



Zero-dispersion limit of the short-wave–long-wave interaction equations

Chi-Kun Lin^{a,b,*}, Yau-Shu Wong^{b,2}

^a Department of Mathematics, National Cheng Kung University, Tainan 701, Taiwan

^b Department of Mathematical & Statistical Sciences, University of Alberta, Edmonton, Canada

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Abstract

The purpose of this paper is to study the zero-dispersion limit of the water wave interaction equations which arise in modelling surface waves in the present of both gravity and capillary modes. This topic is also of interest in plasma physics. For the smooth solution, the limiting equation is given by the compressible Euler equation with a nonlocal pressure caused by the long wave. For weak solution, when the coupling coefficient λ is small order of ε , $\lambda = o(\varepsilon)$, the wave map equation is derived and the scattering sound wave is shown to satisfy a linear wave equation.

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1. Introduction

In this paper we study the behavior of solutions to the water wave interaction equations in the limit $\varepsilon \rightarrow 0^+$, where the parameter ε is analogous to the Planck constant in quantum mechanics. This system has the form

* Corresponding author. Present address: Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan.

E-mail address: cklin@math.nctu.edu.tw (C.-K. Lin).

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$$\varepsilon i \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \partial_{xx} \psi^\varepsilon - (\alpha(|\psi^\varepsilon|^2 - 1) + V^\varepsilon) \psi^\varepsilon = 0, \quad (1.1)$$

$$\partial_t V^\varepsilon = -\lambda \partial_x (|\psi^\varepsilon|^2), \quad (1.2)$$

where the complex-valued function ψ^ε and the real-valued function V^ε represent the envelope of the short wave and the amplitude of the long wave, respectively. The two real parameters α and λ are assumed to be positive for convenience. The initial values are given by

$$\psi^\varepsilon(x, 0) = \psi_0^\varepsilon(x) = A_0(x) \exp\left(\frac{i}{\varepsilon} S_0(x)\right), \quad (1.3)$$

$$V^\varepsilon(x, 0) = V_0^\varepsilon(x), \quad (1.4)$$

where S_0 is a function of $H^s(\mathbb{R})$ (Sobolev spaces) for s large enough, and A_0 is a function, polynomial in ε , with coefficients of Sobolev regularity in x . More precisely, we are concerned with the behavior of solutions to (1.1), (1.2) as ε tends to zero with the rapid oscillating initial data for the short wave. The small parameter ε represents the space and time scales introduced in (1.1), (1.2), as well as the typical wave length of oscillations of the initial data. In the special case of Schrödinger equation with vanishing Planck's constant this is precisely the *semiclassical limit*.

Under the assumptions of long wave–short wave resonance, Benney [4] proposed several systems of dispersive equations. One of the systems is given by (1.1), (1.2) which has frequently been used to model for interactions between long and short waves in a variety of physical settings [4,6,10,11,30,31,34]. For example, Djordjevic and Redekopp [11] derived (1.1), (1.2) for $\alpha = 0$ as a model for the interaction between long gravity waves and capillary waves on the surface of shallow water, in the case when the group velocity of capillary wave coincides with the velocity of the long wave. They pointed out that the physical significance of Eqs. (1.1), (1.2) is such that the dispersion of the short wave is balanced by the nonlinear interaction of the long wave with the short wave, while the evolution of the long wave is driven by the self-interaction of the short wave. When $\alpha = 0$ this model is integrable by the inverse scattering method [26]. Another example arises from the study of resonant ion–acoustic/Langmuir wave interactions in plasma under the assumption that the ion–sound wave is unidirectional. This system has also been employed to substitute for the Davey–Stewartson system due to the effect of resonance, a phenomenon which occurs when the group velocity of the short waves matches the phase velocity of the long waves [11,31]. Note that Eqs. (1.1), (1.2) can be also served as the simplified version of the Zakharov system for the Langmuir turbulence of the plasma physics [1,29,31]. When $V^\varepsilon = 0$, (1.1) uncouples from (1.2) and it becomes a nonlinear Schrödinger equation which has been studied by Zakharov and Shabat [35]. Depending on the sign of α , this equation has soliton or decaying oscillatory solutions. If $\lambda = 0$, V^ε depends on the space variable x only and (1.1) is known as cubic nonlinear Schrödinger equation with stationary potential $V^\varepsilon = V_0^\varepsilon(x)$.

The solvability of (1.1), (1.2) is considered under various settings. When $\varepsilon = 1$, applying the smoothing effects of the free Schrödinger operator, it was shown by Tsutsumi and Hatano [32,33] that the initial value problem is locally and globally well posed in $H^{\frac{1}{2}}(\mathbb{R})$ when $\alpha = 0$ and in $H^{\frac{1}{2}+m}(\mathbb{R})$, $m = 1, 2, \dots$ when $\alpha \neq 0$. For the largest space, i.e., $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ and the conservation laws hold, it was proved by Ogawa [28] (see also [2,3,19] and references therein). To investigate the singular limit employing the structure of the nonlinear Schrödinger equation we will transform the system (1.1), (1.2) into a single equation and serve (1.2) as a constraint

[2,28,32,33]. Integrating (1.2) in t and eliminating V^ε from (1.2), we can rewrite (1.1) and (1.2) as a single equation for ψ^ε

$$\varepsilon i \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \partial_{xx} \psi^\varepsilon = -\lambda \left[\int_0^t \partial_x (|\psi^\varepsilon(x, \tau)|^2) d\tau \right] \psi^\varepsilon + [V_0^\varepsilon(x) + \alpha(|\psi^\varepsilon|^2 - 1)] \psi^\varepsilon, \quad (1.5)$$

$$\psi^\varepsilon(x, 0) = \psi_0^\varepsilon(x), \quad x \in \mathbb{R}. \quad (1.6)$$

The first nonlocal or memory term induced by the long wave on the right-hand side of (1.5) causes the so-called derivative loss phenomenon which prevent one to apply the well-known Segal nonlinear semigroup theory in a simple manner. This difficulty can be overcome by using the smoothing-effects estimates of solutions of linear Schrödinger evolution equations developing by Kenig et al. [17,18] (see also [2,3]).

The semiclassical or small dispersion problem (i.e., small ε) has been the subject of research in the last 20 years. According to the correspondence principle, the classical world should emerge from the quantum world whenever the Planck constant is negligible. But the limit as the Planck constant tends to zero is mathematically singular. This fact complicates the reduction to classical mechanics. Thus the mathematical rigorous analysis of the semiclassical limit for Schrödinger type equations (or more general dispersive equations) is an issue of importance and full of challenge to mathematical analysis. For linear Schrödinger equation or Schrödinger–Poisson, the idea of kinetic formulation to solve it global-in-time is the followings. By applying the Wigner transform, we can obtain a kinetic integro-differential equation the so-called Wigner equation. The investigation of the kinetic structure of the Wigner equation and the application of the moments methods to its solutions, which provide information of macroscopic densities, help us to pass limit as the Planck constant tends to 0 in the Wigner equation and the macroscopic densities. We have the Vlasov (Vlasov–Poisson) equation, which is the quantum (hydrodynamic) limiting system of the linear Schrödinger-type equations [12,13]. The analysis of the limiting system gives us the similar macroscopic densities and results to those obtained by the geometric optics approach to the WKB limit of Schrödinger equations and reveals a close relation between the semiclassical limit of quantum fluid equations and the kinetic equations [13].

However, the situation is quite different for nonlinear Schrödinger-type equations because the theory of Wigner transform and the semiclassical (or zero-dispersion) limit are still under investigation for nonlinear Schrödinger-type equations. Most of the rigorous global analysis of the limiting behavior are restricted to the integrable nonlinear wave equations (see [15,25] and references therein). Thus, any analytical or numerical results related to (1.1), (1.2) should be important in the study of short-wave–long-wave interactions. In particular, the zero-dispersion limit, when one could expect creation of shock waves and interesting limiting dynamics of the conserved quantities. Therefore the study of the semiclassical limit of (1.1), (1.2) will significantly enhance our understanding of the general semiclassical behavior of nonlinear dispersive waves. Our approach to the dispersive limit of (1.1), (1.2) is the WKB analysis. Since the WKB method has the drawback of being local in time, thus we treat the local smooth solutions only [8, 9,14]. This work about the zero-dispersion limit of the short-wave–long-wave interaction equations was motivated by the a natural mathematical question; however, the problem is also of direct importance to water wave and plasma physics.

The plan of the paper is as follows. In Section 2, we derive the hydrodynamical structure and the local conservation laws of the short-wave–long-wave equations (1.1), (1.2). The formal

dispersive limit is also discussed. We employ the modified Madelung transformation to represent the water wave interaction equations as a perturbation of a quasilinear symmetric hyperbolic system in Section 3. For suitable initial data in the Sobolev space $H^s(\mathbb{R})$ with s sufficiently large, the classical solutions of (1.1)–(1.4) exist for a time T independent of ε and converge pointwise together with some number of derivatives to a classical solution of the compressible Euler equation with nonlocal potential. In Section 4, we study the zero-dispersion limit of (1.1), (1.2) directly motivated by the defocussing cubic nonlinear Schrödinger equation [5,25]. In the case when there are no vortices (uniform bounded energy as $\varepsilon \rightarrow 0$), we show that the limit of the wave functions solve the wave map equations and the associated phase functions satisfy a linear wave equation which is the same as the defocussing cubic nonlinear Schrödinger equation. This concludes that the long wave plays no role in the zero-dispersion limit.

2. Hydrodynamical structures and conservation laws

The semiclassical limit of Eqs. (1.1), (1.2) is to determine the limiting dynamics of any functions of the fields ψ^ε as $\varepsilon \rightarrow 0$. However, it is not clear directly from (1.1) what form such a dynamics might take. Insight into this question can be gained by considering the conservation laws associated with (1.1), (1.2). To this end, we make the geometric optic (semiclassical) ansatz [8,9,14]

$$\psi^\varepsilon(x, t) = A^\varepsilon(x, t) \exp\left(\frac{i}{\varepsilon} S^\varepsilon(x, t)\right) = \sqrt{\rho^\varepsilon(x, t)} \exp\left(\frac{i}{\varepsilon} S^\varepsilon(x, t)\right). \tag{2.1}$$

This transformation is usually called the Madelung transformation and was originally introduced in the context of the linear Schrödinger equation for quantum mechanics. The real amplitude A^ε , phase function S^ε and V^ε obey the following equations:

$$\partial_t A^\varepsilon + \partial_x A^\varepsilon \partial_x S^\varepsilon + \frac{1}{2} A^\varepsilon \partial_{xx} S^\varepsilon = 0, \tag{2.2}$$

$$\partial_t S^\varepsilon + \frac{1}{2} (\partial_x S^\varepsilon)^2 + (\alpha(|A^\varepsilon|^2 - 1) + V^\varepsilon) = \frac{\varepsilon^2}{2A^\varepsilon} \partial_{xx} A^\varepsilon, \tag{2.3}$$

$$\partial_t V^\varepsilon + 2\lambda A^\varepsilon \partial_x A^\varepsilon = 0, \tag{2.4}$$

where Eq. (2.3) is the quantum deformation of the Hamilton–Jacobi equation by quantum potential. Consider the new variables

$$\rho^\varepsilon \equiv |A^\varepsilon|^2 = |\psi^\varepsilon|^2, \quad u^\varepsilon \equiv \partial_x S^\varepsilon, \tag{2.5}$$

we have the following two local conservation laws

$$\partial_t \rho^\varepsilon + \partial_x (\rho^\varepsilon u^\varepsilon) = 0, \tag{2.6}$$

$$\partial_t u^\varepsilon + \partial_x \left(\frac{|u^\varepsilon|^2}{2} + \alpha \rho^\varepsilon + V^\varepsilon \right) = \frac{\varepsilon^2}{2} \partial_x \left(\frac{\partial_{xx} \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right). \tag{2.7}$$

Equations (2.6), (2.7) comprise a closed system governing ρ^ε and u^ε , which have the form of a perturbation of the Euler equations with V^ε satisfying

$$\partial_t V^\varepsilon + \lambda \partial_x \rho^\varepsilon = 0, \tag{2.8}$$

which is equivalent to

$$V^\varepsilon(x, t) = V_0^\varepsilon(x) - \lambda \int_0^t \partial_x \rho^\varepsilon(x, \tau) d\tau. \tag{2.8'}$$

This is the local conservation law of mass (or amplitude) of the long wave V^ε . From (2.6) and (2.7) we can also derive the equation for the canonical momentum $\mu^\varepsilon \equiv \rho^\varepsilon u^\varepsilon$

$$\partial_t \mu^\varepsilon + \partial_x \left(\frac{|\mu^\varepsilon|^2}{\rho^\varepsilon} + \frac{\alpha}{2} |\rho^\varepsilon|^2 \right) + \rho^\varepsilon \partial_x V^\varepsilon = \frac{\varepsilon^2}{4} \partial_x (\rho^\varepsilon \partial_{xx} \log \rho^\varepsilon), \tag{2.9}$$

which is not conservative because of the coupling. However, employing (2.8), we still have the local conservation law of momentum in the following form

$$\partial_t \left(\mu^\varepsilon + \frac{1}{2\lambda} |V^\varepsilon|^2 \right) + \partial_x \left(\frac{|\mu^\varepsilon|^2}{\rho^\varepsilon} + \frac{\alpha}{2} |\rho^\varepsilon|^2 + \rho^\varepsilon V^\varepsilon \right) = \frac{\varepsilon^2}{4} \partial_x (\rho^\varepsilon \partial_{xx} \log \rho^\varepsilon), \tag{2.10}$$

where $\mu^\varepsilon + \frac{1}{2\lambda} |V^\varepsilon|^2 = \rho^\varepsilon u^\varepsilon + \frac{1}{2\lambda} |V^\varepsilon|^2$ is the *noncanonical momentum* which means that even if the fluid velocity (short wave) vanishes, i.e., $u^\varepsilon = 0$, the flow still has background momentum caused by the long wave. This implies that solitons of the water wave interaction equation (1.1), (1.2) have nontrivial static limit. In the field of theoretical language, we can say that the spectrum of excitations has always a gap (like in superfluidity). Besides the principles of conservation of mass and momentum upon which Eqs. (1.1), (1.2) are formulated, the conservation of energy is another principle of great physical and mathematical importance. Define the energy density E^ε by

$$E^\varepsilon = E_1^\varepsilon + E_2^\varepsilon + E_3^\varepsilon + E_4^\varepsilon \equiv \frac{|\mu^\varepsilon|^2}{2\rho^\varepsilon} + \frac{\alpha}{2} |\rho^\varepsilon|^2 + \rho^\varepsilon V^\varepsilon + \frac{\varepsilon^2}{8} \frac{|\partial_x \rho^\varepsilon|^2}{\rho^\varepsilon}, \tag{2.11}$$

i.e., the total energy of the short-wave and long-wave interaction equations is constituted by the classical part, E_1^ε the kinetic energy, $E_2^\varepsilon + E_3^\varepsilon$ the potential energy and the quantum part E_4^ε which is of order $O(\varepsilon^2)$. The crossing term $E_3^\varepsilon = \rho^\varepsilon V^\varepsilon$ comes from the interaction of the short-wave and long-wave. They propagate according to

$$\partial_t E_1^\varepsilon + \partial_x (E_1^\varepsilon \cdot u^\varepsilon) + u^\varepsilon \partial_x \left(\frac{\alpha}{2} |\rho^\varepsilon|^2 \right) + \rho^\varepsilon u^\varepsilon \partial_x V^\varepsilon = \frac{\varepsilon^2}{4} u^\varepsilon \partial_x (\rho^\varepsilon \partial_x^2 \log \rho^\varepsilon), \tag{2.12}$$

$$\partial_t E_2^\varepsilon + \partial_x (2E_2^\varepsilon \cdot u^\varepsilon) - u^\varepsilon \partial_x \left(\frac{\alpha}{2} |\rho^\varepsilon|^2 \right) = 0, \tag{2.13}$$

$$\partial_t E_3^\varepsilon + \partial_x (E_3^\varepsilon \cdot u^\varepsilon) - \rho^\varepsilon u^\varepsilon \partial_x V^\varepsilon + \partial_x \left(\frac{\lambda |\rho^\varepsilon|^2}{2} \right) = 0, \tag{2.14}$$

$$\partial_t E_4^\varepsilon + \partial_x (E_4^\varepsilon \cdot u^\varepsilon) + \frac{\varepsilon^2}{4} \partial_x \left(\frac{\partial_x \rho^\varepsilon \partial_x \mu^\varepsilon}{\rho^\varepsilon} - \frac{\mu^\varepsilon \partial_{xx} \rho^\varepsilon}{\rho^\varepsilon} \right) = -\frac{\varepsilon^2}{4} u^\varepsilon \partial_x (\rho^\varepsilon \partial_{xx} \log \rho^\varepsilon). \tag{2.15}$$

Summing (2.12)–(2.15) yields the energy equation

$$\partial_t E^\varepsilon + \partial_x \left((E^\varepsilon + E_2^\varepsilon) \frac{\mu^\varepsilon}{\rho^\varepsilon} + \frac{\lambda}{2} |\rho^\varepsilon|^2 \right) = \frac{\varepsilon^2}{4} \partial_x \left(\frac{\mu^\varepsilon \partial_{xx} \rho^\varepsilon}{\rho^\varepsilon} - \frac{\partial_x \rho^\varepsilon \partial_x \mu^\varepsilon}{\rho^\varepsilon} \right), \tag{2.16}$$

which is conservative. Therefore we obtain the quantum hydrodynamics equations of (1.1), (1.2) (see [12,22,23] for the similar models). The above hydrodynamical structures imply the local conservation laws of the water wave interaction equations (1.1), (1.2).

Theorem 2.1. *Let $\bar{\cdot}$ denote the complex conjugate and $t \in [0, \infty)$. The following quantities are conservation integrals of (1.1), (1.2)*

$$\int_{-\infty}^{\infty} \rho^\varepsilon(x, t) dx = C_1, \tag{2.17}$$

$$\int_{-\infty}^{\infty} u^\varepsilon(x, t) dx = C_2, \tag{2.18}$$

$$\int_{-\infty}^{\infty} V^\varepsilon(x, t) dx = C_3, \tag{2.19}$$

$$\int_{-\infty}^{\infty} \left(\mu^\varepsilon(x, t) + \frac{1}{2\lambda} |V^\varepsilon(x, t)|^2 \right) dx = C_4, \tag{2.20}$$

$$\int_{-\infty}^{\infty} E^\varepsilon(x, t) dx = C_5, \tag{2.21}$$

where the hydrodynamic variables ρ^ε , u^ε , μ^ε , V^ε and E^ε are given in terms of the envelope of the short wave function ψ^ε as follows

$$\rho^\varepsilon(x, t) = |\psi^\varepsilon(x, t)|^2 = \psi^\varepsilon(x, t) \bar{\psi}^\varepsilon(x, t), \tag{2.22}$$

$$u^\varepsilon(x, t) = \frac{i\varepsilon}{2} \left(\frac{\partial_x \bar{\psi}^\varepsilon(x, t)}{\bar{\psi}^\varepsilon(x, t)} - \frac{\partial_x \psi^\varepsilon(x, t)}{\psi^\varepsilon(x, t)} \right), \tag{2.23}$$

$$\mu^\varepsilon(x, t) = \frac{i\varepsilon}{2} (\psi^\varepsilon(x, t) \partial_x \bar{\psi}^\varepsilon(x, t) - \bar{\psi}^\varepsilon(x, t) \partial_x \psi^\varepsilon(x, t)), \tag{2.24}$$

$$V^\varepsilon(x, t) = V_0^\varepsilon(x) - \lambda \int_0^t \partial_x (|\psi^\varepsilon(x, \tau)|^2) d\tau, \tag{2.25}$$

$$E^\varepsilon(x, t) = \frac{\varepsilon^2}{2} |\partial_x \psi^\varepsilon(x, t)|^2 + \frac{\alpha}{2} |\psi^\varepsilon(x, t)|^4 + V^\varepsilon(x, t) |\psi^\varepsilon(x, t)|^2. \tag{2.26}$$

The conservative quantities of the water wave interaction equations may be recast from the action principle. The Lagrangian formulation allows us to systematically derive conserved quantities by means of Noether’s theorem which assigns to each of these symmetries a corresponding conserved quantities by taking the form of the integral of a multinomial in $\psi^\varepsilon, V^\varepsilon$ and their x -derivatives. Equations (1.1), (1.2) are trivially invariant under the phase rotation

$$\psi^\varepsilon(x, t) \mapsto e^{i\theta} \psi^\varepsilon(x, t), \quad \theta \in \mathbb{R},$$

and have

$$C_1 = \int_{\mathbb{R}} |\psi^\varepsilon(x, t)|^2 dx = \int_{\mathbb{R}} \rho^\varepsilon(x, t) dx$$

as the corresponding constant of the motion. The equations have no explicit dependence on x and hence

$$\psi^\varepsilon(x, t) \mapsto \psi^\varepsilon(x - \Delta x, t), \quad V^\varepsilon(x, t) \mapsto V^\varepsilon(x - \Delta x, t)$$

must be symmetry. This space translation invariant gives C_4 as its conserved quantity, which can be thought of as the *momentum* of the equations. Similarly, the equations have no explicit dependence on t so

$$\psi^\varepsilon(x, t) \mapsto \psi^\varepsilon(x, t - \Delta t), \quad V^\varepsilon(x, t) \mapsto V^\varepsilon(x, t - \Delta t)$$

must also be symmetry. The corresponding conserved quantity is the Hamiltonian or the total energy C_5 .

Remark. The Madelung formulation relies on the assumption that the amplitude of ψ^ε is not zero and the phase S^ε is not singular, otherwise the transformation is not well-defined and the system (2.6), (2.7) becomes singular even though (1.1), (1.2) is still regular. Therefore, we treat only the regime of smooth phase functions. However, the conservation laws do not rely on the Madelung transformation, we can still derive the same result from (1.1), (1.2) directly. In contrast to all earlier applications of the Madelung transformation, we can avoid making explicit use of the phase function S^ε and do not work with (2.6), (2.7). By the three conservation laws, mass (2.17), momentum (2.20) and energy (2.21), Ogawa [28] proves the global well-posedness of the system (1.1)–(1.4) for $\varepsilon = 1$ in the largest class of initial data.

In the formal semiclassical limit $\varepsilon \rightarrow 0$, one neglects the contribution from the quantum potential $\partial_{xx}(\sqrt{\rho^\varepsilon})/\sqrt{\rho^\varepsilon}$ in (2.6)–(2.8), and the limiting densities $\rho, \mu = \rho u$ and V satisfy the Euler system

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{2.27}$$

$$\partial_t \mu + \partial_x \left(\frac{|\mu|^2}{\rho} + \frac{\alpha}{2} \rho^2 \right) + \rho \partial_x V = 0, \tag{2.28}$$

with initial conditions inferred from (1.3) given by

$$\rho_0(x) = \rho(x, 0) = |A_0(x)|^2, \quad \mu_0(x) = \mu(x, 0) = |A_0(x)|^2 \partial_x S_0(x), \tag{2.29}$$

where V is formally given by

$$V(x, t) = V_0(x) - \lambda \int_0^t \partial_x \rho(x, \tau) d\tau. \tag{2.30}$$

Here $V_0(x)$ is the limit of $V_0^\varepsilon(x)$. If $\lambda = 0$ then the potential V is stationary, $V(x, t) = V_0(x)$. These are the classical Euler equations of the compressible fluid. This argument is self-consistent only if the solution of (2.22)–(2.25) remains classical (i.e., before the development of the first shock). In that case the limiting energy density will be given by

$$E = \frac{|\mu|^2}{2\rho} + \frac{\alpha}{2} |\rho|^2 + \rho V \tag{2.31}$$

and will satisfy

$$\partial_t E + \partial_x \left((E + E_2(\rho)) \frac{\mu}{\rho} + \frac{\lambda}{2} |\rho|^2 \right) = 0, \tag{2.32}$$

where $E_2(\rho) = \frac{\alpha}{2} \rho^2$ plays the role similar to the pressure.

It is clear from (2.2)–(2.4) that the formal dispersionless limit equations associated with (1.1), (1.2) are given by

$$\partial_t A + \partial_x A \partial_x S + \frac{1}{2} A \partial_{xx} S = 0, \tag{2.33}$$

$$\partial_t S + \frac{1}{2} \partial_x S^2 + (\alpha(A^2 - 1) + V) = 0, \tag{2.34}$$

$$\partial_t V + \lambda \partial_x (A^2) = 0, \tag{2.35}$$

where (A, S, V) is the formal limit of $(A^\varepsilon, S^\varepsilon, V^\varepsilon)$ of (2.2)–(2.4). Note that Eq. (2.34) for the phase S is a classical Hamilton–Jacobi equation for the action of a particle with respect to the potential $\alpha(A^2 - 1) + V$. Introducing the new complex wave function

$$\varphi(x, t) = A(x, t) \exp(iS(x, t)), \tag{2.36}$$

system (2.33)–(2.35) is equivalent to the following modification of the water wave interaction equations (1.1), (1.2):

$$i \partial_t \varphi + \frac{1}{2} \partial_{xx} \varphi - (\alpha(|\varphi|^2 - 1) + V) \varphi = \frac{1}{2} \frac{\partial_{xx} |\varphi|}{|\varphi|} \varphi, \tag{2.37}$$

$$\partial_t V = -\lambda \partial_x (|\varphi|^2), \tag{2.38}$$

with the quantum potential on the right-hand side of (2.37). The quantum potential contribution to the right-hand side with fixed strength completely compensates $U(1)$ gauge invariant contribution to dispersion on the left-hand side. This potential, the so-called Bohm potential or the internal self-potential was introduced by deBroglie and later explored by Bohm to make a hidden-variable theory, is responsible for producing the quantum behavior, so that all quantum features are related to its special properties. The role of the quantum potential is to change the

dispersion of the Schrödinger equation. If the strength of the quantum potential deviates from the critical value as given in dispersionless equations (2.37), (2.38), then we have the deformed wave equations

$$i \partial_t \varphi^\varepsilon + \frac{1}{2} \partial_{xx} \varphi^\varepsilon - (\alpha(|\varphi^\varepsilon|^2 - 1) + V^\varepsilon) \varphi^\varepsilon = (1 + \varepsilon^2) \frac{1}{2} \frac{\partial_{xx} |\varphi^\varepsilon|}{|\varphi^\varepsilon|} \varphi^\varepsilon, \tag{2.39}$$

$$\partial_t V^\varepsilon = -\lambda \partial_x (|\varphi^\varepsilon|^2) \tag{2.40}$$

of the dispersionless system (2.37), (2.38), which is determined by the deformation parameter ε (the Planck constant) and the semiclassical ansatz

$$\varphi^\varepsilon(x, t) = A(x, t) \exp(iS(x, t)/\varepsilon). \tag{2.41}$$

Moreover, for the classically inaccessible regions simulated by analytical continuation of the Planck constant to a pure imaginary value $\varepsilon \mapsto i\varepsilon$, instead of (2.39), (2.40), we have

$$i \partial_t \varphi^\varepsilon + \frac{1}{2} \partial_{xx} \varphi^\varepsilon - (\alpha(|\varphi^\varepsilon|^2 - 1) + V^\varepsilon) \varphi^\varepsilon = (1 - \varepsilon^2) \frac{1}{2} \frac{\partial_{xx} |\varphi^\varepsilon|}{|\varphi^\varepsilon|} \varphi^\varepsilon, \tag{2.42}$$

$$\partial_t V^\varepsilon = -\lambda \partial_x (|\varphi^\varepsilon|^2). \tag{2.43}$$

Furthermore, written in terms of the two *real-valued* functions

$$Q^\pm(x, t) = A(x, t) \exp(\pm S(x, t)/\varepsilon) = \sqrt{\rho(x, t)} \exp(\pm S(x, t)/\varepsilon), \tag{2.44}$$

the diffusion equations in duality analog of (1.1), (1.2)

$$\varepsilon \partial_t Q^\pm \pm \frac{\varepsilon^2}{2} \partial_{xx} Q^\pm + (\alpha(Q^+ Q^- - 1) + V) Q^\pm = 0, \tag{2.45}$$

$$\partial_t V = -\lambda \partial_x (Q^+ Q^-) \tag{2.46}$$

are derived. Thus system (1.1), (1.2) is *intrinsically* in the theory of diffusion processes as an equation in the context of time reversal of diffusion processes, namely, diffusion equations in duality. The origin of the idea of considering diffusion process for quantum mechanics goes back to Schrödinger (1931), in which he formulated Brownian motions in a symmetric form of time reversal. Schrödinger’s time-symmetric theory of diffusion process revealed the deep relation between diffusion theory and quantum theory. We can further represent this system as the decoupled pair of Burgers’ equations by the well known Hopf–Cole transformation which suggests to introduce the pair of velocity fields

$$u^+ = \varepsilon \partial_x (\log Q^+) = \varepsilon \frac{\partial_x Q^+}{Q^+}, \quad u^- = -\varepsilon \partial_x (\log Q^-) = -\varepsilon \frac{\partial_x Q^-}{Q^-}, \tag{2.47}$$

such that instead of (2.38), (2.39), we have the coupled system of Burgers’ equations with negative and positive viscosities

$$\partial_t u^+ + u^+ \partial_x u^+ = -\frac{\varepsilon}{2} \partial_{xx} u^+ - \partial_x \Omega, \tag{2.48}$$

$$\partial_t u^- + u^- \partial_x u^- = \frac{\varepsilon}{2} \partial_{xx} u^- + \partial_x \Omega, \tag{2.49}$$

$$\partial_t V = -\lambda \partial_x (Q^+ Q^-), \tag{2.50}$$

with the potential function given by

$$\Omega = \alpha(Q^+ Q^- - 1) + V. \tag{2.51}$$

These relative velocities

$$u^\pm = u \pm u^* = \partial_x S \pm \frac{\varepsilon}{2} \frac{\partial_x \rho}{\rho} \tag{2.52}$$

characterize two motions, the center of mass motion with velocity $u = \partial_x S$ and internal oscillations in the envelope with velocity $u^* = \frac{\varepsilon}{2} \frac{\partial_x \rho}{\rho}$.

3. Modified Madelung transformation

In this section, we will employ the modified Madelung transformation to transform (1.1), (1.2) into a linear dispersive perturbation of a quasilinear symmetric hyperbolic system to which the Lax–Friedrich–Kato theory can be applied [16,27]. Equations (1.1), (1.2) or (2.6)–(2.8) do not have the explicit form of a first-order hyperbolic system for the variables $(\rho^\varepsilon, u^\varepsilon, V^\varepsilon)$. However, it can be overcome by serving V^ε as a forcing term given by (2.8'). The limit $\varepsilon \rightarrow 0$ cannot be made directly in (2.6)–(2.8) since the phase S^ε or the quotients $1/\sqrt{\rho^\varepsilon}$ may be undefined. As suggested by Grenier [14] (see also [8,9,20–22,24]), the modified Madelung transformation can be utilized in the study of the semiclassical limit. The similar idea had also been used earlier by Schochet and Weinstein to study the nonlinear Schrödinger limit of the Zakharov system [29]. Indeed, we will look for solution ψ^ε of the form

$$\psi^\varepsilon = A^\varepsilon \exp(S^\varepsilon/\varepsilon), \quad A^\varepsilon = a^\varepsilon + ib^\varepsilon. \tag{3.1}$$

Note here we allow the phase function S^ε to depend on the parameter ε . Now inserting (3.1) into (1.1), we obtain

$$\begin{aligned} \varepsilon i \partial_t A^\varepsilon - A^\varepsilon \partial_t S^\varepsilon + \frac{\varepsilon^2}{2} \partial_{xx} A^\varepsilon + \varepsilon i \partial_x A^\varepsilon \partial_x S^\varepsilon - \frac{1}{2} A^\varepsilon (\partial_x S^\varepsilon)^2 + \frac{\varepsilon i}{2} A^\varepsilon \partial_{xx} S^\varepsilon \\ = (\alpha |A^\varepsilon|^2 - \alpha + V^\varepsilon) A^\varepsilon, \end{aligned}$$

it can then split into

$$\partial_t A^\varepsilon + \left(\partial_x A^\varepsilon \partial_x S^\varepsilon + \frac{1}{2} A^\varepsilon \partial_{xx} S^\varepsilon \right) = \frac{\varepsilon}{2} i \partial_{xx} A^\varepsilon \quad \text{and} \tag{3.2}$$

$$\partial_t S^\varepsilon + \frac{1}{2} (\partial_x S^\varepsilon)^2 + (\alpha |A^\varepsilon|^2 - \alpha + V^\varepsilon) = 0. \tag{3.3}$$

Considering the change of variables $v^\varepsilon \equiv \partial_x S^\varepsilon$, we have the equivalent form of (1.1)–(1.4)

$$\partial_t a^\varepsilon + v^\varepsilon \partial_x a^\varepsilon + \frac{1}{2} a^\varepsilon \partial_x v^\varepsilon = -\frac{\varepsilon}{2} \partial_{xx} b^\varepsilon, \tag{3.4}$$

$$\partial_t b^\varepsilon + v^\varepsilon \partial_x b^\varepsilon + \frac{1}{2} b^\varepsilon \partial_x v^\varepsilon = \frac{\varepsilon}{2} \partial_{xx} a^\varepsilon, \tag{3.5}$$

$$\partial_t v^\varepsilon + v^\varepsilon \partial_x v^\varepsilon + 2\alpha(a^\varepsilon \partial_x a^\varepsilon + b^\varepsilon \partial_x b^\varepsilon) + \partial_x V^\varepsilon = 0, \tag{3.6}$$

$$a^\varepsilon(x, 0) = a_0^\varepsilon(x), \quad b^\varepsilon(x, 0) = b_0^\varepsilon(x), \quad v^\varepsilon(x, 0) = v_0^\varepsilon(x) = \partial_x S^\varepsilon(x, 0), \tag{3.7}$$

where, according to (1.2), the potential V^ε is given explicitly by

$$V^\varepsilon(x, t) = V_0^\varepsilon(x) - 2\lambda \int_0^t (a^\varepsilon \partial_x a^\varepsilon + b^\varepsilon \partial_x b^\varepsilon) d\tau. \tag{3.8}$$

The system can be rewritten in the vector form:

$$\partial_t U^\varepsilon + A(U^\varepsilon) \partial_x U^\varepsilon + G^\varepsilon = \frac{\varepsilon}{2} L(U^\varepsilon), \tag{3.9}$$

$$U^\varepsilon(x, 0) = U_0^\varepsilon(x) = (a_0^\varepsilon(x), b_0^\varepsilon(x), v_0^\varepsilon(x))^t, \tag{3.10}$$

where $U^\varepsilon = (a^\varepsilon, b^\varepsilon, v^\varepsilon)^t$, $G^\varepsilon = (0, 0, \partial_x V^\varepsilon)^t$,

$$A(U^\varepsilon) = \begin{bmatrix} v^\varepsilon & 0 & a^\varepsilon/2 \\ 0 & v^\varepsilon & b^\varepsilon/2 \\ 2\alpha a^\varepsilon & 2\alpha b^\varepsilon & v^\varepsilon \end{bmatrix}, \quad L = \begin{bmatrix} 0 & -\partial_{xx} & 0 \\ \partial_{xx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.11}$$

Obviously the matrix $A(U^\varepsilon)$ can be symmetrized by

$$S = \begin{bmatrix} 4\alpha & 0 & 0 \\ 0 & 4\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is symmetric and positive definite for $\alpha > 0$. The antisymmetric operator L in (3.11) reflects the dispersive nature of Eqs. (1.1), (1.2). The special structure of (3.9) will be exploited on the classical solutions. The existence of classical solutions proceeds along the lines of the existence proof for the initial value problem for the quasilinear symmetric hyperbolic system with modification. Indeed, applying the theory of the quasilinear symmetric hyperbolic system, we will obtain the existence of smooth solutions $(\psi^\varepsilon, V^\varepsilon)$ of (1.1), (1.2) on a time interval $[0, T)$ independent of ε . Furthermore, the bounds that we obtained are uniformly bounded in ε on the solution $(\psi^\varepsilon, V^\varepsilon)$ will allow to pass to the limit $\varepsilon \rightarrow 0$ in (3.4)–(3.6) and this justify the WKB hierarchy.

In addition to the linear dispersive perturbation of the quasilinear symmetric hyperbolic system nature, the modified Madelung transformation also give us more information about the phase transportation. Since A^ε is complex-valued, we introduce the polar coordinates:

$$A^\varepsilon = a^\varepsilon + ib^\varepsilon = \sqrt{\rho^\varepsilon} e^{i\theta^\varepsilon}. \tag{3.12}$$

Applying the chain rule, we obtain

$$a^\varepsilon \partial_{xx} b^\varepsilon - b^\varepsilon \partial_{xx} a^\varepsilon = \partial_x (\rho^\varepsilon \partial_x \theta^\varepsilon), \tag{3.13}$$

then from (3.4)–(3.6) we derive the system

$$\partial_t \rho^\varepsilon + \partial_x (\rho^\varepsilon v^\varepsilon + \varepsilon \rho^\varepsilon \partial_x \theta^\varepsilon) = 0, \tag{3.14}$$

$$\partial_t \theta^\varepsilon + v^\varepsilon \partial_x \theta^\varepsilon + \frac{\varepsilon}{2} |\partial_x \theta^\varepsilon|^2 = \frac{\varepsilon}{2} \frac{\partial_{xx} (\sqrt{\rho^\varepsilon})}{\sqrt{\rho^\varepsilon}}, \tag{3.15}$$

$$\partial_t v^\varepsilon + v^\varepsilon \partial_x v^\varepsilon + \partial_x (\alpha \rho^\varepsilon + V^\varepsilon) = 0, \tag{3.16}$$

where V^ε is given by

$$V^\varepsilon(x, t) = V_0^\varepsilon(x) - \lambda \int_0^t \partial_x \rho^\varepsilon(x, \tau) d\tau. \tag{3.17}$$

The quantum effect in this system is of order $O(\varepsilon)$ not $O(\varepsilon^2)$ comparing with (2.7). Note the transport equation for ρ^ε has an extra term of order $O(\varepsilon)$ comparing with the typical equation of continuity. By formally letting $\varepsilon \rightarrow 0$, we have

$$\partial_t \rho + \partial_x (\rho v) = 0, \tag{3.18}$$

$$\partial_t \theta + v \partial_x \theta = 0, \tag{3.19}$$

$$\partial_t v + v \partial_x v + \partial_x (\alpha \rho + V) = 0, \tag{3.20}$$

$$V(x, t) + \lambda \int_0^t \partial_x \rho(x, \tau) d\tau = V_0(x). \tag{3.21}$$

Since (3.19) is the pure transport equation then $\theta = 0$ for the trivial initial data, thus we have the same limit system as (2.22)–(2.24).

We now first establish the existence and uniqueness of the classical solution of the dispersive perturbation of the quasilinear symmetric hyperbolic system (3.8)–(3.11).

Theorem 3.1. *Let $s > \frac{5}{2}$. Suppose $M_0 \geq 1$, M and T are given such that*

$$[M_0 + (cM_0^2 + M)T]e^{cM_0T} \leq 2M_0, \tag{3.22}$$

then

- (i) *if $G^\varepsilon \in L^\infty([0, T]; H^s(\mathbb{R})) \cap C([0, T]; H^{s-2}(\mathbb{R}))$ such that $\|G^\varepsilon\|_{H^s} \leq M$ and the initial data $U_0^\varepsilon = (a_0^\varepsilon, b_0^\varepsilon, v_0^\varepsilon) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R})$ satisfying $\|U_0^\varepsilon\|_{H^s(\mathbb{R})} \leq M_0$ are given, then the IVP for the (3.8), (3.9) has a unique solution $U^\varepsilon \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R}))$ such that $\|U^\varepsilon\|_{H^s} \leq 2M_0$;*

(ii) if $U^\varepsilon = U_{(0)}^\varepsilon, U_{(1)}^\varepsilon$ are the solutions corresponding with $G^\varepsilon = G_{(0)}^\varepsilon, G_{(1)}^\varepsilon$ for the same initial condition U_0^ε satisfying condition (i), then

$$\|U_{(0)}^\varepsilon - U_{(1)}^\varepsilon\|_{H^{s-2}} \leq c e^{cM_0 T} T \|G_{(0)}^\varepsilon - G_{(1)}^\varepsilon\|_{H^{s-2}}; \tag{3.23}$$

(iii) if $\rho_0^\varepsilon(x) = (a_0^\varepsilon)^2 + (b_0^\varepsilon)^2 > 0$, then $\rho^\varepsilon(x, t) > 0$ for all $t \geq 0$; if ρ_0^ε has a compact support, then $\rho^\varepsilon(\cdot, t)$ does too for any $t \in [0, T]$ and

$$R\{\rho^\varepsilon(\cdot, t)\} \leq R\{\rho_0^\varepsilon\} + (1 + \varepsilon)CT, \tag{3.24}$$

where $R\{u\} \equiv \sup\{|x|: u(x) \neq 0\}$.

Proof. The existence of a solution in a sufficiently short time interval is guaranteed by [16, Theorem II]. It suffices only to find the explicit estimates stated in the theorem. The following procedure is attributed to [16,27] (see also [8,9,14,22]).

(i) For further reference, we ignore the superscripts ε . Given

$$U \in L^\infty([0, T]; H^s(\mathbb{R})) \cap C([0, T]; H^{s-2}(\mathbb{R})),$$

such that $\|U\|_{H^s} \leq 2M_0$, the linear problem

$$\partial_t \tilde{U} + A(U)\partial_x \tilde{U} + G = \frac{\varepsilon}{2}L(\tilde{U}), \quad \tilde{U}(x, 0) = U_0^\varepsilon(x) \tag{3.25}$$

has a unique solution $\tilde{U} \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R}))$. Multiplying (3.25) by the matrix S then taking the inner product with \tilde{U} and integrating over \mathbb{R} yields

$$\frac{d}{dt} \|\tilde{U}\|_E = \int_{\mathbb{R}} \langle A_x(U)\tilde{U}, \tilde{U} \rangle dx + \int_{\mathbb{R}} \langle SG\tilde{U}, \tilde{U} \rangle dx + \varepsilon \int_{\mathbb{R}} \langle SL(\tilde{U})\tilde{U}, \tilde{U} \rangle dx,$$

where $\|\tilde{U}\|_E \equiv \int_{\mathbb{R}} \langle S\tilde{U}, \tilde{U} \rangle dx$ is the canonical energy associated with (3.25). The term $\int_{\mathbb{R}} \langle SL(\tilde{U})\tilde{U}, \tilde{U} \rangle dx = 0$ contributes nothing to the estimate, by the antisymmetry of L . This means that the singular perturbation does not create energy. We assume that the matrix A together with its derivatives of any desired order are continuous and bounded uniformly in $[0, T] \times \mathbb{R}$. Since $\|\partial_x A(U)\|_\infty \leq \|\partial_x A(U)\|_{L^2} \leq C_1 M_0$ and $\|S\|_\infty \leq c_2 \equiv \max\{1, \alpha\}$, we have

$$\max_{0 \leq t \leq T} \|\tilde{U}(t)\|_{L^2} \leq \left(\|\tilde{U}_0^\varepsilon\|_{L^2} + \int_0^T \|SG\| dt \right) e^{C_1 M_0 T} \leq (M_0 + c_2 MT) e^{C_1 M_0 T}.$$

The estimates of the higher derivatives of \tilde{U} can be obtained in the same manner. We write $\tilde{U}^{(\nu)} = \partial_x^\nu \tilde{U}$, $\nu \geq 1$. Then

$$\tilde{U}^{(\nu)} \in C([0, T]; L^2(\mathbb{R})) \cap C^1([0, T]; H^{-2}(\mathbb{R})),$$

satisfies

$$\partial_t \tilde{U}^{(\nu)} + A(U)\partial_x \tilde{U}^{(\nu)} + G^{(\nu)} = \frac{\varepsilon}{2}L(\tilde{U}^{(\nu)}), \tag{3.26}$$

where

$$G^{(\nu)} = \partial_x^\nu G - [\partial_x^\nu (A(U)\partial_x \tilde{U}) - (A(U)\partial_x^\nu \partial_x \tilde{U})] = \partial_x^\nu G - [\partial_x^\nu, A(U)]\partial_x \tilde{U}. \tag{3.27}$$

The commutator $[\partial_x^\nu, A(U)]\partial_x \tilde{U}$ consists of terms of the form $\partial^\alpha A_x \cdot \partial^\beta \partial_x \tilde{U}$, $\alpha + \beta \leq \nu - 1 \leq s - 2$. Since $\partial_x A, \partial_x \tilde{U} \in H^s(\mathbb{R})$, we can apply the Moser-type calculus inequality [8,16,27] to estimate the commutator terms:

$$\|\partial^\alpha \partial_x A \cdot \partial^\beta \partial_x \tilde{U}\|_{L^2} \leq C \|\partial_x A\|_{H^s} \|\partial_x \tilde{U}\|_{H^s} \leq CM_0^2, \tag{3.28}$$

provided that $\|\tilde{U}\|_{H^s} \leq 2M_0$. (Note that $\|A(U)\|_{H^s} \leq C\|U\|_{H^s}$.) Thus we have

$$\|G^{(\nu)}\|_{L^2} \leq \|\partial_x^\nu G\|_{L^2} + C_2 M_0^2, \tag{3.29}$$

as long as $\|U\|_{H^s} \leq 2M_0$. This implies

$$\|\tilde{U}\|_{H^s} \leq (M_0 + (C_3 M_0^2 + M)T)e^{C_3 M_0 T} \leq 2M_0, \tag{3.30}$$

for $0 \leq t \leq T$ provided that T is sufficiently small that the last inequality holds. Moreover, from Eq. (3.9) we have the estimate of the time derivative $\partial_t \tilde{U}$

$$\max_{0 \leq t \leq T} \|\partial_t \tilde{U}\|_{H^{s-2}} = \left\| -A(U)\partial_x \tilde{U} + \frac{\varepsilon}{2} L(\tilde{U}) \right\|_{H^{s-2}} \leq C_4 M_0^2 + M = L. \tag{3.31}$$

Note that unlike solutions to the quasilinear hyperbolic system considered in [7,16,27], $\partial_t \tilde{U}^\varepsilon$ is only H^{s-2} and not H^{s-1} due to the presence of the higher order term $L(\tilde{U}^\varepsilon)$ in (3.25). It is interesting to mention that the function space $C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R}))$ is natural from the point of view of dimensional analysis. Both function spaces have the same dimension $\frac{2}{\infty} + \frac{n}{2} - s = \frac{2}{\infty} - 2 + \frac{n}{2} - (s - 2)$. Here we use the fact that the time dimension is 2 other than 1 comparing with the pure quasilinear symmetric hyperbolic system.

Now we consider the fundamental set

$$X = \{U \in L^\infty([0, T]; H^s(\mathbb{R})) \cap C([0, T]; H^{s-2}(\mathbb{R})) : \|U\|_{H^s} \leq 2M_0, \\ \|U(t_1) - U(t_0)\|_{H^{s-2}} \leq L|t_1 - t_0|\},$$

and the mapping $F : U \mapsto \tilde{U}$. Thus we will show the contraction in the lower norm. This is the seminal ideal of Lax and Kato. We have already shown that F maps X into X itself. To apply the fixed point theorem, we make X into a complete metric space with the metric

$$d(U_{(1)}, U_{(0)}) = \|U_{(1)} - U_{(0)}\| \equiv \sup_{0 \leq t \leq T} \|U_{(1)}(t) - U_{(0)}(t)\|_{L^2}. \tag{3.32}$$

We are going to show that F is a contraction if T is sufficiently small. The perturbation $\delta \tilde{U} = \tilde{U}_{(1)} - \tilde{U}_{(0)}$ solves

$$(\partial_t + A(U_{(0)})\partial_x)\delta \tilde{U} = f + \frac{\varepsilon}{2} L(\delta \tilde{U}), \quad \delta \tilde{U}(x, 0) = 0, \tag{3.33}$$

where

$$f = (A(U_{(1)}) - A(U_{(0)}))\partial_x \tilde{U}_{(1)}. \tag{3.34}$$

Keeping in mind that $A(U)$ is linear in U , we see that

$$\|f\|_{L^2} \leq C_5 \|\delta U\| \|\tilde{U}_{(1)}\| \leq 2C_5 M_0 \|\delta U\|.$$

Thus $\|\delta \tilde{U}\| \leq 2C_5 M_0 \|\delta U\| T e^{C_3 M_0 T}$. Therefore, if $2C_5 M_0 T e^{C_3 M_0 T} < 1$, then F is a contraction, so has a fixed point in X , which belongs to the required space $C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R}))$ and solves (3.8), (3.9). This complete the proof of (i).

(ii) The equation satisfied by the perturbation $\delta U = U_{(1)} - U_{(0)}$ is

$$(\partial_t + A(U_{(0)})\partial_x + B(\partial_x U_{(1)}))\delta U + \delta G = \frac{\varepsilon}{2} L(\delta U), \quad \delta U(x, 0) = 0, \tag{3.35}$$

where $\delta G = G_{(1)} - G_{(0)}$ and

$$B(\partial_x U_{(1)}) = \frac{A(U_{(1)}) - A(U_{(0)})}{U_{(1)} - U_{(0)}} \partial_x U_{(1)} = A'((1 - \theta)U_{(1)} + \theta U_{(0)})\partial_x U_{(1)}. \tag{3.36}$$

Keeping in mind that $A(U_{(0)}) \in C([0, T]; H^s(\mathbb{R}))$ with $\|A(U_{(0)})\|_{H^s} \leq C_6 M_0$, and $B(\partial_x U_{(1)}) \in C([0, T]; H^{s-1}(\mathbb{R}))$ with $\|B(\partial_x U_{(1)})\|_{H^{s-1}} \leq C_6 M_0$. Let $v \leq s$. Then $\delta U^{(v)} = \partial_x^v \delta U$ solves

$$(\partial_t + A\partial_x + B)\delta U^{(v)} = h^{(v)} + \frac{\varepsilon}{2} L(\delta U^{(v)}), \tag{3.37}$$

where

$$h^{(v)} = \partial_x^v \delta G - [\partial_x^v, A]\partial_x(\delta U) - [\partial_x^v, B]\delta U. \tag{3.38}$$

Here the bracket $[\cdot, \cdot]$ denotes the commutator. For $v \geq 1$, $[\partial_x^v, A]\partial_x(\delta U)$ consists of terms of the form $\partial_x^\alpha A_x \cdot \partial_x^\beta \partial_x(\delta U)$, $\alpha + \beta \leq s - 1$ with

$$\|\partial_x^\alpha A_x \cdot \partial_x^\beta \partial_x(\delta U)\|_{L^2} \leq C \|A_x\|_{H^s} \|\delta U\|_{H^s} \leq C M_0 \|\delta U\|_{H^s}, \tag{3.39}$$

and $[\partial_x^v, B]\delta U$ consists of terms of the form $\partial_x^\alpha B_x \cdot \partial_x^\beta \delta U$, $\alpha + \beta \leq s - 1$ with

$$\|\partial_x^\alpha B_x \cdot \partial_x^\beta \delta U\|_{L^2} \leq C \|A\|_{H^s} \|\delta U\|_{H^s} \leq C M_0 \|\delta U\|_{H^s}. \tag{3.40}$$

Thus

$$\|h^{(v)}\|_{L^2} \leq \|\partial_x^v \delta G\| + C_7 M_0 \|\delta U\|_{H^s}. \tag{3.41}$$

This implies

$$\|\delta U\|_{H^s} \leq C_8 \|\delta G\|_{H^s} T e^{C_8 M_0 T}, \tag{3.42}$$

which completes the proof of (ii).

(iii) To show that $\rho^\varepsilon(x, t) = (a^\varepsilon(x, t))^2 + (b^\varepsilon(x, t))^2 > 0$ for all $0 \leq t < \infty$, we employ the continuity equation (3.14) for ρ^ε . Let (η, τ) be an arbitrary fixed space–time point in $\Omega \times [0, T]$.

Since $v^\varepsilon + \varepsilon\theta_x^\varepsilon \in C^1(\mathbb{R} \times [0, T])$, the well-known theorem for ordinary differential equations guarantees that the problem

$$\frac{dx}{dt} = v^\varepsilon(x, t) + \varepsilon\partial_x\theta^\varepsilon(x, t), \quad x|_{t=\tau} = \eta, \quad (3.43)$$

has a unique solution $x = \Psi(t) \in C^1([0, T]; \mathbb{R})$. The continuity equation implies

$$\frac{d}{dt}\rho^\varepsilon(\Psi(t), t) = -\partial_x(v^\varepsilon + \varepsilon\partial_x\theta^\varepsilon)\rho^\varepsilon(\Psi(t), t).$$

Integrating over $[0, \tau]$, we have

$$\rho^\varepsilon(\eta, \tau) = \rho^\varepsilon(\Psi(0), 0) \exp\left[-\int_0^\tau \partial_x(v^\varepsilon(\Psi(t), t) + \varepsilon\partial_x\theta^\varepsilon(\Psi(t), t)) dt\right].$$

Thus $\rho^\varepsilon(\eta, \tau) \geq 0$, if $\rho^\varepsilon(\Psi(0), 0) = \rho_0^\varepsilon(\Psi(0)) \geq 0$. If $\rho^\varepsilon(\eta, \tau) \neq 0$, then $\rho_0^\varepsilon(\Psi(0)) \neq 0$ so that $|\Psi(0)| \leq R\{\rho_0^\varepsilon\}$, and

$$\begin{aligned} |\eta| &= |\Psi(\tau)| = \left| \Psi(0) + \int_0^\tau v^\varepsilon(\Psi(t), t) + \varepsilon\nabla\theta^\varepsilon(\Psi(t), t) dt \right| \\ &\leq |\Psi(0)| + \int_0^\tau \|v^\varepsilon\|_\infty + \varepsilon\|\nabla\theta^\varepsilon\|_\infty dt \leq R\{\rho_0^\varepsilon\} + (1 + \varepsilon)C_2\tau. \end{aligned}$$

In order to complete the proof of the theorem, we only need to show that $G^\varepsilon \in L^\infty([0, T]; H^s(\mathbb{R})) \cap C([0, T]; H^{s-2}(\mathbb{R}))$ such that $\|G^\varepsilon\|_{H^s} \leq M$, which is equivalent to show $V^\varepsilon \in L^\infty([0, T]; H^{s+1}(\mathbb{R})) \cap C([0, T]; H^{s-1}(\mathbb{R}))$. Indeed, it follows immediately from the conservation laws (2.19), (2.20) and the assumption $\lambda > 0$ that $V^\varepsilon \in L^\infty([0, T]; L^1 \cap L^2(\mathbb{R}))$ if $V_0^\varepsilon(x) \in L^1 \cap L^2(\mathbb{R})$. Similarly for higher derivative we have $V^\varepsilon \in L^\infty([0, T]; W^{s,1} \cap H^s(\mathbb{R}))$ if $V_0^\varepsilon(x) \in W^{s,1} \cap H^s(\mathbb{R})$. However, the assumption $\lambda > 0$ can be overcome by employing the explicit representation (2.8') or (3.8) of V^ε . Indeed, by Cauchy–Schwarz and Minkowski's integral inequalities and the imbedding $H^1 \hookrightarrow L^2$ we have

$$\begin{aligned} \|V^\varepsilon(t)\|_{L^2} &\leq \|V_0^\varepsilon\|_{L^2} + 2|\lambda| \left(\int_{\mathbb{R}} \left| \int_0^t (a^\varepsilon\partial_x a^\varepsilon + b^\varepsilon\partial_x b^\varepsilon) d\tau \right|^2 dx \right)^{1/2} \\ &\leq \|V_0^\varepsilon\|_{L^2} + 2|\lambda| \int_0^t \left(\int_{\mathbb{R}} (a^\varepsilon\partial_x a^\varepsilon + b^\varepsilon\partial_x b^\varepsilon)^2 dx \right)^{1/2} d\tau \\ &\leq \|V_0^\varepsilon\|_{L^2} + 2|\lambda|T(\|a^\varepsilon\|_{H^1}^2 + \|b^\varepsilon\|_{H^1}^2). \end{aligned}$$

Similarly, for higher derivatives

$$\|V^\varepsilon(t)\|_{H^s} \leq \|V_0^\varepsilon\|_{H^s} + 2|\lambda|T(\|a^\varepsilon\|_{H^{s+1}}^2 + \|b^\varepsilon\|_{H^{s+1}}^2).$$

The same computation also shows that V^ε satisfies for any $0 \leq t_1 < t_2 \leq T$,

$$\|V^\varepsilon(t_2) - V^\varepsilon(t_1)\|_{L^2} \leq |\lambda|C \int_{t_1}^{t_2} \|a^\varepsilon(\tau)\|_{H^1}^2 + \|b^\varepsilon(\tau)\|_{H^1}^2 d\tau \quad \text{and}$$

$$\|V^\varepsilon(t_2) - V^\varepsilon(t_1)\|_{H^{s-1}} \leq |\lambda| \int_{t_1}^{t_2} \|a^\varepsilon(\tau)\|_{H^s}^2 + \|a^\varepsilon(\tau)\|_{H^s}^2 d\tau.$$

Since $\rho^\varepsilon \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R}))$, thus the above inequality implies that $V^\varepsilon \in \text{Lip}([0, T]; H^{s-1}(\mathbb{R}))$. Indeed, we can prove that $V^\varepsilon \in C^1([0, T]; H^{s-1}(\mathbb{R}))$. \square

Theorem 3.2. Assume A_0^ε, S_0 and V_0^ε in $H^s(\mathbb{R})$, $s > 5/2$ then solutions $(\psi^\varepsilon, V^\varepsilon)$ of the (1.1)–(1.4) exist on a small time interval $[0, T]$, T independent of ε . Moreover, $\psi^\varepsilon(x, t) = A^\varepsilon(x, t)e^{iS^\varepsilon(x,t)/\varepsilon}$, with $A^\varepsilon, S^\varepsilon$ and V^ε in $L^\infty([0, T]; H^s)$ uniformly in ε , and $(\rho^\varepsilon, S_x^\varepsilon, V^\varepsilon)$, with $\sqrt{\rho^\varepsilon} = A^\varepsilon$, converges to the solution (ρ, u, V) of (2.22)–(2.25).

Proof. Since $A^\varepsilon = a^\varepsilon + ib^\varepsilon$ and $v^\varepsilon = \partial_x S^\varepsilon$, it follows from the theorem that

$$\begin{aligned} A^\varepsilon &\in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R})), \\ S^\varepsilon &\in C([0, T]; H^{s+1}(\mathbb{R})) \cap C^1([0, T]; H^s(\mathbb{R})), \end{aligned}$$

and thus

$$A^\varepsilon \in C^1([0, T] \times \mathbb{R}) \cap C^1([0, T]; C^2(\mathbb{R})), \quad S^\varepsilon \in C^1([0, T]; C^2(\mathbb{R})).$$

For classical solutions, (1.1)–(1.4) is equivalent to the dispersive quasilinear hyperbolic system (3.4)–(3.8). Applying this equivalent relation, the theorem follows immediately by Theorem 3.1. \square

The limiting system of (3.4)–(3.8) or (3.9), (3.10) is the quasilinear symmetric hyperbolic system (formally letting $\varepsilon \rightarrow 0$)

$$\partial_t U + A(U)\partial_x U + G = 0, \quad U(x, 0) = U_0(x), \tag{3.44}$$

$$V(x, t) + 2\lambda \int_0^t (a\partial_x a + b\partial_x b) d\tau = V_0(x), \tag{3.45}$$

which is equivalent to (2.27)–(2.30) as long as the solutions are smooth. As an immediate consequence, we also prove the existence and uniqueness of the local smooth solutions to the system (2.27)–(2.30).

Theorem 3.3. *Assume the hypothesis of Theorem 3.1. Given initial datum $U_0^\varepsilon, U_0 \in H^s(\mathbb{R})$ and $U_0^\varepsilon(x)$ converges to $U_0(x)$ in $H^s(\mathbb{R})$ as $\varepsilon \rightarrow 0$. Let $[0, T]$ be the fixed interval determined in Theorem 3.1. Then as $\varepsilon \rightarrow 0$, there exists $U \in L^\infty([0, T]; H^s(\mathbb{R}))$ and $V \in C^1([0, T]; H^{s-1}(\mathbb{R}))$ such that for all $\sigma > 0$*

$$U^\varepsilon \rightarrow U \quad \text{in } C([0, T]; H^{s-\sigma}(\mathbb{R})), \tag{3.46}$$

$$V^\varepsilon \rightarrow V \quad \text{in } C^1([0, T]; H^{s-\sigma-1}(\mathbb{R})). \tag{3.47}$$

The function $U(x, t)$ belongs to $C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}))$ and is a classical solution of (2.27)–(2.30).

Proof. Since U^ε is bounded in $C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{R}))$, by the Arzela–Ascoli theorem (applied in the time variable), the Rellich compactness theorem (applied in the space variable) and interpolation, we have that for every sequence of ε 's tending to 0, $\{U^\varepsilon\}_\varepsilon$ has a subsequence that converges in $C([0, T]; H_{\text{loc}}^{s-\sigma}(\mathbb{R}))$ for $\sigma > 0$, to U . Furthermore, from (3.25) the convergence takes place as well in $C^1([0, T]; H_{\text{loc}}^{s-2-\sigma}(\mathbb{R}))$. Since $U_0^\varepsilon(x)$ converges strongly to $U_0(x)$ in $H^s(\mathbb{R})$, this limiting solution has initial data $U_0(x)$. Also $L(U^\varepsilon)$ is uniformly bounded in $H^s(\mathbb{R})$ therefore the perturbation $\frac{\varepsilon}{2}L(U^\varepsilon)$ tends to zero as $\varepsilon \rightarrow 0$. Thus the sequence $\{U^\varepsilon\}$ converges to a solution of the quasilinear hyperbolic system (3.44). Also, after extraction of a subsequence, the above limit converges weakly in $H^s(\mathbb{R})$. Therefore, by the identity of weak and strong limits, $U^\varepsilon \in L^\infty([0, T]; H^s(\mathbb{R})) \cap AC([0, T]; H^{s-2}(\mathbb{R}))$. Since the system admits a unique solution, it then follows that the convergence to U takes place without passing to the subsequence. \square

To ensure the strong convergence of $(\psi^\varepsilon, V^\varepsilon)$ to a classical solution of the (3.44), (3.45) (or equivalently (2.27)–(2.30)), we require the hypothesis that the solution sequences are near the system (2.27)–(2.30) initially. It means that the regularity of solutions of (2.27)–(2.30) controls that of solutions to (1.1)–(1.4).

Theorem 3.4. *Let $T > 0$ be arbitrary and (ρ_0, μ_0, V_0) be such that the IVP of (2.27)–(2.30) has a classical solution (ρ, μ, V) . Then, there is a critical value of $\varepsilon, \varepsilon_c$ and a constant $C > 0$ such that under the hypotheses:*

- (1) A_0^ε and V_0^ε converge strongly respectively to A_0 and V_0 in $H^s(\mathbb{R})$ and $H^{s-1}(\mathbb{R})$ as ε tends to 0,
- (2) $\|\rho_0\|_{H^s} < C, \|\mu_0\|_{H^s} < C$ and $\|V_0\|_{H^{s-1}} < C, s > 3,$
- (3) $0 < \varepsilon < \varepsilon_c,$

the IVP for (1.1)–(1.4) has a unique classical solution $(\psi^\varepsilon, V^\varepsilon)$ on $[0, T]$ with $\psi^\varepsilon(x, t) = A^\varepsilon(x, t) \exp(\int_\varepsilon^t S^\varepsilon(x, t))$. Moreover, A^ε and S_x^ε are bounded in $L^\infty([0, T]; H^s(\mathbb{R}))$ and V^ε is bounded in $C^1([0, T]; H^{s-1}(\mathbb{R}))$ uniformly in ε .

The proof is standard by considering the difference of the two systems and then applying the energy estimate [8,27,29]. We therefore omit the details.

4. Weak solutions and scattering sound

In this section, we consider the semiclassical (WKB) limit of (1.1) directly. For the cubic nonlinear Schrödinger equation, it has been studied by Colin and Soyeur [5] for the case when there are no vortices, and by Lin and Xin [25] when there are vortices in two space dimensions. For the derivative nonlinear Schrödinger equation, we will refer to [9,24].

We multiply (1.1) by $\bar{\psi}^\varepsilon$ and its complex conjugate by ψ^ε , and subtract the latter from the former to obtain the conservation of mass in terms of the wave function:

$$\partial_t (|\psi^\varepsilon|^2) + \partial_x \left(\frac{i\varepsilon}{2} (\psi^\varepsilon \partial_x \bar{\psi}^\varepsilon - \bar{\psi}^\varepsilon \partial_x \psi^\varepsilon) \right) = 0, \tag{4.1}$$

which is the same as (2.6). We rewrite (4.1) as

$$\partial_t \left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \right) + \partial_x W(\psi^\varepsilon) = 0, \tag{4.2}$$

where the linear momentum W is defined by

$$W(\psi^\varepsilon) = W(\psi^\varepsilon, \partial_x \psi^\varepsilon) \equiv \frac{i}{2} (\psi^\varepsilon \partial_x \bar{\psi}^\varepsilon - \bar{\psi}^\varepsilon \partial_x \psi^\varepsilon). \tag{4.3}$$

In the sequel, we assume $\lambda > 0$ satisfying $\lambda = o(\varepsilon)$ and the initial data are taken in such way that

$$\frac{V_0^\varepsilon}{\varepsilon} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \quad W(\psi_0^\varepsilon) \in L^1(\mathbb{R}), \tag{4.4}$$

$$\psi_0^\varepsilon \in H^1(\mathbb{R}), \quad \frac{|\psi_0^\varepsilon|^2 - 1}{\varepsilon} \in L^2(\mathbb{R}), \tag{4.5}$$

then the conservation laws (2.17)–(2.21) imply

$$\int_{-\infty}^{\infty} \frac{V^\varepsilon}{\varepsilon} dx \leq C_0, \quad \int_{-\infty}^{\infty} W(\psi^\varepsilon) + \frac{1}{2\lambda\varepsilon} |V^\varepsilon|^2 dx \leq C_1, \tag{4.6}$$

$$\int_{-\infty}^{\infty} \frac{1}{2} |\partial_x \psi^\varepsilon|^2 + \frac{\alpha}{2} \left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \right)^2 + \frac{1}{\varepsilon^2} |\psi^\varepsilon|^2 V^\varepsilon dx \leq C_2, \tag{4.7}$$

where C_0, C_1 and C_2 are constants independent of ε . The above bounds imply

$$\psi^\varepsilon \text{ is bounded in } L^\infty([0, T]; H^1(\mathbb{R})), \tag{4.8}$$

$$\partial_t \psi^\varepsilon \text{ is bounded in } L^\infty([0, T]; H^{-1}(\mathbb{R})), \tag{4.9}$$

$$\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \text{ is bounded in } L^\infty([0, T]; L^2(\mathbb{R})), \tag{4.10}$$

$$W(\psi^\varepsilon) \text{ is bounded in } L^\infty([0, T]; L^1(\mathbb{R})), \tag{4.11}$$

for the short wave part and similarly for the long wave part we have

$$\frac{V^\varepsilon}{\varepsilon} \text{ is bounded in } L^\infty([0, T]; L^2(\mathbb{R}) \cap L^1(\mathbb{R})), \tag{4.12}$$

$$\partial_t \left(\frac{V^\varepsilon}{\varepsilon} \right) \text{ is bounded in } L^\infty([0, T]; H^{-1}(\mathbb{R}) + W^{-1,\infty}(\mathbb{R})), \tag{4.13}$$

and thus by interpolation

$$\frac{V^\varepsilon}{\varepsilon} \text{ is bounded in } L^\infty([0, T]; L^p(\mathbb{R})), \tag{4.14}$$

$$\partial_t \left(\frac{V^\varepsilon}{\varepsilon} \right) \text{ is bounded in } L^\infty([0, T]; W^{-1,q}(\mathbb{R})), \tag{4.15}$$

for $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$. Although it holds for all $1 \leq p \leq 2$, but $p = 2$ is good enough for our discussion. Moreover, from (4.10)

$$|\psi^\varepsilon|^2 \rightarrow 1 \text{ strongly in } L^2(\mathbb{R}) \text{ and a.e.} \tag{4.16}$$

It follows from these bounds that $\{\psi^\varepsilon\}_\varepsilon$ is strongly compact in $C([0, T]; L^2(\mathbb{R}))$ and weakly compact in $L^\infty([0, T]; H^1(\mathbb{R}))$, and from the classical compactness arguments there exists a subsequence still denoted $\{\psi^\varepsilon\}_\varepsilon$ and a function $\psi \in L^\infty([0, T]; H^1(\mathbb{R}))$ such that

$$\psi^\varepsilon \rightarrow \psi \text{ strongly in } C([0, T]; L^2(\mathbb{R})), \tag{4.17}$$

$$\psi^\varepsilon \rightharpoonup \psi \text{ weakly* in } L^\infty([0, T]; H^1(\mathbb{R})). \tag{4.18}$$

Similarly, there exists $V \in L^\infty([0, T]; L^2(\mathbb{R}))$ such that

$$\frac{V^\varepsilon}{\varepsilon} \rightarrow V \text{ strongly in } C([0, T]; H^{-\eta}(\mathbb{R})), \quad 0 < \eta < 1, \tag{4.19}$$

$$\frac{V^\varepsilon}{\varepsilon} \rightharpoonup V \text{ weakly* in } L^\infty([0, T]; L^2(\mathbb{R})). \tag{4.20}$$

We claim that $\psi^\varepsilon \frac{V^\varepsilon}{\varepsilon}$ converges to ψV in $\mathcal{D}'((0, T) \times \mathbb{R})$. It is easily seen that $\psi^\varepsilon \in C([0, T]; L^2(\mathbb{R}))$ implies that $\psi^\varepsilon \in L^2([0, T] \times (-M, M))$ for any $M > 0$. Indeed, if g is in $\mathcal{D}'((0, T) \times \mathbb{R})$ with compact support Ω , ψ^ε converges strongly in $L^2(\Omega)$ and since $\frac{V^\varepsilon}{\varepsilon}$ converges weakly to V in $L^2([0, T]; L^2(\Omega))$, then

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Omega} \psi^\varepsilon(x, t) \frac{V^\varepsilon(x, t)}{\varepsilon} g(x, t) dx dt = \iint_{\Omega} \psi(x, t) V(x, t) g(x, t) dx dt, \tag{4.21}$$

which proves the claim. Also from (4.2), (4.5) and (4.17), we have

$$\frac{|\psi^\varepsilon(x, t)|^2 - 1}{\varepsilon} = \frac{|\psi_0^\varepsilon(x)|^2 - 1}{\varepsilon} - \int_0^t \partial_x W(\psi^\varepsilon) d\tau \rightharpoonup - \int_0^t \partial_x W(\psi) d\tau, \tag{4.22}$$

in the sense of distributions. We rewrite (1.2) as

$$\partial_t \left(\frac{V^\varepsilon}{\varepsilon} \right) + \frac{\lambda}{\varepsilon} \partial_x (|\psi^\varepsilon|^2) = 0, \tag{4.23}$$

then using the fact that $\lambda = o(\varepsilon)$ and $|\psi^\varepsilon|^2 \rightarrow 1$ a.e., we can conclude that $\partial_t V = 0$ in the sense of distribution thus

$$V = \lim_{\varepsilon \rightarrow 0} \frac{V^\varepsilon(x, t)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{V_0^\varepsilon(x)}{\varepsilon} = V_0(x). \tag{4.24}$$

Therefore, using the above compactness results, $\lambda = o(\varepsilon)$ and the fact that $|\psi^\varepsilon|^2 \rightarrow 1$ a.e. again, as ε tends to zero in (1.1), we have

$$i \partial_t \psi + \alpha \psi \int_0^t \partial_x W(\psi) d\tau - V_0(x) \psi = 0, \tag{4.25}$$

in the sense of distribution. Since $|\psi| = 1$, $\bar{\psi} \partial_x \psi + \psi \partial_x \bar{\psi} = 0$ and

$$W(\psi) = \frac{i}{2} (\psi \partial_x \bar{\psi} - \bar{\psi} \partial_x \psi) = -i \frac{\partial_x \psi}{\psi} = -i \partial_x (\log \psi),$$

(4.25) becomes

$$i \partial_t \psi - \left(i \alpha \int_0^t \partial_{xx} (\log \psi) d\tau + V_0(x) \right) \psi = 0. \tag{4.26}$$

Differentiating (4.26) with respect to t once, we can then derive the wave map equation

$$\partial_{tt} \psi - \alpha \partial_{xx} \psi = (\alpha |\partial_x \psi|^2 - |\partial_t \psi|^2) \psi, \quad |\psi| = 1 \text{ a.e.} \tag{4.27}$$

with initial data

$$\psi(x, 0) = \psi_0(x), \quad i \partial_t \psi(x, 0) = V_0(x) \psi_0(x). \tag{4.28}$$

Using the fact $|\psi| = 1$ again, writing $\psi = e^{i\theta}$ shows

$$\partial_{tt} \theta - \alpha \partial_{xx} \theta = 0, \quad \mathcal{D}'([0, T] \times \mathbb{R}), \tag{4.29}$$

i.e., θ is a distribution solution of the linear wave equation. Moreover, $\theta(x, t) \in H^1(\mathbb{R})$ implies that θ is the unique weak solution of (4.28) with finite energy. Thus we have proved the following theorem.

Theorem 4.1. *Let the positive parameter λ be small order of ε , $\lambda = o(\varepsilon)$ and (4.4), (4.5) be satisfied uniformly in ε . Assume $\psi_0^\varepsilon \rightarrow \psi_0$ in $L^2(\mathbb{R})$, $|\psi_0| = 1$ a.e. and $V_0^\varepsilon/\varepsilon \rightarrow V_0$ in $L^2(\mathbb{R})$, then denoting by $(\psi^\varepsilon, V^\varepsilon)$ a weak solution of (1.1)–(1.4); we have $\psi^\varepsilon \rightarrow \psi$ strongly in $C([0, T]; L^2(\mathbb{R}))$, $V^\varepsilon/\varepsilon \rightharpoonup V \equiv V_0$ weakly* in $L^\infty([0, T]; L^2(\mathbb{R}))$, $\psi^\varepsilon \rightharpoonup \psi$ weakly in $L^\infty([0, T]; H^1(\mathbb{R}))$ and $V^\varepsilon/\varepsilon \rightarrow V \equiv V_0$ strongly in $C([0, T]; H^{-\eta}(\mathbb{R}))$, then ψ satisfies*

$$\begin{aligned} \partial_{tt}\psi - \alpha \partial_{xx}\psi &= (\alpha |\partial_x \psi|^2 - |\partial_t \psi|^2)\psi, \quad |\psi| = 1 \text{ a.e.}, \\ \psi(x, 0) &= \psi_0(x), \quad \partial_t \psi(x, 0) = -iV_0(x)\psi_0(x), \end{aligned}$$

or equivalently $\psi = e^{i\theta}$ with the phase function θ satisfying the wave equation

$$\partial_{tt}\theta - \alpha \partial_{xx}\theta = 0.$$

Remarks. (1) The potential V_0 disappears in (4.27) because it is stationary, $V_0 = V_0(x)$. However, from (4.25) or (4.26) the initial data need to satisfy the compatibility condition $i \partial_t \psi(x, 0) = V_0(x)\psi_0(x)$ which shows the long wave effect. For cubic defocussing NLS equation it vanishes, $\partial_t \psi(x, 0) = 0$ due to the lack of extra potential (see [5,25]).

(2) It is straightforward to pass to the limit from (1.1) because we have the compactness of the long wave $\{V^\varepsilon/\varepsilon\}_\varepsilon$. However we can also consider the limit procedure from (1.5). Since

$$\{V^\varepsilon(\cdot, t)/\varepsilon\}_\varepsilon \text{ is weakly compact in } L^2(\mathbb{R}) \text{ for } t \in [0, T],$$

which according to (1.2) is equivalent to

$$\left\{ \int_0^t \partial_x (|\psi^\varepsilon(x, \tau)|^2/\varepsilon) d\tau \right\}_\varepsilon \text{ is weakly compact in } L^2(\mathbb{R}) \text{ for } t \in [0, T].$$

We also have

$$\{\psi^\varepsilon(\cdot, t)\}_\varepsilon \text{ is strongly compact in } L^2(\mathbb{R}) \text{ for } t \in [0, T].$$

We deduce from the above two statements that

$$\left\{ \left[\int_0^t \partial_x (|\psi^\varepsilon(x, \tau)|^2/\varepsilon) d\tau \right] \psi^\varepsilon(\cdot, t) \right\}_\varepsilon \text{ is weakly compact in } L^1(\mathbb{R}) \text{ for } t \in [0, T].$$

Therefore $\{[\int_0^t \partial_x (|\psi^\varepsilon(x, \tau)|^2/\varepsilon) d\tau] \psi^\varepsilon\}_\varepsilon$ is uniformly bound in $L^\infty([0, T]; L^1(\mathbb{R}))$, hence uniformly bounded in $L^1_{loc}(\mathbb{R}^+; L^1(\mathbb{R}))$. Since $\lambda = o(\varepsilon)$ we can conclude that the nonlocal term $-\lambda[\int_0^t \partial_x (|\psi^\varepsilon(x, \tau)|^2/\varepsilon) d\tau] \psi^\varepsilon$ in (1.5) will tend to zero as $\varepsilon \rightarrow 0$.

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