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Isometric path numbers of graphs☆

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Abstract

An isometric path between two vertices in a graph G is a shortest path joining them. The isometric path number of G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths needed to cover all vertices of G. In this paper, we determine exact values of isometric path numbers of complete r-partite graphs and Cartesian products of 2 or 3 complete graphs.
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Keywords: Isometric path; Complete r-partite graph; Hamming graphs

1. Introduction

An *isometric path* between two vertices in a graph G is a shortest path joining them. The *isometric path number* of G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths required to cover all vertices of G. This concept has a close relationship with the game of cops and robbers described as follows.

The game is played by two players, the *cop* and the *robber*, on a graph. The two players move alternately, starting with the cop. Each player's first move consists of choosing a vertex at which to start. At each subsequent move, a player may choose either to stay at the same vertex or to move to an adjacent vertex. The object for the cop is to catch the robber, and for the robber is to prevent this from happening. Nowakowski and Winkler [7] and Quilliot [9] independently proved that the cop wins if and only if the graph can be reduced to a single vertex by successively removing pitfalls, where a *pitfall* is a vertex whose closed neighborhood is a subset of the closed neighborhood of another vertex.

As not all graphs are cop-win graphs, Aigner and Fromme [1] introduced the concept of *cop-number* of a general graph G, denoted by $c(G)$, which is the minimum number of cops needed to put into the graph in order to catch the robber. On the way to giving an upper bound for the cop-numbers of planar graphs, they showed that a single cop moving on an isometric path P guarantees that after a finite number of moves the robber will be immediately caught if he moves onto P. Observing this fact, Fitzpatrick [3] then introduced the concept of isometric path cover and pointed out that $c(G) \leqslant \mathrm{ip}(G)$.

The isometric path number of the Cartesian product $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_r}$ has been studied in the literature. Fitzpatrick [4] gave bounds for the case when $n_1 = n_2 = \cdots = n_r$. Fisher and Fitzpatrick [2] gave exact values for the case $r = 2$.

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Fitzpatrick et al. [5] gave a lower bound, which is in fact the exact value if $r+1$ is a power of 2, for the case when $n_1 = n_2 = \cdots = n_r = 2$. Pan and Chang [8] gave a linear-time algorithm to solve the isometric path problem on block graphs.

In this paper, we determine exact values of isometric path numbers of all complete r-partite graphs and Cartesian products of 2 or 3 complete graphs. Recall that a *complete r-partite graph* is a graph whose vertex set can be partitioned into disjoint union of r nonempty parts, and two vertices are adjacent if and only if they are in different parts. We use $K_{n_1,n_2,\ldots,n_r}$ to denote a complete r-partite graph whose parts are of sizes $n_1, n_2, \ldots, n_r$, respectively. A *Hamming graph* is the Cartesian product of complete graphs, which is the graph $K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r}$ with vertex set

$$V(K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r}) = \{(x_1, x_2, \ldots, x_r) : 0 \leqslant x_i < n_i \text{ for } 1 \leqslant i \leqslant r\}$$

and edge set $E(K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r})$ is

$$\{(x_1, x_2, \ldots, x_r)(y_1, y_2, \ldots, y_r) : x_i = y_i \in V(K_i) \text{ for all } i \text{ except just one } x_j \neq y_j\}.$$

2. Complete *r*-partite graphs

The purpose of this section is to determine exact values of the isometric path numbers of all complete r-partite graphs.

Suppose G is the complete r-partite graph $K_{n_1,n_2,\cdots,n_r}$ of n vertices, where $r \geqslant 2$, $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ and $n = n_1 + n_2 + \cdots + n_r$. Suppose G have α parts of odd sizes. We note that every isometric path in G has at most three vertices. Consequently,

$$\mathrm{ip}(G) \geqslant \lceil n/3 \rceil.$$

Also, for any path of three vertices in an isometric path cover $\mathscr{C}$, two end vertices of the path are in one part of G and the center vertex is in another part. In case when two paths of three vertices in $\mathscr{C}$ have a common end vertex, we may replace one by a path of two vertices. And, a path of one vertex can be replaced by a path of two vertices. So, without loss of generality, *we may only consider isometric path covers in which every path is of two or three vertices*, *and two* 3-*vertex paths have different end vertices.*

Lemma 1. *If* $3n_1 > 2n$, *then* $\mathrm{ip}(G) = \lceil n_1/2 \rceil$.

Proof. First, $\mathrm{ip}(G) \geqslant \lceil n_1/2 \rceil$ since every isometric path contains at most two vertices in the first part.

On the other hand, we use induction on $n - n_1$ to prove that $\mathrm{ip}(G) \leqslant \lceil n_1/2 \rceil$. When $n - n_1 = 1$, we have $G = K_{n-1,1}$. In this case, it is clear that $\mathrm{ip}(G) \leqslant \lceil n_1/2 \rceil$. Suppose $n - n_1 \geqslant 2$ and the claim holds for $n' - n_1' < n - n_1$. Then we remove two vertices from the first part and one vertex from the second part to form an isometric 3-path P. Since $3n_1 > 2n$, we have $n_1 - 2 > 2(n - n_1 - 1) > 0$ and so $n_1 - 2 > n_2$. Then, the remaining graph G' has $r' \geqslant 2$, $n_1' = n_1 - 2$ and $n' = n - 3$. It then still satisfies $3n_1' > 2n'$. As $n' - n_1' = n - n_1 - 1$, by the induction hypothesis, $\mathrm{ip}(G') \leqslant \lceil n_1'/2 \rceil$ and so $\mathrm{ip}(G) \leqslant \lceil n_1'/2 \rceil + 1 = \lceil n_1/2 \rceil$. □

Lemma 2. *If* $3\alpha > n$, *then* $\mathrm{ip}(G) = \lceil (n+\alpha)/4 \rceil$.

Proof. Suppose $\mathscr{C}$ is an optimum isometric path cover with p_2 paths of two vertices and p_3 paths of three vertices. Then

$$2p_2 + 3p_3 \geqslant n.$$

Note that at most $n - \alpha$ vertices in G can be paired up as the end vertices of the 3-paths in $\mathscr{P}$. Hence $p_3 \leqslant (n-\alpha)/2$ and so

$$2p_2 + 2p_3 \geqslant n - (n-\alpha)/2 = (n+\alpha)/2 \text{ or } \mathrm{ip}(G) = p_2 + p_3 \geqslant \lceil (n+\alpha)/4 \rceil.$$

On the other hand, we use induction on $n - \alpha$ to prove that $\mathrm{ip}(G) \leqslant \lceil (n+\alpha)/4 \rceil$. When $n - \alpha \leqslant 1$, we have $n = \alpha$ and G is the complete graph of order n. So, $\mathrm{ip}(G) = \lceil n/2 \rceil = \lceil (n+\alpha)/4 \rceil$. Suppose $n - \alpha \geqslant 2$ and the claim holds for $n' - \alpha' < n - \alpha$. In this case, $3\alpha > n \geqslant \alpha + 2$ which implies $\alpha > 1$ and $n > 3$. Then we may remove two vertices from the first part and one vertex from an odd part other than the first part to form an isometric 3-path P of G. The remaining graph G' has $n' = n - 3$ and $\alpha' = \alpha - 1$. It then satisfies $3\alpha' > n'$. Note that $r' \geqslant 2$ unless $G = K_{2,1,1}$ in which $n = 4$ and $\alpha = 2$ imply $\mathrm{ip}(G) = 2 = \lceil (n+\alpha)/4 \rceil$. By the induction hypothesis, $\mathrm{ip}(G') \leqslant \lceil (n'+\alpha')/4 \rceil$ and so $\mathrm{ip}(G) \leqslant \lceil (n'+\alpha')/4 \rceil + 1 = \lceil (n+\alpha)/4 \rceil$. □

Lemma 3. *If* $3n_1 \leqslant 2n$ *and* $3\alpha \leqslant n$, *then* $\mathrm{ip}(G) = \lceil n/3 \rceil$.

Proof. Since every isometric path in G has at most three vertices, $\mathrm{ip}(G) \geqslant \lceil n/3 \rceil$.

On the other hand, we use induction on n to prove that $\mathrm{ip}(G) \leqslant \lceil n/3 \rceil$. When $n \leqslant 8$, by the assumptions that $3n_1 \leqslant 2n$ and $3\alpha \leqslant n$ we have $G \in \{K_{2,1}, K_{2,2}, K_{3,2}, K_{2,2,1}, K_{4,2}, K_{4,1,1}, K_{3,3}, K_{3,2,1}, K_{2,2,2}, K_{2,2,1,1}, K_{4,3}, K_{4,2,1}, K_{3,2,2}, K_{2,2,2,1}, K_{5,3}, K_{5,2,1}, K_{4,4}, K_{4,3,1}, K_{4,2,2}, K_{4,2,1,1}, K_{3,3,2}, K_{3,2,2,1}, K_{2,2,2,2}, K_{2,2,2,1,1}\}$. It is straightforward to check that $\mathrm{ip}(G) \leqslant \lceil n/3 \rceil$.

Suppose $n \geqslant 9$ and the claim holds for $n' < n$. We remove two vertices from the first part and one vertex from the jth part to form an isometric 3-path P for G, where j is the largest index such that $j \geqslant 2$ and n_j is odd (when n_i are even for all $i \geqslant 2$, we choose $j = r$). Then, the remaining subgraph G' has $n' = n - 3$ and $\alpha' = \alpha - 1$ or $\alpha' \leqslant 2$. Therefore, $3\alpha \leqslant n$ and $n \geqslant 9$ imply that $3\alpha' \leqslant n'$ in any case. We shall prove that $3n'_1 \leqslant 2n'$ according to the following cases.

Case 1: $n_1 \geqslant n_2 + 2$.

In this case, $n_1 - 2 \geqslant n_2 \geqslant n_i$ for all $i \geqslant 2$ and so $n'_1 = n_1 - 2$. Therefore, $3n'_1 = 3(n_1 - 2) \leqslant 2(n-3) = 2n'$.

Case 2: $n_1 \leqslant n_2 + 1$ and $n_2 \leqslant 4$.

In this case, $n'_1 \leqslant n_2 \leqslant 4$ and $n' \geqslant 6$. Then, $3n'_1 \leqslant 12 \leqslant 2n'$.

Case 3: $n_1 \leqslant n_2 + 1$ and $n_2 \geqslant 5$ and $r = 2$.

In this case, $n'_1 \leqslant n_2 - 1$ and $n' = n - 3 = n_1 + n_2 - 3 \geqslant 2n_2 - 3$. Then, $3n'_1 \leqslant 3n_2 - 3 \leqslant 4n_2 - 8 < 2n'$.

Case 4: $n_1 \leqslant n_2 + 1$ and $n_2 \geqslant 5$ and $r \geqslant 3$.

In this case, $n'_1 \leqslant n_2$ and $n' = n - 3 \geqslant n_1 + n_2 + 1 - 3 \geqslant 2n_2 - 2$. Then, $3n'_1 \leqslant 3n_2 \leqslant 4n_2 - 5 < 2n'$. □

According to Lemmas 1–2, we have the following theorem.

Theorem 4. *Suppose G is the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ on n vertices with $r \geqslant 2$, $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ and $n = n_1 + n_2 + \cdots + n_r$. If there are exactly α indices i with n_i odd, then*

$$\mathrm{ip}(G) = \begin{cases} \lceil n_1/2 \rceil & \text{if } 3n_1 > 2n; \\ \lceil (n+\alpha)/4 \rceil & \text{if } 3\alpha > n; \\ \lceil n/3 \rceil & \text{if } 3\alpha \leqslant n \text{ and } 3n_1 \leqslant 2n. \end{cases}$$

In the proofs of the lemmas above, the essential point for the arguments is not the fact that each partitioning set of the complete r-partite graph is trivial. If we add some edges into the graph but still keep that each partite set can be partitioned into $\lfloor n_i/2 \rfloor$ pairs of two nonadjacent vertices and $n_i - 2\lfloor n_i/2 \rfloor$ vertices, then the same result still holds.

Corollary 5. *Suppose G is the graph obtained from the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ of n vertices by adding edges such that each ith part can be partitioned into $\lfloor n_i/2 \rfloor$ pairs of two nonadjacent vertices and $n_i - 2\lfloor n_i/2 \rfloor$ vertex, where $r \geqslant 2$, $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ and $n = n_1 + n_2 + \cdots + n_r$. If there are exactly α indices i with n_i odd, then*

$$\mathrm{ip}(G) = \begin{cases} \lceil n_1/2 \rceil & \text{if } 3n_1 > 2n; \\ \lceil (n+\alpha)/4 \rceil & \text{if } 3\alpha > n; \\ \lceil n/3 \rceil & \text{if } 3\alpha \leqslant n \text{ and } 3n_1 \leqslant 2n. \end{cases}$$

3. Hamming graphs

This section establishes isometric path numbers of Cartesian products of two or three complete graphs.

Suppose G is the Hamming graph $K_{n_1} \square K_{n_2} \square \ldots \square K_{n_r}$ of n vertices, where $n = n_1 n_2 \ldots n_r$ and $n_i \geqslant 2$ for $1 \leqslant i \leqslant r$. We note that every isometric path in G has at most $r+1$ vertices. Consequently,

$$\mathrm{ip}(G) \geqslant \lceil n/(r+1) \rceil.$$

Recall that the vertex set of $K_{n_1}\Box K_{n_2}\Box\ldots\Box K_{n_r}$ is

$$V(K_{n_1}\Box K_{n_2}\Box\ldots\Box K_{n_r}) = \{(x_1, x_2, \ldots, x_r) : 0\leqslant x_i < n_i \text{ for } 1\leqslant i\leqslant r\}$$

and edge set $E(K_{n_1}\Box K_{n_2}\Box\ldots\Box K_{n_r})$ is

$$\{(x_1, x_2, \ldots, x_r)(y_1, y_2, \ldots, y_r) : x_i = y_i \in V(K_i) \text{ for all } i \text{ except just one } x_j \neq y_j\}.$$

We first consider the case when $r = 2$.

Theorem 6. *If* $n_1 \geqslant 2$ *and* $n_2 \geqslant 2$, *then* $\mathrm{ip}(K_{n_1}\Box K_{n_2}) = \lceil n_1n_2/3\rceil$.

Proof. We only need to prove that $\mathrm{ip}(K_{n_1}\Box K_{n_2}) \leqslant \lceil n_1n_2/3\rceil$. We shall prove this assertion by induction on $n_1 + n_2$. For the case when $n_1 + n_2 \leqslant 6$, the isometric path covers

$$\begin{aligned}
\mathscr{C}_{2,2} &= \{(0, 0)(0, 1), (1, 0)(1, 1)\},\\
\mathscr{C}_{2,3} &= \{(0, 0)(0, 1)(1, 1), (0, 2)(1, 2)(1, 0)\},\\
\mathscr{C}_{2,4} &= \{(0, 0)(0, 1)(1, 1), (0, 2)(1, 2)(1, 0), (0, 3)(1, 3)\} \quad \text{and}\\
\mathscr{C}_{3,3} &= \{(0, 0)(2, 0)(2, 2), (0, 1)(0, 2)(1, 2), (1, 0)(1, 1)(2, 1)\}
\end{aligned}$$

for $K_2\Box K_2$, $K_2\Box K_3$, $K_2\Box K_4$ and $K_3\Box K_3$, respectively, give the assertion.

Suppose $n_1 + n_2 \geqslant 7$ and the assertion holds for $n'_1 + n'_2 < n_1 + n_2$. For the case when all $n_i \leqslant 4$, without loss of generality we may assume that $n_1 = 4$ and $3 \leqslant n_2 \leqslant 4$. As we can partition the vertex set of $K_{n_1}\Box K_{n_2}$ into the vertex sets of two copies of distance invariant induced subgraphs $K_2\Box K_{n_2}$,

$$\mathrm{ip}(K_{n_1}\Box K_{n_2}) \leqslant 2\mathrm{ip}(K_2\Box K_{n_2}) \leqslant 2\lceil 2n_2/3\rceil = \lceil n_1n_2/3\rceil.$$

For the case when there is at least one $n_i \geqslant 5$, say $n_1 \geqslant 5$, again we can partition the vertex set of $K_{n_1}\Box K_{n_2}$ into the vertex sets of two distance invariant induced subgraphs $K_3\Box K_{n_2}$ and $K_{n_1-3}\Box K_{n_2}$. Then,

$$\mathrm{ip}(K_{n_1}\Box K_{n_2}) \leqslant \mathrm{ip}(K_3\Box K_{n_2}) + \mathrm{ip}(K_{n_1-3}\Box K_{n_2}) \leqslant \lceil 3n_2/3\rceil + \lceil (n_1 - 3)n_2/3\rceil = \lceil n_1n_2/3\rceil. \qquad \Box$$

Lemma 7. *If* n_1, n_2 *and* n_3 *are positive even integers*, *then*

$$\mathrm{ip}(K_{n_1}\Box K_{n_2}\Box K_{n_3}) = n_1n_2n_3/4.$$

Proof. We only need to prove that $\mathrm{ip}(K_{n_1}\Box K_{n_2}\Box K_{n_3}) \leqslant n_1n_2n_3/4$. First, the isometric path cover $\mathscr{C}_{2,2,2} = \{(0, 0, 0)(0, 0, 1)\ (0, 1, 1)(1, 1, 1), (1, 0, 1)(1, 0, 0)(1, 1, 0)(0, 1, 0)\}$ for $K_2\Box K_2\Box K_2$ proves the assertion for the case when $n_1 = n_2 = n_3 = 2$. For the general case, as the vertex set of $K_{n_1}\Box K_{n_2}\Box K_{n_3}$ can be partitioned into the vertex sets of $n_1n_2n_3/8$ copies of distance invariant induced subgraphs $K_2\Box K_2\Box K_2$,

$$\mathrm{ip}(K_{n_1}\Box K_{n_2}\Box K_{n_3}) \leqslant (n_1n_2n_3/8)\mathrm{ip}(K_2\Box K_2\Box K_2) \leqslant n_1n_2n_3/4. \qquad \Box$$

Lemma 8. *If* $n_3 \geqslant 3$ *is odd*, *then* $\mathrm{ip}(K_2\Box K_2\Box K_{n_3}) = n_3 + 1$.

Proof. First, we prove that $\mathrm{ip}(K_2\Box K_2\Box K_{n_3}) \geqslant n_3 + 1$. Suppose to the contrary that the graph can be covered by n_3 isometric paths

$$P_i : (x_{i1}, x_{i2}, x_{i3})(y_{i1}, y_{i2}, y_{i3})(z_{i1}, z_{i2}, z_{i3})(w_{i1}, w_{i2}, w_{i3}),$$

$i = 1, 2, \ldots, n_3$. These paths are in fact vertex-disjoint paths of four vertices, each contains exactly one type-j edge for $j = 1, 2, 3$, where an edge $(x_1, x_2, x_3)(y_1, y_2, y_3)$ is type-j if $x_j \neq y_j$. For each P_i we then have $x_{i1} = 1 - w_{i1}$ and $x_{i2} = 1 - w_{i2}$, which imply that $x_{i1} + x_{i2}$ has the same parity as $w_{i1} + w_{i2}$. As P_i has just one type-3 edge, by symmetry, we may assume either $x_{i3} \neq y_{i3} = z_{i3} = w_{i3}$ or $x_{i3} = y_{i3} \neq z_{i3} = w_{i3}$, for which we call P_i type 1-3 or type 2-2, respectively. For a type 2-2 path P_i we may further assume that $x_{i1} \neq y_{i1} = z_{i1} = w_{i1}$.

For $0 \leqslant x_3 < n_3$, the x_3-*square* is the set $S(x_3) = \{(0, 0, x_3), (0, 1, x_3), (1, 0, x_3), (1, 1, x_3)\}$. Note that a type 1-3 path P_i contains one vertex in $S(x_{i3})$ and three vertices in $S(w_{i3})$, while a type 2-2 path P_i contains two vertices in $S(x_{i3})$

and two vertices in $S(w_{i3})$. We call a type 1-3 path P_i *adjacent to* another type 1-3 path P_j if the last three vertices of P_i and the first vertex of P_j form a square. This defines a digraph D whose vertices are all type 1-3 paths, in which each vertex has out-degree one and in-degree at most one. In fact, each vertex then has in-degree one. In other words, each type 1-3 path P_i corresponds to exactly one type 1-3 path P_j such that the last three vertices of P_i and the first vertex of P_j form a square. Consequently, the vertices of all type 1-3 paths together form p squares; and so the vertices of all type 2-2 paths form the other $n_3 - p$ squares.

Since $x_{i1} \neq y_{i1} = z_{i1} = w_{i1}$ for a type 2-2 path P_i, the first two vertices of a type 2-2 path together with the first two vertices of another type 2-2 path form a square. This shows that there is an even number of type 2-2 paths. Therefore, there is an odd number of type 1-3 paths.

On the other hand, in a type 1-3 path P_i we have that $x_{i_1} + x_{i_2} = y_{i_1} + y_{i_2}$ has the different parity as $z_{i_1} + z_{i_3}$, and the same parity as $w_{i_1} + w_{i_2}$. We call the path P_i *even* or *odd* when $x_{i1} + x_{i2}$ is even or odd, respectively. So P_i is adjacent to a type 1-3 path whose parity is the same as $z_{i_1} + z_{i_2}$. That is, a type 1-3 path is adjacent to a type 1-3 path of different parity. Therefore, the digraph D is the union of some even directed cycles. This is a contradiction to the fact that there is an odd number of type 1-3 paths.

The arguments above prove that $\mathrm{ip}(K_2 \square K_2 \square K_{n_3}) \geqslant n_3 + 1$. On the other hand, since the vertex set of $K_2 \square K_2 \square K_{n_3}$ is the union of the vertex sets of $(n_3 + 1)/2$ copies of $K_2 \square K_2 \square K_2$, by the cover $\mathscr{C}_{2,2,2}$ in the proof of Lemma 7, we have $\mathrm{ip}(K_2 \square K_2 \square K_{n_3}) \leqslant n_3 + 1$. □

Theorem 9. *If all* $n_i \geqslant 2$, *then* $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) = \lceil n_1 n_2 n_3 / 4 \rceil$ *except for the case when two* n_i *are* 2 *and the third is odd. In the exceptional case*, $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) = n_1 n_2 n_3 / 4 + 1$.

Proof. The claim for the exceptional case holds according to Lemma 8.

For the main case, by Lemma 7, we may assume that at least one n_i is odd. Again, we only need to prove that $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) \leqslant \lceil n_1 n_2 n_3 / 4 \rceil$. We shall prove the assertion by induction on $\sum_{i=1}^{3} n_i$. For the case when $\sum_{i=1}^{3} n_i \leqslant 10$, the following isometric path covers for $K_2 \square K_3 \square K_3$, $K_2 \square K_3 \square K_4$, $K_3 \square K_3 \square K_3$ and $K_3 \square K_3 \square K_4$, respectively, prove the assertion:

$$
\begin{aligned}
\mathscr{C}_{2,3,3} = \{ & (0,1,1)(0,1,0)(0,0,0)(1,0,0),\ (0,2,2)(0,2,0)(1,2,0)(1,1,0),\\
& (0,2,1)(1,2,1)(1,1,1),\ (0,0,2)(0,1,2)(1,1,2),\\
& (0,0,1)(1,0,1)(1,0,2)(1,2,2)\};\\
\mathscr{C}_{2,3,4} = \{ & (0,1,1)(0,1,0)(0,0,0)(1,0,0),\ (0,2,1)(0,2,0)(1,2,0)(1,1,0),\\
& (0,2,3)(0,2,2)(1,2,2)(1,1,2),\ (0,1,3)(0,1,2)(0,0,2)(1,0,2),\\
& (0,0,1)(1,0,1)(1,1,1)(1,1,3),\ (1,2,1)(1,2,3)(1,0,3)(0,0,3)\};\\
\mathscr{C}_{2,3,5} = \mathscr{C}^*_{2,3,3} \cup \{ & (0,1,4)(0,1,3)(0,2,3)(1,2,3),\ (0,0,3)(0,0,4)(0,2,4)(1,2,4),\\
& (1,0,3)(1,0,4)\} \quad \text{where}\\
\mathscr{C}^*_{2,3,3} = \mathscr{C}_{2,3,3} \backslash \{ & (0,2,1)(1,2,1)(1,1,1),\ (0,0,2)(0,1,2)(1,1,2)\} \cup\\
\{ & (0,2,1)(1,2,1)(1,1,1)(1,1,3),\ (0,0,2)(0,1,2)(1,1,2)(1,1,4)\};\\
\mathscr{C}_{3,3,3} = \{ & (0,0,0)(0,2,0)(1,2,0)(1,2,1),\ (1,1,0)(2,1,0)(2,2,0)(2,2,1),\\
& (0,2,1)(0,1,1)(1,1,1)(1,1,2),\ (1,0,1)(2,0,1)(2,1,1)(2,1,2),\\
& (0,1,0)(0,1,2)(0,2,2)(1,2,2),\ (0,0,1)(0,0,2)(2,0,2)(2,2,2),\\
& (1,0,2)(1,0,0)(2,0,0)\};\\
\mathscr{C}_{3,3,4} = \{ & (0,0,0)(0,2,0)(1,2,0)(1,2,1),\ (1,1,0)(2,1,0)(2,2,0)(2,2,1),\\
& (0,2,1)(0,1,1)(1,1,1)(1,1,2),\ (1,0,1)(2,0,1)(2,1,1)(2,1,2),\\
& (0,1,0)(0,1,2)(0,2,2)(1,2,2),\ (0,0,2)(2,0,2)(2,2,2)(2,2,3),\\
& (0,1,3)(1,1,3)(1,0,3)(1,0,2),\ (1,0,0)(2,0,0)(2,0,3)(2,1,3),\\
& (0,0,1)(0,0,3)(0,2,3)(1,2,3)\}.
\end{aligned}
$$

Suppose $\sum_{i=1}^{3} n_i \geqslant 11$ and the assertion holds for $\sum_{i=1}^{3} n'_i < \sum_{i=1}^{3} n_i$. We shall consider the following cases.

For the case when there is some i, say $i=3$, such that $n_3 \geqslant 7$ or $n_3=6$ with all $n_j \geqslant 3$, we have $\mathrm{ip}(K_{n_1}\square K_{n_2}\square K_{n_3}) \leqslant \mathrm{ip}(K_{n_1}\square K_{n_2}\square K_4) + \mathrm{ip}(K_{n_1}\square K_{n_2}\square K_{n_3-4}) \leqslant \lceil n_1 n_2 4/4\rceil + \lceil n_1 n_2 (n_3-4)/4\rceil = \lceil n_1 n_2 n_3/4\rceil$.

For the case when some n_i, say n_3, is equal to 4, we may assume $n_1 \geqslant n_2$ and so $n_1 \geqslant 4$. Then $\mathrm{ip}(K_{n_1}\square K_{n_2}\square K_4) \leqslant \mathrm{ip}(K_2\square K_{n_2}\square K_4) + \mathrm{ip}(K_{n_1-2}\square K_{n_2}\square K_4) = \lceil 2n_2 4/4\rceil + \lceil (n_1-2)\, n_2 4/4\rceil = \lceil n_1 n_2 n_3/4\rceil$.

There are six remaining cases. The following isometric path covers prove the assertion for $K_2\square K_3\square K_6$, $K_2\square K_5\square K_5$ and $K_3\square K_5\square K_5$, respectively:

$$
\begin{aligned}
\mathscr{C}_{2,3,6} = \mathscr{C}^*_{2,3,3} \cup \{&(0,0,4)(0,0,3)(1,0,3)(1,2,3), (0,1,3)(0,1,4)(0,2,4)(1,2,4),\\
&(0,2,3)(0,2,5)(1,2,5)(1,1,5), (0,1,5)(0,0,5)(1,0,5)(1,0,4)\};\\
\mathscr{C}_{2,5,5} = \mathscr{C}_{2,3,5} &\backslash \{(1,0,3)(1,0,4)\}\\
\cup\, \{&(0,4,1)(0,4,0)(0,3,0)(1,3,0), (1,4,0)(1,4,1)(1,3,1)(0,3,1),\\
&(0,4,3)(0,4,2)(0,3,2)(1,3,2), (1,4,2)(1,4,3)(1,3,3)(0,3,3),\\
&(1,0,3)(1,0,4)(1,4,4), (0,4,4)(0,3,4)(1,3,4)\};\\
\mathscr{C}_{3,5,5} = \mathscr{C}_{2,3,5} &\backslash \{(1,0,3)(1,0,4)\}\\
\cup\, \{&(0,4,0)(2,4,0)(2,0,0)(2,0,1), (0,3,0)(2,3,0)(2,1,0)(2,1,1),\\
&(0,4,1)(0,3,1)(1,3,1)(1,3,0), (1,4,0)(1,4,1)(2,4,1)(2,2,1),\\
&(1,0,3)(2,0,3)(2,2,3)(2,2,0), (1,0,4)(2,0,4)(2,3,4)(2,3,1),\\
&(0,3,2)(2,3,2)(2,1,2)(2,1,3), (0,4,4)(0,4,2)(2,4,2)(2,0,2),\\
&(0,4,3)(1,4,3)(1,3,3)(1,3,2), (0,3,3)(2,3,3)(2,4,3)(2,4,4),\\
&(0,3,4)(1,3,4)(1,4,4)(1,4,2), (2,2,2)(2,2,4)(2,1,4)\}.
\end{aligned}
$$

In the three other cases the claim follows from the following inequalities:

$$\mathrm{ip}(K_2\square K_5\square K_6) \leqslant \mathrm{ip}(K_2\square K_3\square K_6) + \mathrm{ip}(K_2\square K_2\square K_6) \leqslant 9+6=15,$$

$$\mathrm{ip}(K_3\square K_3\square K_5) \leqslant \mathrm{ip}(K_3\square K_3\square K_2) + \mathrm{ip}(K_3\square K_3\square K_3) \leqslant 5+7=12,$$

$$\mathrm{ip}(K_5\square K_5\square K_5) \leqslant \mathrm{ip}(K_5\square K_5\square K_3) + \mathrm{ip}(K_5\square K_5\square K_2) \leqslant 19+13=32. \qquad \square$$

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References

[1] M. Aigner, M. Fromme, A game of cops and robbers, Discrete Appl. Math. 8 (1984) 1–12.
[2] D.C. Fisher, S.L. Fitzpatrick, The isometric number of a graph, J. Combin. Math. Combin. Comput. 38 (2001) 97–110.
[3] S.L. Fitzpatrick, Ph.D. Thesis, Dalhousie University, Nova Scotia, Canada, 1997.
[4] S.L. Fitzpatrick, The isometric path number of the Cartesian product of paths, Congr. Numer. 137 (1999) 109–119.
[5] S.L. Fitzpatrick, R.J. Nowakowski, D. Holton, I. Caines, Covering hypercubes by isometric paths, Discrete Math. 240 (2001) 253–260.
[7] R. Nowakowski, P. Winkler, Vertex to vertex pursuit in a graph, Discrete Math. 43 (1983) 235–239.
[8] J.-J. Pan, G.J. Chang, Isometric path numbers of block graphs, Inform. Process. Letters 93 (2005) 99–102.
[9] A. Quilliot, Thèse de 3e cycle, Université de Paris VI, 1978.

Further reading

[1] F. Harary, A.J. Schwenk, Evolution of the path number of a graph: covering and packing in graphs, II in: R.C. Reed (Ed.), Graph Theory Comput., Academic Press, 1972, pp. 39–45.