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The objective of packing problems is to determine an optimal way of placing a given set of three-dimensional (3D) rectangular cartons within a minimum volume 3D rectangular container. Current packing optimization methods either use too many extra 0–1 variables or find it difficult to obtain a globally optimal solution. This study proposes an efficient method for finding the global optimum of packing problems. First the traditional packing optimization problem is converted into an equivalent program containing many fewer 0–1 variables than contained in current methods. Then the global optimum of the converted program is found by utilizing piecewise linearization techniques. The numerical examples demonstrate that the proposed method is capable of finding the global optimum of a packing problem.

Keywords: Packing; Global optimization; Piecewise linearization

1. Introduction

The objective of packing optimization problems is to seek a minimal volume container which can contain a given set of small three-dimensional (3D) rectangular cartons. All the cartons can be different in size and may be rotated in any orthogonal direction. The problem has many applications in the electronic, manufacturing and distribution industries. Examples include packing several components into a minimal case to form a device, cutting wood or foam rubber into smaller pieces, loading pallets with goods or designing packages. A compact device with a minimal case reduces the manufacturing costs and also increases its competitiveness in the market. Moreover, cutting wood, foam rubber or other materials into smaller pieces with minimal waste leads to lower production costs. An optimal design of a package for packing some specific goods also means high economic relevance in real goods processes. Furthermore, an optimal filling of a container decreases the transportation costs along with ‘side effects’ such as increased traffic activity and negative consequences on environmental resources.

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The packing problem appears in many related studies such as knapsack (Fayard and Zissimopoulos 1995), assortment problems (Beasley 1985, Li and Chang 1998), pallet loading (Liu and Hsiao 1997, Terno *et al.* 2000) and container loading problems (Chen *et al.* 1995, Scheithauer 1999, Bortfeldt and Gehring 2001, Gehring and Bortfeldt 2002, Pisinger 2002, Andreas *et al.* 2003). In addition, researchers have dealt with various approaches to the problem. For instance, Dowsland (1991) proposed a heuristic method for solving 3D packing problems, Chen *et al.* (1995) formulated a mixed integer program for container loading problems, and Li and Chang (1998) developed a method for finding the approximately global optimum of the assortment problem. Additionally, Bortfeldt and Gehring (2001), Gehring and Bortfeldt (2002), Pisinger (2002) and Andreas *et al.* (2003) presented different heuristic algorithms such as the parallel tabu search algorithm, hybrid genetic algorithm and parallel genetic algorithm for solving the container loading problem. Two difficulties of these current methods are now listed. Firstly, methods that utilize bar-relaxation or layer-relaxation techniques (Dowsland 1991, Liu and Hsiao 1997) and heuristic algorithms (Bortfeldt and Gehring 2001, Gehring and Bortfeldt 2002, Pisinger 2002, Andreas *et al.* 2003) may only find locally optimal solutions. Secondly, too many 0–1 variables are used to formulate a packing optimization problem. For instance, Chen *et al.*'s model (Chen *et al.* 1995) contains $3n(n-1)+4n$ 0–1 variables (n is the number of cartons), which might cause a heavy computational burden.

To overcome these difficulties, this study proposes another method to solve packing problems. Comparing with current methods, the proposed method has the following advantages:

(i) It can solve real packing optimization problems without bar-relaxation or layer-relaxation techniques.
(ii) It only uses half the 0–1 variables used in Chen *et al.*'s method (Chen *et al.* 1995) to formulate a packing problem.
(iii) It is guaranteed to find a global optimum of the packing problem within a tolerable error. The generalization to nonlinear objective functions is achieved by the piecewise linearization techniques.

The rest of this article is organized as follows. In the next section, a detailed description of the packing problem is addressed. Then the reformulation problem is proposed to reduce the number of 0–1 variables. The linearization strategy of the nonlinear objective function is discussed in the fourth section. The fifth section presents the solution algorithm. Numerical examples are examined in the sixth section. Finally, some concluding remarks are included.

2. Problem formulation

Given n rectangular cartons with fixed lengths, widths and heights, a packing optimization problem is to allocate these n cartons within a rectangular container which has minimal volume without the need to fix the y and z values in the nonlinear objective function xyz. Denote x, y and z as the width, length and height of the container ($x > 0$, $y > 0$, $z > 0$), respectively; the packing optimization problem is then stated as follows:

Minimize xyz

subject to

1. All of n boxes are non-overlapping.
2. All of n boxes are within the range of x, y and z.

3. $\underline{x} \leq x \leq \bar{x}$, $\underline{y} \leq y \leq \bar{y}$ and $\underline{z} \leq z \leq \bar{z}$ ($\underline{x}$, $\underline{y}$, $\underline{z}$, $\bar{x}$, $\bar{y}$ and $\bar{z}$ are constants and represent the lower and upper bounds of x, y and z, respectively).

The related terminologies used in the packing model, referring to Chen *et al.* (1995), are described below:

(P_i, Q_i, R_i): Parameters indicating the length, width and height of carton i.

(x_i, y_i, z_i): Continuous variables (for location) indicating the coordinates of the front-left-bottom corner of carton i (x_i, y_i and z_i are integers if the given dimensions of cartons are integers).

(l_{xi}, l_{yi}, l_{zi}): Binary variables indicating whether the length of carton i is parallel to the X-axis, Y-axis or Z-axis. For example, the value of l_{xi} is equal to 1 if the length of carton i is parallel to the X-axis; otherwise, it is equal to 0. It is clear that $l_{xi} + l_{yi} + l_{zi} = 1$.

(w_{xi}, w_{yi}, w_{zi}): Binary variables indicating whether the width of carton i is parallel to the X-axis, Y-axis or Z-axis. For example, the value of w_{xi} is equal to 1 if the width of carton i is parallel to the X-axis; otherwise, it is equal to 0. It is clear that $w_{xi} + w_{yi} + w_{zi} = 1$.

(h_{xi}, h_{yi}, h_{zi}): Binary variables indicating whether the height of carton i is parallel to the X-, Y- or Z-axis. For example, the value of h_{xi} is equal to 1 if the height of carton i is parallel to the X-axis; otherwise, it is equal to 0. It is clear that $h_{xi} + h_{yi} + h_{zi} = 1$.

For a pair of cartons (i, k) where $i < k$, there is a set of 0–1 variables $\{A_{ik}, B_{ik}, C_{ik}, D_{ik}, E_{ik}, F_{ik}\}$ defined as:

$A_{ik} = 1$ if carton i is on the left of carton k, otherwise $A_{ik} = 0$.
$B_{ik} = 1$ if carton i is on the right of carton k, otherwise $B_{ik} = 0$.
$C_{ik} = 1$ if carton i is behind carton k, otherwise $C_{ik} = 0$.
$D_{ik} = 1$ if carton i is in front of carton k, otherwise $D_{ik} = 0$.
$E_{ik} = 1$ if carton i is below carton k, otherwise $E_{ik} = 0$.
$F_{ik} = 1$ if carton i is above carton k, otherwise $F_{ik} = 0$.

The front-left-bottom corner of the container is fixed at the origin. The interpretation of these variables is illustrated in figure 1. Figure 1 contains two cartons i and k, where carton i is located with its length parallel to the X-axis and the width parallel to the Z-axis, and carton k is located with its length parallel to the Z-axis and the width parallel to the X-axis. Then l_{xi}, w_{zi},h_{yi}, l_{zk}, w_{xk} and h_{yk} are equal to 1. In addition, since the carton i is located on the left-hand side of and in front of carton k, it is clear that $A_{ik} = D_{ik} = 1$ and $B_{ik} = C_{ik} = E_{ik} = F_{ik} = 0$.

The packing problem can then be formulated below, referring to Chen *et al.* (1995):

Model 1:

$$\text{Minimize } xyz \tag{1}$$

subject to

$$x_i + P_i l_{xi} + Q_i w_{xi} + R_i h_{xi} \leq x_k + (1 - A_{ik})M \quad \text{for all } i, k, i < k, \tag{2}$$

$$x_k + P_k l_{xk} + Q_k w_{xk} + R_k h_{xk} \leq x_i + (1 - B_{ik})M \quad \text{for all } i, k, i < k, \tag{3}$$

$$y_i + P_i l_{yi} + Q_i w_{yi} + R_i h_{yi} \leq y_k + (1 - C_{ik})M \quad \text{for all } i, k, i < k, \tag{4}$$

$$y_k + P_k l_{yk} + Q_k w_{yk} + R_k h_{yk} \leq y_i + (1 - D_{ik})M \quad \text{for all } i, k, i < k, \tag{5}$$

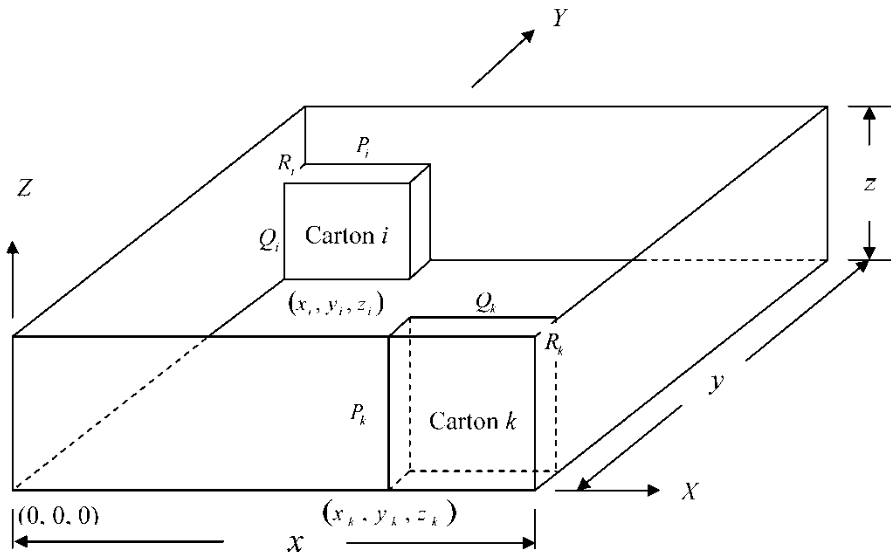

Figure 1. Graphical illustration of variables.

$$z_i + P_i l_{zi} + Q_i w_{zi} + R_i h_{zi} \le z_k + (1 - E_{ik})M \quad \text{for all } i, k, i < k, \tag{6}$$

$$z_k + P_k l_{zk} + Q_k w_{zk} + R_k h_{zk} \le z_i + (1 - F_{ik})M \quad \text{for all } i, k, i < k, \tag{7}$$

$$A_{ik} + B_{ik} + C_{ik} + D_{ik} + E_{ik} + F_{ik} \ge 1 \quad \text{for all } i, k, i < k, \tag{8}$$

$$x_i + P_i l_{xi} + Q_i w_{xi} + R_i h_{xi} \le x \quad \text{for all } i, k, i < k, \tag{9}$$

$$y_i + P_i l_{yi} + Q_i w_{yi} + R_i h_{yi} \le y \quad \text{for all } i, k, i < k, \tag{10}$$

$$z_i + P_i l_{zi} + Q_i w_{zi} + R_i h_{zi} \le z \quad \text{for all } i, k, i < k, \tag{11}$$

where $l_{xi}, l_{yi}, l_{zi}, w_{xi}, w_{yi}, w_{zi}, h_{xi}, h_{yi}, h_{zi}, A_{ik}, B_{ik}, C_{ik}, D_{ik}, E_{ik}$ and F_{ik} are 0–1 variables, $M = \max\{\bar{x}, \bar{y}, \bar{z}\}$, $x_i, y_i, z_i \ge 0$, $0 < \underline{x} \le x \le \bar{x}$, $0 < \underline{y} \le y \le \bar{y}$, $0 < \underline{z} \le z \le \bar{z}$, and $\underline{x}, \underline{y}, \underline{z}, \bar{x}, \bar{y}$ and $\bar{z}$ are constants.

The objective of this model is to minimize the volume of the container. Constraints (2)–(8) are non-overlapping conditions used to ensure that none of these n boxes overlaps with each other. Constraints (9)–(11) guarantee that all boxes are within the enveloping container.

The binary variables, $l_{xi}, l_{yi}, l_{zi}, w_{xi}, w_{yi}, w_{zi}, h_{xi}, h_{yi}$ and h_{zi}, are dependent and have the following relationships:

$$l_{xi} + l_{yi} + l_{zi} = 1 \quad \forall i \in I, \tag{12}$$

$$w_{xi} + w_{yi} + w_{zi} = 1 \quad \forall i \in I, \tag{13}$$

$$h_{xi} + h_{yi} + h_{zi} = 1 \quad \forall i \in I, \tag{14}$$

$$l_{xi} + w_{xi} + h_{xi} = 1 \quad \forall i \in I, \tag{15}$$

$$l_{yi} + w_{yi} + h_{yi} = 1 \quad \forall i \in I, \tag{16}$$

$$l_{zi} + w_{zi} + h_{zi} = 1 \quad \forall i \in I. \tag{17}$$

Constraints (12)–(17) describe the allocation restrictions among logic variables. For instance, (12) implies that the length of carton i is parallel to one of the axes. Constraint (15) implies that only one of the length, the width and the height of carton i is parallel to the

X-axis. Using constraints (12)–(17), the following five variables can be eliminated from the model l_{yi}, w_{xi}, w_{zi}, h_{xi} and h_{yi}. Model 1 is then fully converted into Model 2 below:

Model 2:

$$\text{Minimize } xyz \tag{1}$$

subject to

$$x_i + P_i l_{xi} + Q_i(l_{zi} - w_{yi} + h_{zi}) + R_i(1 - l_{xi} - l_{zi} + w_{yi} - h_{zi}) \leq x_k + (1 - A_{ik})M, \tag{2'}$$

$$x_k + P_k l_{xk} + Q_k(l_{zk} - w_{yk} + h_{zk}) + R_k(1 - l_{xk} - l_{zk} + w_{yk} - h_{zk}) \leq x_i + (1 - B_{ik})M, \tag{3'}$$

$$y_i + P_i(1 - l_{xi} - l_{zi}) + Q_i w_{yi} + R_i(l_{xi} + l_{zi} - w_{yi}) \leq y_k + (1 - C_{ik})M, \tag{4'}$$

$$y_k + P_k(1 - l_{xk} - l_{zk}) + Q_k w_{yk} + R_k(l_{xk} + l_{zk} - w_{yk}) \leq y_i + (1 - D_{ik})M, \tag{5'}$$

$$z_i + P_i l_{zi} + Q_i(1 - l_{zi} - h_{zi}) + R_i h_{zi} \leq z_k + (1 - E_{ik})M, \tag{6'}$$

$$z_k + P_k l_{zk} + Q_k(1 - l_{zk} - h_{zk}) + R_k h_{zk} \leq z_i + (1 - F_{ik})M, \tag{7'}$$

$$A_{ik} + B_{ik} + C_{ik} + D_{ik} + E_{ik} + F_{ik} \geq 1, \tag{8'}$$

$$x_i + P_i l_{xi} + Q_i(l_{zi} - w_{yi} + h_{zi}) + R_i(1 - l_{xi} - l_{zi} + w_{yi} - h_{zi}) \leq x, \tag{9'}$$

$$y_i + P_i(1 - l_{xi} - l_{zi}) + Q_i w_{yi} + R_i(l_{xi} + l_{zi} - w_{yi}) \leq y, \tag{10'}$$

$$z_i + P_i l_{zi} + Q_i(1 - l_{zi} - h_{zi}) + R_i h_{zi} \leq z, \tag{11'}$$

where all variables are the same as defined in Model 1.

Chen *et al.* (1995) solved Model 2 by treating the nonlinear objective function as xy^0z^0 where y^0 and z^0 are fixed values specified by the user. Model 2 then becomes a linear mixed 0–1 program. The following are two disadvantages in Chen *et al.*'s model:

(i) Too many 0–1 variables are included in the constraints of Model 2. This number of 0–1 variables substantially increases the computational effect required to solve the problem.
(ii) Chen *et al.*'s method can only find a local optimum of Model 2 without fixing the y and z values in the objective function xyz.

The next section describes a way to reduce redundant 0–1 variables in Model 2. Then an algorithm for finding the globally optimal solution of the packing problem is developed.

3. Problem reformulation for computational improvement

This section reformulates the packing optimization problem to reduce the number of 0–1 variables contained in Model 2. Consider the set of six 0–1 variables $(A_{ik}, B_{ik}, C_{ik}, D_{ik}, E_{ik}, F_{ik})$ in constraints (2′)–(8′), which is used to express the six types of non-overlapping conditions of left–right, behind–front and below–above between a pair of cartons (i, k). In fact, a set of three 0–1 variables is enough to express these six types of relationships. Consider the following proposition:

PROPOSITION 1 *The conditions of non-overlapping between cartons i and k can be reformulated by introducing three binary variables* α_{ik}, β_{ik} *and* δ_{ik} *with definitions given in table* 1 *with reference to figure* 2, *where* $1 \leq \alpha_{ik} + \beta_{ik} + \delta_{ik} \leq 2$.

Table 1. Improved non-overlapping conditions ($1 \leq \alpha_{ik} + \beta_{ik} + \delta_{ik} \leq 2$).

Condition number	α_{ik}	β_{ik}	δ_{ik}	Meaning
1	0	0	1	Carton i is on the right side of carton k
2	0	1	0	Carton i is on the left side of carton k
3	1	0	0	Carton i is behind carton k
4	0	1	1	Carton i is in front of carton k
5	1	0	1	Carton i is below carton k
6	1	1	0	Carton i is above carton k

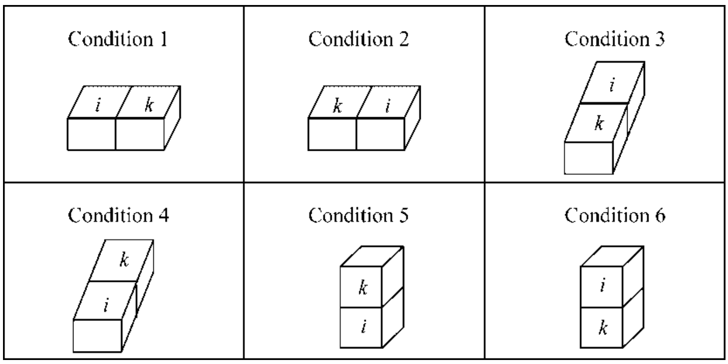

Figure 2. Graphical illustration of non-overlapping conditions.

From the basis of Proposition 1, constraints (2′)–(8′) can be reformulated effectively as follows:

PROPOSITION 2 *The non-overlapping constraints (2′)–(8′) are equivalent to the following inequalities*:

$$x_i + P_i l_{xi} + Q_i(l_{zi} - w_{yi} + h_{zi}) + R_i(1 - l_{xi} - l_{zi} + w_{yi} - h_{zi}) \leq x_k + \alpha_{ik}M + \beta_{ik}M + (1 - \delta_{ik})M, \tag{2''}$$

$$x_k + P_k l_{xk} + Q_k(l_{zk} - w_{yk} + h_{zk}) + R_k(1 - l_{xk} - l_{zk} + w_{yk} - h_{zk}) \leq x_i + \alpha_{ik}M + (1 - \beta_{ik})M + \delta_{ik}M, \tag{3''}$$

$$y_i + P_i(1 - l_{xi} - l_{zi}) + Q_i w_{yi} + R_i(l_{xi} + l_{zi} - w_{yi}) \leq y_k + (1 - \alpha_{ik})M + \beta_{ik}M + \delta_{ik}M, \tag{4''}$$

$$y_k + P_k(1 - l_{xk} - l_{zk}) + Q_k w_{yk} + R_k(l_{xk} + l_{zk} - w_{yk}) \leq y_i + \alpha_{ik}M + (1 - \beta_{ik})M + (1 - \delta_{ik})M, \tag{5''}$$

$$z_i + P_i l_{zi} + Q_i(1 - l_{zi} - h_{zi}) + R_i h_{zi} \leq z_k + (1 - \alpha_{ik})M + \beta_{ik}M + (1 - \delta_{ik})M, \tag{6''}$$

$$z_k + P_k l_{zk} + Q_k(1 - l_{zk} - h_{zk}) + R_k h_{zk} \leq z_i + (1 - \alpha_{ik})M + (1 - \beta_{ik})M + \delta_{ik}M, \tag{7''}$$

$$1 \leq \alpha_{ik} + \beta_{ik} + \delta_{ik} \leq 2, \tag{8''}$$

where all variables are the same as defined before.

Proof Constraint (2′) means that if and only if $A_{ik} = 1$ then i is on the left of k; constraint (2″) implies that for $\alpha_{ik} = \beta_{ik} = 0$ and $\delta_{ik} = 1$, i is also on the left of k. Constraint (2′) therefore is equivalent to constraint (2″). Similarly, constraints (3′)–(7′) are equivalent to constraints (3″)–(7″), respectively. ■

Model 2 can then be transformed into Model 3 below:

Model 3:

Minimize xyz

subject to
constraints (2″)–(8″), (9′)–(11′)
where α_{ik}, β_{ik}, δ_{ik}, l_{xi}, l_{zi}, w_{yi} and h_{zi} are 0–1 variables.

Comparing Model 2 with Model 3 shows that Model 2 contains $3n(n-1)+4n$ 0–1 variables, while Model 3 involves $3/2n(n-1)+4n$ 0–1 variables. Model 3 is therefore computationally less demanding than Model 2. This will be illustrated by some numerical results presented in section 6.

4. Linearization strategy

This article is concerned with packing problems with the nonlinear objective function xyz. Therefore, finding the global optimum of the packing problem is difficult. This section proposes a generalized approach to piecewisely linearize the objective function xyz in Model 3 in order to find its globally optimal solution.

Denote F as a feasible set of Model 3 in which $x \geq y \geq z$, $F = \{(2'')\text{–}(8''), (9')\text{–}(11'), x \geq y \geq z\}$. First, consider the following fact:

An optimization program P1: {Minimize Obj1 $= xyz$, subject to $\underline{x} \leq x \leq \bar{x}$, $\underline{y} \leq y \leq \bar{y}$, $\underline{z} \leq z \leq \bar{z}$, $x, y, z \in F$} is equivalent to the program below.

P2: {Minimize Obj2 $= \ln x + \ln y + \ln z$, subject to $\underline{x} \leq x \leq \bar{x}$, $\underline{y} \leq y \leq \bar{y}$, $\underline{z} \leq z \leq \bar{z}$, $x, y, z \in F$}.

The following propositions discuss the proposed approach of linearizing the logarithmic terms $\ln x$, $\ln y$ and $\ln z$.

Proposition 3 *A logarithm function* $\ln x$, $0 < a_1 \leq x \leq a_m$, *as shown in figure* 3, *can piecewise linearly be approximated*

$$\ln x \doteq \ln \hat{x} = \ln a_1 + s_1(x - a_1) + \sum_{j=2}^{m-1} \frac{s_j - s_{j-1}}{2}(|x - a_j| + x - a_j), \tag{18}$$

where a_j, $j = 1, 2, \ldots, m$, *are the break points of* $\ln x$, $a_j < a_{j+1}$; *and* s_j *are the slopes of line segments between* a_j *and* a_{j+1}, $s_j = (\ln a_{j+1} - \ln a_j)/(a_{j+1} - a_j)$, *for* $j = 1, 2, \ldots, m-1$.

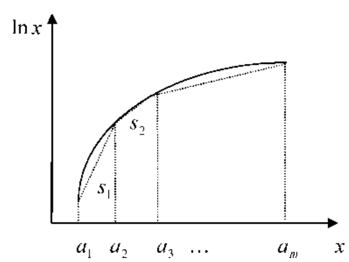

Figure 3. Graphical illustration of piecewise linear approximation of $\ln x$.

This proposition can be examined as follows:

If $x = a_1$ then $\ln x = \ln a_1$ (exact).
If $x \le a_2$ then $\ln \hat{x} = \ln a_1 + s_1(x - a_1)$.
If $x \le a_3$ then

$$\ln \hat{x} = \ln a_1 + s_1(x - a_1) + \frac{s_2 - s_1}{2}(|x - a_2| + x - a_2).$$

Similarly, logarithm functions $\ln y$ and $\ln z$, can be approximately linearized as

$$\ln y \doteq \ln \hat{y} = \ln b_1 + t_1(y - b_1) + \sum_{j=2}^{m-1} \frac{t_j - t_{j-1}}{2}(|y - b_j| + y - b_j), \tag{19}$$

where $t_j = (\ln b_{j+1} - \ln b_j)/(b_{j+1} - b_j)$, $0 < b_1 \le y \le b_m$ and $b_1, b_2, \ldots, b_m$ are its break points, $b_j < b_{j+1}$, for $j = 1, 2, \ldots, m-1$;

$$\ln z \doteq \ln \hat{z} = \ln c_1 + r_1(z - c_1) + \sum_{j=2}^{m-1} \frac{r_j - r_{j-1}}{2}(|z - c_j| + z - c_j), \tag{20}$$

where $r_j = (\ln c_{j+1} - \ln c_j)/(c_{j+1} - c_j)$, $0 < c_1 \le z \le c_m$ and $c_1, c_2, \ldots, c_m$ are its break points, $c_j < c_{j+1}$, for $j = 1, 2, \ldots, m-1$.

Remark 1 Since $\ln x$, $\ln y$ and $\ln z$ are concave functions, it is clear that the approximations bound $\ln x$, $\ln y$ and $\ln z$ from below; $\ln x \ge \ln \hat{x}$, $\ln y \ge \ln \hat{y}$ and $\ln z \ge \ln \hat{z}$.

The following results are then obtained.

Remark 2 (lower bound) Consider the following program:

$$P3 : \{\text{Minimize Obj3} = \ln \hat{x} + \ln \hat{y} + \ln \hat{z},\ \text{subject to } \underline{x} \le x \le \bar{x},\ \underline{y} \le y \le \bar{y},$$
$$\underline{z} \le z \le \bar{z}, x, y, z \in F\}.$$

Program P3 provides a lower bound on Program P2 due to Remark 1.
Consider the following proposition of how to linearize $\ln \hat{x}$:

PROPOSITION 4 (linearization) $\ln \hat{x}$ *in* (18) *can be re-expressed in a linearized form as follows*:

$$\ln \hat{x} = \ln a_1 + s_1(x - a_1) + \sum_{j=2}^{m-1}(s_j - s_{j-1})(a_j u_j + x - a_j - g_j), \tag{21}$$

(i) $-a_m u_j \le x - a_j \le a_m(1 - u_j)$ *for* $j = 2, 3, \ldots, m-1$
(ii) $-a_m u_j \le g_j \le a_m u_j$ *for* $j = 2, 3, \ldots, m-1$
(iii) $a_m(u_j - 1) + x \le g_j \le a_m(1 - u_j) + x$ *for* $j = 2, 3, \ldots, m-1$
(iv) $u_j \ge u_{j-1}$ *for* $j = 2, 3, \ldots, m-1$ *where* $u_j = 0$ *or* 1, *and* $g_j = 1$ *or* x, *respectively.*

Proof If $x - a_j \ge 0$ then $u_j = 0$ and $g_j = 0$ based on (i) and (ii); which results in

$$a_j u_j + x - a_j - g_j = (|x - a_j| + x - a_j)/2.$$

If $x - a_j < 0$ then $u_j = 1$ and $g_j = x$ based on (i) and (iii); which results in

$$a_j u_j + x - a_j - g_j = |x - a_j| + x - a_j.$$

Therefore, $\ln \hat{x}$ in (18) is equivalent to (21). Now consider condition (iv). ■

Since $a_{j-1} < a_j$, if $x < a_j$ (i.e. $u_j = 1$) then $x < a_{j+1}$ and $u_{j+1} = 1$.
If $x > a_{j+1}$ (i.e. $u_{j+1} = 0$) then $x > a_j$ and $u_j = 0$.
Therefore, it is true that $u_j \geq u_{j-1}$.
Condition (iv) is used to accelerate the computational speed of solving the problem.
Similarly, $\ln \hat{y}$ and $\ln \hat{z}$ can be re-expressed in a piecewise linearized form in the same way.

PROPOSITION 5 (range reduction) *Let (x^*, y^*, z^*) be the global optimum of Model 3, the range of the objective function xyz is bounded as*

$$\sum_{i=1}^{n} P_i Q_i R_i \leq x^* y^* z^* \leq \left(\frac{x^\Delta + y^\Delta + z^\Delta}{3} \right)^3 \quad \textit{for any } (x^\Delta, y^\Delta, z^\Delta) \in F.$$

Proof Since $\sum_{i=1}^{n} P_i Q_i R_i \leq xyz$ for any $(x, y, z) \in F$ and

$$\sqrt[3]{x^* y^* z^*} \leq \sqrt[3]{x^\Delta y^\Delta z^\Delta} \leq \frac{x^\Delta + y^\Delta + z^\Delta}{3},$$

the proposition is true. ■

Note that (P_i, Q_i, R_i) is the length, width and height of carton i.

5. Solution algorithm

From the above discussion, the proposed solution algorithm is as follows.

Let S_τ, T_τ and U_τ be respectively a set of break points of $\ln x$, $\ln y$ and $\ln z$ at the τth iteration. Denote ε as a tolerable error (specified later).

Step 1 Range reduction
Solving a linear 0–1 program: {Minimize $x + y + z$, subject to $x, y, z \in F$}.
Let the solution be $(x^\Delta, y^\Delta, z^\Delta)$ and the objective value be $\overline{Obj} = x^\Delta + y^\Delta + z^\Delta$.
According to Proposition 5, the range of finding the global optimum of *xyz* is

$$\sum_{i=1}^{n} P_i Q_i R_i \leq x^* y^* z^* \leq \left(\frac{\overline{Obj}}{3} \right)^3$$

and

$$\ln \sum_{i=1}^{n} P_i Q_i R_i \leq \ln x^* + \ln y^* + \ln z^* \leq 3 \ln \overline{Obj} - 3 \ln 3. \tag{22}$$

Step 2 Linearization
Let iteration $\tau = 1$, $S_1 = \{\underline{x}, \bar{x}\}$, $T_1 = \{\underline{y}, \bar{y}\}$, $U_1 = \{\underline{z}, \bar{z}\}$, $\underline{x} > 0$, $\underline{y} > 0$, $\underline{z} > 0$.

Consider the linear program

$$\underset{(x,y,z)}{Min}\ Obj(x(2)) + Obj(y(2)) + Obj(z(2))$$

where

$$Obj(x(2)) + Obj(y(2)) + Obj(z(2)) = \ln \underline{x} + \frac{\ln \bar{x} - \ln \underline{x}}{\bar{x} - \underline{x}}(x - \underline{x}) + \ln \underline{y} + \frac{\ln \bar{y} - \ln \underline{y}}{\bar{y} - \underline{y}}(y - \underline{y}) + \ln \underline{z} + \frac{\ln \bar{z} - \ln \underline{z}}{\bar{z} - \underline{z}}(z - \underline{z})$$

and variable $x(2)$ denotes the approximation variable of x^* in the current iteration, similarly for $y(2)$ and $z(2)$.

Subject to (22), $(x, y, z) \in F$, $\underline{x} \le x \le \bar{x}$, $\underline{y} \le y \le \bar{y}$, $\underline{z} \le z \le \bar{z}$.

Let the solution be denoted by the specific values of (x, y, z), namely, $(x(2), y(2), z(2))$, the approximation to (x^*, y^*, z^*) in the current iteration.

If $|(Obj(x(2)) - \ln x(2))/\ln x(2)| < \varepsilon$, $|(Obj(y(2)) - \ln y(2))/\ln y(2)| < \varepsilon$ and $|(Obj(z(2)) - \ln z(2))/\ln z(2)| < \varepsilon$ then terminate the process. The optimal solution is $(x(2), y(2), z(2))$.

Otherwise, go to Step 3.

Step 3

Iteration $\tau = \tau + 1$.

Let $S_\tau = S_{\tau-1} \cup \{x(\tau)\}$, $T_\tau = T_{\tau-1} \cup \{y(\tau)\}$ and $U_\tau = U_{\tau-1} \cup \{z(\tau)\}$, where '∪' means union of sets.

Denote the number of elements (break points) in S_τ, T_τ and U_τ as m_τ. Consider the following linear mixed 0–1 program:

$$\underset{(x,y,z)}{Min}\ Obj(x(\tau+1)) + Obj(y(\tau+1)) + Obj(z(\tau+1))$$

where

$$\begin{aligned} &Obj(x(\tau+1)) + Obj(y(\tau+1)) + Obj(z(\tau+1)) \\ &= \ln a_1 + s_1(x - a_1) + \sum_{j=2}^{m_\tau - 1}(s_j - s_{j-1})(a_j u_j + x - a_j - g_j) + \ln b_1 + t_1(y - b_1) \\ &+ \sum_{j=2}^{m_\tau - 1}(t_j - t_{j-1})(b_j v_j + y - b_j - q_j) + \ln c_1 + r_1(z - c_1) + \sum_{j=2}^{m_\tau - 1}(r_j - r_{j-1}) \\ &\times (c_j o_j + z - c_j - p_j). \end{aligned}$$

Subject to (22), $(x, y, z) \in F$, for all j, the following constraints should be satisfied:

$$\begin{aligned} &-\bar{x}u_j \le x - a_j \le \bar{x}(1 - u_j), \quad -\bar{x}u_j \le g_j \le \bar{x}u_j, \\ &\bar{x}(u_j - 1) + x \le g_j \le \bar{x}(1 - u_j) + x, \quad u_j \ge u_{j-1}, \\ &-\bar{y}v_j \le y - b_j \le \bar{y}(1 - v_j), \quad -\bar{y}v_j \le q_j \le \bar{y}v_j, \\ &\bar{y}(v_j - 1) + y \le q_j \le \bar{y}(1 - v_j) + y, \quad v_j \ge v_{j-1}, \\ &-\bar{z}o_j \le z - c_j \le \bar{z}(1 - o_j), \quad -\bar{z}o_j \le p_j \le \bar{z}o_j, \\ &\bar{z}(o_j - 1) + z \le p_j \le \bar{z}(1 - o_j) + z, \quad o_j \ge o_{j-1}, \end{aligned}$$

where u_j, v_j and o_j are 0–1 variables, g_j, q_j and p_j are 1 or, respectively, x, y and z, $a_1, a_2, \ldots, a_{m_\tau} \in S_\tau$, $a_1 = \underline{x} < a_2 < \cdots < a_{m_\tau} = \bar{x}$, $b_1, b_2, \ldots, b_{m_\tau} \in T_\tau$, $b_1 = \underline{y} <$

$b_2 < \cdots < b_{m_\tau} = \bar{y}, c_1, c_2, \ldots, c_{m_\tau} \in U_\tau, c_1 = \underline{z} < c_2 < \cdots < c_{m_\tau} = \bar{z}, s_j, t_j$ and r_j are the slopes in the piecewise linearization of $\ln x, \ln y$ and $\ln z$, respectively, for $j = 2, 3, \ldots, m_\tau - 1$.

Let the solution be $(x(\tau+1), y(\tau+1), z(\tau+1))$.

If $|(Obj(x(\tau+1)) - \ln x(\tau+1))/\ln x(\tau+1)| < \varepsilon$, $|(Obj(y(\tau+1)) - \ln y(\tau+1))/\ln y(\tau+1)| < \varepsilon$ and $|(Obj(z(\tau+1)) - \ln z(\tau+1))/\ln z(\tau+1)| < \varepsilon$ then terminate the process, and $(x(\tau+1), y(\tau+1), z(\tau+1))$ is the optimal solution.

Otherwise, repeat Step 3.

Development (convergence). The above algorithm (run with $\varepsilon \doteq 0$) terminates with the incumbent solution $(\hat{x}^*, \hat{y}^*, \hat{z}^*)$ being optimum to Model 3 when $\tau \to \infty$.

Explanation. For iteration τ, let $\{[l_x^\tau, u_x^\tau]\}$ express the sequence $[a_1^\tau, a_2^\tau]$, $[a_2^\tau, a_3^\tau], \ldots, [a_{m_\tau-1}^\tau, a_{m_\tau}^\tau]$, $\{[l_y^\tau, u_y^\tau]\}$ express the sequence $[b_1^\tau, b_2^\tau], [b_2^\tau, b_3^\tau], \ldots, [b_{m_\tau-1}^\tau, b_{m_\tau}^\tau]$ and $\{[l_z^\tau, u_z^\tau]\}$ express the sequence $[c_1^\tau, c_2^\tau]$, $[c_2^\tau, c_3^\tau], \ldots, [c_{m_\tau-1}^\tau, c_{m_\tau}^\tau]$ where $a_1^\tau < a_2^\tau < \cdots < a_{m_\tau}^\tau$, $b_1^\tau < b_2^\tau < \cdots < b_{m_\tau}^\tau$ and $c_1^\tau < c_2^\tau < \cdots < c_{m_\tau}^\tau$.

Since sequences $\{a_\kappa^\tau\}$, $\{b_\kappa^\tau\}$ and $\{c_\kappa^\tau\}$ are monotone and bounded, where $\kappa = 1, \ldots, m_\tau$ and $\tau \to \infty$, it is obvious that $\{[l_x^\tau, u_x^\tau]\}$, $\{[l_y^\tau, u_y^\tau]\}$ and $\{[l_z^\tau, u_z^\tau]\}$ converge to some intervals $[l_x^\tau, u_x^\tau]$, $[l_y^\tau, u_y^\tau]$ and $[l_z^\tau, u_z^\tau]$. Also, when $\tau \to \infty$, by the concavity of $\ln \hat{x}$, $\ln \hat{y}$ and $\ln \hat{z}$ in (18) and the Mean Value Theorem, $\hat{x}^* = l_x^* = u_x^*$, $\hat{y}^* = l_y^* = u_y^*$ and $\hat{z}^* = l_z^* = u_z^*$. Which means $\ln \hat{x}^* = \ln x^*$, $\ln \hat{y}^* = \ln y^*$ and $\ln \hat{z}^* = \ln z^*$. By referring to Remark 2, $\ln \hat{x}^*$, $\ln \hat{y}^*$ and $\ln \hat{z}^*$ are the lower bounds of Program P2; $(\hat{x}^*, \hat{y}^*, \hat{z}^*)$ is then the optimal solution to Model 3.

6. Numerical examples

To validate the proposed method, two tests are performed. Test 1 is used to demonstrate that the reformulation of packing problems can substantially improve the computational efficiency of Chen *et al.*'s model. Test 2 comprises packing problems with cubes in which the problem model is simpler than Model 3; carton rotation is not considered. All test problems are solved by LINGO (2002) on a Pentium III 1000 personal computer.

Test 1: The test problems taken from Chen *et al.* (1995) are solved and the CPU times are compared with those of the proposed method. Table 2 indicates that, compared with

Table 2. Comparison of computation results.

Problem number	Carton number i	P_i	Q_i	R_i	CPU time (hh:mm:ss)		Objective value	
					Chen *et al.*	Proposed method	Chen *et al.*	Proposed method
1	1	25	8	6	00:04:43	00:00:18	4368	4368
	2	20	10	5				
	3	16	7	3				
	4	15	12	6				
2	1	25	8	6	03:12:07	00:02:19	5040	5040
	2	20	10	5				
	3	16	7	3				
	4	15	12	6				
	5	22	8	3				
3	1	25	8	6	46:38:29	00:45:02	5880	5880
	2	20	10	5				
	3	16	7	3				
	4	15	12	6				
	5	22	8	3				
	6	20	10	4				

Table 3. Optimal solutions obtained by the proposed method.

Problem number	Carton number i	x_i	y_i	z_i	(x, y, z)
1	1	0	0	0	(28, 26, 6)
	2	8	0	0	
	3	8	10	2	
	4	16	11	0	
2	1	0	3	0	(30, 28, 6)
	2	20	8	0	
	3	14	0	0	
	4	8	8	0	
	5	8	0	3	
3	1	10	20	0	(35, 28, 6)
	2	25	0	0	
	3	0	0	3	
	4	10	8	0	
	5	5	0	0	
	6	0	8	0	

Chen *et al.*'s method, the proposed method uses much less CPU time to reach the same objective values. This implies that the reformulation of packing optimization problems improves the computational efficiency. The optimal packing solutions for Problems 1, 2 and 3 of table 2 are listed in table 3.

Test 2: Packing problems with cubes are tested by the proposed method and the results of four problems are presented in table 4. For Problem 4, with eight cubes, the proposed method spends 8 seconds finding the optimal solution. For Problem 5, with 10 cubes, it takes about 2.5 minutes to get the global solution. The other two problems are also solved to obtain the global optima with the tolerable error $\varepsilon = 0.01$.

Taking the test Problem 1 for instance, the detailed results obtained by the developed algorithm are listed in table 5. The globally optimal solution is found in the sixth iteration with six break points. From the sixth iteration to the ninth iteration, one of the break points (28, 26, 6) is also the obtained optimal solution. Accordingly, the values of $\varepsilon(x)$, $\varepsilon(y)$ and $\varepsilon(z)$, which represent the errors of $\ln(x)$, $\ln(y)$ and $\ln(z)$, respectively, are 0. The convergence

Table 4. Computational results; all cartons are cubes.

Problem number	Number of cubes	Cube side length	(x, y, z)	CPU time (hh:mm:ss)	Objective value
4	3	1	(8, 5, 5)	00:00:08	200
	3	2			Global optimum
	1	3			
	1	5			
5	4	1	(8, 6, 5)	00:02:26	240
	3	2			Global optimum
	2	3			
	1	5			
6	4	1	(8, 6, 3)	00:04:44	144
	3	2			Global optimum
	4	3			
7	4	1	(9, 8, 5)	00:56:38	360
	3	2			Global optimum
	3	3			
	1	4			
	1	5			

Table 5. Detailed results of Problem 1.

Number of break points	Objective value	(x, y, z)	$\varepsilon(x)$	$\varepsilon(y)$	$\varepsilon(z)$
1	6.4738	(25, 18, 11)	0.034	0.1129	0.3383
2	7.3158	(41, 12, 10)	0.1144	0.2373	0.2851
3	7.6211	(41, 12, 10)	0.1144	0.2373	0.2851
4	7.8335	(41, 12, 10)	0.1144	0.2373	0.2851
5	8.0188	(60, 12, 7)	0.2287	0.2373	0.086
6	8.0925	(28, 26, 6)	0	0	0
7	8.1208	(28, 26, 6)	0	0	0
8	8.1626	(28, 26, 6)	0	0	0
9	8.1832	(28, 26, 6)	0	0	0

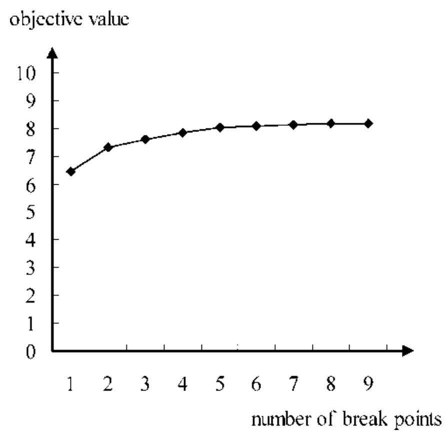

Figure 4. Convergence graph of objective value against number of break points.

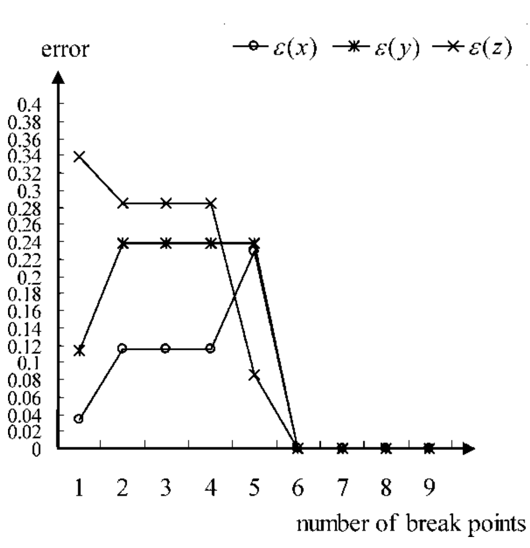

Figure 5. Sensitivity analysis of $\varepsilon(\cdot)$ to number of break points.

graph of objective value against the number of break points is shown in figure 4. Figure 4 reveals that the objective value increases stably and converges to the globally optimal solution while the number of break points increases. The sensitivity analysis of the errors of $\ln(x)$, $\ln(y)$ and $\ln(z)$ against the number of break points is also depicted in figure 5.

7. Conclusions

This article proposes a new method to solve packing problems. First the computational efficiency is improved by reducing the number of 0–1 variables of the problem model and by finding the minimal range of the objective function. Then a piecewise linearization technique is applied to linearize the nonlinear objective function xyz. By solving the linear mixed 0–1 program iteratively the proposed method can find a global optimum within the tolerable error. Numerical examples demonstrate that the proposed method can obtain the global optimum of a packing problem. To further improve the computational efficiency a direction for future development is to implement the proposed method in a distributed computation system. Another direction of development is to use a heuristic method such as tabu search, simulated annealing or genetic algorithm to find a sub-optimal solution which is treated as a lower bound of the objective. Based on this solution, the search region of the global optimum can be reduced and the computational time decreased.

References

Andreas, B., Hermann, G. and Daniel, M., A parallel tabu search algorithm for solving the container loading problem. *Parallel Computing*, 2003, **29**(5), 641–662.

Beasley, J.E., An algorithm for the two-dimensional assortment problem. *Eur. J. Oper. Res.*, 1985, **19**, 253–261.

Bortfeldt, A. and Gehring, H., A hybrid genetic algorithm for the container loading problem. *Eur. J. Oper. Res.*, 2001, **131**(1), 143–161.

Chen, C.S., Lee, S.M. and Shen, Q.S., An analytical model for the container loading problem. *Eur. J. Oper. Res.*, 1995, **80**, 68–76.

Dowsland, W.B., Three-dimensional packing-solution approaches and heuristic development. *Int. J. Prod. Res.*, 1991, **29**(8), 1673–1685.

Fayard, D. and Zissimopoulos, V., An approximation algorithm for solving unconstrained two-dimensional knapsack problems. *Eur. J. Oper. Res.*, 1995, **84**, 618–632.

Gehring, H. and Bortfeldt, A., A parallel genetic algorithm for solving the container loading problem. *Int. Trans. Oper. Res.*, 2002, **9**, 497–511.

Li, H.L. and Chang, C.T., An approximately global optimization method for assortment problems. *Eur. J. Oper. Res.*, 1998, **105**, 604–612.

LINGO 8.0, LINDO SYSTEMS INC., Chicago, 2002.

Liu, F.H. and Hsiao, C.J., A three-dimensional pallet loading method for single-size boxes. *J. Oper. Res. Soc.*, 1997, **48**(6), 726–735.

Pisinger, D., Heuristics for the container loading problem. *Eur. J. Oper. Res.*, 2002, **141**(2), 382–392.

Scheithauer, G., LP-based bounds for the container and multi-container loading problem. *Int. Trans. Oper. Res.*, 1999, **6**, 199–213.

Terno, J., Scheithauer, G., Sommerweiß, U. and Riehme, J., An efficient approach for the multi-pallet loading problem. *Eur. J. Oper. Res.*, 2000, **123**, 372–381.