J Glob Optim (2006) 36:21–31
DOI 10.1007/s10898-006-9025-0

ORIGINAL PAPER

The mean-partition problem

Fei-Hwang Chang · Frank K. Hwang · Uriel G. Rothblum

Received: 4 July 2005 / Accepted: 13 March 2006 /
Published online: 4 July 2006

Abstract In mean-partition problems the goal is to partition a finite set of elements, each associated with a d-vector, into p disjoint parts so as to optimize an objective, which depends on the averages of the vectors that are assigned to each of the parts. Each partition is then associated with a $d \times p$ matrix whose columns are the corresponding averages and a useful approach in studying the problem is to explore the mean-partition polytope, defined as the convex hull of the set of matrices associated with feasible partitions.

Keywords Partition Problems · Combinatorial Optimization · Means

1 Introduction

Consider a finite set $N = \{1, \ldots, n\}$ where each element i in N is associated with a vector $A^i \in R^d$. A *partition* of N is a finite ordered collection $\pi = (\pi_1, \ldots, \pi_p)$ where $\pi_1, \ldots, \pi_p$ are disjoint sets whose union is N. In this case, p is called the *size* of π, $\pi_1, \ldots, \pi_p$ are called the *parts* of π and $\langle \pi \rangle \equiv (|\pi_1|, \ldots, |\pi_p|)$ is called the *shape* of π. A *p-partition* is a partition of size p. Throughout, we assume that p, n and $A^1, \ldots, A^n$ are given.

In a *constrained-shape partition problem*, one is to select a partition π of N whose shape is in a given set Γ of integer p-vectors with coordinate-sum n so as to maximize an objective function $F(.)$ that is defined over partitions. Special cases

F.-H. Chang · F. K. Hwang
Department of Applied Mathematics, National Chiaotung University, Hsinchu, Taiwan, ROC 300.

U. G. Rothblum (✉)
Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa 32000, Israel
Fax: 972-4829-5688
e-mail: rothblum@ie.technion.ac.il

include *single-shape*, *bounded-shape* and *size partition problems* in which the set Γ consists of a single vector, is defined by lower and upper bounds and is unrestricted.

Partition problems are further classified by their objective function $F(.)$. For a subset S of $\{1, \ldots, n\}$, let

$$A^S = \sum_{i \in S} A^i \in R^d \tag{1.1}$$

and for a partition $\pi = (\pi_1, \ldots, \pi_p)$, let

$$A^\pi = (A^{\pi_1}, \ldots, A^{\pi_p}) \in R^{d \times p}. \tag{1.2}$$

A predominant class of partition problems is the *sum-partition problem* in which

$$F(\pi) = f(A^\pi), \tag{1.3}$$

where $f(.)$ is a real-valued function on $R^{d \times p}$. A useful approach in addressing partition problems in this class is to study the corresponding *sum-partition polytope* defined to be the convex hull of the A^πs, with π ranging over the set of feasible partitions. The sum-partition polytope corresponding to a set of partitions Π is denoted P^Π. If f is guaranteed to attain an optimum over P^Π at a vertex of that polytope then there must exist an optimal partition π with A^π being a vertex of the P^Π (see [10] for a sufficient condition for the optimality of vertices, which generalizes the classic conditions of convexity and quasi-convexity). In such cases, it is useful to identify properties of partitions π for which A^π is a vertex of the sum-partition polytope. For $d = 1$, Hwang et al. [9] gave an explicit solution of the bounded-shape sum-partition problem when f is Schur convex and a majorization shape exists. Chang et al. [5] extended the result to the general case that no majorizing shape exists; see, Sect. 4.

We next introduce the mean-partition problem, which is the subject of the current paper. For a nonempty subset S of $\{1, \ldots, n\}$, let

$$\bar{A}^S = \frac{1}{|S|} \sum_{i \in S} A^i \in R^d \tag{1.4}$$

and for a partition $\pi = (\pi_1, \ldots, \pi_p)$ (with nonempty parts) let

$$\bar{A}^\pi = (\bar{A}^{\pi_1}, \ldots, \bar{A}^{\pi_p}). \tag{1.5}$$

We next consider the *mean-partition problem*, which is the class of partition problems with

$$F(\pi) = g(\bar{A}^\pi), \tag{1.6}$$

where $g(.)$ is a real-valued function on $R^{d \times p}$; with $d = 1$, this problem was first explored by Anily and Federgruen [2]; see, Sect. 2 for details about their results. Also, the study of the mean-partition problem motivated Chang and Hwang [4] to study the supermodularity property of a function related to the mean-partition polytope; see, Sect. 3.

As in the case of the sum-partition problem, given a set of p-partitions Π, the *mean-partition polytope* $\bar{P}^\Pi$ is defined as the convex hull of $\{\bar{A}^\pi : \pi \in \Pi\}$. And when g is guaranteed to attain an optimum over $\bar{P}^\Pi$ at a vertex, there exists an optimal partition with $\bar{A}^\Pi$ being a vertex of the $\bar{P}^\Pi$. It is then useful to identify properties of partitions π for which $\bar{A}^\pi$ is a vertex of the mean-partition polytope.

The purpose of the current paper is to study the mean-partition problem. Our first result (in Sect. 2) is the observation that the single-shape mean-partition problem can be reduced to a corresponding sum-partition problem. We use the reduction as a tool for deducing properties of optimal partitions of mean-partition problems from corresponding results about sum-partition problems. We also observe that every single-shape mean-partition polytope is the image of a corresponding sum-partition polytope under a one-to-one linear transformation with the sets of partitions corresponding to the vertices of the two polytopes coinciding. Consequently, properties of partitions π with A^{π} being a vertex of P^{Π} hold for partitions with $\bar{A}^{\pi}$ being a vertex of $\bar{P}^{\Pi}$. While the above tools do not extend to bounded-shape problems, the existence of properties of optimal partitions of single-shape partition problems extends to bounded-shape problems.

In Sect. 3, we explore the single-shape mean-partition polytopes with $d = 1$. We derive an explicit representation of these polytopes and review alternative approaches to address the mean partition problem.

In Sect. 4, we continue the examination of mean-partition problems with $d = 1$ and establish geometric properties (reverse size-consecutiveness) for optimal partitions of single-shape problems, providing more structure of optimal partitions than is obtainable from the transformation approach. For the bounded-shape problem, we are able to shrink the set of consecutive partitions and still preserve the existence of an optimal partition in the shrunk set.

2 Reduction of single-shape mean-partition problems to sum-partition problems

We start by recording the observation that single-shape mean-partition problems are reducible to corresponding sum-partition problems.

Lemma 2.1 *Let $n_1, \ldots, n_p$ be positive integers whose coordinate-sum is n. Then the single-shape mean-partition problem with prescribed-shape $(n_1, \ldots, n_p)$ and objective function given by* (1.6) *coincides with the corresponding sum-partition problem with objective function given by* (1.3) *where f satisfies*

$$f(x_1, \ldots, x_p) = g\left(\frac{x_1}{n_1}, \ldots, \frac{x_p}{n_p}\right) \quad \text{for } x \in R^p. \tag{2.1}$$

□

Lemma 2.1 implies that properties of optimal solutions for single-shape mean-partition problems are deducible from properties of optimal solutions of corresponding sum-partition problems. For example, it is known that when the A^is are distinct, every single-shape sum-partition problem with f (quasi-) convex has at least one *disjoint* optimal partition, that is, an optimal partition for which the convex hulls of the vectors A^i corresponding to distinct parts are disjoint (see [3]); further, the set of disjoint partitions has at most $O[n^{d\binom{p}{2}}]$ partitions and these can be enumerated in polynomial time (see [1] or [8]). These results establish the polynomial solvability of the single-shape sum-partition problem when the function f is (quasi-) convex. (For the relaxation of the assumption that the A^is are distinct see [8].) Now, as a function g is (quasi-) convex if and only if so is the function f that is defined through (2.1), we obtain the following corollary of Lemma 2.1.

Corollary 2.2 *Suppose the A^is are distinct. Then every single-shape mean-partition problem with objective function given by* (1.6) *where g is* (*quasi-*) *convex has at least one disjoint optimal solution, and such problems are solvable in polynomial time.* □

The following standard observation allows one to extend conclusions about the presence of (geometric and combinatorial) properties in optimal partitions from single-shape to constrained-shape problems [see, Lemma 1 of Golany et al. (2005, submitted)]: Consider a cost function F over p-partitions and a property Q of p-partitions such that for each single-shape mean-partition problem with cost function F, Q is satisfied by some (every) optimal partition. Then, for every constrained-shape partition problem with cost function F, Q is satisfied by some (every) optimal partition.

The above arguments combine with Lemma 2.1 to show that any property that is present in optimal solutions of single-shape sum-partition problem, is present in optimal solutions of corresponding constrained-shape mean-partition problems. But, these conclusions cannot be reached by using (2.1) to map constrained-shape mean-partition problems onto corresponding sum-partition problems. The above observation is demonstrated in the next corollary.

Corollary 2.3 *Suppose the A^is are distinct. Then every constrained-shape mean-partition problem with objective function given by* (1.6) *where g is* (*quasi-*) *convex has at least one disjoint optimal solution. Further*, *assuming efficient* (*that is*, *polynomial*) *verifiability of the shape-constraints*, *such problems are solvable in polynomial time.*

Proof The existence of disjoint optimal partitions follows from the observation preceding the statement of the corollary and Corollary 2.2, and the polynomial solvability follows from the polynomial enumerability of disjoint partitions (established in [8]). □

Anily and Federgruen [2] studied the bounded-shape mean-partition problem for $d = 1$ under the objective function $f(\pi) = \sum_{i=1}^{p} h(\bar{A}^{\pi}, n_i)$. They proved that if for each n_i, $h(X, n_i)$ is convex and nondecreasing in X, then there exists a disjoint optimal partition. Their result follows from the above discussion with $d = 1$ and with $f(\pi)$ as a special type of (quasi-) convex function. We note that with stronger assumptions on $h(X, y)$, Anily and Federgruen obtained additional, tighter, results, which are not available from our approach.

We next go back to single-shape problems and record an isomorphism between single-shape mean-partition polytopes and the corresponding sum-partition polytopes. With $n_1, \ldots, n_p$ as the given positive integers, let P^{Π} be the set of partitions with shape $(n_1, \ldots, n_p)$ and let $D^{n_1,\ldots,n_p}$ be the $p \times p$ diagonal matrix whose diagonal elements are, respectively, $n_1, \ldots, n_p$. For every partition $\pi \in \Pi$, $\bar{A}^{\pi} = (\frac{A^{\pi_1}}{n_1}, \ldots, \frac{A^{\pi_p}}{n_p}) = (D^{n_1,\ldots,n_p})^{-1} A^{\pi}$, and therefore

$$\bar{P}^{\Pi} = \text{conv}\{(D^{n_1,\ldots,n_p})^{-1} A^{\pi} : \pi \in \Pi\} = \{(D^{n_1,\ldots,n_p})^{-1} X : X \in P^{\Pi}\}. \tag{2.2}$$

Thus, the one-to-one linear transformation

$$X = (X^1, \ldots, X^p) \to (D^{n_1,\ldots,n_p})^{-1} X = \left(\frac{X^1}{n_1}, \ldots, \frac{X^p}{n_p} \right) \tag{2.3}$$

maps P^{Π} onto $\bar{P}^{\Pi}$, that is, the single-shape mean-partition polytope is the one-to-one linear image of the corresponding single-shape sum-partition polytope. A virtue of this transformation is that it preserves vertices. Consequently, any bound on the number of vertices of P^{Π} is a bound on the number of vertices of $\bar{P}^{\Pi}$ and any algorithm for generating the vertices of P^{Π} can be used to generate the vertices of $\bar{P}^{\Pi}$.

3 Mean-partition polytopes with $d = 1$

In this section, we consider the mean-partition polytope with $d = 1$. We review known results about single-shape sum-partition polytopes and show how these are transformable to mean-partition polytopes by the one-to-one transformation outlined in Sect. 2. We also review difficulties in the direct simulation to mean-partition problems of the approach that has been implemented successfully for the sum-partition problem. Following standard notation, we use the notation $\theta^1, \ldots, \theta^n$ for the scalars $A^1, \ldots, A^n$, respectively.

Gao et al. [6] developed an effective approach to study sum-partition polytopes with $d = 1$ by deriving explicit representations for the corresponding sum-partition polytopes through systems of linear inequalities (in fact, [6] considers only single-shape problems and the general case is developed in Hwang and Rothblum [11]). To present the approach, let Π be a set of p-partitions and consider the real-valued function λ_*^{Π} on subsets of $\{1, \ldots, p\}$, where for each nonempty $I \subseteq \{1, \ldots, p\}$

$$\lambda_*^{\Pi}(I) = \min_{\pi=(\pi_1,\ldots,\pi_p)\in\Pi} \sum_{j\in I} \theta^{\pi_j} \tag{3.1}$$

and $\lambda_*^{\Pi}(\emptyset) = 0$; in particular, $\lambda_*^{\Pi}(\{1, \ldots, p\}) = \sum_{j=1}^{n} \theta^j$.

A real-valued function λ over subset of $\{1, \ldots, p\}$ is used to define two polytopes. First, C^{λ} is defined to be the set of vectors x in R^p satisfying

$$\sum_{i\in I} x_i \geq \lambda(I) \quad \text{for all } I \subseteq \{1, \ldots, p\} \tag{3.2}$$

and

$$\sum_{i=1}^{p} x_i = \lambda(\{1. \ldots, p\}). \tag{3.3}$$

Also, for each permutation $\sigma = (\sigma_1, \ldots, \sigma_p)$ of $\{1, \ldots, p\}$ and $k \in \{1, \ldots, p\}$, let $j_\sigma(k)$ denote the index for which $\sigma_{j_\sigma(k)} = k$ and let λ_σ be the vector $((\lambda_\sigma)_1, \ldots, (\lambda_\sigma)_p)$ where for each $k = 1, \ldots, p$, $(\lambda_\sigma)_k = \lambda(\{\sigma_1, \ldots, \sigma_{j_\sigma(k)}\}) - \lambda(\{\sigma_1, \ldots, \sigma_{j_\sigma(k-1)}\})$. With Σ as the set of permutations of $\{1, \ldots, p\}$, the second polytope corresponding to λ, denoted H^{λ}, is defined as the convex hull of $\{\lambda_\sigma : \sigma \in \Sigma\}$. The function λ is called *supermodular* if for every pair of subsets I and J of $\{1, \ldots, p\}$, $\lambda(I \cup J) + \lambda(I \cap J) \geq \lambda(I) + \lambda(J)$. Shapley [13] proved that when λ is supermodular, $H^{\lambda} = C^{\lambda}$ and the λ_σs are the vertices of this polytope.

Given a set of p-partitions Π and the corresponding function λ_*^{Π} defined by (3.1), it is immediately verified that $P^{\Pi} \subseteq C^{\lambda_*^{\Pi}}$. The set of partitions Π is called *consistent* if for every permutation σ of $\{1, \ldots, p\}$ there exists a partition π in Π with $\theta^{\pi} = (\lambda_*^{\Pi})_\sigma$. It follows immediately from the definition that $H^{\lambda_*^{\Pi}} \subseteq P^{\Pi}$ whenever Π is consistent; further, it is proved in [11] that consistency of Π implies that λ_*^{Π} is supermodular, implying that in this case $H^{\lambda_*^{\Pi}} = P^{\Pi} = C^{\lambda_*^{\Pi}}$. The equality $P^{\Pi} = C^{\lambda_*^{\Pi}}$ provides a representation of the sum-partition polytope P^Π as the feasible set of a corresponding system of linear inequalities. And the equality $P^\Pi = H^{\lambda_*^\Pi}$, together with Shapley characterization of vertices of the polytope corresponding to a supermodular function, provide a characterization of the vertices of the sum-partition polytope P^Π through the $p!$ $(\lambda_*^\Pi)_\sigma$s. Examples of consistent sets of partitions include single-shape problems and bounded-shape problems where the θ^is are one-sided, that is, either nonnegative or nonpositive; see [11]. We recall from [7] that for any bounded-shape set of partitions Π, λ_*^Π is supermodular, implying that $H^{\lambda_*^\Pi} = C^{\lambda_*^\Pi} (\supseteq P^\Pi)$; but, for arbitrary bounded-shape problems, without the assumption that the θ^is are one-sided, the supermodularity of λ_*^Π does not imply the representation $P^\Pi = C^{\lambda_*^\Pi}$.

We next combine the representation of single-shape sum-partition polytopes through (3.2)–(3.3) with the transformation (2.3) to obtain a representation of single-shape mean-partition polytopes. We also obtain a representation of the vertices of such polytopes. For the latter, recall that for each permutation $\sigma = (\sigma_1, \ldots, \sigma_p)$ of $\{1, \ldots, p\}$ and $k \in \{1, \ldots, p\}$, $j_\sigma(k)$ denotes the index for which $\sigma_{j_\sigma(k)} = k$.

Lemma 3.1 *Let $n_1, \ldots, n_p$ be positive integers whose coordinate-sum is n and let Π be the set of partitions with shape $(n_1, \ldots, n_p)$. Then $\bar{P}^\Pi$ is the set of vectors $y \in R^p$ that satisfy*

$$\sum_{i \in I} n_i y_i \geq \lambda(I) \quad \textit{for all } I \subseteq \{1, \ldots, p\} \tag{3.4}$$

and

$$\sum_{i=1}^{p} n_i y_i = \lambda(\{1, \ldots, p\}). \tag{3.5}$$

Further, the vertices of $\bar{P}^\Pi$ are available from the $p!$ permutations of $\{1, \ldots, p\}$ with permutation σ corresponding to the vector v^σ having

$$(v^\sigma)_k = \frac{1}{n_k}[(\lambda_*^\Pi)(\{\sigma_1, \ldots, \sigma_{j_\sigma(k)}\}) - (\lambda_*^\Pi)(\{\sigma_1, \ldots, \sigma_{j_\sigma(k-1)}\})] \quad \text{for } k = 1, \ldots, p. \tag{3.6}$$

Proof Let $D^{n_1,\ldots,n_p}$ be the $p \times p$ diagonal matrix whose diagonal elements are, respectively, $n_1, \ldots, n_p$. We observe from (2.2) that

$$\bar{P}^\Pi = \{(D^{n_1,\ldots,n_p})^{-1} x : x \in P^\Pi\} = \{y : (D^{n_1,\ldots,n_p}) y \in P^\Pi\}. \tag{3.7}$$

Using the representation of P^Π through (3.2)–(3.3) we get the representation of $\bar{P}^\Pi$ as the set of vectors $y \in R^p$ that satisfy (3.4)–(3.5). Finally, the representation of the vertices of $\bar{P}^\Pi$ follows from the representation of the vertices of the single-shape sum-partition polytope P^Π mentioned earlier, and that observation made in Sect. 2 that the transformation $x \to (D^{n_1,\ldots,n_p})^{-1} x$ (described in (2.3)) maps vertices of P^Π onto vertices of $\bar{P}^\Pi$. □

Chang and Hwang [4] tried to develop a direct approach for studying the mean-partition problem with $d = 1$. Given a set of partitions Π, with no partition in Π having empty parts, consider the real-valued function $\bar{\lambda}$ on subsets of $\{1, \ldots, p\}$ where for each nonempty $I \subseteq \{1, \ldots, p\}$

$$\bar{\lambda}_*^\Pi(I) = \min_{\pi = (\pi_1, \ldots, \pi_p) \in \Pi} \sum_{j \in I} \bar{\theta}^{\pi_j} \tag{3.8}$$

and $\bar{\lambda}_*^{\Pi}(\emptyset) = 0$. Chang and Hwang [4] proved that for a single-shape set of partitions Π, $\bar{\lambda}_*^{\Pi}$ is supermodular; it then follows from the result of [13] mentioned in the Introduction that $H^{\bar{\lambda}_*^{\Pi}} = C^{\bar{\lambda}_*^{\Pi}}$. But, the following example demonstrates that this polytope can be different from $\bar{P}^{\Pi}$.

Example 3.1 Let $n = 3, \theta^1 = 1, \theta^2 = 2, \theta^3 = 3, p = 2$ and consider the mean-partition problem corresponding to the set Π of partitions with shape $(1, 2)$. The set Π contains the three partitions $(\{1\}, \{2, 3\}), (\{2\}, \{1, 3\})$ and $(\{3\}, \{1, 2\})$ whose corresponding vectors are, respectively, $(1, 2.5), (2, 2)$ and $(3, 1.5)$. The mean-partition polytope is then the line-segment connecting $(1, 2.5)$ and $(3, 1.5)$. Also, we have that $\bar{\lambda}_*^{\Pi}(\{1\}) = 1/1 = 1$, $\bar{\lambda}_*^{\Pi}(\{2\}) = 1 + 2/2 = 1.5$ and $\bar{\lambda}_*^{\Pi}(\{1, 2\}) = \min\{1/1 + 2 + 3/2 = 3.5, 2/1 + 1 + 3/2 = 4,\ 3/1 + 1 + 2/2 = 4.5\} = 3.5$. So, $C^{\bar{\lambda}_*^{\Pi}}$ is the polytope defined by the inequalities $x_1 \geq 1, x_2 \geq 1.5, x_1 + x_2 = 3.5$, that is, it is the line-segment connecting $(1, 2.5)$ and $(2, 1.5)$. Finally, the two permutations $(1, 2)$ and $(2, 1)$ of $\{1, 2\}$ correspond, respectively, to the vectors $(\bar{\lambda}_*^{\Pi})_{(1,2)} = (\bar{\lambda}_*^{\Pi}(\{1\}), \bar{\lambda}_*^{\Pi}(\{1, 2\}) - \bar{\lambda}_*^{\Pi}(\{1\}) = (1, 2.5)$ and $(\bar{\lambda}_*^{\Pi})_{(2,1)} = (\bar{\lambda}_*^{\Pi}(\{1, 2\}) - \bar{\lambda}_*^{\Pi}(\{2\}),\ \bar{\lambda}_*^{\Pi}(\{2\})) = (2, 1.5)$, and $H^{\bar{\lambda}_*^{\Pi}}$ is the line-segment connecting these points (see Fig. 1 for an example of $\bar{P}^{\Pi}, C^{\bar{\lambda}_*^{\Pi}}$ and $H^{\bar{\lambda}_*^{\Pi}}$). Notice that the equality $C^{\bar{\lambda}_*^{\Pi}} = H^{\bar{\lambda}_*^{\Pi}}$ is consistent with the conclusion of [4].

One explanation for $C^{\bar{\lambda}_*^{\Pi}}$ being different from $\bar{P}^{\Pi}$ is the fact that the coordinate-sums of the points in the mean-partition polytope need not be constant, hence, it seems natural to relax the equality constraint (3.3) in the definition of $C^{\bar{\lambda}_*^{\Pi}}$. But, such a relaxation will typically result in an unbounded polyhedron (consider Example 3.1). With the goal of augmenting the constraints of (3.2) with upper bounds on the variables, consider the real-valued function on subsets of $\{1, \ldots, p\}$ defined by

$$(\bar{\lambda}^*)^{\Pi}(I) = \max_{\pi=(\pi_1,\ldots,\pi_p)\in\Pi} \sum_{j\in I} \bar{\theta}^{\pi_j} \quad \text{for each } I \subseteq \{1, \ldots, p\} \tag{3.9}$$

(with $(\bar{\lambda}^*)^{\Pi}(\emptyset) = 0$). Further, let K^{Π} be the set of vectors x in R^p that satisfy

$$(\bar{\lambda}_*)^{\Pi}(I) \leq \sum_{i\in I} x_i \leq (\bar{\lambda}^*)^{\Pi}(I) \quad \text{for all } I \subseteq \{1, \ldots, p\}. \tag{3.10}$$

Evidently, K^{Π} contains the corresponding mean-partition polytope $\bar{P}^{\Pi}$. But, the following continuation of Example 3.1 demostrates that the inclusion may be strict.

Example 3.1 (***Continued***) Reconsider the data of Example 3.1. The modification of $C^{\bar{\lambda}_*^{\Pi}}$ obtained through the relaxation of (3.3) is the (unbounded) polyhedron, which

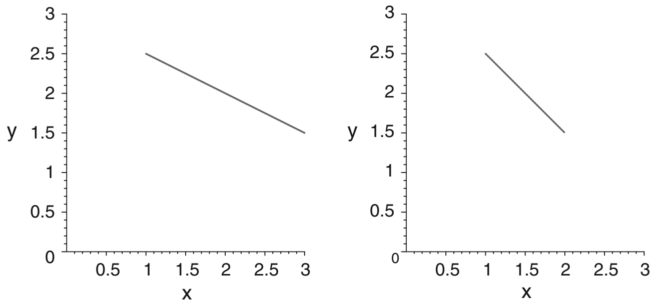

Fig. 1 $\bar{P}^{\Pi}$ and $C^{\bar{\lambda}_*^{\Pi}} = H^{\bar{\lambda}_*^{\Pi}}$ in Example 3.1

is defined by the constraints $x_1 \geq 1, x_2 \geq 1.5$ and $x_1 + x_2 \geq 3.5$; see Fig. 2. Also, $(\bar{\lambda}^*)^\Pi(\{1\}) = \frac{3}{1} = 3$, $(\bar{\lambda}^*)^\Pi(\{2\}) = 2 + 3/2 = 2.5$ and $(\bar{\lambda}^*)^\Pi(\{1,2\}) = 3/1 + 1 + 2/2 = 4.5$. So, K^Π is the polytope defined by the inequalities $1 \leq x_1 \leq 3, 1.5 \leq x_2 \leq 2.5$ and $3.5 \leq x_1 + x_2 \leq 4.5$, which equals the convex hull of $\{1, 2.5\}, \{2, 1.5\}, \{3, 1.5\}, \{2, 2.5\}$; see Fig. 2. As is always the case, the polytope K^Π includes the mean-partition polytope $\bar{P}^\Pi$. □

We next consider the variant of K^Π that corresponds to the sum-partition problem and show that it coincides with C^Π. Specifically, let $(\lambda^*)^\Pi(I)$ be defined by the right-hand side of (3.9) with θ replacing $\bar{\theta}$, and consider the system of linear inequalities given by

$$(\lambda_*)^\Pi(I) \leq \sum_{i \in I} x_i \leq (\lambda^*)^\Pi(I) \quad \text{for all } I \subseteq \{1, \ldots, p\}. \tag{3.11}$$

Evidently, $(\lambda^*)^\Pi(\{1, \ldots, p\}) = (\lambda_*)^\Pi(\{1, \ldots, p\}) = \sum_{j=1}^n \theta^j$ and the pair of constraints of (3.11) corresponding to $I = \{1, \ldots, p\}$, together, coincide with the constraint $\sum_{i=1}^p x_i = \sum_{j=1}^n \theta^j$, that is, with (3.3). Also, for each $I \subset \{1, \ldots, p\}$, $(\lambda^*)^\Pi(I) = \sum_{j=1}^n \theta^j - (\lambda_*)^\Pi(I^c)$, implying that the constraint $\sum_{i \in I} x_i \leq (\lambda^*)^\Pi(I)$ coincides with the constraint $\sum_{i \in I^c} x_i \geq (\lambda_*)^\Pi(I^c)$. Thus, indeed, the set of vectors in R^p that satisfy (3.11) coincides with C^Π.

4 Mean-partition problems with $d = 1$

In this section, we give some results about single-, bounded- and constrained-shape mean-partition problems with $d = 1$ that are not obtainable through the techniques described in Sect. 2 and 3.

Throughout this section we continue to let $d = 1$ and use the notation $\theta^1, \ldots, \theta^n$ for the scalars $A^1, \ldots, A^n$, respectively. Further, for simplicity, we assume that these scalars are distinct and that (by possibly reindexing them) $\theta^1 < \theta^2 < \cdots < \theta^n$. Also, let $n_1, \ldots, n_p$ be positive integers, which sum to n and consider the single-shape partition problems with prescribed shape $(n_1, \ldots, n_p)$. A *reverse-size-consecutive* partition is a consecutive partition with the smallest elements being assigned to the larger parts. Henceforth, we assume that the parts are labelled so that $n_1 \leq n_2 \leq \cdots \leq n_p$, in this case, up to index-permutation of parts having the same size, there is a unique

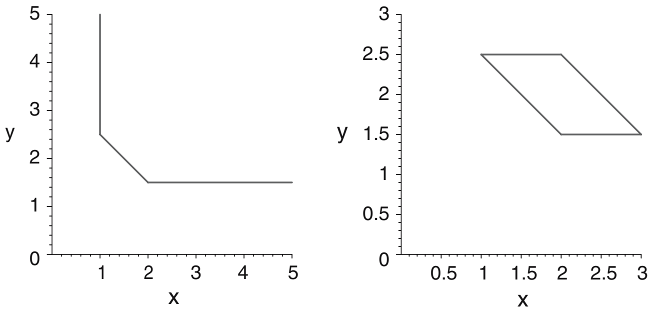

Fig. 2 The Modification of $C^{\bar{\lambda}_*^\Pi}$ by relaxing (3.3) and K^Π in Example 3.1

reverse-size-consecutive partition π^* with shape $(n_1,\ldots,n_p)$ and it is given by $\pi^*_p = \{1,\ldots,n_p\}, \pi^*_{p-1} = \{n_p+1,\ldots,n_p+n_{p-1}\},\ldots,\pi^*_1 = \{n-n_1+1,\ldots,n\}$.

For a vector a in R^p and $i = 1,\ldots,p$, let $a_{[i]}$ be the ith largest member of $\{a_1,\ldots,a_p\}$. Given vectors a and b in R^p, we say that *a weakly submajorizes b*, written $a \succeq_w b$ if

$$\sum_{i=1}^{k} a_{[i]} \geq \sum_{i=1}^{k} b_{[i]} \quad \text{for } k = 1,\ldots, p-1. \tag{4.1}$$

If further

$$\sum_{i=1}^{p} a_i = \sum_{i=1}^{p} b_i\,, \tag{4.2}$$

then a is said to majorize b, written $a \succeq b$.

A real-valued function f on R^p is *Schur convex* if $f(a) \geq f(b)$ whenever a majorizes b. A Schur convex function is known to be symmetric (that is, invariant under coordinate-premutation); see [12] for further details about majorization and Schur convexity. In particular, the following result is well-known

Proposition 4.1 *If f is nondecreasing Schur convex on R^p and a and b are vectors in R^p satisfying $a \succeq_w b$, then $f(a) \geq f(b)$.* □

The next lemma establishes an important property of reverse size-consecutive partitions.

Lemma 4.2 *Consider the case with $p = 2$. Then for every partition $\pi = (\pi_1,\pi_2)$, $(\bar\theta_{\pi^*_1},\bar\theta_{\pi^*_2}) \succeq_w (\bar\theta_{\pi_1},\bar\theta_{\pi_2})$.*

Proof Consider a partition $\pi = (\pi_1,\pi_2)$. We first prove that

$$\max\left\{\bar\theta_{\pi^*_1},\bar\theta_{\pi^*_2}\right\} \geq \max\left\{\bar\theta_{\pi_1},\bar\theta_{\pi_2}\right\},$$

by proving that $\bar\theta_{\pi^*_1} \geq \max\left\{\bar\theta_{\pi_1},\bar\theta_{\pi_2}\right\}$. It is trivial that $\bar\theta_{\pi_1} \leq \bar\theta_{\pi^*_1}$. Similarly, with π'_2 as the set of the n_2 largest indices, $\bar\theta_{\pi_2} \leq \bar\theta_{\pi'_2}$. Further, $\theta_{\pi'_2} = \theta_{\pi^*_1} + a$ where a is the sum of (n_2-n_1) θ^i's, each of which is smaller then $\bar\theta_{\pi^*_1}$. Consequently,

$$\bar\theta_{\pi_2} \leq \bar\theta_{\pi'_2} = \frac{\theta_{\pi^*_1}+a}{n_1+(n_2-n_1)} \leq \max\left\{\frac{\theta_{\pi^*_1}}{n_1},\frac{a}{n_2-n_1}\right\} = \bar\theta_{\pi^*_1}$$

and therefore $\max\{\bar\theta_{\pi^*_1},\bar\theta_{\pi^*_2}\} \geq \bar\theta_{\pi^*_1} \geq \max\left\{\bar\theta_{\pi_1},\bar\theta_{\pi_2}\right\}$,

We next consider the single-shape sum-partition problem with prescribed shape (n_1,n_2) and objective given by (1.3) with the function f being the linear function mapping $x \in R^2$ into $x_1/n_1 + x_2/n_2$. As $1/n_1 \geq 1/n_2$, it follows from Theorem 2.1 of [9] that the consecutive partition under which the n_1 indices associated with the smallest θ^is are assigned to the second part and the n_2 indices associated with the larget θ^is are assigned to the first part, that is, the reverse size-consecutive partition is optimal. This proves that $\bar\theta_{\pi^*_1} + \bar\theta_{\pi^*_2} \geq \bar\theta_{\pi_1} + \bar\theta_{\pi_2}$, completing the proof that $(\bar\theta_{\pi^*_1},\bar\theta_{\pi^*_2}) \succeq_w (\bar\theta_{\pi_1},\bar\theta_{\pi_2})$. □

From Proposition 4.1 and Lemma 4.2 we have.

Theorem 4.3 *Every single-shape mean-partition problem with objective function given by* (1.6) *where g is nondecreasing and Schur convex has a reverse size-consecutive optimal partition.*

Proof By [5], (reverse) size-consecutiveness is strongly 2-shape-sortable, which implies that in order to prove that the property exists in some optimal partition, it suffices to consider the case where $p = 2$. For that case, Lemma 4.2 implies that for every partition $\pi = (\pi_1, \pi_2)$, $(\bar{\theta}_{\pi_1^*}, \bar{\theta}_{\pi_2^*}) \succeq_w (\bar{\theta}_{\pi_1}, \bar{\theta}_{\pi_2})$, and therefore Proposition 4.1 assures that $F(\pi^*) = g(\bar{\theta}_{\pi^*}) \geq g(\bar{\theta}_{\pi}) = F(\pi)$, establishing the optimality of π^*. □

As there is essentially a unique reverse size-consecutive partition for each shape, the constrained-shape mean-partition problem corresponding to a set Δ of shapes we need to compare only $|\Delta|$ partitions, one for each shape in Δ. For bounded-shape problems, $|\Delta|$ is available through lower and upper bounds on the part-sizes. One can then use recent results of Chang et al. (2005, submitted) to observe that it suffices to consider only those shapes in Δ, which are not majorized by any other shape in Δ. Further a bound of 2^{p-1} has been established on the number of such unmajorized shapes of bounded-shape problem along with enumeration scheme. Thus we get the following result

Proposition 4.4 *For every bounded-shape mean-partition problem with objective function given by* (1.6) *where g is nondecreasing and Schur convex it is possible to construct efficiently a set of* 2^{p-1} *reverse size-consecutive partitions that contain an optimal partition.* □

Although we do not know how to efficiently describe constrained-shape mean-partition polytopes, we can bound the number of vertices of such polytopes by the sum of the number of vertices of the corresponding single-shape polytopes for each shape in Δ. Since there is a one-to-one mapping between the vertices of the single-shape mean-partition polytopes and the vertices of the corresponding single-shape sum-partition polyotpes which are generated by the disjoint partitons and correspond to the $p!$ permutations of $\{1, \ldots, p\}$ (see Sect. 3), we obtain a bound of $|\Delta|p!$ on the number of vertices of the constrained-shape mean-partition polytope corresponding to Δ. Further, if objective function is given by (1.6) where g is (quasi-) convex, one can enumerate a list of $|\Delta|p!$ (disjoint) partitions which contains an optimal one.

Theorem 4.5 *Suppose the* A^i*'s are distinct. Then the constrained-shape mean-partition problem corresponding to a set* Δ *of shapes with objective function given by* (1.6) *where g is* (*quasi-*) *convex can be solved with effort that is proportional to* $|\Delta|$, *with p considered fixed.*

The upper bound on the number of vertices of constrained-shape mean-partition polytopes, derived in the paragraph preceding Theorem 4.5, is not tight, as the following example demonstrates.

Example 4.1 Let $n = 4, \theta^i = i$ *for* $i = 1, \ldots, 4, p = 2$ *and* $\Delta = \{(1,3), (2,2), (3,1)\}$. Each shape defines two disjoint partitions with two associated vectors—the three shapes, contribute respectively the vectors $(1,3), (4,2), (1.5, 3.5), (3.5, 1.5)$ *and* $(3,1)$, $(2,4)$. The mean-partition polytope is then the convex hull of the above 6 points, but only $(1,3), (4,2), (3,1)$ and $(1,4)$ are vertices. □

Acknowledgements Fei-Hwang Chang and F.K. Hwang's research was partially supported by ROC National Science Grant NSC 93-2115-M-009-013. U.G. Rothblum's research was supported by a grant from ISF—the Israel Science Foundation, by a VPR grant at the Technion, and by the Fund for the Promotion of Research at the Technion.

References

1. Alon, N., Onn, S.: Separable partitions. Discrete Appl. Math. **91**, 39–51 (1999)
2. Anily, S., Federgruen, A.: Structured partitioning problems. Oper. Res. **39**, 130–149 (1991)
3. Barnes, E.R., Hoffman, A.J., Rothblum, U.G.: Optimal partitions having disjoint convex and conic hulls. Math. Prog. **54**, 69–86 (1992)
4. Chang, F.H., Hwang, F.K.: Supermodularity in mean-partition problems. J. Global Optim. **33**, 337–347 (2005)
5. Chang, G.J., Chen, F.L., Huang, L.L., Hwang, F.K., Nuan, S.T., Rothblum, U.G., Sun, I.F., Wang, J.W., Yeh, H.G.: Sortabilities of partition properties. J. Comb. Optim. **2**, 413–427 (1999)
6. Gao, B., Hwang, F.K., Li, W.W-C., Rothblum, U.G.: Partition-polytopes over 1-dimensional points. Math. Prog. **85**, 335–362 (1999)
7. Hwang, F.K., Liao, J.S., Chen, C.Y.: Supermodularity of various partition problems. J. Global Optim. **18**, 275–282 (2000)
8. Hwang, F.K., Onn, S., Rothblum, U.G.: A polynomial time algorithm for shaped partition problems. SIAM J. of Opt. **10**, 70–81 (1999)
9. Hwang, F.K., Onn, S., Rothblum, U.G.: Explicit solution of partitioning problems over a one-dimensional parameter space. Nav. Res. Logist. **47**, 531–540 (2000)
10. Hwang, F.K., Rothblum, U.G.: Directional-quasi-convexity, asymmetric Schur-convexity and optimality of consecutive partitions. Math. Oper. Res. **21**, 540–554 (1996)
11. Hwang, F.K., Rothblum, U.G.: Partitions: optimality and clustering. World Scientific, London, New York, Singapore (2005), to appear
12. Marshall, A.W., Olkin, I.: Inequalties, Theory of Majorization and its Applications. Academic Press, New York (1979)
13. Shapley, L.S.: Cores of convex games. Int. J. Game Theory **1**, 11–26 (1971)