1}^{n_2} r_i \cdot |s_i| \tag{8}$$

where s_i denotes the ith entry of the sliding vector $\mathbf{s}$. Equation (8) implies that the system states will reach the sliding surface in a finite time and remain there [11]. Then, by Assumption 2, the reduced-order dynamics on sliding surface makes the states slide toward the origin.

In addition to the design of actuators in Ω', as described by (7), we now suppose that actuators in Ω are also available. The governing equation in this case is given by (1)–(2). From (1)–(2), and (5), $\mathbf{u}_{\Omega'} = \mathbf{u}_{\Omega'}^*$ given in (7), and (8) we have

$$\mathbf{s}^T \dot{\mathbf{s}} \leq -\sum_{i=1}^{n_2} r_i \cdot |s_i| + \mathbf{s}^T \Gamma(\mathbf{x}) \mathbf{u}_\Omega \tag{9}$$

where $\Gamma(\mathbf{x}) := G_{\Omega_2}(\mathbf{x}) - ((\partial\phi)/(\partial\mathbf{x}_1)) \cdot G_{\Omega_1}(\mathbf{x})$. Clearly, an intuitive candidate of $\mathbf{u}_\Omega$ to make $\mathbf{s}^T\mathbf{s}$ more negative than the case of $\mathbf{u}_\Omega = \mathbf{0}$ has the form

$$\mathbf{u}_\Omega^* = -\Lambda_\Omega \cdot \operatorname{sgn}(\Gamma^T(\mathbf{x})\mathbf{s}) \tag{10}$$

where $\Lambda_\Omega := \operatorname{diag}(\mu_1, \ldots, \mu_{m_1})$ and $\mu_i \geq 0$ for all $i = 1, \ldots, m_1$. In practical applications, actuators might experience a change in control gain magnitude which covers the cases of normal operation, degradation, amplification and outage. Therefore, $\mathbf{u}_\Omega$ has the form of (11)

$$\mathbf{u}_\Omega = N_\Omega \mathbf{u}_\Omega^* \tag{11}$$

where $N_\Omega \in \mathbb{R}^{m_1 \times m_1}$ is a nonsingular diagonal matrix which denotes the change in gain magnitude of $\mathbf{u}_\Omega^*$. Clearly, the two cases $N_\Omega = I$ and $N_\Omega = 0$ correspond to the situations where all actuators in Ω are in normal operation and are in outage, respectively. It follows from (9)–(11) that

$$\mathbf{s}^T \dot{\mathbf{s}} \leq -\sum_{i=1}^{n_2} r_i \cdot |s_i| - \sum_{j=1}^{m_1} (N_\Omega)_{jj} \cdot \mu_j \cdot |(\Gamma^T(\mathbf{x})\mathbf{s})_j| \tag{12}$$

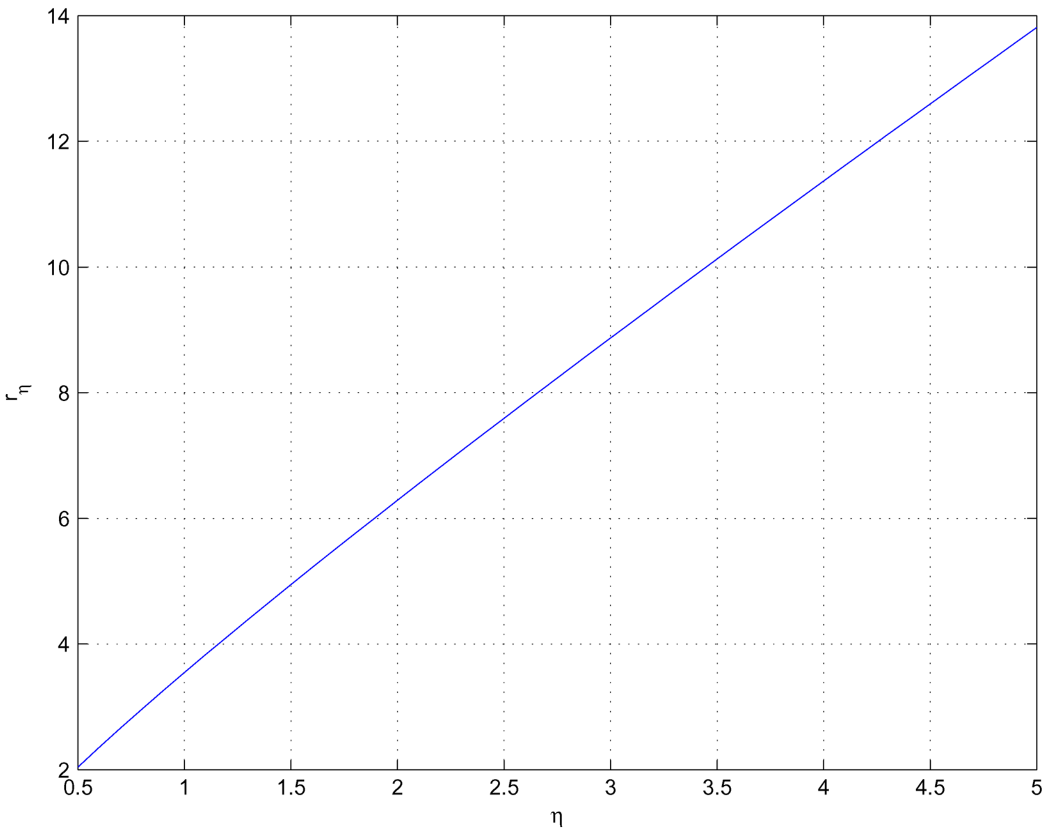

Fig. 1. Relation between $\boldsymbol{r}_\eta$ and $\boldsymbol{\eta}$ in the ROS estimation of VSC reliable design.

where $(\cdot)_j$ and $(\cdot)_{ij}$ denote the jth entry of a vector and the (i,j)-entry of a matrix, respectively. Equation (12) implies that, when some or all of actuators in Ω are healthy and $\mathbf{u}^*_{\Omega'}$ has been chosen in the form of (7), system states towards the sliding surface are faster than the case when all actuators in Ω fail to operate. These lead to the next result.

Theorem 1: Suppose that Assumptions 1–3 hold. Then the origin of system (1)–(2) is locally asymptotically stable (AS) under the controls (7) and (10) even when some or all of actuators in Ω experience abnormal operation in the sense of (11).

B. Reliability Analysis

System (1)–(2) with controls given by (7) and (10) discussed above has been shown to be able to tolerate any abnormal operation of actuators in Ω. In this section, we will also estimate the allowable changes in control gain magnitude of actuators in Ω' that still guarantee asymptotic stability performance of the system with control (7) and (10). For this purpose, we suppose that $G_{\Omega'}(\mathbf{x})$ is a diagonal matrix and the actual effective controls in Ω' have the form

$$\mathbf{u}_{\Omega'} = N_{\Omega'}\mathbf{u}^*_{\Omega'} \tag{13}$$

where $N_{\Omega'} \in \mathbb{R}^{n_2\times n_2}$ is a nonnegative diagonal matrix which denotes the change in gain magnitude of $\mathbf{u}^*_{\Omega'}$. Under the effective controls (11) and (13), the overall system becomes

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}) + G_{\Omega_1}(\mathbf{x})N_\Omega \mathbf{u}^*_\Omega \tag{14}$$

and

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}) + G_{\Omega_2}(\mathbf{x})N_\Omega \mathbf{u}^*_\Omega + G_{\Omega'}(\mathbf{x})N_{\Omega'}\mathbf{u}^*_{\Omega'} + \mathbf{d}. \tag{15}$$

Since both $G_{\Omega'}(\mathbf{x})$ and $N_{\Omega'}$ are diagonal matrices, we then have

$$\begin{aligned}\mathbf{s}^T\dot{\mathbf{s}} = \mathbf{s}^T \Bigg[(N_{\Omega'} - I)\left(\frac{\partial\phi}{\partial\mathbf{x}_1}\mathbf{f}_1(\mathbf{x}) - \mathbf{f}_2(\mathbf{x})\right) \\ - N_{\Omega'}\Lambda_{\Omega'}\mathrm{sgn}(\mathbf{s}) + \mathbf{d}\Bigg] \\ - \sum_{j=1}^{m_1}\mu_j(N_\Omega)_{jj}|(\Gamma(\mathbf{x})^T\mathbf{s})_j|.\end{aligned} \tag{16}$$

Similar to the derivation of robust controllers in [11], we have the next result, which addresses the reliability of the design.

Theorem 2: Suppose that Assumptions 1–3 hold and $G_{\Omega'}(\mathbf{x})$ is a diagonal matrix. Then, the origin of system (1)–(2) is locally AS under the effective controls given by (11) and (13) if

$$\rho_i(\mathbf{x},t) + \left|\left((N_{\Omega'} - I)\left(\frac{\partial\phi}{\partial\mathbf{x}_1}\cdot\mathbf{f}_1(\mathbf{x}) - \mathbf{f}_2(\mathbf{x})\right)\right)_i\right| < (N_{\Omega'}\Lambda_{\Omega'})_{ii}, \text{ for all } i = 1,\ldots,n_2. \tag{17}$$

To compare the result with those given in [9] and [13], we consider the special case where $\mathbf{d} = 0$ and $N_{\Omega'} \geq I/2$. The latter implies that $|(N_{\Omega'} - I)_{ii}| \leq (N_{\Omega'})_{ii}$. Condition (17) can then be simplified as (18).

Corollary 1: Suppose that $N_{\Omega'} \geq I/2, \mathbf{d} = 0, G_{\Omega'}(\mathbf{x})$ is a diagonal matrix, and Assumptions 1 and 2 hold. Then, the origin of system (1)–(2) is locally AS under the control given by (13) if

$$\left|\left(\frac{\partial\phi}{\partial\mathbf{x}_1}\mathbf{f}_1(\mathbf{x}) - \mathbf{f}_2(\mathbf{x})\right)_i\right| < (\Lambda_{\Omega'})_{ii}, \quad \text{for all } i = 1,\ldots,n_2. \tag{18}$$

Remark 1: If all states are available for feedback, then the control parameters $(\Lambda_{\Omega'})_{ii}$ (or η_i) in (18) can be assigned dynamically

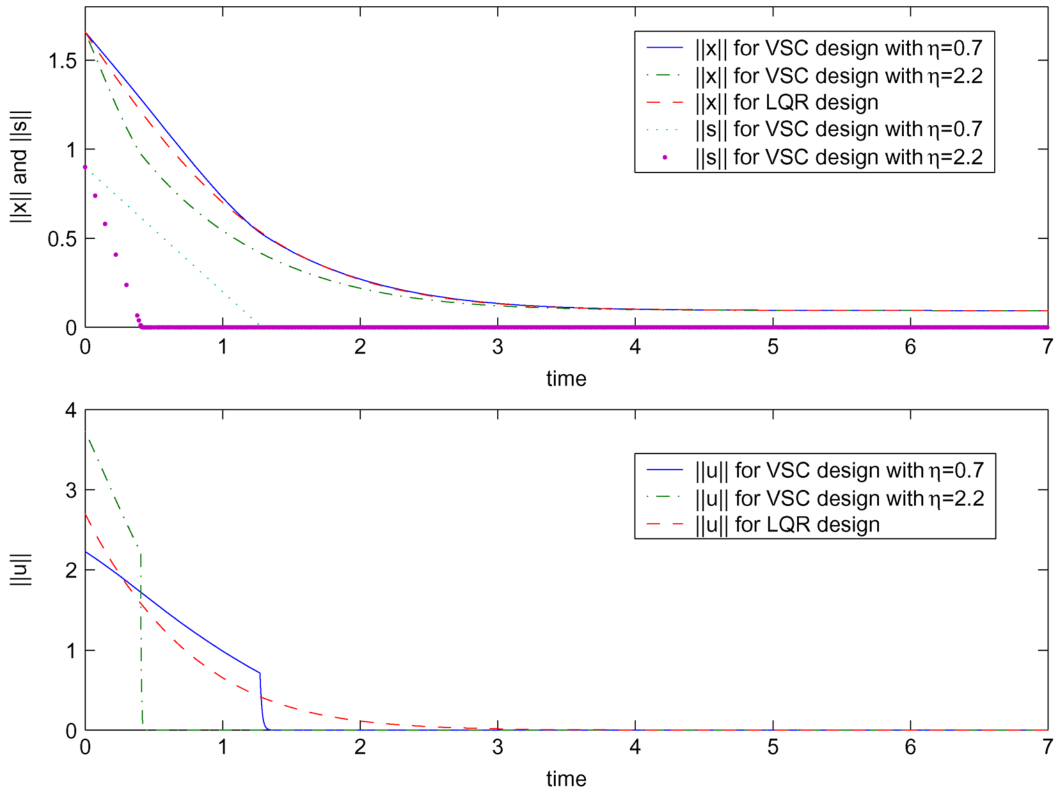

Fig. 2. Norm of states, sliding vector, and control inputs for $\boldsymbol{N}_\Omega = \mathbf{0}$.

as $|(((\partial\phi)/(\partial\mathbf{x}_1))\mathbf{f}_1(\mathbf{x}) - \mathbf{f}_2(\mathbf{x}))_i| + c_i$ for some positive constant c_i. Otherwise, they can be tuned as large as possible to increase the response but fulfill the maximum control magnitude constraint. Clearly, (18) will establish a region of stability (ROS) that depends on the choice of η_i. ■

Remark 2: From Corollary 1, we know that, when the control parameters $(\Lambda_{\Omega'})_{ii}$ (or η_i) for $i = 1, \ldots, n_2$ are selected to satisfy (18), the ranges $N_{\Omega'} \geq I/2$ and $N_\Omega \geq 0$ are sufficient to guarantee the asymptotic stability performance of the closed-loop system. Since this note only deals with passive reliable control (i.e., without requiring fault information), the conditions $N_{\Omega'} \geq I/2$ and $N_\Omega \geq 0$ then characterize the reliability level of the closed-loop system. That is, the asymptotic stability is preserved even when the actuators in Ω experience abnormal operation in any order and in any combination. A larger region for $N_{\Omega'}$ may also be allowed if $\Lambda_{\Omega'}$ is chosen to satisfy (17). Thus, the acceptable regions of N_Ω and $N_{\Omega'}$ for system's stability can be made larger than those given by [9] and [13]. However, the enlargement of the gain magnitude $\Lambda_{\Omega'}$ might come at the price of increased chatter in the sliding mode. ■

Remark 3: It is noted that $N_\Omega \mathbf{u}_\Omega^* = 0$ if system states keep staying on sliding surface. This implies that the actuators in Ω have no effect on reduced-order dynamics no matter whether or not they are in normal operation. In order to promote system performance on sliding surface and keep the same reliability level as in Theorem 2 when the actuators in Ω are available, $\mathbf{u}_\Omega^*$ can be modified as $\mathbf{u}_\Omega^{**}$ given by (19) if a Lyapunov function $V(\mathbf{x}_1)$ of the reduced-order system given in Assumption 2 is known

$$\mathbf{u}_\Omega^{**} = -\Lambda_\Omega \cdot \mathrm{sgn}(\Gamma^T(\mathbf{x})\mathbf{s}) - K_\Omega \cdot G_{\Omega_1}^T(\mathbf{x}_1, \phi(\mathbf{x}_1)) \left(\frac{\partial V}{\partial \mathbf{x}_1}\right)^T \quad (19)$$

where $K_\Omega = \mathrm{diag}(k_1, \ldots, k_{m_1})$ and $k_i \geq 0$ for all $i = 1, \ldots, m_1$. The derivatives of $V(\mathbf{x}_1)$ along the reduced-order system $\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \phi(\mathbf{x}_1)) + G_{\Omega_1}(\mathbf{x}_1, \phi(\mathbf{x}_1))N_\Omega \mathbf{u}_\Omega$ with $\mathbf{u}_\Omega = \mathbf{u}_\Omega^{**}$ and $\mathbf{u}_\Omega = \mathbf{u}_\Omega^*$ are found to be $\dot{V}|_{\mathbf{u}_\Omega = \mathbf{u}_\Omega^{**}} = ((\partial V)/(\partial \mathbf{x}_1)) \cdot \mathbf{f}_1(\mathbf{x}_1, \phi(\mathbf{x}_1)) - \sum_{i=1}^{m_1} k_i \cdot (N_\Omega)_i \cdot (((\partial V)/(\partial \mathbf{x}_1))G_{\Omega_1}(\mathbf{x}_1, \phi(\mathbf{x}_1)))_i^2$ and $\dot{V}|_{\mathbf{u}_\Omega = \mathbf{u}_\Omega^*} = ((\partial V)/(\partial \mathbf{x}_1)) \cdot \mathbf{f}_1(\mathbf{x}_1, \phi(\mathbf{x}_1))$, respectively, since $N_\Omega \mathbf{u}_\Omega^* = 0$ whenever system states keep staying on sliding surface. Clearly, $\dot{V}|_{\mathbf{u}_\Omega^{**}} \leq \dot{V}|_{\mathbf{u}_\Omega^*} < 0$. This implies that the convergence speed of the reduced-order system with $\mathbf{u}_\Omega = \mathbf{u}_\Omega^{**}$ is faster than that with $\mathbf{u}_\Omega = \mathbf{u}_\Omega^*$. Next, we investigate the reliability of (1)–(2) under controls (13) with $\mathbf{u}_\Omega^*$ being replaced by $\mathbf{u}_\Omega^{**}$. In this case, $\mathbf{s}^T\dot{\mathbf{s}}$ is modified from (16) as

$$\begin{aligned}\mathbf{s}^T\dot{\mathbf{s}} &= \mathbf{s}^T\left[(N_{\Omega'} - I)\left(\frac{\partial\phi}{\partial\mathbf{x}_1}\mathbf{f}_1(\mathbf{x}) - \mathbf{f}_2(\mathbf{x})\right)\right. \\ &\quad \left. - N_{\Omega'}\Lambda_{\Omega'} \cdot \mathrm{sgn}(\mathbf{s}) + \mathbf{d}\right] \\ &\quad - \sum_{j=1}^{m_1}(N_\Omega)_j\left\{\mu_j \cdot |(\Gamma(\mathbf{x})^T\mathbf{s})_j|\right. \\ &\quad \left. - k_j\left(\frac{\partial V}{\partial \mathbf{x}_1}G_{\Omega_1}(\mathbf{x}_1, \phi(\mathbf{x}_1))\right)_j\left(\Gamma^T(\mathbf{x})\mathbf{s}\right)_j\right\}. \end{aligned} \quad (20)$$

To guarantee the same reliability level as those given by Theorem 2 and Corollary 1, the control parameters μ_j and k_j should be selected to satisfy

$$k_j \cdot \left(\frac{\partial V}{\partial \mathbf{x}_1}G_{\Omega_1}(\mathbf{x}_1, \phi(\mathbf{x}_1))\right)_j \leq \mu_j, \text{ for } j = 1, \ldots, m_1 \quad (21)$$

in addition to (17). In particular, if $\Gamma(\mathbf{x}) = 0$, then (16) and (20) become the same. This implies that the requirement of (21) can be removed without affecting the reliability level. ■

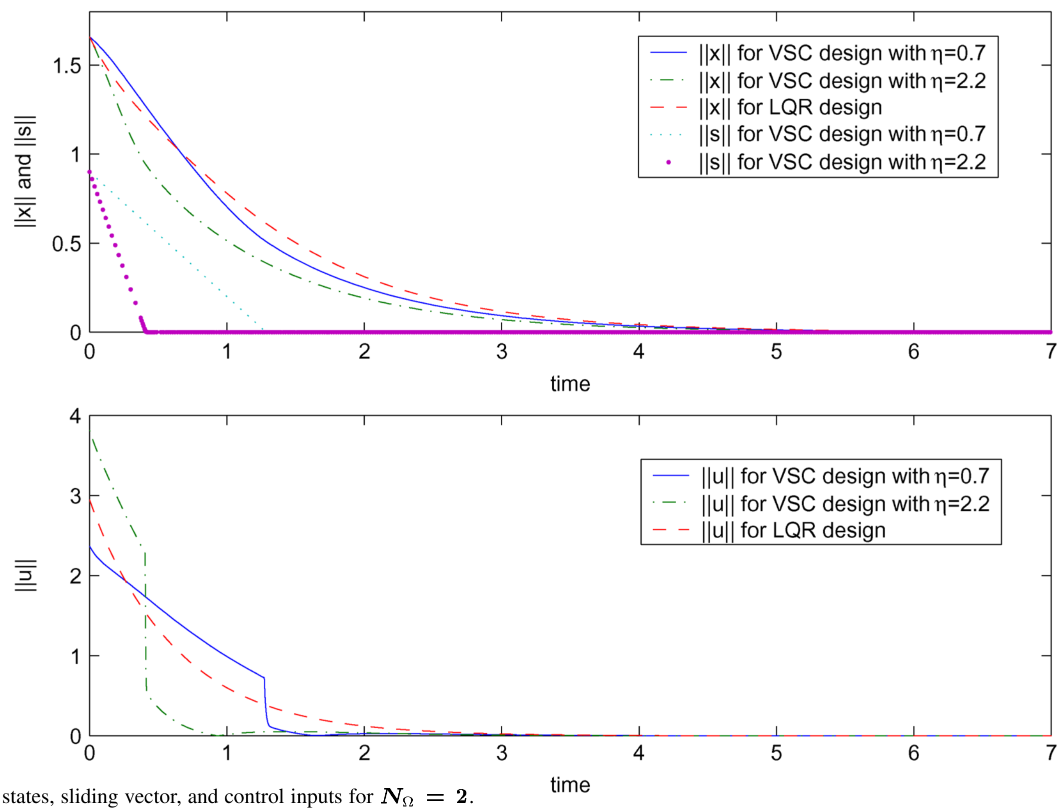

Fig. 3. Norm of states, sliding vector, and control inputs for $\boldsymbol{N}_\Omega = \mathbf{2}$.

C. An Illustrative Example

Consider a nonlinear control system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) + u_1 \cdot \mathbf{g}_1(\mathbf{y}) + u_2 \cdot \mathbf{g}_2(\mathbf{y})$ from [9] with $\mathbf{f}(\mathbf{y}) = (-y_1^3, -y_2 + y_3 y_4, -y_3 + y_4^2, y_4 + y_3 y_4)^T, \mathbf{g}_1(\mathbf{y}) = (0, y_1, 0, 1)^T$ and $\mathbf{g}_2(\mathbf{y}) = (1, 0, y_3, 0)^T$. It was pointed out in [9] that $(\mathbf{f}, \mathbf{g}_2)$ is not a stabilizable pair, while $(\mathbf{f}, \mathbf{g}_1)$ is asymptotically stabilizable. This means that the first actuator can not be taken as the susceptible input. Thus, in this example, we consider $\Omega' = u_1$ and $\Omega = u_2$. A change of coordinate $\mathbf{x} = T(\mathbf{y}) = (y_1, y_2 - y_1 y_4, y_3, y_4)^T$ with $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T)^T, \mathbf{x}_1 = (x_1, x_2, x_3)^T$ and $\mathbf{x}_2 = x_4$ leads to the form of (1)–(2) with $\mathbf{f}_1(\mathbf{x}) = (-x_1^3, x_1^3 x_4 - 2x_1 x_4 + x_3 x_4 - x_1 x_3 x_4 - x_2, x_4^2 - x_3)^T, \mathbf{f}_2(\mathbf{x}) = x_4 + x_3 x_4, G_{\Omega_1}(\mathbf{x}) = (1, -x_4, x_3)^T, G_{\Omega_2}(\mathbf{x}) = 0$ and $G_{\Omega'}(\mathbf{x}) = 1$. Clearly, the function $\mathbf{x}_2 = \phi(\mathbf{x}_1) = 0$ fulfills the requirement of Assumption 2, and a Lyapunov function for the reduced-order system $\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, 0)$ is found to be $V(\mathbf{x}_1) = (x_1^2 + x_2^2 + x_3^2)/2$. According to (5), the sliding surface has the form $s = x_4 = 0$, and the VSC laws given by (7) have the form $u_1^* = -x_4 - x_3 x_4 - \eta \cdot \mathrm{sgn}(x_4)$. Since in this example $\Gamma(\mathbf{x}) = G_{\Omega_2}(\mathbf{x}) - ((\partial\phi)/(\partial\mathbf{x}_1)) \cdot G_{\Omega_1}(\mathbf{x}) = 0$, it follows from Remark 3 that u_2 may be selected as $u_2^{**} = -k \cdot G_{\Omega_1}^T(\mathbf{x}_1, 0) \cdot ((\partial V)/(\partial\mathbf{x}_1))^T = -k(x_1 + x_3^2), k \geq 0$, to promote system stability on sliding surface and maintain the same reliability level as those of Theorem 2 without the need of (21). The overall control in terms of original variables $\mathbf{y}$ then has the form

$$\mathbf{u} = -\left(y_4 + y_3 y_4 + \eta \cdot \mathrm{sgn}(y_4), k y_1 + k y_3^2\right)^T. \tag{22}$$

In case u_1 is healthy, the dynamics of y_4 decouples from the others as $\dot{y}_4 = -\eta \cdot \mathrm{sgn}(y_4)$. This implies that y_4 approaches zero (i.e., the states reach the sliding surface) in a finite time $y_4(0)/\eta$. After reaching the sliding surface we have $\mathbf{f}_2(\mathbf{x}) = \mathbf{0}$. Condition (18) is then fulfilled since $\phi(\mathbf{x}_1) = 0$. However, the state might move out of its ROS before reaching the sliding surface. As a matter of fact, it still needs the information of ROS, depending on η, for stability. An estimation of the ROS for u_1 being healthy can be derived as $D_\eta = \{\mathbf{y} | y_1^2 + y_3^2 + y_4^2 < r_\eta\}$ from the analysis given in [9] with slight modifications, where r_η is the solution of (23)

$$\begin{cases} \text{minimize} & y_1^2 + y_3^2 + y_4^2 \\ \text{subject to} & -y_1^4 - y_3^2 + y_3 y_4^2 - \eta |y_4| = 0 \ \& \ (y_1, y_3, y_4) \neq (0, 0, 0). \end{cases} \tag{23}$$

Clearly, r_η is a function of η, and the relation between r_η and η is described in Fig. 1.

To compare the performances between VSC and LQR reliable designs, the LQR reliable laws are recalled from [9] as $\mathbf{u} = (-k y_4, -y_3^2)^T$ with $k > \sqrt{2} + 1$. However, under these laws, the closed-loop dynamics of y_1 is short of linear terms. It follows that y_1 will converge slowly for small y_1 even when u_2 is available. To promote the convergence speed of y_1, we modify the LQR reliable laws as (24) below:

$$\mathbf{u} = \left(-k y_4, -y_1 - y_3^2\right)^T \quad \text{with } k > \sqrt{2} + 1. \tag{24}$$

Clearly, these modified laws result in exponential convergence of y_1 near the origin when u_2 is available, and they are also the LQR reliable laws associated with the class of positive semidefinite solutions $V(\mathbf{x}) = y_1^2 + y_3^2 + k y_4^2, k > \sqrt{2} + 1$, of the same HJ-inequality given in [9].

In this example, the LQR and VSC reliable laws are adopted from (24) with $k = 3$ and (22) with $k = 1$, respectively. The initial states are selected as $\mathbf{y}_0 = (0.1, 1.2, 0.7, 0.9)^T$. To emphasize the relation between control magnitude and speed of response, the value of control parameter η in the VSC law is set to be 0.7 and 2.2. Clearly, $\mathbf{y}_0$ is inside the estimated ROS for $\eta = 0.7$ and 2.2. Furthermore, the sign function is replaced by saturation function with boundary layer width 0.01 to avoid chattering. To examine the influences of the change in control gain magnitude, we also consider the two cases of which $N_2 = 0$ and

2. These two cases correspond to the second loop gain being broken and amplified, respectively.

Numerical simulations are given in Figs. 2–3, which correspond to $N_2 = 0$ and 2, respectively. In these figures, the dashed, solid and dash-dotted lines denote the timing responses of norms of states and controls by LQR and VSC designs with $\eta = 0.7$ and 2.2, respectively. The dotted line and starred line denote the norm of the sliding vector by VSC design with $\eta = 0.7$ and 2.2, respectively. From these figures, system states are observed convergent to zero for all the two cases, which agree with the theoretic results. When u_2 fails to operate, the system is found to have a linear uncontrollable mode $\lambda = 0$ and the associated closed-loop dynamics of y_1 decouples from the others as $\dot{y}_1 = -y_1^3$. It means that y_1 will approach zero but the convergence rate will be progressively smaller as $|y_1|$ gets smaller. This is why the norm of system states in Fig. 2 converges to zero very slowly. In addition, since in this example $\Gamma(\mathbf{x}) = 0$, u_1^* is then the main force to make system states reach the sliding surface. It follows that the first reaching time of system states to the sliding surface depends only on the choice of control parameter η. The first reaching time observed from $||\mathbf{s}||$ in Figs. 2–3 for $\eta = 0.7$ and $\eta = 2.2$ are around $t_{\text{reach}} \approx 1.2$ and 0.4. This implies that the larger the value of η is, the shorter the first reaching time t_{reach} is. These phenomena can also be told from the abrupt change of the control magnitude, where the VSC reliable system is driven mainly by the equivalent part of $\mathbf{u}$ after the first reaching time. This example verifies that the control parameters of the VSC reliable design can be tuned as large as possible to increase the response while fulfilling the maximum control magnitude constraint.

IV. Concluding Remarks

Variable structure type stabilizing control laws have been proposed in this note to study reliable control issues. This approach can alleviate the computational burden for solving the HJ equation or inequality. In addition, the regions of acceptable change in control gain magnitude that guarantees system's stability can be made larger than those obtained by LQR reliable design by the choice of control parameters. As a matter of fact, the control parameters can be tuned as large as possible in practical applications to promote the responding performances while fulfilling the maximum control magnitude constraint.

Acknowledgment

The authors would like to thank Prof. D. Nesic for his assistance and the anonymous reviewers for their constructive comments and helpful suggestions which have led to substantial improvements to this note.

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Comments and Remarks on "On Improved Delay-Dependent Robust Control for Uncertain Time-Delay Systems"

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***Abstract*—The purpose of this note is to correct some statements and numerical examples' results in the above paper. A few remarks are also given to clarify the facts that for systems with *small* delay, the results in Han *et al.* (2003) are much less conservative than those in Kwon and Park (2004); for systems with *nonsmall* delay, the criterion in Kwon and Park (2004) fails to make any conclusion, while the criterion in Han *et al.* (2003) can be applicable to these kinds of systems.**

I. Comments and Remarks

Consider the following system [1]:

$$\begin{cases} \dot{x}(t) = (A+\Delta A)x(t) + (A_1+\Delta A_1)x(t-h) + (B+\Delta B)u(t) \\ x(s) = \phi(s), \ s \in [-h, 0] \end{cases} \tag{1}$$

where ΔA, ΔA_1, and ΔB are uncertain matrices satisfying

$$\Delta A = D_1 F_1(t) E_1, \Delta A_1 = D_2 F_2(t) E_2, \Delta B = D_3 F_3(t) E_3$$

Manuscript received April 17, 2005; revised December 20, 2005 and April 19, 2006. Recommended by Associate Editor L. Magni. This work was supported in part by Central Queensland University for the Research Advancement Awards Scheme Project "Robust Fault Detection, Filtering and Control for Uncertain Systems with Time-Varying Delay" (January 2006–December 2008) and the Strategic Research Project "Delay Effects: Analysis, Synthesis and Applications" (2003–2006).

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Digital Object Identifier 10.1109/TAC.2006.883032