Interaction between a two-dimensional pancake vortex and a circular nonsuperconducting defect

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The interaction energy $U_{\rm int}(r)$ between a two-dimensional (2D) pancake vortex and a circular dielectric defect with radius a in a superconducting thin film is calculated in the London limit $\kappa \gg 1$. We obtain a general expression formula of $U_{\rm int}(r)$ for all range $r>a$. Our theory predicts that $U_{\rm int}(r) \propto \ln(1-a^2/r^2)$ at a short distance, which is in the same form as the 3D bulk superconductor. But $U_{\rm int}(r)$ decreases with r^{-4} at a long distance, unlike decreases exponentially in the 3D bulk one. [S0163-1829(96)06025-0]

I. INTRODUCTION

The investigation of the pinning structures for vortices is very significant to the potential applications of the high-T_c superconductors. The understanding of the interaction between vortices and defects appears to be an important task from both fundamental and technological points of view. Some techniques have been applied to increase the concentration of defects, such as heavy ion irradiation[1–3] and the inclusion of an insulating phase.[4] These nonsuperconducting defects, which interact with vortices, induce the screening currents and lead to the so-called electromagnetic pinning.[5]

For three-dimensional (3D) bulk type-II superconductors, the simple case of a single straight vortex in the presence of a cylindrical defect has been calculated by Mkrtchyan and Shmidt in the London limit $\kappa \gg 1$.[6] Their theory has been generalized to a periodical structure of columnar defects.[7] The numerical results of the Ginzberg-Landau theory for various κ have been obtained by Takezawa and Fukushima.[8]

Pearl[9–11] was the first to point out that the distinctive properties of thin-film vortices arise from their electromagnetic long-range interaction. Then it may be deduced that the interaction between a pancake vortex and a circular defect in a superconducting thin film is a long-range force, unlike a short-range force in 3D bulk system.

Buzdin and Feinberg[12] analyzed the interaction of a vortex with a cylindrical defect of radius a by employing the image method. They have derived the interaction energy $U_{\rm int}(r) \propto \ln(1-a^2/r^2)$ at a short distance both in the superconducting thin film and 3D bulk material. Unfortunately their theory could not be applied to the interaction at a long distance, because the image method is valid only in that at a short range.

In this paper, with the help of the Hankel transformation, we develop a method of constructing the solution of the vector potential $\mathbf{A}_H$ caused by the screening currents of a circular defect in the presence of a single 2D pancake vortex. We obtain a general expression of $U_{\rm int}(r)$ and the pinning potential $U_{\rm pin}(a)$, and simplify the expression of $U_{\rm int}(r)$ by considering the actual condition of $a/\Lambda \ll 1$ (Λ is Pearl's 2D screening length[9–11]). Our theory cannot only verify the result of Buzdin and Feinberg's[12] work under consideration of the interaction at a short distance, but also predicts that $U_{\rm int}(r)$ at a long-range interaction is decreased with r^{-4}.

II. SOLUTION OF THE 2D PANCAKE VORTEX IN A THIN FILM WITH A CIRCULAR DEFECT

Let us consider that a circular defect with radius a in an infinite type-II superconducting thin film ($\kappa \gg 1$) lies in the x-y plane, and a 2D pancake vortex locates at a distance r from the center of the defect (as shown in Fig. 1). If the external magnetic field is in the z direction, there will induce the supercurrent density $\mathbf{j}_s$ in the superconducting thin film. Then the vector potential $\mathbf{A}$ satisfies the equations[13]

$$\nabla\times\nabla\times\mathbf{A} = \frac{4\pi}{c} j_s = \frac{2\delta(z)}{\Lambda}\left[\frac{\phi_0 \hat{e}_\phi}{2\pi R} - \mathbf{A}\right] \quad \text{for } \rho > a$$
$$\nabla\times\nabla\times\mathbf{A} = \mathbf{0} \quad \text{for } \rho < a, \quad (1)$$

where the origin of the cylindrical coordinate (R,ϕ,z) is located at $(r,0,0)$, and the origin of the other cylindrical coordinate (ρ,θ,z) is located at $(0,0,0)$. The screening length $\Lambda = 2\lambda^2/d$ plays the role of an effective penetration depth in the superconducting thin film.[9–11] λ is the London penetration depth, and d is the thickness of the superconducting thin film ($d \ll \lambda$). ϕ_0 is flux quantum and is equal to $hc/2e$.

The solution of Eq. (1) includes two parts, and can be written as

$$\mathbf{A}(\rho,\theta,z) = \mathbf{A}_0 + \mathbf{A}_H, \quad (2)$$

where $\mathbf{A}_0$ is the particular solution of Eq. (1) and represents the vector potential of the single pancake vortex. $\mathbf{A}_H$ is the homogeneous solution of Eq. (1) and is caused by the screen-

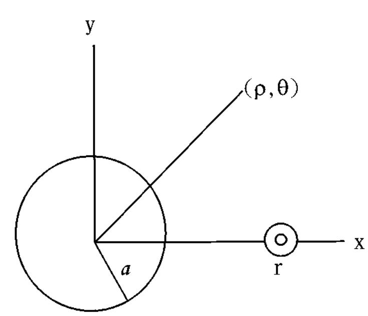

FIG. 1. A circular defect with radius a in an infinite superconducting thin film lies in the x-y plane, and a 2D pancake vortex locates at a distance r from the center of the defect.

ing current of the circular defect. $\mathbf{A}_0$ has been solved with a Hankel transform by Pearl,[11,13] who found

$$\mathbf{A}_0(R,\phi,z)=\frac{\phi_0\hat{e}_\phi}{2\pi}\int_0^\infty dk\,\frac{J_1(kR)}{\Lambda k+1}\,e^{-k|z|}. \tag{3}$$

If we transform the coordinate from (R,ϕ,z) to (ρ,θ,z), the two following mathematic formulas can be obtained (proved in the Appendix, in (i)]:

$$\frac{\hat{e}_\phi}{R}=\int_0^\infty dk\;J_1(kR)\hat{e}_\phi \tag{4}$$

and

$$J_1(kR)\hat{e}_\phi=-\hat{e}_\rho\sum_{n=1}^\infty[J_{n+1}(k\rho)+J_{n-1}(k\rho)]J_n(kr)\sin n\theta+\hat{e}_\theta\left\{J_1(k\rho)J_0(kr)+\sum_{n=1}^\infty[J_{n+1}(k\rho)-J_{n-1}(k\rho)]J_n(kr)\cos n\theta\right\}, \tag{5}$$

where $J_n(x)s$ are the Bessel functions.

From Eq. (5), $\mathbf{A}_0$ can be rewritten as

$$\mathbf{A}_0(\rho,\theta,z)=\frac{\phi_0}{2\pi}\int_0^\infty dk\,\frac{e^{-k|z|}}{\Lambda k+1}\left\{-\hat{e}_\rho\sum_{n=1}^\infty[J_{n+1}(k\rho)+J_{n-1}(k\rho)]J_n(kr)\sin n\theta+\hat{e}_\theta\left[J_1(k\rho)J_0(kr)+\sum_{n=1}^\infty[J_{n+1}(k\rho)-J_{n-1}(k\rho)]J_n(kr)\cos n\theta\right]\right\}. \tag{6}$$

Let $\mathbf{A}'=\phi_0\hat{e}_\phi/2\pi R-\mathbf{A}_0$. From Eqs. (4)–(6), we find that $\mathbf{A}'(\rho,\theta,z)$ can be expanded in a series of the orthogonal functions $\mathbf{U}_n(\rho,\theta,z;r)$, where $\mathbf{U}_n(\rho,\theta,z;r)$ are the solutions of the equation: $\nabla\times\nabla\times\mathbf{U}_n=-2\delta(z)\mathbf{U}_n/\Lambda$, and can be written as

$$\mathbf{U}_n(\rho,\theta,z;r)=\frac{\phi_0}{2\pi}\int_0^\infty dk\left(1-\frac{e^{-k|z|}}{\Lambda k+1}\right)\{-\hat{e}_\rho[J_{n+1}(k\rho)+J_{n-1}(k\rho)]J_n(kr)\sin n\theta+\hat{e}_\theta[J_{n+1}(k\rho)-J_{n-1}(k\rho)]J_n(kr)\cos n\theta\},\quad n=0,1,2,\ldots\,. \tag{7}$$

Set $r=a$ in Eq. (7). Then we can construct the homogeneous solution of Eq. (1), $\mathbf{A}_H$, in the region of interest $\rho>a$ with the expansions of $\mathbf{U}_n(\rho,\theta,z;a)$ in series as follows:

$$\begin{aligned}\mathbf{A}_H(\rho,\theta,z)&=\frac{\alpha_0(r)}{2}\mathbf{U}_0(\rho,\theta,z;a)+\sum_{n=1}^\infty\alpha_n(r)\mathbf{U}_n(\rho,\theta,z;a)\\&=\frac{\phi_0}{2\pi}\int_0^\infty dk\left(1-\frac{e^{-k|z|}}{\Lambda k+1}\right)\left\{\alpha_0(r)J_0(ka)J_1(k\rho)\hat{e}_\theta+\sum_{n=1}^\infty\alpha_n(r)J_n(ka)\{-\hat{e}_\rho[J_{n+1}(k\rho)+J_{n-1}(k\rho)]\sin n\theta\right.\\&\quad\left.+\hat{e}_\theta[J_{n+1}(k\rho)-J_{n-1}(k\rho)]\cos n\theta\}\right\}\quad\text{for }\rho>a,\end{aligned} \tag{8}$$

and the supercurrent density $\mathbf{j}_s$ is given by

$$\begin{aligned}\mathbf{j}_s=\frac{c\,\delta(z)}{2\pi\Lambda}\left[\frac{\phi_0\hat{e}_\phi}{2\pi R}-\mathbf{A}\right]&=\frac{\phi_0 c\,\delta(z)}{4\pi^2}\int_0^\infty dk\,\frac{k}{\Lambda k+1}\left\{-\hat{e}_\theta[\alpha_0(r)J_0(ka)-J_0(kr)]J_1(k\rho)+\sum_{n=1}^\infty[\alpha_n(r)J_n(ka)-J_n(kr)]\right.\\&\quad\left.\times\{\hat{\mathbf{e}}_\rho[J_{n+1}(k\rho)+J_{n-1}(k\rho)]\sin n\theta-\hat{\mathbf{e}}_\theta[J_{n+1}(k\rho)-J_{n-1}(k\rho)]\cos n\theta\}\right\}\quad\text{for }\rho>a,\end{aligned} \tag{9}$$

where $\alpha_n(r)s$ are the coefficients which can be determined by boundary conditions.

Because the radial component of $\mathbf{j}_s$ vanishes at $\rho=a^+$, where $a^+=a+\varepsilon$, ε is in the order of the Ginzberg-Landau coherence length $\xi(T)$. The coefficient $\alpha_n(r)$ for $n\neq0$ can be determined as

$$\alpha_n(r)=\frac{\displaystyle\int_0^\infty dk\,\frac{J_n(ka)J_n(kr)}{\Lambda k+1}}{\displaystyle\int_0^\infty dk\,\frac{J_n(ka)J_n(ka^+)}{\Lambda k+1}},\quad n=1,2,3,\ldots\quad\text{for }r>a. \tag{10}$$

The coefficient $\alpha_0(r)$ can be determined by considering the condition: the total magnetic flux trapped in the superconducting thin film (including the defect) is equal to ϕ_0. Integrating Eq. (1) along the circular contour of the defect in the superconducting thin film, just as is done in Ref. 6, we have

$$\pi a^2H_z^0(\rho=a^+)+\oint_{\rho\to\infty}\mathbf{A}\cdot\mathbf{d}\boldsymbol{l}_1-\oint_{\rho=a}\mathbf{A}\cdot\mathbf{d}\boldsymbol{l}_2=\phi_0\quad\text{for }z=0, \tag{11}$$

where H_z^0 is the term for $n=0$ of the z-component of magnetic field $\mathbf{H}$ and $\mathbf{H}=\nabla\times\mathbf{A}$. $H_z^0(\rho\leqslant a,z=0)$ is distributed uniformly because $\nabla\times H=4\pi\mathbf{j}_s/c=0$ in the region of $\rho\leqslant a$ and $z=0$. The contour integral $\mathbf{dl}_1$ for $\rho\to\infty$ in Eq. (11) means the total flux in the superconducting thin film which is equal to ϕ_0. Substituting Eqs. (2), (6), and (8) into Eq. (11), we obtain

$$\alpha_0(r)=\frac{\displaystyle\int_0^\infty dk\,\frac{aJ_0(kr)}{\Lambda k+1}\left[\frac{ka}{2}\,J_0(ka)-J_1(ka)\right]}{\displaystyle\int_0^\infty dk\,\frac{aJ_0(ka)}{\Lambda k+1}\left[\frac{ka}{2}\,J_0(ka^+)-J_1(ka^+)\right]+1}\quad\text{for } r>a. \tag{12}$$

According to Ref. 14, the free energy $U(r)$ is obtained by integrating the supercurrent density $\mathbf{j}_s$, and the result is

$$U(r)=\frac{\phi_0}{2c}\int \mathbf{j}_s\cdot d\sigma=U_{\text{self}}+U_{\text{int}}(r), \tag{13}$$

where $U_{\text{self}}=(\phi_0^2/8\pi^2\Lambda)\ln[\Lambda/\xi(T)]$ is the self-energy of a single pancake vortex, and the interaction energy $U_{\text{int}}(r)$ between the pancake vortex and the circular defect is given by

$$U_{\text{int}}(r)=-\frac{\phi_0^2}{8\pi^2}\int_0^\infty \frac{dk}{\Lambda k+1}\left[\alpha_0(r)J_0(ka)J_0(kr)+2\sum_{n=1}^{\infty}\alpha_n(r)J_n(ka)J_n(kr)\right]. \tag{14}$$

The pinning potential for a 2D pancake vortex in our system, $U_{\text{pin}}(a)$, in neglecting the core energy, is defined by

$$U_{\text{pin}}(a)=U(a^+)-U(\infty). \tag{15}$$

Substituting Eqs. (10) and (12)–(14) into Eq. (15), we obtain the following expression for $U_{\text{pin}}(a)$:

$$U_{\text{pin}}(a)=\frac{\displaystyle\frac{\phi_0^2}{8\pi^2}\int_0^\infty dk\,\frac{J_0(ka)J_0(ka^+)}{\Lambda k+1}}{\displaystyle\int_0^\infty dk\,\frac{aJ_0(ka)}{\Lambda k+1}\left[\frac{ka}{2}\,J_0(ka^+)-J_1(ka^+)\right]+1}-\frac{\phi_0^2}{8\pi^2\Lambda}\ln\left(\frac{\Lambda}{\xi}\right). \tag{16}$$

The first term of Eq. (16) on the right-hand side approaches $\phi_0^2\ln(\Lambda/a)/8\pi^2\Lambda$ for $a/\Lambda\ll1$, and vanishes for $a/\Lambda\gg1$.

Then we have

$$U_{\text{pin}}(a)=\begin{cases}-\dfrac{\phi_0^2}{8\pi^2\Lambda}\ln\left(\dfrac{a}{\xi}\right) & \text{for } a/\Lambda\ll1\\[2ex] -\dfrac{\phi_0^2}{8\pi^2\Lambda}\ln\left(\dfrac{\Lambda}{\xi}\right) & \text{for } a/\Lambda\gg1.\end{cases} \tag{17}$$

We show the numerical result for the dependence of the pinning potential U_{pin}, on the defect radius a in Fig. 2. U_{pin} is negative. This means that the interaction between the vortex and defect is attractive. As a increases in the region of $a\ll\Lambda$, the depth of U_{pin} increases rapidly. U_{pin} reaches a saturation value when $a\gg\Lambda$. The 2D behavior of the pinning energy, $U_{\text{pin}}(a)\approx-\phi_0^2\ln(a/\xi)/8\pi^2\Lambda$ for $a\ll\Lambda$, has been observed in $Bi_2Sr_2CaCu_2O_x$ crystals.[2]

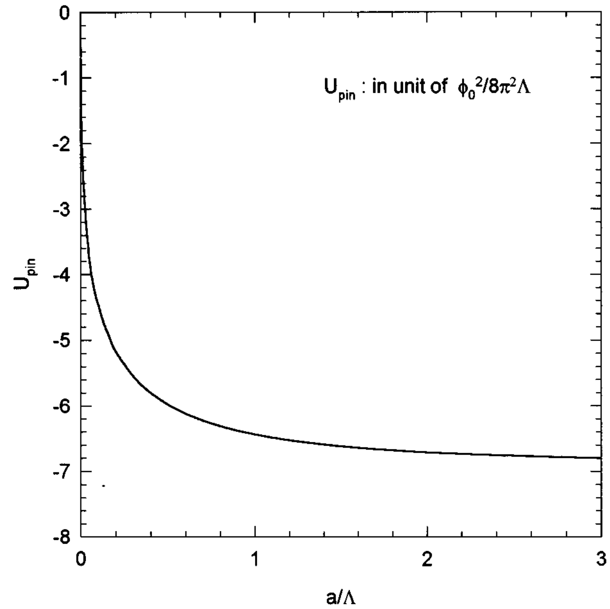

FIG. 2. The pinning potential U_{pin} as a function of the defect radius a for $\Lambda/\xi=1000$ on the basis of Eq. (16).

III. DISCUSSION

In Sec. II we derived the expression formula Eq. (14) to explain the interaction energy between a 2D pancake vortex and a circular defect. Now let us compare the order of magnitude of Λ with that of a. In general a is less than 1000 Å and $\Lambda=2\lambda^2/d>10^4$ Å. Therefore, it is very reasonable to take the limitation for $a/\Lambda\ll1$ in our theory. Then we can obtain a good approximation for the following integral [proved in the Appendix, in (ii)]:

$$\int_0^\infty dk\,\frac{J_n(ka)J_n(kr)}{k+1/\Lambda}\approx\frac{\pi}{2}\,[H_{-n}(r/\Lambda)-Y_{-n}(r/\Lambda)]J_{-n}(a/\Lambda)\quad\text{for } a\ll\Lambda, r>a, \tag{18}$$

where $H_n(x)s$ are the Sturve functions, and $Y_n(x)s$ are the Neumann functions. From Eq. (18), by taking the limitation for $a/\Lambda\ll1$, the first term of Eq. (14) on the right-hand side is much smaller than the second term, and can be neglected. Finally we obtain the interaction energy $U_{\text{int}}(r)$ in the limit of $a/\Lambda\ll1$ as follows:

$$U_{\text{int}}(r)=-\frac{\phi_0^2}{8\pi\Lambda}\sum_{n=1}^{\infty}\alpha_n(r)[H_{-n}(r/\Lambda)-Y_{-n}(r/\Lambda)]\times J_{-n}(a/\Lambda) \tag{19}$$

and

$$\alpha_n(r)=\frac{H_{-n}(r/\Lambda)-Y_{-n}(r/\Lambda)}{H_{-n}(a/\Lambda)-Y_{-n}(a/\Lambda)}. \tag{20}$$

If we take two limits (i) $r/\Lambda \ll 1$ and (ii) $r/\Lambda \gg 1$, we can obtain[15]

$$H_{-n}(r/\Lambda)-Y_{-n}(r/\Lambda) = \begin{cases} \dfrac{(n-1)!}{\pi}\left(-\dfrac{2\Lambda}{r}\right)^n & \text{for } r\ll\Lambda \\ -\dfrac{2\cdot(2n-1)!!}{\pi}\left(-\dfrac{\Lambda}{r}\right)^{n+1} & \text{for } r\gg\Lambda. \end{cases} \tag{21}$$

Substituting Eqs. (20) and (21) and $J_{-n}(a/\Lambda) \approx(-a/2\Lambda)^n/n!$ into Eq. (19), we obtain the following expressions for $U_{\rm int}(r)$:

(i) For $a<r\ll\Lambda$,

$$U_{\rm int}(r)=-\frac{\phi_0^2}{8\pi^2\Lambda}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^{2n} =\frac{\phi_0^2}{8\pi^2\Lambda}\ln(1-a^2/r^2) \quad \text{for } a<r\ll\Lambda. \tag{22}$$

(ii) For $a\ll\Lambda\ll r$, taking the lowest-order term of Λ/r and neglecting the higher-order terms, we can obtain

$$U_{\rm int}(r)=-\frac{\phi_0^2}{8\pi^2\Lambda}\left(\frac{\Lambda^2a^2}{r^4}\right) \quad \text{for } a\ll\Lambda\ll r. \tag{23}$$

On the basis of Eqs. (19) and (20), we successfully obtain a general formula for expressing the interaction energy $U_{\rm int}(r)$ between the pancake vortex and a circular defect ($a\ll\Lambda$), and this formula is valid for all ranges $r>a$. Our theory cannot only be reduced to the result of Buzdin and Feinberg,[12] i.e., $U_{\rm int}(r)\propto\ln(1-a^2/r^2)$ for $r\ll\Lambda$, derived by the image method, but can also obtain the prediction of $U_{\rm int}(r)\propto r^{-4}$ for $r\gg\Lambda$, which cannot be derived from the image method. The numerical result of $U_{\rm int}(r)$ in Eqs. (19) and (20) is shown in Fig. 3 for $a/\Lambda=0.1$ and $\Lambda/\xi=1000$.

On the other hand, the vortex-defect interaction energy in the 3D bulk, $U_{\rm int}^{\rm 3D}(r)$, derived by Mkrtchyan and Shmidt,[6] is in the following form:

$$U_{\rm int}^{\rm 3D}(r)\approx\begin{cases}\left(\dfrac{\phi_0}{4\pi\lambda}\right)^2\ln\left(1-\dfrac{a^2}{r^2}\right) & \text{for } a<r\ll\lambda \\ -\left(\dfrac{\phi_0}{4\pi\lambda}\right)^2\dfrac{\pi a^2}{2\lambda r}e^{-2r/\lambda} & \text{for } a\ll\lambda\ll r.\end{cases} \tag{24}$$

Comparing Eqs. (22) and (23) with Eq. (24), we obtain that the vortex-defect interaction energies for small r have the same form $\ln(1-a^2/r^2)$ in both 2D and 3D systems. However, when r is large, the vortex-defect interaction energy decreases with r^{-4} in the 2D system, and decreases exponentially in the 3D system.

IV. CONCLUSIONS

In this paper we have presented an analytic method to calculate the perturbation of supercurrents caused by a circular defect in the superconducting thin film. A Hankel transformation technique was used, whereby we can construct the homogeneous solution of Eq. (1), $\mathbf{A}_H$. The interaction energy $U_{\rm int}(r)$ between a 2D pancake vortex and a circular defect is consequently derived for all ranges $r>a$. The pinning potential $U_{\rm pin}(a)$ is also obtained.

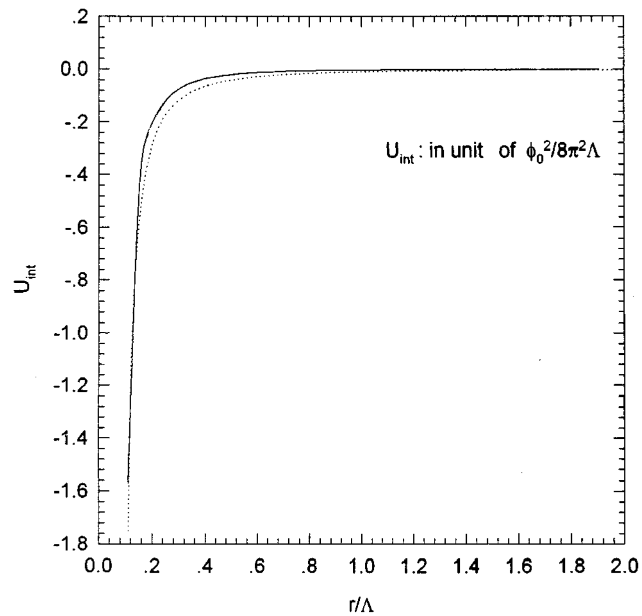

FIG. 3. The interaction energy $U_{\rm int}$ as a function of r for $a/\Lambda=0.1$ and $\Lambda/\xi=1000$ on the basis of Eqs. (19) and (20) (solid line). The result of Ref. 12, $U_{\rm int}=\phi_0^2/8\pi^2\Lambda\ln(1-a^2/r^2)$, is also shown (dashed line).

It is worthy to emphasize that our theory can be reduced to the result of Buzdin and Feinberg,[12] i.e., $U_{\rm int}(r) \propto\ln(1-a^2/r^2)$ for $a<r\ll\Lambda$. Moreover, our theory predicts that $U_{\rm int}(r)$ decreases with r^{-4} for $r\gg\Lambda\gg a$, unlike exponential decreases in 3D bulk superconductor. Therefore we may conclude that the interaction between a 2D pancake vortex and a circular defect in the superconducting thin film is a long-range force, unlike a short-range force in the 3D bulk material.

ACKNOWLEDGMENT

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APPENDIX

(i) Prove

$$J_1(kR)\hat{e}_\phi=-\hat{e}_\rho\sum_{n=1}^{\infty}[J_{n+1}(k\rho)+J_{n-1}(k\rho)]J_n(kr)\sin n\theta +\hat{e}_\theta\left\{J_1(k\rho)J_0(kr)+\sum_{n=1}^{\infty}[J_{n+1}(k\rho) -J_{n-1}(k\rho)]J_n(kr)\cos n\theta\right\}.$$

By the summation theorem, the Bessel function $J_0(kR)$ can be expanded in the following form:[16]

$$J_0(kR)=J_0(k\rho)J_0(kr)+2\sum_{n=1}^{\infty} J_n(k\rho)J_n(kr)\cos n\theta. \quad \text{(A1)}$$

Taking the partial derivatives $\partial/\partial\rho$ and $\partial/\partial\theta$ on both sides of Eq. (A1), we have

$$\left(\frac{\rho-r\cos\theta}{R}\right)J_1(kR)=J_1(k\rho)J_0(kr)+\sum_{n=1}^{\infty}[J_{n+1}(k\rho)-J_{n-1}(k\rho)]J_n(kr)\cos n\theta, \quad \text{(A2)}$$

$$\frac{r\sin\theta}{R}J_1(kR)=\sum_{n=1}^{\infty}[J_{n+1}(k\rho)+J_{n-1}(k\rho)]J_n(kr)\sin n\theta. \quad \text{(A3)}$$

Considering the transformation of the coordinates from (R,ϕ,z) to (ρ,θ,z), the angular unit vector in coordinate system (R,ϕ,z), $\hat{e}_\phi$, can be expressed as

$$\hat{e}_\phi=\frac{-r\sin\theta\hat{e}_\rho+(\rho-r\cos\theta)\hat{e}_\theta}{R}. \quad \text{(A4)}$$

From Eqs. (A2)–(A4), we have

$$J_1(kR)\hat{e}_\phi=-\hat{e}_\rho\sum_{n=1}^{\infty}[J_{n+1}(k\rho)+J_{n-1}(k\rho)]J_n(kr)\sin n\theta+\hat{e}_\theta\left\{J_1(k\rho)J_0(kr)+\sum_{n=1}^{\infty}[J_{n+1}(k\rho)-J_{n-1}(k\rho)]J_n(kr)\cos n\theta\right\}. \quad \text{(A5)}$$

(ii) Prove

$$\int_0^\infty dk\,\frac{J_n(ka)J_n(kr)}{k+1/\Lambda}\approx\frac{\pi}{2}\left[H_{-n}\left(\frac{r}{\Lambda}\right)-Y_{-n}\left(\frac{r}{\Lambda}\right)\right]J_{-n}\left(\frac{a}{\Lambda}\right)$$

for $a\ll\Lambda, r>a$.

Changing variable k into x/Λ, the integration of Bessel functions over k can be rewritten as

$$\int_0^\infty dk\,\frac{J_n(ka)J_n(kr)}{k+1/\Lambda}=\int_0^\infty dx\,\frac{J_n(\tilde{a}x)J_n(\tilde{r}x)}{x+1}, \quad \text{(A6)}$$

where $\tilde{a}=a/\Lambda$ and $\tilde{r}=r/\Lambda$. We can expand $J_n(\tilde{a}x)$ in terms of power series of $\tilde{a}x$, where $\tilde{a}$ is a small quantity. Then Eq. (A6) can be rewritten as

$$\int_0^\infty dk\,\frac{J_n(ka)J_n(kr)}{k+1/\Lambda}=\lim_{\varepsilon\to0}\frac{1}{n!}\left(\frac{\tilde{a}}{2}\right)^n\int_0^\infty dx\,e^{-\varepsilon x}x^n\frac{J_n(\tilde{r}x)}{x+1}+O(\tilde{a}^{n+2}), \quad \text{(A7)}$$

where the term $O(\tilde{a}^{n+2})$ can be neglected for $\tilde{a}\ll1$. Now we would like to prove the following integral formula by using the mathematical induction:

$$\lim_{\varepsilon\to0}\int_0^\infty dx\,e^{-\varepsilon x}x^n\frac{J_n(\tilde{r}x)}{x+1}=(-1)^n\frac{\pi}{2}[H_{-n}(\tilde{r})-Y_{-n}(\tilde{r})]. \quad \text{(A8)}$$

(1) For $n=1$, it is true for the following integral formula[16]

$$\lim_{\varepsilon\to0}\int_0^\infty dx\,e^{-\varepsilon x}x\frac{J_1(\tilde{r}x)}{x+1}=(-1)\frac{\pi}{2}[H_{-1}(\tilde{r})-Y_{-1}(\tilde{r})]. \quad \text{(A9)}$$

(2) Assume that Eq. (A8) is true for some positive integer m; that is, assume that

$$\lim_{\varepsilon\to0}\int_0^\infty dx\,e^{-\varepsilon x}x^m\frac{J_m(\tilde{r}x)}{x+1}=(-1)^m\frac{\pi}{2}[H_{-m}(\tilde{r})-Y_{-m}(\tilde{r})]. \quad \text{(A10)}$$

Now taking the operator $(\partial/\partial\tilde{r})\tilde{r}^{-m}$ on both sides of Eq. (A10), we have

$$\lim_{\varepsilon\to0}\int_0^\infty dx\,e^{-\varepsilon x}x^{m+1}\frac{J_{m+1}(\tilde{r}x)}{x+1}=(-1)^{m+1}\frac{\pi}{2}[H_{-m-1}(\tilde{r})-Y_{-m-1}(\tilde{r})]. \quad \text{(A11)}$$

Hence Eq. (A8) is true for $n=m+1$.

Since (1) and (2) are both true, Eq. (A8) is true for all positive integers n by the principle of mathematical induction. In addition, Eq. (A8) is also true for $n=0$.[16] Substituting Eq. (A8) into Eq. (A7), we have

$$\int_0^\infty dk\,\frac{J_n(ka)J_n(kr)}{k+1/\Lambda}\approx\frac{1}{n!}\left(\frac{-\tilde{a}}{2}\right)^n\frac{\pi}{2}[H_{-n}(\tilde{r})-Y_{-n}(\tilde{r})]\approx\frac{\pi}{2}\left[H_{-n}\left(\frac{r}{\Lambda}\right)-Y_{-n}\left(\frac{r}{\Lambda}\right)\right]J_{-n}\left(\frac{a}{\Lambda}\right)$$

for $a\ll\Lambda,\ r>a$. (A12)

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