Mathematical Models and Methods in Applied Sciences
Vol. 16, No. 10 (2006) 1627–1651

World Scientific
www.worldscientific.com

HOMOGENIZATION OF TWO-PHASE FLOW IN FRACTURED MEDIA

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Received 17 November 2004
Revised 22 November 2005
Communicated by G. Dal Maso

In a fractured medium, there is an interconnected system of fracture planes dividing the porous rock into a collection of matrix blocks. The fracture planes, while very thin, form paths of high permeability. Most of the fluids reside in matrix blocks, where they move very slow. Let ε denote the size ratio of the matrix blocks to the whole medium and let the width of the fracture planes and the porous block diameter be in the same order. If permeability ratio of matrix blocks to fracture planes is of order ε^2, microscopic models for two-phase, incompressible, immiscible flow in fractured media converge to a dual-porosity model as ε goes to 0. If the ratio is smaller than order ε^2, the microscopic models approach a single-porosity model for fracture flow. If the ratio is greater than order ε^2, then microscopic models tend to another type of single-porosity model. In this work, these results will be proved by a two-scale method.

Keywords: Homogenization; fractured media; dual-porosity model; two-scale convergence.

AMS Subject Classification: 35B27, 35B45, 35K55, 35K65, 76S05

1. Introduction

We discuss the homogenization for two-phase, incompressible, immiscible flow in fractured media with small-sized matrix blocks. The two phases are oil "o" phase and water "w" phase. Within a fractured medium, there is an interconnected system of fracture planes dividing the porous rock into a collection of matrix blocks. The fracture planes, while very thin, form paths of high permeability. Most of the fluids reside in matrix blocks, where they move very slow. Primary flow in the medium occurs within the fractures with local exchange of fluids between the fractures and the matrix blocks. No flow is allowed between blocks, and fluids in matrix blocks must enter the fracture planes to move great distance.

Let ε be the size ratio of the matrix blocks to the whole medium and let the width of the fracture planes and the porous block diameter be in the same order. If permeability ratio of matrix blocks to fracture planes is assumed to be of order ε^2, from physical point of view, microscopic models for the two-phase flow converge to a dual-porosity model as ε tends to 0.[7,14] In this limit model, the whole medium is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of matrix blocks. Matrix blocks play the role of global sources, representing the exchange of fluids between matrix blocks and the fractures. Flow equations are formulated by mass conservation principles for each continuum, and global sources are included in fracture equations. Fracture quantities are used to define boundary conditions for the equations in the matrix blocks (see Refs. 8, 11, 16, 17 and references therein). Numerical simulation shows that saturation evolutions of the model in fracture system and matrix blocks are in different time scales. [15] If the permeability ratio is smaller than order ε^2, flow in matrix blocks contributes very little to the fracture system. The microscopic models converge to equations for fracture flow as ε goes to 0. If the ratio is greater than order ε^2, fluid flow in matrix blocks moves very fast. Saturations in matrix blocks are almost constant. In this case, microscopic models converge to a special type of single-porosity model. Some problems in similar situation (for example, homogenization of heat equation in fractured media based on permeability ratio) were studied in Refs. 10 and 19. Their results indicate that if the ratio is smaller than order ε^2, the corresponding macroscopic equation is an equation for fracture flow. If the ratio is of order ε^2, it is a fracture flow equation with a source due to the flow in matrix blocks and is a time-delay equation. If the ratio is greater than order ε^2, it is a fracture flow equation plus a source from matrix blocks but not a time-delay equation. These results are consistent with ours.

Our intention is to show the convergence of the microscopic models for the above three cases in two-scale sense rigorously. Two-scale method was initially defined and had been applied to a diffusion process in highly heterogeneous media.[3,8] The method was also used to derive a model for flow in a partially fissured medium,[13] and used to prove the convergence of microscopic models to a dual-porosity model in a reduced pressure formulation[11] for ratio ε^2 case.

The rest of the paper is organized as follows: In Sec. 2, we state microscopic model for two-phase flow in fractured media. Notation and assumption will be given in Sec. 3. Then in Sec. 4, we present main results, i.e. the convergence of microscopic models in two-scale sense. Some known results needed for the proof of main results will be recalled in Sec. 5. The main result is proved in Sec. 6.

2. Microscopic Model for Small-Sized Matrix Blocks

We consider a porous medium $\Omega \subset \mathbb{R}^3$, which is a two-connected domain with a periodic structure. Let $Y := [0,1]^3$ be a cell consisting of a matrix block domain Y_m completely surrounded by a connected fracture domain Y_f, and we denote by Γ the matrix-fracture interface in the cell Y. Let $\mathcal{X}(y)$ be the characteristic function of Y_m extended Y-periodically to $\mathbb{R}^3$. The medium Ω contains two subdomains, Ω_f^ε

and Ω_m^ε, representing the system of fracture planes and matrix blocks respectively, and satisfying $\Omega_m^\varepsilon \subset \{x \in \Omega | \mathcal{X}(x/\varepsilon) = 1\}, \Omega_f^\varepsilon = \Omega \setminus \overline{\Omega_m^\varepsilon}$. Let $\Gamma^\varepsilon := \partial\Omega_f^\varepsilon \cap \partial\Omega_m^\varepsilon \cap \Omega$ be that part of the interface of Ω_m^ε with Ω_f^ε that is interior to Ω.

For the fracture subdomain Ω_f^ε, we denote porosity by Φ^ε, absolute permeability by K^ε, saturation of oil phase by $S^\varepsilon \in [0,1]$, capillary pressure by $\Upsilon(S^\varepsilon)$, the relative permeability by $\Lambda_\alpha(S^\varepsilon)$, phase pressure by P_α^ε, and a function depending on gravity by G_α^ε for $\alpha = \mathrm{w}, \mathrm{o}$. $\phi^\varepsilon, k^\varepsilon, s^\varepsilon, \upsilon(s^\varepsilon), \lambda_\alpha(s^\varepsilon), p_\alpha^\varepsilon, g_\alpha^\varepsilon$ for $\alpha = \mathrm{w}, \mathrm{o}$, in subdomain Ω_m^ε represent the same quantities as those denoted by upper case symbol in the fracture subdomain. Let $\varpi(> 0)$ be a constant. The conservation of mass in each phase, with the Darcy's law, can be written as, in Ω_f^ε, $t > 0$,

$$-\Phi^\varepsilon(x)\partial_t S^\varepsilon - \nabla \cdot (K^\varepsilon(x)\Lambda_{\mathrm{w}}(S^\varepsilon)\nabla(P_{\mathrm{w}}^\varepsilon - G_{\mathrm{w}}^\varepsilon)) = 0, \tag{2.1}$$

$$\Phi^\varepsilon(x)\partial_t S^\varepsilon - \nabla \cdot (K^\varepsilon(x)\Lambda_{\mathrm{o}}(S^\varepsilon)\nabla(P_{\mathrm{o}}^\varepsilon - G_{\mathrm{o}}^\varepsilon)) = 0, \tag{2.2}$$

$$\Upsilon(S^\varepsilon) = P_{\mathrm{o}}^\varepsilon - P_{\mathrm{w}}^\varepsilon, \tag{2.3}$$

in Ω_m^ε, $t > 0$,

$$-\phi^\varepsilon(x)\partial_t s^\varepsilon - \varepsilon^{2\varpi}\nabla \cdot (k^\varepsilon(x)\lambda_{\mathrm{w}}(s^\varepsilon)\nabla(p_{\mathrm{w}}^\varepsilon - g_{\mathrm{w}}^\varepsilon)) = 0, \tag{2.4}$$

$$\phi^\varepsilon(x)\partial_t s^\varepsilon - \varepsilon^{2\varpi}\nabla \cdot (k^\varepsilon(x)\lambda_{\mathrm{o}}(s^\varepsilon)\nabla(p_{\mathrm{o}}^\varepsilon - g_{\mathrm{o}}^\varepsilon)) = 0, \tag{2.5}$$

$$\upsilon(s^\varepsilon) = p_{\mathrm{o}}^\varepsilon - p_{\mathrm{w}}^\varepsilon. \tag{2.6}$$

Phase fluxes and pressures are required to be continuous on interface Γ^ε, $t > 0$, $\alpha = \mathrm{w}, \mathrm{o}$,

$$K^\varepsilon(x)\Lambda_\alpha(S^\varepsilon)\nabla(P_\alpha^\varepsilon - G_\alpha^\varepsilon)\boldsymbol{\nu} = \varepsilon^{2\varpi}k^\varepsilon(x)\lambda_\alpha(s^\varepsilon)\nabla(p_\alpha^\varepsilon - g_\alpha^\varepsilon)\boldsymbol{\nu}, \tag{2.7}$$

$$P_\alpha^\varepsilon = p_\alpha^\varepsilon, \tag{2.8}$$

where $\boldsymbol{\nu}$ is the unit vector outer normal to Γ^ε. The boundary $\partial\Omega$ of Ω includes Γ_1, Γ_2, which satisfying $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. The boundary conditions are given by, for $\alpha = \mathrm{w}, \mathrm{o}$,

$$K^\varepsilon(x)\Lambda_\alpha(S^\varepsilon)\nabla(P_\alpha^\varepsilon - G_\alpha^\varepsilon)\mathbf{n} = 0, \quad \text{on } \Gamma_1, \tag{2.9}$$

$$P_\alpha^\varepsilon = P_{b,\alpha}, \qquad \text{on } \Gamma_2, \tag{2.10}$$

where $\mathbf{n}$ is the unit vector outer normal to Γ_1. Initial conditions are

$$S^\varepsilon(x,0) = S_0^\varepsilon(x), \qquad \text{in } \Omega_f^\varepsilon, \tag{2.11}$$

$$s^\varepsilon(x,0) = s_0^\varepsilon(x), \qquad \text{in } \Omega_m^\varepsilon. \tag{2.12}$$

3. Notation and Assumption

The notations used in this paper are:

$\mathcal{Q} := \Omega \times Y,\ \mathcal{Q}_i := \Omega \times Y_i$ if $i = f, m$.

$\Omega(2\varepsilon) := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 2\varepsilon\}$,

$\Omega_m^\varepsilon := \{x : x \in \varepsilon(Y_m + j) \subset \Omega(2\varepsilon) \text{ for } j \in \mathbb{Z}^3\}$,

$\Omega_f^\varepsilon := \Omega \setminus \overline{\Omega_m^\varepsilon}$, and
$\Omega^\varepsilon := \{x : x \in \varepsilon(Y+j) \cap \Omega, \varepsilon(Y_m + j) \subset \Omega(2\varepsilon) \text{ for } j \in \mathbb{Z}^3\}$.
$\mathcal{Q}_m^\varepsilon := \Omega^\varepsilon \times Y_m$. $\mathcal{B}^t := \mathcal{B} \times (0,t)$ if $\mathcal{B} = \Omega, \mathcal{Q}_i, \Omega_i^\varepsilon, \mathcal{Q}_m^\varepsilon$ and $i = f, m$.

$\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. $\partial^h \psi(t) := \frac{\psi(t+h)-\psi(t)}{h}$ is time difference. For any set $\mathcal{B}$, $\mathcal{X}_\mathcal{B}$ is the characteristic function of $\mathcal{B}$, *dual* X, is the dual space of X, s_l (resp. $1 - s_r$) is the residual matrix oil (resp. water) saturation.

For $1 < r < \infty$, $L^r_{\text{per}}(Y)$ is the Banach space of functions in $L^r_{\text{loc}}(\mathbb{R}^3)$ which are Y-periodic. $W^{1,r}_{\text{per}}(Y)$ is the Banach space of Y-periodic extensions to $\mathbb{R}^3$ of the functions in $W^{1,r}(Y)$ for which the boundary values agree on the opposite sides of the boundary ∂Y, and its norm is the usual norm of $W^{1,r}(Y)$. $C^\infty_{\text{per}}(Y)$ is the space of Y-periodic and infinitely differentiable functions in $\mathbb{R}^3$. If $\mathbf{B}$ is a Banach space and X is a measure space, then $L^r(X;\mathbf{B})$ denotes the space of rth power norm-summable functions on X with values in $\mathbf{B}$ and $C_0^\infty(X;\mathbf{B})$ is the infinitely differentiable $\mathbf{B}$-valued functions with support in X. $L^r(\Omega^T; L^r_{\text{per}}(Y_i)) := \{\psi \in L^r(\Omega^T; L^r_{\text{per}}(Y)) : \psi(x,y,t) = 0 \text{ if } y \in Y \backslash Y_i\}$ for $i = f, m$. $L^r(\Omega^T; W^{1,r}(Y_m)) := \{\psi : \psi, \nabla_y \psi \in L^r(\Omega^T; L^r(Y_m))\}$. $L^2(\Omega^T; H^1(Y_m)) := L^2(\Omega^T; W^{1,2}(Y_m))$. $L^2(\Omega^T; H^1_0(Y_m)) := \{\psi \in L^2(\Omega^T; H^1(Y_m)) : \psi = 0 \text{ on } \partial Y_m\}$. $\mathcal{W}_0^{i,r}(\Omega) := \{\psi \in W^{i,r}(\Omega) : \psi|_{\Gamma_2} = 0\}$ for $i \in \mathbf{N}$ and $r > 1$. $\mathcal{U} := \mathcal{W}_0^{1,2}(\Omega)$.

If $\Upsilon : [0,1) \to \mathbb{R}_0^+$ (resp. $\upsilon : [s_l, s_r) \to \mathbb{R}_0^+$) is onto and strictly increasing, we denote by Υ^{-1} (resp. υ^{-1}) the inverse function of Υ (resp. υ). We also define $\mathcal{J} : [0,1) \to [s_l, s_r)$ by $\mathcal{J}(z) := \upsilon^{-1}(\Upsilon(z))$, and denote by $\mathcal{J}^{-1}$ the inverse function of $\mathcal{J}$.

$$\begin{cases} P_{b,c} := P_{b,\text{o}} - P_{b,\text{w}}, \\ \Lambda(z) := \Lambda_\text{w}(z) + \Lambda_\text{o}(z), & \text{for } z \in [0,1], \\ \lambda(z) := \lambda_\text{w}(z) + \lambda_\text{o}(z), & \text{for } z \in [0,1], \\ \mathbf{R}(z) := \displaystyle\int_0^z \frac{\Lambda_\text{w}\Lambda_\text{o}}{\Lambda} \frac{d\Upsilon}{dS}(\xi) d\xi, & \text{for } z \in [0,1), \\ \mathcal{A}(z) := \displaystyle\int_0^z \sqrt{\frac{\Lambda_\text{w}\Lambda_\text{o}}{\Lambda}(\Upsilon^{-1}(\xi))} d\xi, & \text{for } z \in [0,\infty), \\ \mathcal{M}(z) := \displaystyle\int_{s_l}^z \frac{\lambda_\text{w}\lambda_\text{o}}{\lambda} \frac{d\upsilon}{ds}(\xi) d\xi, & \text{for } z \in [s_l, s_r). \end{cases} \tag{3.1}$$

Next let us make the following assumptions: For $\alpha = \text{w}, \text{o}$,

A1. $\Lambda_\text{w}, \lambda_\text{w}$ (resp. $\Lambda_\text{o}, \lambda_\text{o}$) : $[0,1] \to [0,1]$ are continuous and decreasing (resp. increasing), and $\Lambda_\text{w}(1-z) \propto z^{\mathbf{d}_1}, \lambda_\text{w}(s_r - z) \propto z^{\mathbf{d}_2}$, $\Lambda_\text{o}(z) \propto z^{\mathbf{d}_3}, \lambda_\text{o}(s_l + z) \propto z^{\mathbf{d}_4}$ for $z \ll 1$,

A2. $\Upsilon : [0,1) \to \mathbb{R}_0^+$ ($\upsilon : [s_l, s_r) \to \mathbb{R}_0^+$) is onto, increasing and a locally Lipschitz continuous function, and $\inf_{z\in[0,1)} \frac{\frac{d\Upsilon}{dS}(z)}{\frac{d\upsilon}{ds}(\mathcal{J}(z))} \times \inf_{z\in[s_l,s_r)} \frac{d\upsilon}{ds}(z) > 0$,

A3. $\sqrt{\Lambda_\text{w}\Lambda_\text{o}} \frac{d\Upsilon}{dS} \in L^\infty((0,1))$, $\lambda_\text{w}\lambda_\text{o} \frac{d\upsilon}{ds} \in L^1((s_l, s_r))$, A4. $\Lambda_{\rm o}^{3/2}(z) \le \int_z^{2z}(\mathcal{A}(\Upsilon(2z)) - \mathcal{A}(\Upsilon(\xi)))d\xi$ for $z \ll 1$ as well as $\Lambda_{\rm w}^{3/2}(1-z) \le \int_{1-2z}^{1-z}(\mathcal{A}(\Upsilon(\xi)) - \mathcal{A}(\Upsilon(1-2z)))d\xi$ for $z \ll 1$,

A5. $\frac{\Lambda_\alpha}{\Lambda}(z) = \frac{\lambda_\alpha}{\lambda}(\mathcal{J}(z))$ for $z \in (0,1)$,

A6. $\partial_t P_{b,c} \in L^1(\Omega^T), P_{b,\alpha} \in L^2(0,T;H^1(\Omega))$, $G_\alpha^\varepsilon, g_\alpha^\varepsilon \in L^\infty(0,T;W^{1,\infty}(\Omega))$,

A7. $\phi^\varepsilon = \phi(\frac{x}{\varepsilon})$, $k^\varepsilon = k(\frac{x}{\varepsilon})$, where ϕ, k are smooth Y-periodic functions,

A8. $\Phi^\varepsilon, K^\varepsilon, s_0^\varepsilon \to \Phi^H, K^H, s_0$ in $L^2(\Omega)$, and $\nabla G_\alpha^\varepsilon \to \nabla G_\alpha$ in $L^2(\Omega^T)$,

A9. $\gamma_1 \le \Phi^\varepsilon, K^\varepsilon, \phi, k \le \gamma_2$, and $\gamma_3 \le \Upsilon^{-1}(P_{b,c}), S_0^\varepsilon, \mathcal{J}^{-1}(s_0^\varepsilon) \le 1 - \gamma_3$,

A10. $\Gamma_2 \ne \emptyset$, and $\Omega \subset \mathbb{R}^3$ (resp. $Y_m \subset Y$) is open, bounded, and connected with Lipschitz boundary $\Gamma_1 \cup \Gamma_2$ (resp. ∂Y_m), $\Omega_m^\varepsilon := \{z : z \in \varepsilon(Y_m + j) \subset \Omega(2\varepsilon)$ for $j \in \mathbb{Z}^3\}$ and $\Omega_f^\varepsilon = \Omega \backslash \overline{\Omega_m^\varepsilon}$,

where $\mathbf{d}_i, \gamma_i, i = 1, \ldots, 4$ are positive constants.

Remark 3.1. In A1, relative permeability $\Lambda_{\rm o}$ (resp. $\lambda_{\rm o}$) is assumed to be an increasing function of oil saturation, and in the neighborhood of oil residue saturation 0 (resp. s_l) it is proportional to a power function. $\Lambda_{\rm w}, \lambda_{\rm w}$ have similar properties in the neighborhood of water residue saturations as $\Lambda_{\rm o}, \lambda_{\rm o}$. A2 says capillary pressures Υ, υ are increasing with respect to saturation. Usually, the derivative of capillary pressure $\Upsilon'(z)$ (resp. $\upsilon'(z)$) tends to ∞ as $z \to 0$ or 1 (resp. s_l or s_r). It also requires fracture capillary pressure increases faster than capillary pressure of matrix blocks. A3 allows the differential equations with degeneracy at two ends (see also Refs. 15–17), a characteristic of a porous medium equation. A4 is the restriction on relative permeability and capillary pressure in fractures. In fact, if $\mathbf{d}_i, i = 1, \ldots, 4$ (see A1) are large enough (depending on capillary pressure), A4 holds. A5 requires that relative phase mobility functions in fractures and matrix blocks are almost the same. Initial and boundary saturations are away from both ends, see A9. By A10, Ω_f^ε is an open, bounded and connected with Lipschitz boundary. By A1–3 and (3.1), $\mathcal{M}$ is bounded and strictly increasing in $[s_l, s_r)$, so one can extend $\mathcal{M}$ to $\mathbb{R}$ such that it is still continuous and strictly increasing.

4. The Main Result

We study the convergence of the microscopic models for two-phase flow in fractured media as ε goes to 0. If $\varpi = 1$, the limit model is a dual-porosity model. In this case, domain acts as a porous medium consisting of two superimposed continua, a continuous fracture system Ω and a discontinuous system of matrix blocks $\mathcal{Q}_m$. Primary flow occurs in fracture system. Flow in matrix blocks plays the role of global sources in the whole fracture system. The model includes two systems of equations, one for flow in fracture system and the other for flow in matrix block system. The two systems are coupled through global sources. If $\varpi > 1$, flow in matrix blocks moves so slow that it does not enter fracture system much. So the limit model is a single-porosity model and contains equations for fracture flow only. Contrary to

$\varpi > 1$, flow in matrix blocks spreads very fast for $0 < \varpi < 1$ case. Saturations are constant in space in the limit model, which is another type of single-porosity model. Details are described below:

4.1. For $\varpi = 1$ case

Let $\Omega \subset \mathbb{R}^3$ be the medium. Equations for fracture flow are, for $x \in \Omega,\ t > 0$,

$$-\Phi\partial_t S - \nabla_x \cdot (K\Lambda_{\rm w}(S)\nabla_x(P_{\rm w} - G_{\rm w})) = q_{\rm w}, \tag{4.1}$$

$$\Phi\partial_t S - \nabla_x \cdot (K\Lambda_{\rm o}(S)\nabla_x(P_{\rm o} - G_{\rm o})) = q_{\rm o}, \tag{4.2}$$

$$\Upsilon(S) = P_{\rm o} - P_{\rm w}. \tag{4.3}$$

Φ is porosity, K is permeability field, S is oil saturation, $\Upsilon(S)$ is capillary pressure, Λ_α ($\alpha = {\rm w, o}$) is relative permeability of α-phase, P_α denotes phase pressure, G_α is a function depending on density, gravity and position, and q_α is the matrix-fracture source.

Above each point $x \in \Omega$ is suspended topologically a matrix block $Y_m \subset \mathbb{R}^3$. Equations for flow in a matrix block are, for $x \in \Omega,\ y \in Y_m,\ t > 0$,

$$-\phi\partial_t s - \nabla_y \cdot (k\lambda_{\rm w}(s)\nabla_y p_{\rm w}) = 0, \tag{4.4}$$

$$\phi\partial_t s - \nabla_y \cdot (k\lambda_{\rm o}(s)\nabla_y p_{\rm o}) = 0, \tag{4.5}$$

$$\upsilon(s) = p_{\rm o} - p_{\rm w}. \tag{4.6}$$

Each lower case symbol denotes the quantity on Y_m corresponding to that denoted by an upper case symbol in the fracture system.

The matrix-fracture sources are given by, for $x \in \Omega,\ t > 0$,

$$q_{\rm o}(x,t) = \frac{-1}{|Y_m|}\int_{Y_m} \phi\partial_t s(x,y,t)dy = -q_{\rm w}(x,t), \tag{4.7}$$

where $|Y_m|$ is the volume of Y_m. The boundary conditions for fracture system are, for $t > 0$, $\alpha = {\rm w, o}$,

$$K\Lambda_\alpha(S)\nabla_x(P_\alpha - G_\alpha)\cdot\mathbf{n} = 0, \quad \text{for } x \in \Gamma_1, \tag{4.8}$$

$$P_\alpha = P_{b,\alpha}, \quad \text{for } x \in \Gamma_2, \tag{4.9}$$

where $\mathbf{n}$ is the unit vector outward normal to Γ_1. On interface, pressures are continuous, i.e. for $t > 0,\ x \in \Omega,\ y \in \partial Y_m,\ \alpha = {\rm w, o}$,

$$p_\alpha(x,y,t) = P_\alpha(x,t). \tag{4.10}$$

Initial conditions are

$$S(x,0) = S_0(x), \quad \text{for } x \in \Omega, \tag{4.11}$$

$$s(x,y,0) = s_0(x), \quad \text{for } x \in \Omega, \quad y \in Y_m. \tag{4.12}$$

Next we give a definition of a weak solution of (4.1)–(4.12). $\{S, s, P_\alpha, p_\alpha, \alpha = {\rm w, o}\}$ is a weak solution of (4.1)–(4.12) if $\mathbf{R}(S) \in L^2(0,T;H^1(\Omega))$,

$\mathcal{M}(s) \in L^2(\Omega^T; H^1(Y_m))$, $P_\alpha \in L^r(0,T;W^{1,r}(\Omega))$, $p_\alpha \in L^r(\Omega^T; W^{1,r}(Y_m))$ for $1 < r < 2$ as well as (4.3), (4.6), (4.8)–(4.12) and the following equations hold:

$$\int_{\Omega^T} \Phi\partial_t S\ \zeta_1 - \int_{\Omega^T} K\Lambda_{\rm w}(S)\nabla(P_{\rm w} - G_{\rm w})\nabla\zeta_1 = \int_{\mathcal{Q}_m^T} \frac{-\phi\partial_t s}{|Y_m|}dy\ \zeta_1,$$

$$\int_{\Omega^T} \Phi\partial_t S\ \zeta_2 + \int_{\Omega^T} K\Lambda_{\rm o}(S)\nabla(P_{\rm o} - G_{\rm o})\nabla\zeta_2 = \int_{\mathcal{Q}_m^T} \frac{-\phi\partial_t s}{|Y_m|}dy\ \zeta_2,$$

$$\int_{\mathcal{Q}_m^T} \phi\partial_t s\ \eta_1 - \int_{\mathcal{Q}_m^T} k\lambda_{\rm w}(s)\nabla_y p_{\rm w}\nabla_y\eta_1 = 0,$$

$$\int_{\mathcal{Q}_m^T} \phi\partial_t s\ \eta_2 + \int_{\mathcal{Q}_m^T} k\lambda_{\rm o}(s)\nabla_y p_{\rm o}\nabla_y\eta_2 = 0,$$

for any $\zeta_1, \zeta_2 \in L^2(0,T;\mathcal{U})$ and $\eta_1, \eta_2 \in L^2(\Omega^T; H_0^1(Y_m))$.

Theorem 4.1. *Under* A1$-$10 *and* $\varpi = 1$, *a subsequence of weak solutions of* (2.1)–(2.12) *converges in two-scale sense to a weak solution of* (4.1)–(4.12) (*convergence in two-scale sense is given in Sec.* 5).

A two-phase flow problem for ratio ε^2 case was also considered in Ref. 11. Their result is basically the reduced pressure formulation of (4.1)–(4.12). However, their interface condition (see (4.51) of Ref. 11) is $\int_0^S \sqrt{\frac{\Lambda_{\rm w}\Lambda_{\rm o}}{\Lambda}\frac{d\Upsilon}{dS}(\xi)}d\xi = \int_0^s \sqrt{\frac{\lambda_{\rm w}\lambda_{\rm o}}{\lambda}\frac{dv}{ds}(\xi)}d\xi$ for $x \in \Omega$, $y \in \partial Y_m$, which is different from (4.10).

4.2. *For* $\varpi > 1$ *case*

Equations in Ω are

$$-\Phi\partial_t S - \nabla\cdot(K\Lambda_{\rm w}(S)\nabla(P_{\rm w} - G_{\rm w})) = 0, \tag{4.13}$$

$$\Phi\partial_t S - \nabla\cdot(K\Lambda_{\rm o}(S)\nabla(P_{\rm o} - G_{\rm o})) = 0, \tag{4.14}$$

$$\Upsilon(S) = P_{\rm o} - P_{\rm w}, \tag{4.15}$$

where $\Phi, K, S, \Upsilon(S), \Lambda_\alpha(S), P_\alpha, G_\alpha (\alpha = {\rm w}, {\rm o})$ are the same quantities as those in $\varpi = 1$ case. The boundary and initial conditions are, for $\alpha = {\rm w}, {\rm o}$,

$$K\Lambda_\alpha(S)\nabla(P_\alpha - G_\alpha)\cdot\mathbf{n} = 0, \quad \text{for } x \in \Gamma_1, \tag{4.16}$$

$$P_\alpha = P_{b,\alpha}, \quad \text{for } x \in \Gamma_2, \tag{4.17}$$

$$S(x,0) = S_0(x), \quad \text{for } x \in \Omega, \tag{4.18}$$

where $\mathbf{n}$ is the unit vector outward normal to Γ_1. $\{S, P_\alpha, \alpha = {\rm w}, {\rm o}\}$ is called a weak solution of (4.13)–(4.18) if $\mathbf{R}(S) \in L^2(0,T;H^1(\Omega))$, $P_\alpha \in L^r(0,T;W^{1,r}(\Omega))$ for $1 < r < 2$, (4.15)–(4.18) and the following equations hold

$$\int_{\Omega^T} \Phi\partial_t S\ \zeta_1 - \int_{\Omega^T} K\Lambda_{\rm w}(S)\nabla(P_{\rm w} - G_{\rm w})\nabla\zeta_1 = 0,$$

$$\int_{\Omega^T} \Phi\partial_t S\ \zeta_2 + \int_{\Omega^T} K\Lambda_{\rm o}(S)\nabla(P_{\rm o} - G_{\rm o})\nabla\zeta_2 = 0,$$

for any $\zeta_1, \zeta_2 \in L^2(0,T;\mathcal{U})$.

Theorem 4.2. *Under* A1–10 *and* $\varpi > 1$, *a subsequence of weak solutions of* (2.1)–(2.12) *converges in two-scale sense to a weak solution of* (4.13)–(4.18).

4.3. *For* $0 < \varpi < 1$ *case*

Equations in Ω are

$$-\Phi\partial_t S - \nabla\cdot(K\Lambda_{\rm w}(S)\nabla(P_{\rm w}-G_{\rm w})) = \frac{\partial_t s(x,t)}{|Y_m|}\int_{Y_m}\phi(y)dy, \tag{4.19}$$

$$\Phi\partial_t S - \nabla\cdot(K\Lambda_{\rm o}(S)\nabla(P_{\rm o}-G_{\rm o})) = \frac{-\partial_t s(x,t)}{|Y_m|}\int_{Y_m}\phi(y)dy, \tag{4.20}$$

$$\Upsilon(S) = P_{\rm o} - P_{\rm w} = \upsilon(s), \tag{4.21}$$

where $\Phi, \phi, K, S, s, \Upsilon(S), \upsilon(s), \Lambda_\alpha(S), P_\alpha, G_\alpha(\alpha = {\rm w}, {\rm o})$ are the same quantities as those in $\varpi = 1$ case. The boundary and initial conditions are, for $\alpha = {\rm w}, {\rm o}$,

$$K\Lambda_\alpha(S)\nabla(P_\alpha - G_\alpha)\cdot\mathbf{n} = 0, \quad \text{for } x\in\Gamma_1, \tag{4.22}$$

$$P_\alpha = P_{b,\alpha}, \quad \text{for } x\in\Gamma_2, \tag{4.23}$$

$$S(x,0) = S_0(x), \quad \text{for } x\in\Omega, \tag{4.24}$$

where $\mathbf{n}$ is the unit vector outward normal to Γ_1. $\{S, P_\alpha, \alpha = {\rm w}, {\rm o}\}$ is called a weak solution of (4.19)–(4.24) if $\mathbf{R}(S) \in L^2(0,T;H^1(\Omega))$, $P_\alpha \in L^r(0,T;W^{1,r}(\Omega))$ for $1 < r < 2$, (4.21)–(4.24) and the following equations hold:

$$\int_{\Omega^T}\Phi\partial_t S\,\zeta_1 - \int_{\Omega^T}K\Lambda_{\rm w}(S)\nabla(P_{\rm w}-G_{\rm w})\nabla\zeta_1 = \int_{\Omega^T}\frac{-\partial_t s(x,t)}{|Y_m|}\int_{Y_m}\phi(y)dy\,\zeta_1,$$
$$\int_{\Omega^T}\Phi\partial_t S\,\zeta_2 + \int_{\Omega^T}K\Lambda_{\rm o}(S)\nabla(P_{\rm o}-G_{\rm o})\nabla\zeta_2 = \int_{\Omega^T}\frac{-\partial_t s(x,t)}{|Y_m|}\int_{Y_m}\phi(y)dy\,\zeta_2,$$

for any $\zeta_1, \zeta_2 \in L^2(0,T;\mathcal{U})$. Let us make one more assumption:

A11. $\lambda_{\rm w}\lambda_{\rm o}\frac{d\upsilon}{ds} \in L^\infty((s_l, s_r))$.

Theorem 4.3. *Under* A1–11 *and* $0 < \varpi < 1$, *a subsequence of weak solutions of* (2.1)–(2.12) *converges in two-scale sense to a weak solution of* (4.19)–(4.24).

5. Some Known Results

Let us recall some results from Refs. 1–3, 8, 11 and 13. By A10, Ω_f^ε is an open, bounded, and connected with Lipschitz boundary. So we know

Lemma 5.1.[1] *Let* $1 \le r < \infty$ *and* A10 *hold. There is a constant* $\gamma_5(Y_f, r)$ *independent of* ε, *and a linear continuous extension operator* $\Pi_\varepsilon : W^{1,r}(\Omega_f^\varepsilon) \to W^{1,r}(\Omega)$ *such that*

$$\begin{cases}\Pi_\varepsilon\varphi = \varphi \quad \textit{in } \Omega_f^\varepsilon \textit{ almost everywhere},\\ \|\Pi_\varepsilon\varphi\|_{W^{1,r}(\Omega)} \le \gamma_5\|\varphi\|_{W^{1,r}(\Omega_f^\varepsilon)}.\end{cases}$$

In addition, if $\gamma_6 < \varphi < \gamma_7$, *then* $\gamma_6 < \Pi_\varepsilon\varphi < \gamma_7$.

Definition 5.1. For a given $\varepsilon > 0$ and $1 \le r < \infty$, we define a dilation operator "$-$" mapping a measurable function $\varphi \in L^r(\Omega_m^{\varepsilon,T})$ to a measurable function $\overline{\varphi} \in L^r(\mathcal{Q}_m^T)$ by, for $(x, y, t) \in \mathcal{Q}_m^T$,

$$\overline{\varphi}(x,y,t) := \begin{cases} \varphi(\ell^\varepsilon(x) + \varepsilon y, t), & if\ \ell^\varepsilon(x) + \varepsilon y \in \Omega_m^\varepsilon, \\ 0, & elsewhere, \end{cases}$$

where $\ell^\varepsilon(x) := \varepsilon j$ if $x \in \varepsilon(Y + j),\ j \in \mathbb{Z}^3$, denoting the lattice translation point of ε-cell domain containing x.

Lemma 5.2.[8] *Let* $1 \le r < \infty$.

$$\begin{cases} \nabla_y \overline{\varphi} = \varepsilon \overline{\nabla \varphi} \quad in\ \mathcal{Q}_m^T\ almost\ everywhere\ if\ \varphi \in L^r(0,T; W^{1,r}(\Omega_m^\varepsilon)), \\ \|\overline{\varphi}\|_{L^r(\mathcal{Q}_m^T)} = \|\varphi\|_{L^r(\Omega_m^{\varepsilon,T})} \quad if\ \varphi \in L^r(\Omega_m^{\varepsilon,T}). \end{cases}$$

Definition 5.2. A sequence of functions φ^ε in $L^r(\Omega^T), 1 < r < \infty$, is said to be two-scale converge to φ in $L^r(\Omega^T; L^r_{\text{per}}(Y))$ if, for any function $\psi \in C_0^\infty(\Omega^T; C^\infty_{\text{per}}(Y))$, we have

$$\lim_{\varepsilon \to 0} \int_{\Omega^T} \varphi^\varepsilon(x,t)\psi\left(x, \frac{x}{\varepsilon}, t\right) dxdt = \int_{\mathcal{Q}^T} \varphi(x,y,t)\psi(x,y,t) dydxdt,$$

denoted by $\varphi^\varepsilon \overset{2}{\rightharpoonup} \varphi \in L^r(\Omega^T; L^r_{\text{per}}(Y))$. Besides $\lim_{\varepsilon\to 0} \|\varphi^\varepsilon\|_{L^r(\Omega^T)} = \|\varphi\|_{L^r(\mathcal{Q}^T)}$, φ^ε is said to be two-scale converge to φ in $L^r(\Omega^T; L^r_{\text{per}}(Y))$ strongly, and denoted by $\varphi^\varepsilon \overset{2}{\to} \varphi \in L^r(\Omega^T; L^r_{\text{per}}(Y))$ strongly.

By Refs. 11, 13 and Lemma 5.2, we have

Lemma 5.3. *If* $1 < r < \infty$ *and if* φ^ε *is a bounded sequence in* $L^r(\Omega_m^{\varepsilon,T})$ *satisfying*

$$\begin{cases} \overline{\varphi^\varepsilon} \rightharpoonup \varphi_0\ weakly\ in\ L^r(\mathcal{Q}_m^T), \\ \mathcal{X}_{\Omega_m^\varepsilon} \varphi^\varepsilon \overset{2}{\rightharpoonup} \varphi_1 \in L^r(\Omega^T; L^r_{\text{per}}(Y_m)), \end{cases} \quad as\ \varepsilon \to 0,$$

where $\mathcal{X}_{\Omega_m^\varepsilon}$ *is a characteristic function (see Sec.* 3*), then* $\varphi_0 = \varphi_1$ *in* $\mathcal{Q}_m^T$ *almost everywhere.*

By Theorem 2.28 of Ref. 2 and Lemmas 5.2–5.3 we obtain

Corollary 5.1. *If* $\varphi^\varepsilon \in L^r(\Omega_m^{\varepsilon,T})$ *and* $\mathcal{X}_{\Omega_m^\varepsilon}\varphi^\varepsilon \overset{2}{\to} \varphi \in L^r(\Omega^T; L^r_{\text{per}}(Y_m))$ *strongly for* $1 < r < \infty$, *then* $\overline{\varphi^\varepsilon}$ *converges to* φ *strongly in* $L^r(\mathcal{Q}_m^T)$.

Tracing the proof of Theorem 1.8 of Ref. 3, we have

Lemma 5.4. *If* $u^\varepsilon \overset{2}{\to} u \in L^r(\Omega^T; L^r_{\text{per}}(Y))$ *strongly for* $1 < r < \infty$ *and if* v^ε *is a bounded sequence in* $L^{r'}(\Omega^T)$ *satisfying* $v^\varepsilon \overset{2}{\rightharpoonup} v \in L^{r'}(\Omega^T; L^{r'}_{\text{per}}(Y))$ *and* $\frac{1}{r} + \frac{1}{r'} = 1$, *then*

$$\lim_{\varepsilon \to 0} \int_{\Omega^T} u^\varepsilon(x,t) v^\varepsilon(x,t) \psi\left(x, \frac{x}{\varepsilon}, t\right) dxdt = \int_{\mathcal{Q}^T} u(x,y,t)v(x,y,t)\psi(x,y,t) dydxdt,$$

for any $\psi \in C_0^\infty(\Omega^T; C^\infty_{\text{per}}(Y))$.

6. Proof of the Main Result

A1–10 are assumed throughout this section. We first derive a weak formulation for (2.1)–(2.12). Multiply (2.1), (2.4) by η and (2.2), (2.5) by ζ, integrate over Ω^T, and use (2.7), (2.9) to obtain

$$-\int_{\Omega_f^{\varepsilon,T}} \Phi^\varepsilon \partial_t S^\varepsilon \eta + \int_{\Omega_f^{\varepsilon,T}} K^\varepsilon \Lambda_{\mathrm{w}}(S^\varepsilon)\nabla(P_{\mathrm{w}}^\varepsilon - G_{\mathrm{w}}^\varepsilon)\nabla\eta$$
$$-\int_{\Omega_m^{\varepsilon,T}} \phi^\varepsilon \partial_t s^\varepsilon \eta + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,T}} k^\varepsilon \lambda_{\mathrm{w}}(s^\varepsilon)\nabla(p_{\mathrm{w}}^\varepsilon - g_{\mathrm{w}}^\varepsilon)\nabla\eta = 0, \tag{6.1}$$

$$\int_{\Omega_f^{\varepsilon,T}} \Phi^\varepsilon \partial_t S^\varepsilon \zeta + \int_{\Omega_f^{\varepsilon,T}} K^\varepsilon \Lambda_{\mathrm{o}}(S^\varepsilon)\nabla(P_{\mathrm{o}}^\varepsilon - G_{\mathrm{o}}^\varepsilon)\nabla\zeta$$
$$+\int_{\Omega_m^{\varepsilon,T}} \phi^\varepsilon \partial_t s^\varepsilon \zeta + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,T}} k^\varepsilon \lambda_{\mathrm{o}}(s^\varepsilon)\nabla(p_{\mathrm{o}}^\varepsilon - g_{\mathrm{o}}^\varepsilon)\nabla\zeta = 0, \tag{6.2}$$

for functions $\eta, \zeta \in L^2(0,T;\mathcal{U})$. Next we define global pressures[12] as

$$P^\varepsilon := \frac{1}{2}\left(P_{\mathrm{o}}^\varepsilon + P_{\mathrm{w}}^\varepsilon + \int_0^{\Upsilon(S^\varepsilon)}\left(\frac{\Lambda_{\mathrm{o}}}{\Lambda}(\Upsilon^{-1}(\xi)) - \frac{\Lambda_{\mathrm{w}}}{\Lambda}(\Upsilon^{-1}(\xi))\right)d\xi\right), \tag{6.3}$$

$$p^\varepsilon := \frac{1}{2}\left(p_{\mathrm{o}}^\varepsilon + p_{\mathrm{w}}^\varepsilon + \int_0^{\upsilon(s^\varepsilon)}\left(\frac{\lambda_{\mathrm{o}}}{\lambda}(\upsilon^{-1}(\xi)) - \frac{\lambda_{\mathrm{w}}}{\lambda}(\upsilon^{-1}(\xi))\right)d\xi\right). \tag{6.4}$$

Then $\nabla P^\varepsilon = \frac{\Lambda_{\mathrm{w}}}{\Lambda}(S^\varepsilon)\nabla P_{\mathrm{w}}^\varepsilon + \frac{\Lambda_{\mathrm{o}}}{\Lambda}(S^\varepsilon)\nabla P_{\mathrm{o}}^\varepsilon$ and $\nabla p^\varepsilon = \frac{\lambda_{\mathrm{w}}}{\lambda}(s^\varepsilon)\nabla p_{\mathrm{w}}^\varepsilon + \frac{\lambda_{\mathrm{o}}}{\lambda}(s^\varepsilon)\nabla p_{\mathrm{o}}^\varepsilon$ by (2.3) and (2.6). So (6.2) can be rewritten as

$$\int_{\Omega_f^{\varepsilon,T}} \Phi^\varepsilon \partial_t S^\varepsilon \zeta + \int_{\Omega_f^{\varepsilon,T}} K^\varepsilon\big(\Lambda_{\mathrm{o}}(S^\varepsilon)\nabla(P^\varepsilon - G_{\mathrm{o}}^\varepsilon) + \nabla\mathbf{R}(S^\varepsilon)\big)\nabla\zeta$$
$$+\int_{\Omega_m^{\varepsilon,T}} \phi^\varepsilon \partial_t s^\varepsilon \zeta + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,T}} k^\varepsilon\big(\lambda_{\mathrm{o}}(s^\varepsilon)\nabla(p^\varepsilon - g_{\mathrm{o}}^\varepsilon) + \nabla\mathcal{M}(s^\varepsilon)\big)\nabla\zeta = 0. \tag{6.5}$$

Adding (6.1) and (6.2) to obtain, for $\eta \in L^2(0,T;\mathcal{U})$,

$$\int_{\Omega_f^{\varepsilon,T}} K^\varepsilon\big(\Lambda(S^\varepsilon)\nabla(P^\varepsilon - G_{\mathrm{o}}^\varepsilon) - \Lambda_{\mathrm{w}}(S^\varepsilon)\nabla(G_{\mathrm{w}}^\varepsilon - G_{\mathrm{o}}^\varepsilon)\big)\nabla\eta$$
$$+\varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,T}} k^\varepsilon\big(\lambda(s^\varepsilon)\nabla(p^\varepsilon - g_{\mathrm{o}}^\varepsilon) - \lambda_{\mathrm{w}}(s^\varepsilon)\nabla(g_{\mathrm{w}}^\varepsilon - g_{\mathrm{o}}^\varepsilon)\big)\nabla\eta = 0. \tag{6.6}$$

For $\zeta \in L^2(0,T;\mathcal{U}) \cap H^1(\Omega^T)$, (2.11)–(2.12) imply

$$\int_{\Omega_f^{\varepsilon,T}} \Phi^\varepsilon \partial_t(S^\varepsilon\zeta) + \int_{\Omega_m^{\varepsilon,T}} \phi^\varepsilon \partial_t(s^\varepsilon\zeta)$$
$$= \int_{\Omega_f^\varepsilon} \Phi^\varepsilon S^\varepsilon\zeta(T) - \int_{\Omega_f^\varepsilon} \Phi^\varepsilon S_0^\varepsilon\zeta(0) + \int_{\Omega_m^\varepsilon} \phi^\varepsilon s^\varepsilon\zeta(T) - \int_{\Omega_m^\varepsilon} \phi^\varepsilon s_0^\varepsilon\zeta(0). \tag{6.7}$$

(6.1)–(6.7), (2.3), (2.6), (2.8), (2.10) form a weak formulation of (2.1)–(2.12). Let us define

$$\mathcal{A}^\varepsilon := \begin{cases} \mathcal{A}(\Upsilon(S^\varepsilon)), & \text{if } x \in \Omega_f^\varepsilon, \\ \mathcal{A}(\upsilon(s^\varepsilon)), & \text{if } x \in \Omega_m^\varepsilon. \end{cases}$$

Lemma 6.1. *Under* A1–10, *there exist functions* $S^\varepsilon, P^\varepsilon, P^\varepsilon_\alpha$ *in* Ω^ε_f *and* $s^\varepsilon, p^\varepsilon, p^\varepsilon_\alpha$ *in* Ω^ε_m *for* $\alpha = \mathrm{w}, \mathrm{o}$ *satisfying* (6.1)–(6.7), (2.3), *and* (2.6)–(2.10). *Moreover,* $0 < S^\varepsilon < 1, s_l < s^\varepsilon < s_r$, *and*

$$\sum_{\alpha=\mathrm{w,o}} \left(\left\| \sqrt{\Lambda_\alpha(S^\varepsilon)} \nabla P^\varepsilon_\alpha \right\|_{L^2(\Omega^{\varepsilon,T}_f)} + \varepsilon^\varpi \left\| \sqrt{\lambda_\alpha(s^\varepsilon)} \nabla p^\varepsilon_\alpha \right\|_{L^2(\Omega^{\varepsilon,T}_m)} \right)$$
$$+ \left\| \, |P^\varepsilon| + |\nabla P^\varepsilon| + |\nabla \mathbf{R}(S^\varepsilon)| + |\nabla \mathcal{A}^\varepsilon| \, \right\|_{L^2(\Omega^{\varepsilon,T}_f)}$$
$$+ \varepsilon^\varpi \left\| \, |p^\varepsilon| + |\nabla p^\varepsilon| + |\nabla \mathcal{M}(s^\varepsilon)| + |\nabla \mathcal{A}^\varepsilon| \, \right\|_{L^2(\Omega^{\varepsilon,T}_m)} \le c,$$

where c *is a constant independent of* ε.

Proof. Proof of the lemma can be found in Refs. 5, 6, 9, 18 and 22. □

Lemma 6.2. $\|p^\varepsilon\|_{L^2(0,T;H^1(\Omega^\varepsilon_m))} \le c$, *where* c *is a constant independent of* ε.

Proof. A5, (2.8), (6.3)–(6.4) and Lemmas 5.1, 6.1 imply $\eta := \mathcal{X}_{\Omega^\varepsilon_m}(p^\varepsilon - \Pi_\varepsilon P^\varepsilon) \in L^2(0, T; \mathcal{U})$. Take η above in (6.6) to get

$$\int_{\Omega^{\varepsilon,T}_m} k^\varepsilon \big(\lambda(s^\varepsilon)\nabla(p^\varepsilon - g^\varepsilon_\mathrm{o}) - \lambda_\mathrm{w}(s^\varepsilon)\nabla(g^\varepsilon_\mathrm{w} - g^\varepsilon_\mathrm{o})\big)\nabla(p^\varepsilon - \Pi_\varepsilon P^\varepsilon) = 0.$$

By A1, A6, A9 and Lemmas 5.1, 6.1, we obtain

$$\|\nabla p^\varepsilon\|_{L^2(\Omega^{\varepsilon,T}_m)} \le c, \tag{6.8}$$

where c is a constant independent of ε. By (6.8),

$$\|p^\varepsilon\|_{L^2(\Omega^{\varepsilon,T}_m)} \le \|\Pi_\varepsilon P^\varepsilon\|_{L^2(\Omega^{\varepsilon,T}_m)} + \|p^\varepsilon - \Pi_\varepsilon P^\varepsilon\|_{L^2(\Omega^{\varepsilon,T}_m)} \le c,$$

where c is a constant independent of ε. □

Lemma 6.3. *For sufficiently small* δ,

$$\int_\delta^T \int_{\Omega^\varepsilon_f} \big(S^\varepsilon(t) - S^\varepsilon(t-\delta)\big)\big(\mathcal{A}^\varepsilon(t) - \mathcal{A}^\varepsilon(t-\delta)\big) \le c\delta,$$

where c *is independent of* ε, δ.

Proof. By Lemma 6.1, (2.8) and A6, A9,

$$\zeta(x,t) := \int_{\max(t,\delta)}^{\min(t+\delta,T)} \delta\, \partial^{-\delta}\big(\mathcal{A}^\varepsilon - \mathcal{A}(P_{b,c})\big)(x,\tau)d\tau \in L^2(0,T;\mathcal{U}).$$

Take ζ above in (6.2) as well as employ Fubini's theorem, A6 and Lemma 6.1 to get

$$\int_\delta^T \int_{\Omega^\varepsilon_f} \Phi^\varepsilon \delta^2 \partial^{-\delta} S^\varepsilon \; \partial^{-\delta}\mathcal{A}^\varepsilon(x,\tau) + \int_\delta^T \int_{\Omega^\varepsilon_m} \phi^\varepsilon \delta^2 \partial^{-\delta} s^\varepsilon \partial^{-\delta}\mathcal{A}^\varepsilon(x,\tau)$$
$$= \int_{\Omega^{\varepsilon,T}_f} \Phi^\varepsilon \partial_t S^\varepsilon(x,t)\zeta + \int_{\Omega^{\varepsilon,T}_m} \phi^\varepsilon \partial_t s^\varepsilon(x,t)\zeta$$

$$+\int_\delta^T\int_{\Omega_f^\varepsilon} \Phi^\varepsilon \delta^2 \partial^{-\delta} S^\varepsilon\ \partial^{-\delta}\mathcal{A}(P_{b,c}) + \int_\delta^T\int_{\Omega_m^\varepsilon} \phi^\varepsilon \delta^2 \partial^{-\delta} s^\varepsilon \partial^{-\delta}\mathcal{A}(P_{b,c})$$

$$= -\int_{\Omega_f^{\varepsilon,T}} K^\varepsilon \Lambda_{\mathrm{o}}(S^\varepsilon)\nabla(P_{\mathrm{o}}^\varepsilon - G_{\mathrm{o}}^\varepsilon)\nabla\zeta - \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,T}} k^\varepsilon \lambda_{\mathrm{o}}(s^\varepsilon)\nabla(p_{\mathrm{o}}^\varepsilon - g_{\mathrm{o}}^\varepsilon)\nabla\zeta$$

$$+\int_\delta^T\int_{\Omega_f^\varepsilon} \Phi^\varepsilon \delta^2 \partial^{-\delta} S^\varepsilon\ \partial^{-\delta}\mathcal{A}(P_{b,c}) + \int_\delta^T\int_{\Omega_m^\varepsilon} \phi^\varepsilon \delta^2 \partial^{-\delta} s^\varepsilon \partial^{-\delta}\mathcal{A}(P_{b,c}) \le c\delta,$$

where c is independent of ε and δ. So the proof is complete. □

Lemma 6.4. *There is a subsequence of* $\Pi_\varepsilon(\mathcal{A}^\varepsilon|_{\Omega_f^\varepsilon})$ *converging to* $\mathcal{A}^*$ *in* $L^2(\Omega^T)$.

Proof. This is due to A3, Lemmas 5.1, 6.1, 6.3, and Corollary 3.6 of Chap. II of Ref. 21. □

Lemma 6.5. *For any* $\tau(\le T)$ *and* $\beta, \beta_0(\in \mathbb{N})$ *satisfying* $2 \le \beta_0 \le \beta - 2$, *the following inequalities hold*:

$$\sup_{t\le\tau}\left|\left\{x\in\Omega : S^\varepsilon(x,t)\le\mu \ \textit{or}\ \rho^\varepsilon(x,t)\le\mu\right\}\right| \le \frac{c_0|c_0\tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}}, \tag{6.9}$$

$$\sup_{t\le\tau}\left|\left\{x\in\Omega : 1-\mu\le S^\varepsilon(x,t) \ \textit{or}\ 1-\mu\le\rho^\varepsilon(x,t)\right\}\right| \le \frac{c_0|c_0\tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}}, \tag{6.10}$$

where $\mu := \frac{\gamma_3}{2^\beta}$, $\rho^\varepsilon := \mathcal{J}^{-1}(s^\varepsilon)$, $\lim_{\beta\to\infty}\mathbf{f}_\beta = 1$, *and* c_0 *is a constant independent of* $\tau, \beta, \varepsilon, \mu$. *See* A9 *for* γ_3.

Proof. Define $\widetilde{\mathcal{X}_\mu}, \mathcal{K}_\mu, \widetilde{\mathcal{K}_\mu}$ as

$$\widetilde{\mathcal{X}_\mu}(z) := \begin{cases} 1, & \text{if } \mu\le z\le 2\mu, \\ 0, & \text{elsewhere}, \end{cases}$$

$$\mathcal{K}_\mu(z) := \int_{\mathcal{A}(\Upsilon(2\mu))}^z \widetilde{\mathcal{X}_\mu}\left(\Upsilon^{-1}(\mathcal{A}^{-1}(\xi))\right)d\xi, \qquad \text{for } z\in[0,\mathcal{A}(\infty)),$$

$$\widetilde{\mathcal{K}_\mu}(z) := \int_{\mathcal{A}(\Upsilon(2\mu))}^z \widetilde{\mathcal{X}_\mu}\frac{\Lambda_{\mathrm{o}}}{\Lambda}\left(\Upsilon^{-1}(\mathcal{A}^{-1}(\xi))\right)d\xi, \quad \text{for } z\in[0,\mathcal{A}(\infty)).$$

Note $\mathcal{K}_\mu(\mathcal{A}^\varepsilon), \widetilde{\mathcal{K}_\mu}(\mathcal{A}^\varepsilon) \in L^2(0,T;\mathcal{U})$ by Lemma 6.1, $2\mu \le \frac{\gamma_3}{2}$ and A9. Take $\zeta = \mathcal{K}_\mu(\mathcal{A}^\varepsilon)$ in (6.5) and $\eta = \widetilde{\mathcal{K}_\mu}(\mathcal{A}^\varepsilon)$ in (6.6) to obtain, by A1, A5, A6,

$$\int_{\Omega_f^{\varepsilon,\tau}} \Phi^\varepsilon\mathcal{K}_\mu(\mathcal{A}^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega_f^{\varepsilon,\tau}} K^\varepsilon\Lambda_{\mathrm{o}}(S^\varepsilon)\widetilde{\mathcal{X}_\mu}(S^\varepsilon)\nabla\Upsilon(S^\varepsilon)\nabla\mathcal{A}^\varepsilon$$
$$+\int_{\Omega_m^{\varepsilon,\tau}} \phi^\varepsilon\mathcal{K}_\mu(\mathcal{A}^\varepsilon)\partial_t s^\varepsilon + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} k^\varepsilon\Lambda_{\mathrm{o}}(\rho^\varepsilon)\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)\nabla\upsilon(s^\varepsilon)\nabla\mathcal{A}^\varepsilon$$
$$\le c_1\left(\int_{\Omega_f^{\varepsilon,\tau}} K^\varepsilon\Lambda_{\mathrm{o}}(S^\varepsilon)\widetilde{\mathcal{X}_\mu}(S^\varepsilon)|\nabla\mathcal{A}^\varepsilon| + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} k^\varepsilon\Lambda_{\mathrm{o}}(\rho^\varepsilon)\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)|\nabla\mathcal{A}^\varepsilon|\right), \tag{6.11}$$

where constant c_1 is independent of ε, μ. If

$$\int_{\Omega_f^{\varepsilon,\tau}} \Phi^\varepsilon \mathcal{K}_\mu(\mathcal{A}^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega_m^{\varepsilon,\tau}} \phi^\varepsilon \mathcal{K}_\mu(\mathcal{A}^\varepsilon)\partial_t s^\varepsilon \geq 0, \tag{6.12}$$

then (6.11)–(6.12) imply

$$\begin{aligned}
&\int_{\Omega_f^{\varepsilon,\tau}} K^\varepsilon \Lambda_{\mathrm{o}}(S^\varepsilon)\widetilde{\mathcal{X}_\mu}(S^\varepsilon)|\nabla\mathcal{A}^\varepsilon| + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} k^\varepsilon \Lambda_{\mathrm{o}}(\rho^\varepsilon)\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)|\nabla\mathcal{A}^\varepsilon| \\
&\quad\leq c_2\sqrt{\int_{\Omega_f^{\varepsilon,\tau}} K^\varepsilon \Lambda_{\mathrm{o}}^{3/2}\widetilde{\mathcal{X}_\mu}(S^\varepsilon)}\sqrt{\int_{\Omega_f^{\varepsilon,\tau}} K^\varepsilon \Lambda_{\mathrm{o}}(S^\varepsilon)\widetilde{\mathcal{X}_\mu}(S^\varepsilon)\nabla\Upsilon(S^\varepsilon)\nabla\mathcal{A}^\varepsilon} \\
&\qquad+ c_2\sqrt{\varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} k^\varepsilon \Lambda_{\mathrm{o}}^{3/2}\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)}\sqrt{\varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} k^\varepsilon \Lambda_{\mathrm{o}}(\rho^\varepsilon)\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)\nabla\upsilon(s^\varepsilon)\nabla\mathcal{A}^\varepsilon},
\end{aligned} \tag{6.13}$$

where constant c_2 is independent of ε, μ. A9 and (6.11)–(6.13) imply

$$\begin{aligned}
&\int_{\Omega_f^{\varepsilon,\tau}} \Phi^\varepsilon \mathcal{K}_\mu(\mathcal{A}^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega_m^{\varepsilon,\tau}} \phi^\varepsilon \mathcal{K}_\mu(\mathcal{A}^\varepsilon)\partial_t s^\varepsilon \\
&\quad\leq c_3\left(\int_{\Omega_f^{\varepsilon,\tau}} \Lambda_{\mathrm{o}}^{3/2}\widetilde{\mathcal{X}_\mu}(S^\varepsilon) + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} \Lambda_{\mathrm{o}}^{3/2}\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)\right).
\end{aligned} \tag{6.14}$$

Define

$$\mathcal{Z}(S^\varepsilon, s^\varepsilon, \mu) := \begin{cases} \Phi^\varepsilon \displaystyle\int_{2\mu}^{S^\varepsilon} \mathcal{K}_\mu(\mathcal{A}(\Upsilon(\xi)))d\xi, & \text{in } \Omega_f^\varepsilon, \\ \phi^\varepsilon \displaystyle\int_{\mathcal{J}(2\mu)}^{s^\varepsilon} \mathcal{K}_\mu(\mathcal{A}(\upsilon(\xi)))d\xi, & \text{in } \Omega_m^\varepsilon. \end{cases}$$

(6.14) implies

$$\int_{\Omega^\tau} \partial_t \mathcal{Z}(S^\varepsilon, s^\varepsilon, \mu) \leq c_3\left(\int_{\Omega_f^{\varepsilon,\tau}} \Lambda_{\mathrm{o}}^{3/2}\widetilde{\mathcal{X}_\mu}(S^\varepsilon) + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} \Lambda_{\mathrm{o}}^{3/2}\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)\right). \tag{6.15}$$

(6.15) and A2, A4 yield that, if $0 \leq t_1 \leq t_2 \leq T$,

$$\int_{t_1}^{t_2}\int_\Omega \partial_t \mathcal{Z}(S^\varepsilon, s^\varepsilon, \mu) \leq c_4 \int_{t_1}^{t_2}\int_\Omega \mathcal{Z}(S^\varepsilon, s^\varepsilon, 2\mu), \tag{6.16}$$

where c_4 is independent of $t_1, t_2, \mu, \varepsilon$. Define

$$\mathcal{F}^\varepsilon(\mu, \tau) := \frac{1}{\Lambda_{\mathrm{o}}(\mu)^{3/2}} \sup_{t\leq\tau} \int_\Omega \mathcal{Z}(S^\varepsilon, s^\varepsilon, \mu).$$

A1 and (6.16) imply that, for $0 \leq t_1 \leq t_2 \leq T$,

$$\mathcal{F}^\varepsilon(\mu, t_2) - \mathcal{F}^\varepsilon(\mu, t_1) \leq c_5(t_2 - t_1)\mathcal{F}^\varepsilon(2\mu, t_2),$$

where c_5 is independent of $t_1, t_2, \mu, \varepsilon$. By induction and A9, one obtains, for $j \in \mathbb{N},\ jh \le T$,

$$\mathcal{F}^\varepsilon\left(\frac{\gamma_3}{2^\beta}, jh\right) \le (\beta-\beta_0+1)^{j-1}|c_5 h|^{\beta-\beta_0}\mathcal{F}^\varepsilon\left(\frac{\gamma_3}{2^{\beta_0}}, jh\right). \tag{6.17}$$

If $j = \frac{\beta-\beta_0}{\log(\beta-\beta_0)}$ and $\tau = jh$ in (6.17), then

$$\mathcal{F}^\varepsilon\left(\frac{\gamma_3}{2^\beta}, \tau\right) \le \frac{|c_5\tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}}\mathcal{F}^\varepsilon\left(\frac{\gamma_3}{2^{\beta_0}}, \tau\right), \tag{6.18}$$

where $\mathbf{f}_\beta \to 1$ as $\beta \to \infty$. Define

$$\mathcal{B}(t) := \left\{x \in \Omega : S^\varepsilon(x,t) \le \frac{\gamma_3}{2^\beta} \text{ or } \rho^\varepsilon(x,t) \le \frac{\gamma_3}{2^\beta}\right\}.$$

A4 and (6.18) imply

$$\sup_{t\le\tau}|\mathcal{B}(t)| \le c_6\mathcal{F}^\varepsilon\left(\frac{\gamma_3}{2^\beta}, \tau\right) \le \frac{c_6|c_5\tau|^{\beta-\beta_0}}{(\beta-\beta_0)^{(\beta-\beta_0)\mathbf{f}_\beta}}\mathcal{F}^\varepsilon\left(\frac{\gamma_3}{2^{\beta_0}}, \tau\right),$$

where constant c_6 is independent of $\tau, \beta, \varepsilon, \mu$. So proof of (6.9) is done.

Proof of (6.10) is similar to that of (6.9). In this case, the quantities $\widetilde{\mathcal{X}_\mu},\ \mathcal{K}_\mu,\ \widetilde{\mathcal{K}_\mu},\ \mathcal{Z}$ are modified as follows:

$$\widetilde{\mathcal{X}_\mu}(z) := \begin{cases} 1, & \text{if } 1-2\mu \le z \le 1-\mu, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{K}_\mu(z) := \int_{\mathcal{A}(\Upsilon(1-2\mu))}^z \widetilde{\mathcal{X}_\mu}\left(\Upsilon^{-1}(\mathcal{A}^{-1}(\xi))\right)d\xi, \quad \text{for } z \in [0, \mathcal{A}(\infty)),$$

$$\widetilde{\mathcal{K}_\mu}(z) := \int_{\mathcal{A}(\Upsilon(1-2\mu))}^z \widetilde{\mathcal{X}_\mu}\frac{\Lambda_{\rm o}}{\Lambda}\left(\Upsilon^{-1}(\mathcal{A}^{-1}(\xi))\right)d\xi, \quad \text{for } z \in [0, \mathcal{A}(\infty)),$$

$$\mathcal{Z}(S^\varepsilon, s^\varepsilon, \mu) := \begin{cases} \Phi^\varepsilon \displaystyle\int_{1-2\mu}^{S^\varepsilon} \mathcal{K}_\mu(\mathcal{A}(\Upsilon(\xi)))d\xi, & \text{in } \Omega_f^\varepsilon, \\ \phi^\varepsilon \displaystyle\int_{\mathcal{J}(1-2\mu)}^{s^\varepsilon} \mathcal{K}_\mu(\mathcal{A}(\upsilon(\xi)))d\xi, & \text{in } \Omega_m^\varepsilon. \end{cases}$$

By $2\mu \le \frac{\gamma_3}{2}$, we take $\zeta = \mathcal{K}_\mu(\mathcal{A}^\varepsilon)$ in (6.5) and $\eta = \widetilde{\mathcal{K}_\mu}(\mathcal{A}^\varepsilon)$ in (6.6) to obtain

$$\begin{aligned}
&\int_{\Omega_f^{\varepsilon,\tau}} \Phi^\varepsilon\mathcal{K}_\mu(\mathcal{A}^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega_f^{\varepsilon,\tau}} K^\varepsilon\Lambda_{\rm w}(S^\varepsilon)\widetilde{\mathcal{X}_\mu}(S^\varepsilon)\nabla\Upsilon(S^\varepsilon)\nabla\mathcal{A}^\varepsilon \\
&\quad + \int_{\Omega_m^{\varepsilon,\tau}} \phi^\varepsilon\mathcal{K}_\mu(\mathcal{A}^\varepsilon)\partial_t s^\varepsilon + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} k^\varepsilon\Lambda_{\rm w}(\rho^\varepsilon)\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)\nabla\upsilon(s^\varepsilon)\nabla\mathcal{A}^\varepsilon \\
&\le c_1\left(\int_{\Omega_f^{\varepsilon,\tau}} K^\varepsilon\Lambda_{\rm w}(S^\varepsilon)\widetilde{\mathcal{X}_\mu}(S^\varepsilon)|\nabla\mathcal{A}^\varepsilon| + \varepsilon^{2\varpi}\int_{\Omega_m^{\varepsilon,\tau}} k^\varepsilon\Lambda_{\rm w}(\rho^\varepsilon)\widetilde{\mathcal{X}_\mu}(\rho^\varepsilon)|\nabla\mathcal{A}^\varepsilon|\right),
\end{aligned}$$

where constant c_1 is independent of ε, μ. Then following the argument of (6.9), one can get (6.10). □

Lemma 6.6. *If* $1 < r < 2$,

$$\sum_{\alpha=\mathrm{w,o}} \left(\|P_\alpha^\varepsilon\|_{L^r(0,T;W^{1,r}(\Omega_f^\varepsilon))} + \|\varepsilon^\varpi \nabla p_\alpha^\varepsilon\|_{L^r(\Omega_m^{\varepsilon,T})}\right) \le c,$$

where c is independent of ε. *In addition to* $\varpi = 1$, $\sum_{\alpha=w,o} \|p_\alpha^\varepsilon\|_{L^r(\Omega_m^{\varepsilon,T})} \le c$.

Proof. We define, for $2 \le \beta_0 \in \mathbb{N}$,

$$\begin{cases} \mathcal{B}_{1+\beta_0} := \left\{(x,t) \in \Omega_f^{\varepsilon,T} :\ \dfrac{\gamma_3}{2^{2+\beta_0}} \le S^\varepsilon\right\}, \\ \mathcal{B}_\beta := \left\{(x,t) \in \Omega_f^{\varepsilon,T} :\ \dfrac{\gamma_3}{2^{\beta+1}} \le S^\varepsilon < \dfrac{\gamma_3}{2^\beta}\right\}, \quad \text{if } 2+\beta_0 \le \beta \in \mathbb{N}. \end{cases}$$

A1, Lemmas 6.1, 6.5, and Hölder inequality imply

$$\begin{aligned} \int_{\Omega_f^{\varepsilon,T}} |\nabla P_\mathrm{o}^\varepsilon|^r &\le \left(\int_{\Omega_f^{\varepsilon,T}} \Lambda_\mathrm{o}(S^\varepsilon)|\nabla P_\mathrm{o}^\varepsilon|^2\right)^{\frac{r}{2}} \left(\int_{\Omega_f^{\varepsilon,T}} |\Lambda_\mathrm{o}(S^\varepsilon)|^{\frac{-r}{2-r}}\right)^{\frac{2-r}{2}} \\ &\le c_1 \left(\int_{\Omega_f^{\varepsilon,T}} |\Lambda_\mathrm{o}(S^\varepsilon)|^{\frac{-r}{2-r}}\right)^{\frac{2-r}{2}} = c_1 \left(\int_{\Omega_f^{\varepsilon,T}} |\Lambda_\mathrm{o}(S^\varepsilon)|^{\frac{-r}{2-r}} \sum_{\beta=1+\beta_0}^{\infty} \mathcal{X}_{\mathcal{B}_\beta}\right)^{\frac{2-r}{2}} \\ &\le c_2 \text{ (independent of } \varepsilon). \end{aligned} \tag{6.19}$$

Similar argument as (6.19) will give

$$\int_{\Omega_f^{\varepsilon,T}} |\nabla P_\mathrm{w}^\varepsilon|^r + \sum_{\alpha=\mathrm{w,o}} \int_{\Omega_m^{\varepsilon,T}} |\varepsilon^\varpi \nabla p_\alpha^\varepsilon|^r \le c. \tag{6.20}$$

By boundary condition A6, $\|P_\alpha^\varepsilon\|_{L^r(\Omega_f^{\varepsilon,T})} \le c,\ \alpha = \mathrm{w,o}$. In addition to $\varpi = 1$, by Lemma 5.1, (2.8), (6.19)–(6.20),

$$\int_{\Omega_m^{\varepsilon,T}} |p_\alpha^\varepsilon - \Pi_\varepsilon P_\alpha^\varepsilon|^r \le \varepsilon^r \int_{\Omega_m^{\varepsilon,T}} |\nabla(p_\alpha^\varepsilon - \Pi_\varepsilon P_\alpha^\varepsilon)|^r \le c.$$

So $\|p_\alpha^\varepsilon\|_{L^r(\Omega_m^{\varepsilon,T})}$ is bounded. □

Lemma 6.7. $\overline{s^\varepsilon}, \overline{p^\varepsilon}, \overline{p_\alpha^\varepsilon}$ $(\alpha = \mathrm{w,o})$ *satisfy, for almost all* $x \in \Omega^\varepsilon$,

$$\phi(y)\partial_t \overline{s^\varepsilon} - \varepsilon^{2(\varpi-1)}\nabla_y \cdot \left(k(y)\nabla_y \mathcal{M}(\overline{s^\varepsilon}) + \varepsilon k(y)\lambda_\mathrm{o}(\overline{s^\varepsilon})\overline{\nabla(p^\varepsilon - g_\mathrm{o}^\varepsilon)}\right) = 0, \tag{6.21}$$

in $L^2(0,T;H^{-1}(Y_m))$*. Moreover, if* $\varpi = 1$*, we have, for almost all* $x \in \Omega^\varepsilon$,

$$\phi(y)\partial_t\overline{s^\varepsilon} - \nabla_y \cdot \big(k(y)\lambda_{\rm o}(\overline{s^\varepsilon})(\nabla_y\overline{p^\varepsilon_{\rm o}} - \varepsilon\overline{\nabla g^\varepsilon_{\rm o}})\big) = 0, \tag{6.22}$$

$$-\phi(y)\partial_t\overline{s^\varepsilon} - \nabla_y \cdot \big(k(y)\lambda_{\rm w}(\overline{s^\varepsilon})(\nabla_y\overline{p^\varepsilon_{\rm w}} - \varepsilon\overline{\nabla g^\varepsilon_{\rm w}})\big) = 0, \tag{6.23}$$

in $L^2(0,T;H^{-1}(Y_m))$.

Proof. Let $\widetilde{\zeta} \in H^1(0,T;L^2(Y_m)) \cap L^2(0,T;C_0^\infty(Y_m))$. For $x \in \Omega$, $z \in \mathbb{R}^3$, we define

$$\check{\zeta}(x,z,t) := \begin{cases} \widetilde{\zeta}\Big(\dfrac{z-\ell^\varepsilon(x)}{\varepsilon}, t\Big), & \text{for } z \in \varepsilon Y_m + \ell^\varepsilon(x), \\ 0, & \text{elsewhere.} \end{cases}$$

Then we take $\zeta(x,t) = \mathcal{X}_{\varepsilon(Y_m+j)}(x)\check{\zeta}(x,x,t)$ in (6.5) and (6.7) for $j \in \mathbb{Z}^3$ and $\varepsilon(Y_m+j) \subset \Omega^\varepsilon_m$. Since supp $\zeta \subset \varepsilon(Y_m+j) \times (0,T)$ and the components of Ω^ε_m are disjoint,

$$\int_0^T\!\!\int_{\varepsilon(Y_m+j)} \phi^\varepsilon s^\varepsilon \partial_t\zeta - \varepsilon^{2\varpi}\int_0^T\!\!\int_{\varepsilon(Y_m+j)} k^\varepsilon\big(\nabla\mathcal{M}(s^\varepsilon) + \lambda_{\rm o}(s^\varepsilon)\nabla(p^\varepsilon - g^\varepsilon_{\rm o})\big)\nabla\zeta$$
$$= \int_{\varepsilon(Y_m+j)} \phi^\varepsilon s^\varepsilon(T)\zeta(T) - \int_{\varepsilon(Y_m+j)} \phi^\varepsilon s^\varepsilon_0\zeta(0).$$

Since $x \in \varepsilon(Y_m+j)$, $\ell^\varepsilon(x) = \varepsilon j$. Changing of variable $y = \frac{x-\ell^\varepsilon(x)}{\varepsilon}$ gives, by Lemma 5.2,

$$\int_0^T\!\!\int_{Y_m} \phi\overline{s^\varepsilon}\partial_t\widetilde{\zeta} - \varepsilon^{2(\varpi-1)}\int_0^T\!\!\int_{Y_m} k\big(\nabla_y\mathcal{M}(\overline{s^\varepsilon}) + \varepsilon\lambda_{\rm o}(\overline{s^\varepsilon})\overline{\nabla(p^\varepsilon - g^\varepsilon_{\rm o})}\big)\nabla_y\widetilde{\zeta}$$
$$= \int_{Y_m} \phi\overline{s^\varepsilon}(T)\widetilde{\zeta}(T) - \int_{Y_m} \phi\overline{s^\varepsilon_0}\widetilde{\zeta}(0), \tag{6.24}$$

for almost all $x \in \varepsilon(Y_m+j) \subset \Omega^\varepsilon_m$, $j \in \mathbb{Z}^3$. By Definition 5.1, (6.24) actually holds for $x \in \Omega^\varepsilon$. So we get (6.21). (6.22)–(6.23) can be obtained from (6.1)–(6.2) by a similar argument as above. □

Let us define

$$S^\varepsilon := \Upsilon^{-1}\big(\mathcal{A}^{-1}(\Pi_\varepsilon(\mathcal{A}^\varepsilon|_{\Omega^\varepsilon_f}))\big),$$
$$S := \begin{cases} \Upsilon^{-1}(\mathcal{A}^{-1}(\mathcal{A}^*)), & \text{if } \mathcal{A}^* < \mathcal{A}(\infty), \\ 1, & \text{if } \mathcal{A}^* = \mathcal{A}(\infty). \end{cases}$$

Then $0 \le S^\varepsilon, S \le 1$ by Lemmas 5.1 and 6.4. Next we shall consider $\varpi = 1$, $\varpi > 1$, and $0 < \varpi < 1$ cases separately.

6.1. *For $\varpi = 1$ case*

Lemma 6.8. *There is an $r \in (1,2)$ and a subsequence of $\{S^\varepsilon, s^\varepsilon, S_0^\varepsilon, s_0^\varepsilon, \mathcal{A}^\varepsilon, \phi^\varepsilon, k^\varepsilon, P_\alpha^\varepsilon, p_\alpha^\varepsilon, \alpha = \mathrm{w}, \mathrm{o}\}$ such that, as $\varepsilon \to 0$,*

$$\begin{cases} \mathcal{X}_{\Omega_f^\varepsilon} P_\alpha^\varepsilon \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y) P_\alpha(x,t), \ \textit{where } P_\alpha \in L^r(0,T;W^{1,r}(\Omega)), P_\alpha = P_{b,\alpha} \textit{ in } \Gamma_2, \\ \mathcal{X}_{\Omega_f^\varepsilon} \nabla P_\alpha^\varepsilon \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y)(\nabla P_\alpha + \nabla_y P_{\alpha,1}(x,y,t)), \ \textit{where } P_{\alpha,1} \in L^r(\Omega^T; W^{1,r}_{\mathrm{per}}(Y)), \\ \mathcal{X}_{\Omega_f^\varepsilon} S_0^\varepsilon \overset{2}{\rightharpoonup} S_0 \in L^2(\Omega; L^2_{\mathrm{per}}(Y_f)), \\ \Pi_\varepsilon(\mathcal{A}^\varepsilon|_{\Omega_f^\varepsilon}), \ S^\varepsilon \to \mathcal{A}^*, \ S \textit{ strongly in } L^2(\Omega^T) \textit{ and pointwise}, \\ \mathcal{X}_{\Omega_f^\varepsilon} S^\varepsilon \overset{2}{\to} \mathcal{X}_{Y_f}(y) S(x,t) \textit{ strongly}, \\ \mathcal{X}_{\Omega_m^\varepsilon} s^\varepsilon, \mathcal{X}_{\Omega_m^\varepsilon} \mathcal{M}(s^\varepsilon), \mathcal{X}_{\Omega_m^\varepsilon} \varepsilon \nabla \mathcal{M}(s^\varepsilon) \overset{2}{\rightharpoonup} s, \mathcal{M}^*, \nabla_y \mathcal{M}^* \in L^2(\Omega^T; L^2_{\mathrm{per}}(Y_m)), \\ \mathcal{X}_{\Omega_m^\varepsilon} p_\alpha^\varepsilon \overset{2}{\rightharpoonup} p_\alpha \in L^r(\Omega^T; L^r_{\mathrm{per}}(Y_m)), \\ \mathcal{X}_{\Omega_m^\varepsilon} s^\varepsilon(T) \overset{2}{\rightharpoonup} s^* \in L^2(\Omega; L^2_{\mathrm{per}}(Y_m)), \\ \mathcal{X}_{\Omega_m^\varepsilon} s_0^\varepsilon \overset{2}{\to} s_0 \in L^2(\Omega; L^2_{\mathrm{per}}(Y_m)) \textit{ strongly}, \\ \phi^\varepsilon, k^\varepsilon \overset{2}{\to} \phi, k \in L^2(\Omega; L^2_{\mathrm{per}}(Y) \textit{ strongly}, \\ \mathcal{M}(\overline{s^\varepsilon}) \rightharpoonup \mathcal{M}^* \quad \textit{weakly in } L^2(\Omega^T; H^1(Y_m)), \\ \overline{s^\varepsilon} \rightharpoonup s \quad \textit{weakly in } L^2(\mathcal{Q}_m^T), \\ \overline{p_\alpha^\varepsilon} \rightharpoonup p_\alpha \quad \textit{weakly in } L^r(\Omega^T; W^{1,r}(Y_m)). \end{cases}$$

Proof. We note that $\Pi_\varepsilon P_\alpha^\varepsilon$ is bounded in $L^r(0,T;W^{1,r}(\Omega))$ by Lemmas 5.1 and 6.6. So there is a subsequence of $\Pi_\varepsilon P_\alpha^\varepsilon$ converging weakly to $P_\alpha \in L^r(0,T;W^{1,r}(\Omega))$. Since $\Pi_\varepsilon P_\alpha^\varepsilon = P_{b,\alpha}$ in Γ_2, $P_\alpha = P_{b,\alpha}$ in Γ_2. The lemma is then a direct consequence of the above result, A3, A6–A9, Lemmas 5.2, 5.3, 6.1, 6.4, 6.6, and Refs. 3 and 13. □

Remark 6.1. Let us define

$$\mathcal{G}^\varepsilon := \upsilon^{-1}\big(\mathcal{A}^{-1}(\Pi_\varepsilon(\mathcal{A}^\varepsilon|_{\Omega_f^\varepsilon}))\big),$$
$$\mathcal{G} := \begin{cases} \upsilon^{-1}(\mathcal{A}^{-1}(\mathcal{A}^*)), & \text{if } \mathcal{A}^* < \mathcal{A}(\infty), \\ s_r, & \text{if } \mathcal{A}^* = \mathcal{A}(\infty). \end{cases}$$

By A1, A3, Corollary 5.1, Lemma 6.8. and Ref. 3, it is easy to see that

$$\begin{cases} \|\mathcal{M}(\mathcal{G}^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \text{ are bounded independently of } \varepsilon, \\ \mathcal{M}(\overline{\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon}}) \to \mathcal{M}(\mathcal{G}) \quad \text{strongly in } L^2(\mathcal{Q}_m^T), \\ \nabla_y \mathcal{M}(\mathcal{G}) = 0, \\ \mathcal{M}(\overline{\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon}}) - \mathcal{M}(\overline{s^\varepsilon}), \mathcal{M}(\mathcal{G}) - \mathcal{M}^* \in L^2(\Omega^T; H_0^1(Y_m)). \end{cases} \tag{6.25}$$

Lemma 6.9. *$s(0) = s_0$ and $\overline{s^\varepsilon}$ converges strongly to s in $L^2(\mathcal{Q}_m^T)$.*

Proof. **Step 1:** Let $\psi(t) \in C^1[0,T]$, $\check{\eta} \in C_0^\infty(\Omega; C^\infty_{\text{per}}(Y))$ with $\check{\eta} = 0$ for $y \in Y_f$. We plug $\zeta = \check{\eta}(x, \frac{x}{\varepsilon})\psi(t)$ into (6.5) and (6.7), then

$$\begin{aligned} &-\int_{\Omega_m^{\varepsilon,T}} \phi^\varepsilon s^\varepsilon \check{\eta}\partial_t\psi(t) + \varepsilon^2 \int_{\Omega_m^{\varepsilon,T}} k^\varepsilon \lambda_{\rm o}(s^\varepsilon)\nabla(p^\varepsilon - g^\varepsilon_{\rm o})\left(\nabla_x\check{\eta} + \frac{1}{\varepsilon}\nabla_y\check{\eta}\right)\psi(t) \\ &\quad + \varepsilon^2 \int_{\Omega_m^{\varepsilon,T}} k^\varepsilon \nabla\mathcal{M}(s^\varepsilon)\left(\nabla_x\check{\eta} + \frac{1}{\varepsilon}\nabla_y\check{\eta}\right)\psi(t) \\ &= \int_{\Omega_m^\varepsilon} \phi^\varepsilon s_0^\varepsilon \check{\eta}\psi(0) - \int_{\Omega_m^\varepsilon} \phi^\varepsilon s^\varepsilon(T)\check{\eta}\psi(T). \end{aligned} \tag{6.26}$$

Passing to two-scale limit, we get, by Lemmas 5.4, 6.1, 6.2 and 6.8,

$$-\int_{\mathcal{Q}_m^T} \phi s\check{\eta}\partial_t\psi(t) + \int_{\mathcal{Q}_m^T} k\nabla_y\mathcal{M}^*\nabla_y\check{\eta}\psi(t) = \int_{\mathcal{Q}_m} \phi s_0\check{\eta}\psi(0) - \int_{\mathcal{Q}_m} \phi s^*\check{\eta}\psi(T), \tag{6.27}$$

for $\psi(t) \in C^1[0,T]$, $\check{\eta} \in L^2(\Omega; H^1_0(Y_m))$. Applying Green's theorem for (6.27) in the t variable yields

$$\begin{aligned} &\int_{\mathcal{Q}_m^T} \phi\partial_t s\ \check{\eta}\psi(t) + \int_{\mathcal{Q}_m^T} k\nabla_y\mathcal{M}^*\nabla_y\check{\eta}\psi(t) \\ &\quad = \int_{\mathcal{Q}_m} \phi(s_0 - s(0))\check{\eta}\psi(0) - \int_{\mathcal{Q}_m} \phi(s^* - s(T))\check{\eta}\psi(T). \end{aligned} \tag{6.28}$$

(6.28) implies $\partial_t s \in L^2(\Omega^T; H^{-1}(Y_m))$, $s(0) = s_0$, and

$$s(T) = s^*, \tag{6.29}$$

$$\int_{\mathcal{Q}_m^T} \phi\partial_t s\ \eta + \int_{\mathcal{Q}_m^T} k\nabla_y\mathcal{M}^*\nabla_y\eta = 0, \quad \text{for } \eta \in L^2(\Omega^T; H^1_0(Y_m)). \tag{6.30}$$

Step 2: We claim that $\overline{s^\varepsilon}$ converges to s in $L^2(\mathcal{Q}_m^T)$. Let us find $\varphi^\varepsilon, \varphi \in L^2(\Omega^T; H^1_0(Y_m))$ by solving, for all $(x,t) \in \Omega^T$,

$$\begin{cases} -\nabla_y(k\nabla_y\varphi^\varepsilon) = \phi\overline{s^\varepsilon}, & y \in Y_m, \\ \varphi^\varepsilon|_{\partial Y_m} = 0, \end{cases} \qquad \begin{cases} -\nabla_y(k\nabla_y\varphi) = \phi s, & y \in Y_m, \\ \varphi|_{\partial Y_m} = 0. \end{cases} \tag{6.31}$$

(6.21) for $\varpi = 1$, (6.25), (6.31), and Green's theorem imply

$$\begin{aligned} \int_{\mathcal{Q}_m^{\varepsilon,T}} \phi\mathcal{M}(\overline{s^\varepsilon})\overline{s^\varepsilon} &= \int_{\mathcal{Q}_m^{\varepsilon,T}} \phi\mathcal{M}(\overline{\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon}})\overline{s^\varepsilon} - \int_{\mathcal{Q}_m^{\varepsilon,T}} (\mathcal{M}(\overline{s^\varepsilon}) - \mathcal{M}(\overline{\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon}}))\nabla_y(k\nabla_y\varphi^\varepsilon) \\ &= \int_{\mathcal{Q}_m^{\varepsilon,T}} \phi\mathcal{M}(\overline{\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon}})\overline{s^\varepsilon} - \int_{\mathcal{Q}_m^{\varepsilon,T}} k\nabla_y\mathcal{M}(\overline{\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon}})\nabla_y\varphi^\varepsilon \\ &\quad - \int_{\mathcal{Q}_m^{\varepsilon,T}} \phi\partial_t\overline{s^\varepsilon}\ \varphi^\varepsilon - \int_{\mathcal{Q}_m^{\varepsilon,T}} \varepsilon k\lambda_{\rm o}(\overline{s^\varepsilon})\overline{\nabla(p^\varepsilon - g^\varepsilon_{\rm o})}\nabla_y\varphi^\varepsilon. \end{aligned} \tag{6.32}$$

Note

$$-\int_{\mathcal{Q}_m^{\varepsilon,T}} \phi\partial_t\overline{s^\varepsilon}\varphi^\varepsilon = -\int_{\mathcal{Q}_m^\varepsilon} k\frac{|\nabla_y\varphi^\varepsilon|^2}{2}(T) + \int_{\mathcal{Q}_m^\varepsilon} k\frac{|\nabla_y\varphi^\varepsilon|^2}{2}(0), \tag{6.33}$$

$$\int_{\mathcal{Q}_m^{\varepsilon,T}} k\nabla_y\mathcal{M}(\overline{\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon}})\nabla_y\varphi^\varepsilon = \varepsilon\int_{\mathcal{Q}_m^{\varepsilon,T}} k\overline{\nabla\mathcal{M}(\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon})}\nabla_y\varphi^\varepsilon. \tag{6.34}$$

By (6.29) and Lemmas 5.3, 6.8, $\overline{s^\varepsilon}(T)$ converges weakly to $s(T)$ in $L^2(\mathcal{Q}_m)$. That implies, by (6.31), Hölder inequality, and Green's theorem,

$$\int_{\mathcal{Q}_m} k|\nabla_y\varphi|^2(T) \le \varliminf_{\varepsilon\to 0}\int_{\mathcal{Q}_m} k|\nabla_y\varphi^\varepsilon|^2(T). \tag{6.35}$$

Since $\lim_{\varepsilon\to 0}|\Omega\setminus\Omega^\varepsilon| = 0$, we take limit supremum on both sides of (6.32) to obtain, by (6.25), (6.33)–(6.35), and Lemmas 6.2, 6.8,

$$\varlimsup_{\varepsilon\to 0}\int_{\mathcal{Q}_m^T}\phi\mathcal{M}(\overline{s^\varepsilon})\overline{s^\varepsilon} \le \int_{\mathcal{Q}_m^T}\phi\mathcal{M}(\mathcal{G})s - \int_{\mathcal{Q}_m}\left(\frac{k|\nabla_y\varphi|^2}{2}(T) - \frac{k|\nabla_y\varphi|^2}{2}(0)\right). \tag{6.36}$$

Take $\eta = \varphi$ of (6.31) in (6.30) to obtain, by (6.25),

$$0 = \int_{\mathcal{Q}_m}\frac{k|\nabla_y\varphi|^2}{2}(T) - \int_{\mathcal{Q}_m}\frac{k|\nabla_y\varphi|^2}{2}(0) + \int_{\mathcal{Q}_m^T}\phi(\mathcal{M}^* - \mathcal{M}(\mathcal{G}))s. \tag{6.37}$$

By (6.36) and (6.37),

$$\varlimsup_{\varepsilon\to 0}\int_{\mathcal{Q}_m^T}\phi\mathcal{M}(\overline{s^\varepsilon})\overline{s^\varepsilon} \le \int_{\mathcal{Q}_m^T}\phi\mathcal{M}^*s. \tag{6.38}$$

By Remark 3.1, $\mathcal{M}$ is strictly increasing in $\mathbb{R}$. Employing (6.38) and monotonicity argument,[21] one can easily obtain

$$\mathcal{M}(s) = \mathcal{M}^*. \tag{6.39}$$

By (6.38) and (6.39),

$$\lim_{\varepsilon\to 0}\int_{\mathcal{Q}_m^T}\phi(\mathcal{M}(\overline{s^\varepsilon}) - \mathcal{M}(s))(\overline{s^\varepsilon} - s) = 0.$$

Then by monotonicity of $\mathcal{M}$, we see that $\overline{s^\varepsilon}$ converges to s pointwise almost everywhere in $\mathcal{Q}_m^T$. So $\overline{s^\varepsilon}$ converges to s in $L^2(\mathcal{Q}_m^T)$. □

Lemma 6.10. $0 < S < 1, s_l < s < s_r, p_{\mathrm{o}} - p_{\mathrm{w}} = \upsilon(s), P_{\mathrm{o}} - P_{\mathrm{w}} = \Upsilon(S)$, *and* $p_\alpha(x,y,t) = P_\alpha(x,t)$ *for* $x \in \Omega, y \in \partial Y_m, \alpha = \mathrm{w}, \mathrm{o}$.

Proof. $0 < S < 1$, $s_l < s < s_r$ are due to Egoroff's theorem[20] and Lemmas 6.1, 6.5, 6.8, 6.9. Since $\overline{p_{\mathrm{o}}^\varepsilon} - \overline{p_{\mathrm{w}}^\varepsilon} = \upsilon(\overline{s^\varepsilon})$ in $\mathcal{Q}_m^{\varepsilon,T}$, we get $p_{\mathrm{o}} - p_{\mathrm{w}} = \upsilon(s)$ by Lemmas 6.8, 6.9. Similarly, one has $P_{\mathrm{o}} - P_{\mathrm{w}} = \Upsilon(S)$. By Lemmas 5.1, 6.1, 6.6, we have $\overline{(\Pi_\varepsilon P_\alpha^\varepsilon)|_{\Omega_m^\varepsilon}} - \overline{p_\alpha^\varepsilon} \in L^r(\Omega^T; W_0^{1,r}(Y_m)), 1 < r < 2$. So there is a subsequence of $\overline{(\Pi_\varepsilon P_\alpha^\varepsilon)|_{\Omega_m^\varepsilon}} - \overline{p_\alpha^\varepsilon}$ converging weakly to $\mathcal{X}_{Y_m}(y)P_\alpha(x,t) - p_\alpha \in L^r(\Omega^T; W_0^{1,r}(Y_m))$ by Lemma 6.8. So, $p_\alpha(x,y,t) = P_\alpha(x,t)$ for $y \in \partial Y_m$. □

Next we give the proof of Theorem 4.1.

Proof. We substitute into (6.1) a test function of the form

$$\eta = \widetilde{\zeta}(x,t) + \varepsilon\widetilde{\eta}\left(x, \frac{x}{\varepsilon}, t\right),$$

where $\widetilde{\zeta} \in C_0^\infty(\Omega^T), \widetilde{\eta} \in C_0^\infty(\Omega^T; C_{\text{per}}^\infty(Y))$. Then

$$\begin{aligned} 0 &= \int_{\Omega_f^{\varepsilon,T}} \Phi^\varepsilon S^\varepsilon(\partial_t\widetilde{\zeta} + \varepsilon\partial_t\widetilde{\eta}) + \int_{\Omega_f^{\varepsilon,T}} K^\varepsilon\Lambda_{\text{w}}(S^\varepsilon)\nabla(P_{\text{w}}^\varepsilon - G_{\text{w}}^\varepsilon)(\nabla\widetilde{\zeta} + \varepsilon\nabla_x\widetilde{\eta} + \nabla_y\widetilde{\eta}) \\ &\quad + \int_{\Omega_m^{\varepsilon,T}} \phi^\varepsilon s^\varepsilon(\partial_t\widetilde{\zeta} + \varepsilon\partial_t\widetilde{\eta}) + \varepsilon^2\int_{\Omega_m^{\varepsilon,T}} k^\varepsilon\lambda_{\text{w}}(s^\varepsilon)\nabla(p_{\text{w}}^\varepsilon - g_{\text{w}}^\varepsilon)(\nabla\widetilde{\zeta} + \varepsilon\nabla_x\widetilde{\eta} + \nabla_y\widetilde{\eta}). \end{aligned}$$

By A8 and Lemma 6.8, $K^\varepsilon\Lambda_{\text{w}}(S^\varepsilon)$ converges to $K^H\Lambda_{\text{w}}(S)$ strongly in $L^r(\Omega^T)$, $r < \infty$. Passing to two-scale limit, we get, by A7–8, and Lemmas 5.4, 6.6 and 6.8,

$$\begin{aligned} &\int_{\mathcal{Q}_f^T} \Phi^H S\partial_t\widetilde{\zeta} + \int_{\mathcal{Q}_f^T} K^H\Lambda_{\text{w}}(S)(\nabla P_{\text{w}} + \nabla_y P_{w,1} - \nabla G_{\text{w}})(\nabla\widetilde{\zeta} + \nabla_y\widetilde{\eta}) \\ &\quad = -\int_{\mathcal{Q}_m^T} \phi s\partial_t\widetilde{\zeta}. \end{aligned}$$

The choice of $\widetilde{\eta} = 0$ gives, in Ω^T,

$$|Y_f|\Phi^H\partial_t S + \nabla_x\int_{Y_f} K^H\Lambda_{\text{w}}(S)(\nabla P_{\text{w}} + \nabla_y P_{\text{w},1} - \nabla G_{\text{w}}) = -\int_{Y_m} \phi\partial_t s dy, \quad (6.40)$$

where $|Y_f|$ is the volume of Y_f. The choice of $\widetilde{\zeta} = 0$ gives, by A9 and Lemmas 6.8, 6.10,

$$\begin{cases} \Delta_y P_{\text{w},1} = 0, & \text{in } \mathcal{Q}_f, \\ (\nabla P_{\text{w}} + \nabla_y P_{\text{w},1} - \nabla G_{\text{w}})\cdot\boldsymbol{\nu} = 0, & \text{on } \partial Y_m, \end{cases} \quad (6.41)$$

where $\boldsymbol{\nu}$ is the unit vector outward normal to ∂Y_m. Let $\boldsymbol{e}_j$ be the unit vector in the jth direction. We denote by Ξ the tensor whose (i,j) component is $\partial\varphi_j/\partial y_i$, where φ_j is a periodic solution in Y of the auxiliary problem

$$\begin{cases} \Delta_y\varphi_j = 0, & \text{in } Y_f, \\ \nabla_y\varphi_j\cdot\boldsymbol{\nu} = -\boldsymbol{e}_j\cdot\boldsymbol{\nu}, & \text{on } \partial Y_m. \end{cases}$$

$P_{\text{w},1}$ of (6.41) is given by

$$P_{\text{w},1} = \sum_j \varphi_j(y)\partial_{x_j}(P_{\text{w}} - G_{\text{w}}).$$

So (6.40) becomes

$$\Phi\partial_t S + \nabla\cdot(K\Lambda_{\text{w}}(S)\nabla(P_{\text{w}} - G_{\text{w}})) = \frac{-1}{|Y_m|}\int_{Y_m} \phi\partial_t s dy, \quad (6.42)$$

where $\Phi := \frac{|Y_f|}{|Y_m|}\Phi^H$, $K := \frac{K^H}{|Y_m|}\int_{Y_f}(I+\Xi(y))dy$, and $|Y_m|$ is the volume of Y_m. Proceeding as in the proof of (6.42), we obtain, by (6.2),

$$-\Phi\partial_t S + \nabla\cdot(K\Lambda_{\rm o}(S)\nabla(P_{\rm o}-G_{\rm o})) = \frac{1}{|Y_m|}\int_{Y_m}\phi\partial_t s dy. \tag{6.43}$$

By (6.42) and arguing as in step 1 of Lemma 6.9, one gets

$$S(0) = S_0. \tag{6.44}$$

By (6.22), we have, for any $\eta \in L^2(\Omega^T; H_0^1(Y_m))$,

$$\int_{\mathcal{Q}_m^{\varepsilon,T}}\phi\partial_t\overline{s^\varepsilon}\eta + \int_{\mathcal{Q}_m^{\varepsilon,T}}k\lambda_{\rm o}(\overline{s^\varepsilon})\nabla_y\overline{p_{\rm o}^\varepsilon}\nabla_y\eta - \varepsilon\int_{\mathcal{Q}_m^{\varepsilon,T}}k\lambda_{\rm o}(\overline{s^\varepsilon})\overline{\nabla g_{\rm o}^\varepsilon}\nabla_y\eta = 0.$$

As $\varepsilon \to 0$, by Lemmas 6.8 and 6.9, one obtains

$$\int_{\mathcal{Q}_m^T}\phi\partial_t s\ \eta + \int_{\mathcal{Q}_m^T}k\lambda_{\rm o}(s)\nabla_y p_{\rm o}\nabla_y\eta = 0. \tag{6.45}$$

Using the same reasoning as (6.45), we obtain, by (6.23),

$$\int_{\mathcal{Q}_m^T}\phi\partial_t s\ \eta - \int_{\mathcal{Q}_m^T}k\lambda_{\rm w}(s)\nabla_y p_{\rm w}\nabla_y\eta = 0. \tag{6.46}$$

By (6.42)–(6.46) and Lemmas 6.8–6.10, we complete the proof of Theorem 4.1. □

6.2. *For $\varpi > 1$ case*

Arguing as Lemmas 6.8, 6.10, we also have the following result:

Lemma 6.11. *There is a $r \in (1,2)$ and a subsequence of $\{S^\varepsilon, s^\varepsilon, S_0^\varepsilon, P_\alpha^\varepsilon, \alpha = {\rm w}, {\rm o}\}$ such that, as $\varepsilon \to 0$,*

$$\begin{cases} \mathcal{X}_{\Omega_f^\varepsilon}P_\alpha^\varepsilon \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y)P_\alpha(x,t),\ \textit{where}\ P_\alpha \in L^r(0,T;W^{1,r}(\Omega)), P_\alpha = P_{b,\alpha}\ \textit{in}\ \Gamma_2,\\ \mathcal{X}_{\Omega_f^\varepsilon}\nabla P_\alpha^\varepsilon \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y)(\nabla P_\alpha + \nabla_y P_{\alpha,1}(x,y,t)),\ \textit{where}\ P_{\alpha,1} \in L^r(\Omega^T; W^{1,r}_{\rm per}(Y)),\\ \mathcal{X}_{\Omega_f^\varepsilon}S_0^\varepsilon \overset{2}{\rightharpoonup} S_0 \in L^2(\Omega; L^2_{\rm per}(Y_f)),\\ S^\varepsilon \to S\ \textit{strongly in}\ L^2(\Omega^T)\ \textit{and pointwise},\\ \mathcal{X}_{\Omega_f^\varepsilon}S^\varepsilon \overset{2}{\to} \mathcal{X}_{Y_f}(y)S(x,t)\ \textit{strongly},\\ \mathcal{X}_{\Omega_m^\varepsilon}s^\varepsilon \overset{2}{\rightharpoonup} s \in L^2(\Omega^T; L^2_{\rm per}(Y_m)), \end{cases}$$

and $P_{\rm o} - P_{\rm w} = \Upsilon(S)$.

Lemma 6.12. *$\phi\partial_t s = 0$ in $\mathcal{Q}_m^T$.*

Proof. We introduce $F_\varepsilon := \varepsilon^\varpi k^\varepsilon(\lambda_o(s^\varepsilon)\nabla(p^\varepsilon - g_o^\varepsilon) + \nabla\mathcal{M}(s^\varepsilon))$. Then, by Lemmas 6.1–6.2,

$$\|F_\varepsilon\|_{L^2(\Omega_m^{\varepsilon,T})} \le c. \tag{6.47}$$

So $\mathcal{X}_{\Omega_m^\varepsilon} F_\varepsilon \overset{2}{\rightharpoonup} F^*$ in two-scale sense.[3] Let $\zeta \in C_0^\infty(\Omega^T; C_{\rm per}^\infty(Y))$ with $\zeta = 0$ for $y \in Y_f$. We plug $\zeta(x, \frac{x}{\varepsilon}, t)$ into (6.5) as a test function to obtain

$$0 = -\int_{\Omega_m^{\varepsilon,T}} \phi^\varepsilon s^\varepsilon \partial_t \zeta + \varepsilon^\varpi \int_{\Omega_m^{\varepsilon,T}} F_\varepsilon\left(\nabla_x\zeta + \frac{1}{\varepsilon}\nabla_y\zeta\right).$$

Passing to two-scale limit, we get, by A7, (6.47) and Lemma 6.11,

$$0 = \int_{\mathcal{Q}_m^T} \phi s \partial_t \zeta.$$

So $\phi\partial_t s = 0$ and we complete the proof. □

Repeating the process for the proof of Theorem 4.1, one obtains Theorem 4.2 by Lemmas 6.11 and 6.12.

6.3. *For* $0 < \varpi < 1$ *case*

Lemma 6.13. *There is an* $r \in (1,2)$ *and a subsequence of* $\{S^\varepsilon, s^\varepsilon, S_0^\varepsilon, \mathcal{A}^\varepsilon, P_\alpha^\varepsilon, \alpha = \mathrm{w}, \mathrm{o}\}$ *such that, as* $\varepsilon \to 0$,

$$\begin{cases} \mathcal{X}_{\Omega_f^\varepsilon} P_\alpha^\varepsilon \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y) P_\alpha(x,t), \ \textit{where } P_\alpha \in L^r(0,T;W^{1,r}(\Omega)), P_\alpha = P_{b,\alpha} \textit{ in } \Gamma_2, \\ \mathcal{X}_{\Omega_f^\varepsilon} \nabla P_\alpha^\varepsilon \overset{2}{\rightharpoonup} \mathcal{X}_{Y_f}(y)(\nabla P_\alpha + \nabla_y P_{\alpha,1}(x,y,t)), \ \textit{where } P_{\alpha,1} \in L^r(\Omega^T; W_{\rm per}^{1,r}(Y)), \\ \mathcal{X}_{\Omega_f^\varepsilon} S_0^\varepsilon \overset{2}{\rightharpoonup} S_0 \in L^2(\Omega; L_{\rm per}^2(Y_f)), \\ \Pi_\varepsilon(\mathcal{A}^\varepsilon|_{\Omega_f^\varepsilon}), \ S^\varepsilon \to \mathcal{A}^*, \ S \textit{ strongly in } L^2(\Omega^T) \textit{ and pointwise}, \\ \mathcal{X}_{\Omega_f^\varepsilon} S^\varepsilon \overset{2}{\to} \mathcal{X}_{Y_f}(y) S(x,t) \ \textit{strongly}, \\ \mathcal{X}_{\Omega_m^\varepsilon} s^\varepsilon, \mathcal{X}_{\Omega_m^\varepsilon}\mathcal{M}(s^\varepsilon), \mathcal{X}_{\Omega_m^\varepsilon}\varepsilon^\varpi\nabla\mathcal{M}(s^\varepsilon) \overset{2}{\rightharpoonup} s, \mathcal{M}^*, \mathcal{M}_1 \in L^2(\Omega^T; L_{\rm per}^2(Y_m)), \\ \mathcal{M}(\overline{s^\varepsilon}) \rightharpoonup \mathcal{M}^* \quad \textit{weakly in } L^2(\Omega^T; H^1(Y_m)). \end{cases}$$

Proof. This lemma can be proved by the same argument as Lemma 6.8. □

Remark 6.2. By Lemmas 6.1 and 6.13, it is easy to see

$$\mathcal{M}^* = \mathcal{M}^*(x,t), \tag{6.48}$$

that is, $\mathcal{M}^*$ is independent of $y \in Y_m$. If we define

$$\mathcal{G}^\varepsilon := \upsilon^{-1}\big(\mathcal{A}^{-1}(\Pi_\varepsilon(\mathcal{A}^\varepsilon|_{\Omega_f^\varepsilon}))\big),$$

then A1, A3. Corollary 5.1, Lemma 6.13, and Ref. 3 imply that $\mathcal{M}(\overline{\mathcal{G}^\varepsilon|_{\Omega_m^\varepsilon}})$ is a Cauchy sequence in $L^2(\mathcal{Q}_m^T)$.

Lemma 6.14. $\overline{s^\varepsilon}$ *converges to* s *in* $L^2(\mathcal{Q}_m^T)$. *Indeed,* $s = s(x,t) \in L^2(\Omega^T)$.

Proof. We assume that $\{\overline{s^{\varepsilon_1}}, \overline{p^{\varepsilon_1}}\}$ (resp. $\{\overline{s^{\varepsilon_2}}, \overline{p^{\varepsilon_2}}\}$) is a solution of (6.21) for $\varepsilon = \varepsilon_1$ (resp. $\varepsilon = \varepsilon_2$) case, and ζ is a smooth function satisfying $\zeta|_{\partial Y_m} = 0$. For convenience, $\varepsilon_1 < \varepsilon_2$. Let $x \in \Omega^{\varepsilon_1} \cap \Omega^{\varepsilon_2}$. Subtracting one solution from the other and doing integration by parts, we have

$$\begin{aligned}
&\int_0^T\!\!\int_{Y_m} \big(\mathcal{M}(\overline{s^{\varepsilon_1}}) - \mathcal{M}(\overline{s^{\varepsilon_2}})\big)\nabla_y(k\nabla_y\zeta) \\
&\quad = -\varepsilon_1^{2(1-\varpi)} \int_0^T\!\!\int_{Y_m} (\overline{s^{\varepsilon_1}} - \overline{s^{\varepsilon_2}})\phi\partial_t\zeta + \int_0^T\!\!\int_{Y_m} \big(\mathcal{M}(\overline{\mathcal{G}^{\varepsilon_1}|_{\Omega_m^{\varepsilon_1}}}) \\
&\qquad - \mathcal{M}(\overline{\mathcal{G}^{\varepsilon_2}|_{\Omega_m^{\varepsilon_2}}})\big)\nabla_y(k\nabla_y\zeta) \\
&\qquad + \int_0^T\!\!\int_{Y_m} k\nabla_y\big(\mathcal{M}(\overline{\mathcal{G}^{\varepsilon_1}|_{\Omega_m^{\varepsilon_1}}}) - \mathcal{M}(\overline{\mathcal{G}^{\varepsilon_2}|_{\Omega_m^{\varepsilon_2}}})\big)\nabla_y\zeta \\
&\qquad - \left(\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{2(1-\varpi)} - 1\right) \int_0^T\!\!\int_{Y_m} k\nabla_y\mathcal{M}(\overline{s^{\varepsilon_2}})\nabla_y\zeta \\
&\qquad - \varepsilon_2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{2(1-\varpi)} \int_0^T\!\!\int_{Y_m} k\lambda_{\mathrm{o}}(\overline{s^{\varepsilon_2}})\overline{\nabla(p^{\varepsilon_2} - g_{\mathrm{o}}^{\varepsilon_2})}\nabla_y\zeta \\
&\qquad + \varepsilon_1 \int_0^T\!\!\int_{Y_m} k\lambda_{\mathrm{o}}(\overline{s^{\varepsilon_1}})\overline{\nabla(p^{\varepsilon_1} - g_{\mathrm{o}}^{\varepsilon_1})}\nabla_y\zeta + \varepsilon_1^{2(1-\varpi)} \int_{Y_m} (\overline{s^{\varepsilon_1}} - \overline{s^{\varepsilon_2}})\phi\zeta\Big|_0^T.
\end{aligned} \tag{6.49}$$

Next we select ζ as the solution of

$$\begin{cases} \nabla_y(k\nabla_y\zeta) = \phi(\overline{s^{\varepsilon_1}} - \overline{s^{\varepsilon_2}}) & \text{for } y \in Y_m, \\ \zeta|_{\partial Y_m} = 0. \end{cases} \tag{6.50}$$

We also note that, by Lemmas 5.1, 5.2, 6.1,

$$\|\nabla_y\mathcal{M}(\overline{\mathcal{G}^{\varepsilon_i}|_{\Omega_m^{\varepsilon_i}}})\|_{L^2(\mathcal{Q}_m^T)} = \varepsilon_i\|\overline{\nabla\mathcal{M}(\mathcal{G}^{\varepsilon_i}|_{\Omega_m^{\varepsilon_i}})}\|_{L^2(\mathcal{Q}_m^T)} \le c\varepsilon_i, \quad i = 1, 2, \tag{6.51}$$

$$\|\nabla_y\mathcal{M}(\overline{s^{\varepsilon_2}})\|_{L^2(\mathcal{Q}_m^T)} = \varepsilon_2^{1-\varpi}\|\overline{\varepsilon_2^{\varpi}\nabla\mathcal{M}(s^{\varepsilon_2})}\|_{L^2(\mathcal{Q}_m^T)} \le c\,\varepsilon_2^{1-\varpi}, \tag{6.52}$$

$$\int_0^T\!\!\int_{Y_m} (\overline{s^{\varepsilon_1}} - \overline{s^{\varepsilon_2}})\phi\partial_t\zeta = \int_{Y_m} \frac{k}{2}|\nabla_y\zeta|^2(0) - \int_{Y_m} \frac{k}{2}|\nabla_y\zeta|^2(T), \tag{6.53}$$

where c is a constant independent of $\varepsilon_1, \varepsilon_2$. So (6.49)–(6.53), A11, Remark 6.2, and Lemmas 6.1, 6.2, 6.13 imply that $\mathcal{M}(\overline{s^{\varepsilon}})$ is a Cauchy sequence in $L^2(\mathcal{Q}_m^T)$ with limit $\mathcal{M}^*$. So $\overline{s^{\varepsilon}}$ converges to $s = \mathcal{M}^{-1}(\mathcal{M}^*)$ in $L^2(\mathcal{Q}_m^T)$. By (6.48), $s = s(x, t)$. □

Arguing as in the proof of Theorem 4.1, one obtains Theorem 4.3 from Lemmas 6.13 and 6.14.

Acknowledgments

The author would like to thank the referee for the valuable suggestions in preparing this paper. This research was supported by grant No. NSC 90-2115-M-009-018 from the research program of National Science Council of Taiwan.

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