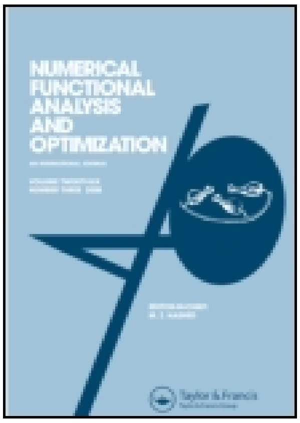

Numerical Functional Analysis and Optimization
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lnfa20

Analysis of least squares finite element methods for a parameter-dependent first-order system

Suh-Yuh Yang [a] & Jinn-Liang Liu [a]
[a] Department of Applied Mathematics , National Chiao Tung University , Taiwan, Hsinchu 30050 E-mail: www.math nctu edu tw/~jinnliu
Published online: 15 May 2007.

NUMER. FUNCT. ANAL. AND OPTIMIZ., 19(1&2), 191–213 (1998)

Analysis of Least Squares Finite Element Methods for A Parameter-Dependent First-Order System *

Suh-Yuh Yang and Jinn-Liang Liu

Department of Applied Mathematics
National Chiao Tung University, Hsinchu 30050, Taiwan
E-mails: syyang; jinnliu@math.nctu.edu.tw
http://www.math.nctu.edu.tw/~jinnliu

ABSTRACT

A parameter-dependent first-order system arising from elasticity problems is introduced. The system corresponds to the 2D isotropic elasticity equations under a stress-pressure-displacement formulation in which the nonnegative parameter measures the material compressibility for the elastic body. Standard and weighted least squares finite element methods are applied to this system, and analyses for different values of the parameter are performed in a unified manner. The methods offer certain advantages such as they need not satisfy the Babuška-Brezzi condition, a single continuous piecewise polynomial space can be used for the approximation of all the unknowns, the resulting algebraic system is symmetric and positive definite, accurate approximations of all the unknowns can be obtained simultaneously, and, especially, computational results do not exhibit any significant numerical locking as the parameter tends to zero which corresponds to the incompressible elasticity problem (or equivalently, the Stokes problem). With suitable boundary conditions, it is shown that both methods achieve optimal rates of convergence in the H^1-norm and in the L^2-norm for all the unknowns. Numerical experiments with various values of the parameter are given to demonstrate the theoretical estimates.

Key words. least squares, finite elements, convergence, error estimates, elasticity equations, Poisson's ratios, Stokes equations

AMS(MOS) Subject Classifications. 65N30, 73V05, 76M10

* This work was supported by NSC-grant 86-2115-M-009-007, Taiwan, R.O.C.

1. INTRODUCTION

We are interested in the finite element approximations to the following parameter-dependent first-order differential system in a bounded domain $\Omega \subset \mathbf{R}^2$ with a smooth boundary $\partial\Omega$,

$$2\mu\left\{-\frac{\partial\varphi_1}{\partial x}-\frac{1}{2}\frac{\partial\varphi_2}{\partial y}-\frac{1}{2}\frac{\partial\varphi_3}{\partial y}+\frac{\partial p}{\partial x}\right\}=f_1 \qquad \text{in } \Omega, \tag{1.1}$$

$$2\mu\left\{\frac{\partial\varphi_1}{\partial y}-\frac{1}{2}\frac{\partial\varphi_2}{\partial x}-\frac{1}{2}\frac{\partial\varphi_3}{\partial x}+(1+\epsilon)\frac{\partial p}{\partial y}\right\}=f_2 \qquad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial\varphi_1}{\partial y}-\frac{\partial\varphi_2}{\partial x}=0 \qquad \text{in } \Omega, \tag{1.3}$$

$$\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_3}{\partial y}+\epsilon\frac{\partial p}{\partial x}=0 \qquad \text{in } \Omega, \tag{1.4}$$

$$\epsilon p+\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0 \qquad \text{in } \Omega, \tag{1.5}$$

$$\varphi_2-\varphi_3-\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}=0 \qquad \text{in } \Omega, \tag{1.6}$$

where $U=(\varphi_1,\varphi_2,\varphi_3,p,u_1,u_2)^T$ is the unknown vector field, $\mathbf{f}=(f_1,f_2)^T$ is the density of a body force, μ is a positive constant, and ϵ is a nonnegative parameter measuring the material compressibility for the elastic body. In general, (1.1)-(1.2), (1.3)-(1.4), and (1.5)-(1.6) respectively express the equilibrium equations, the compatibility equations, and the constitutive equations.

The first-order system arises in problems from the field of continuum mechanics. For $\epsilon > 0$, it corresponds to the 2D elasticity equations in a stress-pressure-displacement formulation (see Section 2 for more details). As ϵ decreases to zero, the material of the elastic body is nearly incompressible which is more important in practical applications, and it is well-known that various finite element schemes result in poor performance as $\epsilon \to 0^+$ due to the so-called locking phenomenon [2, 6, 7]. For the particular case, $\epsilon = 0$, the first-order system corresponds to the 2D Stokes equations for the incompressible viscous flow in a stress-pressure-velocity formulation studied in [20].

In the analysis of elasticity problems, the knowledge of the stresses is often of greater interest than the knowledge of the displacements. It is well-known that the approximation of the stresses can be recovered from the displacements by postprocessing in the standard finite element formulation. Their computation, however, requires the derivatives of the displacement field. From a numerical point of view, differentiating implies a loss of precision. The most widely used approach for obtaining a better approximation of the stresses is based on the mixed finite element formulation which allows the stresses as new variables along with the primary variables (see [12] and many references contained therein). Consequently, the accurate stresses can be obtained directly from the discretized problem. However, the approximation spaces in the mixed method must be required to satisfy the Babuška-Brezzi condition (i.e., the inf-sup condition) which precludes the application of many seemingly natural finite elements.

We provide herein an alternate way to avoid these difficulties by exploiting the least squares principles applied on the new first-order formulation (1.1)-(1.6) of the elasticity problem. This new formulation is different from the usual standard displacement-stress formulation which is extensively studied in the mixed finite element method (see, e.g., [12, 26, 27, 28, 34, 35] etc.), but is similar to the third formulation proposed in [28]. We prove that the first-order formulation is an elliptic system in the sense of Petrovski, and that, with the displacement boundary conditions, it satisfies the Lopatinski condition [36]. As a result, the problem can then be solved by using least squares finite element methods (LSFEMs).

During the past decade, increasing attention has been drawn to the use of least squares principles in connection with finite element applications in the field of computational fluid dynamics (see, e.g., [3, 8, 9, 14, 17, 18, 19, 20, 21, 29, 30] etc.). The least squares approach represents a fairly general methodology that can produce a variety of algorithms. In the present paper, according to the boundary treatment, we shall study two types of these methods. The first method is based on the minimization of a least squares functional that involves only the sum of the squared L^2-norms of the residuals in the differential equations. We refer it as the *standard* least squares finite element method (SLSFEM) (cf. [8, 13, 14, 15, 17, 19, 20, 21, 24, 25, 30, 33] etc.). The other is based on the minimization of a least squares functional which consists of the sum of the squared L^2-norms of the residuals in the differential equations and the boundary conditions with the same weight h^{-1}, where h is the mesh parameter. This method will be referred as the *weighted* least squares finite element method (WLSFEM) (cf. [3, 4, 10, 16, 36] etc.).

The present LSFEMs offer many advantageous features as follows.

- Both methods lead to minimization problems rather than a saddle point problem by the mixed method. Thus they are not subject to the restriction of the Babuška-Brezzi condition, and a single continuous piecewise polynomial space can be used for the approximation of all the unknowns.
- The resulting linear algebraic systems are symmetric and positive definite. Thus efficient solvers, such as the general SOR or conjugate gradient methods, can be used to solve the corresponding large linear systems.
- Since the stresses and the pressure serve as additional dependent variables, accurate approximations of the displacements, stresses, and the pressure are obtained simultaneously. Furthermore, with suitable regularity assumptions, convergence results of both least squares approximations can be established for general boundary conditions in the natural norms associated with the least squares bilinear forms.
- It is shown that, with the displacement boundary conditions, both LSFEMs achieve optimal rates of convergence in the H^1-norm and in the L^2-norm for the sufficiently smooth unknowns. As a special case, the analysis proves a conjecture made in [20] in which the SLSFEM is applied to the Stokes equations with the displacement boundary conditions (cf. Remark 5.3).
- Numerical experiments with various values of the parameter support the theoretical analysis. Especially, computational results do not exhibit any significant numerical locking as the parameter ϵ tends to zero.

The remainder of the paper is organized as follows. We introduce the stress-pressure-displacement formulation (1.1)-(1.6) for the elasticity problem in Section 2. The LSFEMs are given in Section 3, as well as their fundamental properties. A priori estimates with the displacement boundary conditions are derived in Section 4. Error analyses are presented in Section 5. Finally, some numerical experiments are examined in Section 6 to demonstrate the approach.

2. THE STRESS-PRESSURE-DISPLACEMENT ELASTICITY

Consider the following 2D elasticity problem in the usual displacement formulation,

$$-2\mu\Big\{\nabla\cdot\varepsilon(\mathbf{u})+\frac{\nu}{1-2\nu}\nabla(\nabla\cdot\mathbf{u})\Big\}=\mathbf{f} \qquad \text{in } \Omega, \tag{2.1}$$

$$\mathbf{u}=0 \qquad \text{on } \Gamma_1, \tag{2.2}$$

$$2\mu\Big\{\varepsilon(\mathbf{u})+\frac{\nu}{1-2\nu}(\nabla\cdot\mathbf{u})\mathbf{I}\Big\}\cdot\mathbf{n}=\mathbf{g} \qquad \text{on } \Gamma_2, \tag{2.3}$$

with the following notation:

- $\Omega\subset\mathbf{R}^2$ is a bounded domain representing the region occupied by an elastic body.
- $\partial\Omega=\Gamma_1\cup\Gamma_2$ the smooth boundary of Ω is partitioned into two disjoint parts Γ_1 and Γ_2 with the measure of Γ_1 being strictly positive.
- μ is the shear modulus given by

$$\mu=\frac{E}{2(1+\nu)}>0,$$

  where ν is the Poisson ratio, $0<\nu<0.5$, and $E>0$ is the Young modulus. The upper limit of the Poisson ratio, i.e., $\nu\to 0.5^-$, corresponds to an incompressible material.
- $\mathbf{u}=(u_1,u_2)^T$ is the displacement vector field.
- $\mathbf{f}=(f_1,f_2)^T$ is the density of a body force acting on the body.
- $\mathbf{g}=(g_1,g_2)^T$ is the density of a surface force acting on Γ_2.
- $\mathbf{n}=(n_1,n_2)^T$ is the outward unit normal vector to $\partial\Omega$.
- $\varepsilon(\mathbf{u})$ is the strain tensor given by

$$\varepsilon(\mathbf{u})=\Big(\varepsilon_{ij}(\mathbf{u})\Big)_{2\times 2}=\Big(\frac{1}{2}(\partial_j u_i+\partial_i u_j)\Big)_{2\times 2}.$$

- $\mathbf{I}$ is the 2×2 identity matrix.

  Introducing the auxiliary variables

$$\varphi_1=\frac{\partial u_1}{\partial x}, \tag{2.4}$$

$$\varphi_2=\frac{\partial u_1}{\partial y}, \tag{2.5}$$

$$\varphi_3 = \frac{\partial u_2}{\partial x}, \tag{2.6}$$

$$p = -\frac{\nu}{1-2\nu}\nabla\cdot\mathbf{u} \tag{2.7}$$

defined on $\overline{\Omega}$ and letting

$$\epsilon = \frac{1-2\nu}{\nu} > 0, \qquad 0 < \nu < \frac{1}{2},$$

(2.1) can also be written as

$$2\mu\Big\{-\frac{\partial\varphi_1}{\partial x} - \frac{1}{2}\frac{\partial\varphi_2}{\partial y} - \frac{1}{2}\frac{\partial\varphi_3}{\partial y} + \frac{\partial p}{\partial x}\Big\} = f_1 \qquad \text{in } \Omega, \tag{2.8}$$

$$2\mu\Big\{\frac{\partial\varphi_1}{\partial y} - \frac{1}{2}\frac{\partial\varphi_2}{\partial x} - \frac{1}{2}\frac{\partial\varphi_3}{\partial x} + (1+\epsilon)\frac{\partial p}{\partial y}\Big\} = f_2 \qquad \text{in } \Omega. \tag{2.9}$$

We call φ_i the stresses and p the artificial pressure, and remark that the "pressure" p gives the hydrostatic pressure only in the incompressible limit, $\epsilon = 0$ (cf. Remark 2.1). Note that a combination of φ_i, $i = 1, 2, 3$, and p can represent the actual stresses σ_{ij}, $i, j = 1, 2$, which are given by

$$\sigma(\mathbf{u}) = \Big(\sigma_{ij}(\mathbf{u})\Big)_{2\times 2} = 2\mu\Big\{\varepsilon(\mathbf{u}) + \frac{\nu}{1-2\nu}(\nabla\cdot\mathbf{u})\mathbf{I}\Big\}.$$

Also, by (2.4)-(2.7), we obtain the following two compatibility equations

$$\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x} = 0 \qquad \text{in } \Omega, \tag{2.10}$$

$$\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_3}{\partial y} + \epsilon\frac{\partial p}{\partial x} = 0 \qquad \text{in } \Omega. \tag{2.11}$$

To recover the displacements, we have the equations

$$\epsilon p + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \qquad \text{in } \Omega, \tag{2.12}$$

$$\varphi_2 - \varphi_3 - \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = 0 \qquad \text{in } \Omega. \tag{2.13}$$

Equations (2.8)-(2.13) are the so-called stress-pressure-displacement formulation for the elasticity equations (2.1) and may be written in the matrix form

$$\mathcal{L}U := AU_x + BU_y + DU = F \qquad \text{in } \Omega, \tag{2.14a}$$

where

$$A = \begin{pmatrix} -2\mu & 0 & 0 & 2\mu & 0 & 0 \\ 0 & -\mu & -\mu & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\mu & -\mu & 0 & 0 & 0 \\ 2\mu & 0 & 0 & 2\mu(1+\epsilon) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ p \\ u_1 \\ u_2 \end{pmatrix}, \text{ and } F = \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Remark 2.1. *For the incompressible limit, $\epsilon = 0$, using the equations* (2.10) *and* (2.11), *the first-order system* (2.14a) *can be rewritten in form of the so-called stress-pressure-velocity Stokes equations which have been studied in* [20]. *In the context,* **u** *represents the velocity for the Stokes flow, p expresses the pressure with appropriate scaling, and μ denotes the inverse of the Reynolds number.* ∎

The system of differential equations (2.14a) will be also supplemented with the boundary conditions (2.2)-(2.3) which may be written as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} U = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1, \tag{2.15}$$

$$\begin{pmatrix} 2\mu n_1 & \mu n_2 & \mu n_2 & -2\mu n_1 & 0 & 0 \\ -2\mu n_2 & \mu n_1 & \mu n_1 & -2\mu(1+\epsilon)n_2 & 0 & 0 \end{pmatrix} U = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \qquad \text{on } \Gamma_2. \tag{2.16}$$

The boundary conditions (2.15) imply that the tangential derivatives of u_i, $i = 1, 2$, vanish

$$n_2\varphi_1 - n_1\varphi_2 = 0 \qquad \text{on } \Gamma_1,$$
$$n_1\varphi_1 + n_2\varphi_3 + \epsilon n_1 p = 0 \qquad \text{on } \Gamma_1,$$

and that

$$n_1 u_1 + n_2 u_2 = 0 \qquad \text{on } \Gamma_1.$$

So, we have

$$\begin{pmatrix} n_2 & -n_1 & 0 & 0 & 0 & 0 \\ n_1 & 0 & n_2 & \epsilon n_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_1 & n_2 \end{pmatrix} U = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1. \tag{2.17}$$

Conversely, we can show that (2.17) with (2.4)-(2.7) imply (2.15) as well. The boundary conditions (2.17) will play an important role in the later analyses. Rewrite (2.16) and (2.17) as the following operator form:

$$\mathcal{R}U = G \qquad \text{on } \partial\Omega. \tag{2.14b}$$

In the sequel, we shall always assume that problem (2.14a/b) has a unique strong solution $U \in [H^1(\Omega)]^6$ with the given functions $F \in [L^2(\Omega)]^6$ and $\mathbf{g} \in$

$[L^2(\Gamma_2)]^2$. Hereinafter, it is also understood that we further require $(p,1)_{0,\Omega}=0$, as well as in the approximations, when $\epsilon = 0$.

3. LEAST SQUARES FINITE ELEMENT METHODS

Throughout this paper, the classical Sobolev spaces $H^s(\Omega)$, $s \geq 0$ integer, $L^2(\Gamma_1)$, and $L^2(\Gamma_2)$ with their associated inner products $(\cdot,\cdot)_{s,\Omega}$, $(\cdot,\cdot)_{0,\Gamma_1}$, $(\cdot,\cdot)_{0,\Gamma_2}$ and norms $\|\cdot\|_{s,\Omega}$, $\|\cdot\|_{0,\Gamma_1}$, $\|\cdot\|_{0,\Gamma_2}$ are employed [22, 32]. As usual, $L^2(\Omega) = H^0(\Omega)$. For the product spaces $[H^s(\Omega)]^6$, $[L^2(\Gamma_1)]^3$, and $[L^2(\Gamma_2)]^2$, the corresponding inner products and norms are also denoted by $(\cdot,\cdot)_{s,\Omega}$, $(\cdot,\cdot)_{0,\Gamma_1}$, $(\cdot,\cdot)_{0,\Gamma_2}$, and $\|\cdot\|_{s,\Omega}$, $\|\cdot\|_{0,\Gamma_1}$, $\|\cdot\|_{0,\Gamma_2}$, respectively, when there is no chance for confusion.

Let $H_0^s(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$, where $\mathcal{D}(\Omega)$ denotes the linear space of infinitely differentiable functions on Ω with compact support. We denote by $H^{-s}(\Omega)$ the dual space of $H_0^s(\Omega)$ normed by

$$\|u\|_{-s,\Omega} = \sup_{0\neq v\in H_0^s(\Omega)} \frac{< u,v >}{\|v\|_{s,\Omega}},$$

where $< \cdot,\cdot >$ denotes the duality pairing.

Since the boundary $\partial\Omega$ of the bounded domain Ω is smooth, there exists an operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, linear and continuous, such that

$$\gamma_0 v = \text{restriction of } v \text{ on } \partial\Omega \text{ for every } v \in C^1(\overline{\Omega}).$$

The space $\gamma_0(H^1(\Omega))$ is not the whole space $L^2(\partial\Omega)$, which is denoted by $H^{\frac{1}{2}}(\partial\Omega)$ with norm defined by

$$\|\varphi\|_{\frac{1}{2},\partial\Omega} = \inf\Big\{\|v\|_{1,\Omega};\ v \in H^1(\Omega),\ \gamma_0 v = \varphi\Big\}$$

which makes it a Hilbert space. Its dual is denoted by $H^{-\frac{1}{2}}(\partial\Omega)$ with the norm $\|\cdot\|_{-\frac{1}{2},\partial\Omega}$. Also, the associated norms of the product spaces $[H^{\frac{1}{2}}(\partial\Omega)]^3$ and $[H^{-\frac{1}{2}}(\partial\Omega)]^3$ are still denoted by $\|\cdot\|_{\frac{1}{2},\partial\Omega}$ and $\|\cdot\|_{-\frac{1}{2},\partial\Omega}$, respectively.

We now introduce the standard and the weighted LSFEMs for solving problem (2.14a/b) in the following two subsections. For simplicity, we assume that $G = \mathbf{0}$ on $\partial\Omega$, that is, $\mathbf{g} = \mathbf{0}$ on Γ_2.

3.1. *SLSFEM*

Introduce the function space

$$\mathcal{V}^s = \Big\{V \in [H^1(\Omega)]^6;\ \mathcal{R}V = \mathbf{0}\Big\}, \tag{3.1}$$

and then define a standard least squares energy functional $\mathcal{E}^s : \mathcal{V}^s \to \mathbf{R}$ by

$$\mathcal{E}^s(V) = \|\mathcal{L}V - F\|_{0,\Omega}^2. \tag{3.2}$$

Obviously, the exact solution $U \in \mathcal{V}^s$ of problem $(2.14a/b)$ is the unique zero minimizer of the functional $\mathcal{E}^s$ on $\mathcal{V}^s$, that is,

$$\mathcal{E}^s(U) = 0 = \min\Big\{\mathcal{E}^s(V); V \in \mathcal{V}^s\Big\}. \tag{3.3}$$

Applying the techniques of variations, we can find that (3.3) is equivalent to

$$\mathcal{B}^s(U,V) = \mathcal{F}^s(V) \qquad \forall\, V \in \mathcal{V}^s, \tag{3.4}$$

where the bilinear form and the linear form are defined, respectively, by

$$\mathcal{B}^s(V,W) = \int_\Omega \mathcal{L}V \cdot \mathcal{L}W, \tag{3.5}$$

$$\mathcal{F}^s(V) = \int_\Omega F \cdot \mathcal{L}V, \tag{3.6}$$

for all $V, W \in \mathcal{V}^s$. The SLSFEM for problem $(2.14a/b)$ is therefore to determine $U_h^s \in \mathcal{V}_h^s$ such that

$$\mathcal{B}^s(U_h^s, V_h) = \mathcal{F}^s(V_h) \qquad \forall\, V_h \in \mathcal{V}_h^s, \tag{3.7}$$

where the finite element space $\mathcal{V}_h^s \subset \mathcal{V}^s$ is assumed to satisfy the following approximation property. For any $V \in \mathcal{V}^s \cap [H^{p+1}(\Omega)]^6$, $p \geq 0$ integer, there exists $V_h \in \mathcal{V}_h^s$ such that

$$\|V - V_h\|_{0,\Omega} + h\|V - V_h\|_{1,\Omega} \leq Ch^{p+1}\|V\|_{p+1,\Omega}, \tag{3.8}$$

where the positive constant C is independent of V and the mesh parameter h. Throughout this paper, in any estimate or inequality the quantity C will denote a generic positive constant and need not necessarily be the same constant in different places.

3.2. *WLSFEM*

Similar to the standard least squares case, we define

$$\mathcal{V}^w = [H^1(\Omega)]^6, \tag{3.9}$$

and define a weighted least squares energy functional $\mathcal{E}^w : \mathcal{V}^w \to \mathbf{R}$ by

$$\mathcal{E}^w(V) = \|\mathcal{L}V - F\|_{0,\Omega}^2 + h^{-1}\|\mathcal{R}V\|_{0,\partial\Omega}^2. \tag{3.10}$$

The exact solution $U \in \mathcal{V}^w$ of problem $(2.14a/b)$ is the unique zero minimizer of the weighted least squares functional $\mathcal{E}^w$ on $\mathcal{V}^w$, i.e.,

$$\mathcal{E}^w(U) = 0 = \min\Big\{\mathcal{E}^w(V); V \in \mathcal{V}^w\Big\}. \tag{3.11}$$

Taking the first variation, we can find that (3.11) is equivalent to

$$\mathcal{B}^w(U,V) = \mathcal{F}^w(V) \qquad \forall\, V \in \mathcal{V}^w, \tag{3.12}$$

where the bilinear form and the linear form are defined, respectively, by

$$\mathcal{B}^w(V,W) = \int_\Omega \mathcal{L}V \cdot \mathcal{L}W + h^{-1}\int_{\partial\Omega} \mathcal{R}V \cdot \mathcal{R}W, \tag{3.13}$$

$$\mathcal{F}^w(V) = \int_\Omega F \cdot \mathcal{L}V, \tag{3.14}$$

for all $V, W \in \mathcal{V}^w$. The WLSFEM for problem $(2.14a/b)$ is then to determine $U_h^w \in \mathcal{V}_h^w$ such that

$$\mathcal{B}^w(U_h^w, V_h) = \mathcal{F}^w(V_h) \qquad \forall\, V_h \in \mathcal{V}_h^w, \tag{3.15}$$

where the finite element space $\mathcal{V}_h^w \subset \mathcal{V}^w$ is also required to satisfy the following approximation property. For any $V \in \mathcal{V}^w \cap [H^{p+1}(\Omega)]^6$, $p \geq 0$ integer, there exists $V_h \in \mathcal{V}_h^w$ such that

$$\|V - V_h\|_{0,\Omega} + h\|V - V_h\|_{1,\Omega} \leq Ch^{p+1}\|V\|_{p+1,\Omega}, \tag{3.16}$$

where C is a positive constant independent of V and h.

3.3. *SOME FUNDAMENTAL PROPERTIES*

In this subsection, we shall discuss the unique solvability of the numerical schemes (3.7), (3.15), and some of their fundamental properties. Before presenting these properties, it is of interest to note that the trial and test functions in the WLSFEM (3.15) need not satisfy the boundary conditions. In contrast, in the SLSFEM (3.7), both the trial and test functions are required to fulfill the boundary requirements. Moreover, since the original system of second-order equations (2.1) is transformed into the system of first-order equations $(2.14a)$, the same C^0 piecewise polynomials can be used to approximate all the unknown functions.

It is clearly that $\mathcal{B}^s(\cdot,\cdot)$ and $\mathcal{B}^w(\cdot,\cdot)$ define inner products on $\mathcal{V}^s \times \mathcal{V}^s$ and $\mathcal{V}^w \times \mathcal{V}^w$, respectively, since the positive-definiteness is implied by the fact that problem $(2.14a/b)$ possesses the unique solution $U = 0$ for $F = 0$ and $G = 0$. Denote the associated natural norms by

$$\|V\|_s = \left\{\mathcal{B}^s(V,V)\right\}^{\frac{1}{2}} \qquad \forall\, V \in \mathcal{V}^s, \tag{3.17}$$

$$\|V\|_w = \left\{\mathcal{B}^w(V,V)\right\}^{\frac{1}{2}} \qquad \forall\, V \in \mathcal{V}^w. \tag{3.18}$$

We first state the fundamental properties of the SLSFEM (3.7).

Theorem 3.1. *Let $U \in [H^1(\Omega)]^6$ be the exact solution of $(2.14a/b)$ with the given functions $F \in [L^2(\Omega)]^6$ and $G = 0$.*

(i) *Problem* (3.7) *has a unique solution* $U_h^s \in \mathcal{V}_h^s$ *which satisfies the following stability estimate,*

$$\|U_h^s\|_s \le \|F\|_{0,\Omega}. \tag{3.19}$$

(ii) *The matrix of the linear system associated with problem* (3.7) *is symmetric and positive definite.*

(iii) *The following orthogonality relation holds,*

$$\mathcal{B}^s(U - U_h^s, V_h) = 0 \qquad \forall\, V_h \in \mathcal{V}_h^s. \tag{3.20}$$

(iv) *The approximate solution* U_h^s *is a best approximation of* U *in the* $\|\cdot\|_s$*-norm, that is,*

$$\|U - U_h^s\|_s = \inf_{V_h \in \mathcal{V}_h^s} \|U - V_h\|_s. \tag{3.21}$$

(v) *If* $U \in [H^{p+1}(\Omega)]^6$, $p \ge 0$ *integer, then there exists a positive constant* C *independent of* h *such that*

$$\|U - U_h^s\|_s \le Ch^p \|U\|_{p+1,\Omega}. \tag{3.22}$$

Proof. To prove the unique solvability, it suffices to prove the uniqueness of solution since the finite dimensionality of $\mathcal{V}_h^s$. Let U_h^s be a solution of (3.7), then, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\|U_h^s\|_s^2 &= \mathcal{B}^s(U_h^s, U_h^s) = (F, \mathcal{L}U_h^s)_{0,\Omega} \\ &\le \|F\|_{0,\Omega}\|\mathcal{L}U_h^s\|_{0,\Omega} \\ &\le \|F\|_{0,\Omega}\|U_h^s\|_s\end{aligned}$$

which implies (3.19). Consequently, the solution U_h^s of (3.7) is unique.

Assertion (ii) follows from the fact that the bilinear form $\mathcal{B}^s(\cdot,\cdot)$ is symmetric and positive definite. (iii) is obtained by subtracting equation (3.7) from equation (3.4). Using (3.20) and the Cauchy-Schwarz inequality,

$$\begin{aligned}\|U - U_h^s\|_s^2 &= \mathcal{B}^s(U - U_h^s, U - U_h^s) \\ &= \mathcal{B}^s(U - U_h^s, U - V_h) \qquad \forall\, V_h \in \mathcal{V}_h^s \\ &\le \|U - U_h^s\|_s \|U - V_h\|_s \qquad \forall\, V_h \in \mathcal{V}_h^s,\end{aligned}$$

we prove (iv).

Finally, assume that $U \in [H^{p+1}(\Omega)]^6$. Let $V_h \in \mathcal{V}_h^s$ be such that (3.8) holds with V replaced by U. Then, by (3.21) and the fact that $\mathcal{L}$ is a first-order differential operator, we have

$$\|U - U_h^s\|_s \le \|U - V_h\|_s \le C\|U - V_h\|_{1,\Omega} \le Ch^p\|U\|_{p+1,\Omega}.$$

∎

Similarly, we have the following results for the WLSFEM (3.15).

Theorem 3.2. *Let $U \in [H^1(\Omega)]^6$ be the exact solution of (2.14a/b) with the given functions $F \in [L^2(\Omega)]^6$ and $G = 0$.*

(i) *Problem (3.15) has a unique solution $U_h^w \in \mathcal{V}_h^w$ which satisfies the following stability estimate,*

$$\|U_h^w\|_w \le \|F\|_{0,\Omega}. \tag{3.23}$$

(ii) *The matrix of the linear system associated with problem (3.15) is symmetric and positive definite.*

(iii) *The following orthogonality relation holds,*

$$\mathcal{B}^w(U - U_h^w, V_h) = 0 \qquad \forall\, V_h \in \mathcal{V}_h^w. \tag{3.24}$$

(iv) *The approximate solution U_h^w is a best approximation of U in the $\|\cdot\|_w$-norm, that is,*

$$\|U - U_h^w\|_w = \inf_{V_h \in \mathcal{V}_h^w} \|U - V_h\|_w. \tag{3.25}$$

(v) *If $U \in [H^{p+1}(\Omega)]^6$, $p \ge 0$ integer, then there exists a positive constant C independent of h such that*

$$\|U - U_h^w\|_w \le C h^p \|U\|_{p+1,\Omega}. \tag{3.26}$$

Proof. The proofs for (i), (ii), (iii), and (iv) are similar to the standard least squares case. For proving part (v), we need the following result whose proof can be found in [11]: there exists a positive constant C such that, for any $V \in [H^1(\Omega)]^6$ and any $\delta > 0$,

$$\|V\|_{0,\partial\Omega} \le C\Big(\delta\|V\|_{1,\Omega} + \frac{1}{\delta}\|V\|_{0,\Omega}\Big).$$

Taking $\delta = h^{\frac{1}{2}}$ and replacing V by $U - V_h$, where $V_h \in \mathcal{V}_h^w$ is chosen such that (3.16) holds with V replaced by U, we have

$$\begin{aligned}\|U - V_h\|_{0,\partial\Omega} &\le C\Big(h^{\frac{1}{2}}\|U - V_h\|_{1,\Omega} + h^{-\frac{1}{2}}\|U - V_h\|_{0,\Omega}\Big)\\ &\le C h^{p+\frac{1}{2}}\|U\|_{p+1,\Omega}.\end{aligned}$$

Thus,

$$\begin{aligned}\|U - U_h^w\|_w^2 &\le \|U - V_h\|_w^2\\ &\le \|\mathcal{L}(U - V_h)\|_{0,\Omega}^2 + h^{-1}\|\mathcal{R}(U - V_h)\|_{0,\partial\Omega}^2\\ &\le C\Big(\|U - V_h\|_{1,\Omega}^2 + h^{-1}\|U - V_h\|_{0,\partial\Omega}^2\Big)\\ &\le C h^{2p}\|U\|_{p+1,\Omega}^2.\end{aligned}$$

This completes the proof. ∎

As a consequence of part (v) in the above theorems, the consistency of the approximations follows.

Corollary 3.3. *Let U be the exact solution of problem (2.14a/b) with the given functions $F \in [L^2(\Omega)]^6$ and $G = 0$. If $U \in [H^{p+1}(\Omega)]^6$, $p \ge 0$ integer, then there* *exists a positive constant C independent of h such that*

$$\|\mathcal{L}U_h^s - F\|_{0,\Omega} \le Ch^p\|U\|_{p+1,\Omega}, \tag{3.27}$$

$$\|\mathcal{L}U_h^w - F\|_{0,\Omega} \le Ch^p\|U\|_{p+1,\Omega}, \tag{3.28}$$

$$\|\mathcal{R}U_h^w - G\|_{0,\partial\Omega} \le Ch^{p+\frac{1}{2}}\|U\|_{p+1,\Omega}. \tag{3.29}$$

∎

4. A PRIORI ESTIMATES

The error estimates of the previous approximations in the H^1- and L^2-norm are primarily based on the thories of Agmon-Douglis-Nirenberg [1] and of Dikanskij [23]. Our approach in exploiting these thories principally follows that of Wendland [36, Section 3.1 and Chapter 8] for two-dimensional first-order elliptic systems in the sense of Petrovski. The application of the theories to our problem involves some unavoidable difficulties concerning the Lopatinski condition if the boundary condition (2.14*b*) is taken to be such general. For simplicity, we only consider the displacement boundary conditions

$$\mathcal{R}U = \begin{pmatrix} n_2 & -n_1 & 0 & 0 & 0 & 0 \\ n_1 & 0 & n_2 & \epsilon n_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_1 & n_2 \end{pmatrix} U = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{on } \Gamma_1 = \partial\Omega. \tag{2.14b'}$$

We first show that $\mathcal{L}$ is an elliptic operator in the sense of Petrovski, and that the boundary operator $\mathcal{R}$ in (2.14*b*′) satisfies the Lopatinski condition. So (2.14*a*/*b*′) is a regular elliptic boundary value problem and then $(\mathcal{L}, \mathcal{R})$ is a Fredholm operator with zero nullity. This enables us to get the coercive type a priori estimates (see Theorem 4.1).

For all $(\xi, \eta) \in \mathbf{R}^2$ and $(\xi, \eta) \ne (0,0)$,

$$\begin{aligned} \det(\xi A + \eta B) &= 2\mu^2(1+\epsilon)(\xi^2+\eta^2)^3 \\ &\ne 0. \end{aligned}$$

Thus (2.14*a*) is an elliptic system in the sense of Petrovski. Obviously, by taking $(\xi, \eta) = (1,0)$, the matrix A is nonsingular and its inverse is

$$A^{-1} = \begin{pmatrix} -\frac{\epsilon}{2\mu(1+\epsilon)} & 0 & 0 & \frac{1}{1+\epsilon} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\mu} & 1 & 0 & 0 & 0 \\ \frac{1}{2\mu(1+\epsilon)} & 0 & 0 & \frac{1}{1+\epsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the original elliptic system (2.14*a*) can be transformed into the following form:

$$U_x + \widetilde{B}U_y + \widetilde{D}U = \widetilde{F} \qquad \text{in } \Omega,$$

where

$$\widetilde{B} = A^{-1}B = \begin{pmatrix} 0 & \frac{\epsilon}{2(1+\epsilon)} & \frac{2+\epsilon}{2(1+\epsilon)} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -2(1+\epsilon) & 0 & 0 \\ 0 & -\frac{1}{2(1+\epsilon)} & \frac{1}{2(1+\epsilon)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\widetilde{D} = A^{-1}D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{F} = A^{-1}F = \begin{pmatrix} -\frac{\epsilon f_1}{2\mu(1+\epsilon)} \\ 0 \\ -\frac{f_2}{\mu} \\ \frac{f_1}{2\mu(1+\epsilon)} \\ 0 \\ 0 \end{pmatrix}.$$

We now check the Lopatinski condition as follows. After elementary operations, we find that the eigenvalues of the matrix $\widetilde{B}^T$ are the imaginary numbers i and $-i$ both with multiplicities three. Consider the eigenvalue $\tau_+ = i$ in the upper halfplane, to which there exists a chain of linearly independent generalized eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of $\widetilde{B}^T$ defined by

$$\widetilde{B}^T\mathbf{p}_1 - \tau_+\mathbf{p}_1 = \mathbf{0},$$
$$\widetilde{B}^T\mathbf{p}_2 - \tau_+\mathbf{p}_2 = \mathbf{p}_1,$$

and a third eigenvector $\mathbf{p}_3$ is given by

$$\widetilde{B}^T\mathbf{p}_3 - \tau_+\mathbf{p}_3 = \mathbf{0},$$

where

$$\mathbf{p}_1 = \Big(0, 1, -1, -2(1+\epsilon)i, 0, 0\Big)^T,$$
$$\mathbf{p}_2 = \Big(-\frac{4(1+\epsilon)}{2+\epsilon}, 0, \frac{4(1+\epsilon)}{2+\epsilon}i, -\frac{2(1+\epsilon)(2+3\epsilon)}{2+\epsilon}, 0, 0\Big)^T,$$
$$\mathbf{p}_3 = \Big(0, 0, 0, 0, 1, -i\Big)^T.$$

Notice that

$$\mathcal{P} = \Big(\mathbf{p}_1, \overline{\mathbf{p}}_1, \mathbf{p}_2, \overline{\mathbf{p}}_2, \mathbf{p}_3, \overline{\mathbf{p}}_3\Big)^T$$

is nonsingular. Let

$$\mathcal{Q} = \left(\mathbf{q}_1, \overline{\mathbf{q}}_1, \mathbf{q}_2, \overline{\mathbf{q}}_2, \mathbf{q}_3, \overline{\mathbf{q}}_3\right)$$

be the inverse matrix of $\mathcal{P}$, then

$$\mathcal{Q} = \begin{pmatrix} -\frac{2+3\epsilon}{8(1+\epsilon)}i & \frac{2+3\epsilon}{8(1+\epsilon)}i & -\frac{2+\epsilon}{8(1+\epsilon)} & -\frac{2+\epsilon}{8(1+\epsilon)} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{2+\epsilon}{8(1+\epsilon)}i & \frac{2+\epsilon}{8(1+\epsilon)}i & 0 & 0 \\ 0 & 0 & -\frac{2+\epsilon}{8(1+\epsilon)}i & \frac{2+\epsilon}{8(1+\epsilon)}i & 0 & 0 \\ \frac{1}{4(1+\epsilon)}i & -\frac{1}{4(1+\epsilon)}i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2}i & -\frac{1}{2}i \end{pmatrix}.$$

Now we have

$$\det\left\{2\begin{pmatrix} n_2 & -n_1 & 0 & 0 & 0 & 0 \\ n_1 & 0 & n_2 & \epsilon n_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_1 & n_2 \end{pmatrix}\left(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\right)\right\}$$
$$= -\frac{(2+\epsilon)^2}{16(1+\epsilon)^2}(n_1 + n_2 i)^3 \neq 0,$$

for all $\epsilon \geq 0$ and $(n_1, n_2) \neq (0,0)$. That is, the Lopatinski condition is satisfied for the boundary conditions (2.14b'). The following estimates then follow the standard results of [36].

Theorem 4.1. *For the boundary value problem* (2.14a/b'), (2.14a) *is an elliptic system in the sense of Petrovski, and the boundary conditions* (2.14b') *satisfy the Lopatinski condition. Thus we have the a priori estimates: for each* $l \geq 0$ *there is a constant* $C > 0$ *such that if* $V \in [H^{l+1}(\Omega)]^6$, *then*

$$\|V\|_{l+1,\Omega} \leq C\Big(\|\mathcal{L}V\|_{l,\Omega} + \|\mathcal{R}V\|_{l+\frac{1}{2},\partial\Omega}\Big). \tag{4.1}$$

∎

By an interpolation argument [23] (cf. [36, Lemma 8.2.1]), the estimates (4.1) can be extended to the case $l \geq -1$. Taking $l = 1$, $l = 0$, and $l = -1$ in (4.1), we have

$$\|V\|_{2,\Omega} \leq C\|\mathcal{L}V\|_{1,\Omega} \qquad \forall\, V \in \mathcal{V}^s \cap [H^2(\Omega)]^6, \tag{4.2}$$
$$\|V\|_{1,\Omega} \leq C\|\mathcal{L}V\|_{0,\Omega} \qquad \forall\, V \in \mathcal{V}^s, \tag{4.3}$$
$$\|V\|_{0,\Omega} \leq C\|\mathcal{L}V\|_{-1,\Omega} \qquad \forall\, V \in \mathcal{V}^s, \tag{4.4}$$
$$\|V\|_{2,\Omega} \leq C\Big(\|\mathcal{L}V\|_{1,\Omega} + \|\mathcal{R}V\|_{\frac{3}{2},\partial\Omega}\Big) \qquad \forall\, V \in \mathcal{V}^w \cap [H^2(\Omega)]^6, \tag{4.5}$$
$$\|V\|_{1,\Omega} \leq C\Big(\|\mathcal{L}V\|_{0,\Omega} + \|\mathcal{R}V\|_{\frac{1}{2},\partial\Omega}\Big) \qquad \forall\, V \in \mathcal{V}^w, \tag{4.6}$$
$$\|V\|_{0,\Omega} \leq C\Big(\|\mathcal{L}V\|_{-1,\Omega} + \|\mathcal{R}V\|_{-\frac{1}{2},\partial\Omega}\Big) \qquad \forall\, V \in \mathcal{V}^w. \tag{4.7}$$

Two sets of estimates (4.2)-(4.4) and (4.5)-(4.7) imply respectively the error estimates for SLSFEM and WLSFEM in the following two subsections.

Remark 4.2. *It is unclear whether the constant C in* (4.2)-(4.7) *is independent of the nonnegative parameter ϵ since the constant in* (4.1) *is not explicitly known (cf.* [1], *page* 74, *Remark* 2*).* ∎

5. ERROR ANALYSES

5.1. *ERROR ESTIMATES FOR THE SLSFEM*

For the standard least squares case, by (4.3), we have the coercivity

$$\mathcal{B}^s(V,V) = \|\mathcal{L}V\|_{0,\Omega}^2 \geq C\|V\|_{1,\Omega}^2 \qquad \forall\, V \in \mathcal{V}^s. \tag{5.1}$$

Thus, by using the standard argument, we obtain the following theorem.

Theorem 5.1. *Let $U \in \mathcal{V}^s \cap [H^{p+1}(\Omega)]^6$, $U_h^s \in \mathcal{V}_h^s$ be the solutions of* $(2.14a/b')$ *and* (3.7), *respectively. Then*

$$\|U - U_h^s\|_{1,\Omega} \leq Ch^p\|U\|_{p+1,\Omega}. \tag{5.2}$$

Proof. Utilizing (5.1) and (3.20), we have

$$\begin{aligned}\|U - U_h^s\|_{1,\Omega}^2 &\leq C\mathcal{B}^s(U - U_h^s, U - U_h^s)\\ &= C\mathcal{B}^s(U - U_h^s, U - V_h) \qquad \forall\, V_h \in \mathcal{V}_h^s\\ &\leq C\|U - U_h^s\|_{1,\Omega}\|U - V_h\|_{1,\Omega},\end{aligned}$$

Thus,

$$\|U - U_h^s\|_{1,\Omega} \leq C\|U - V_h\|_{1,\Omega} \qquad \forall\, V_h \in \mathcal{V}_h^s.$$

Taking $V_h \in \mathcal{V}_h^s$ such that (3.8) holds with V replaced by U, we obtain (5.2). ∎

Theorem 5.1 shows that the SLSFEM (3.7) achieves optimal convergence in the H^1-norm. For deriving the optimal L^2-estimates, we need the following regularity assumption. Assume that, for any $V \in [H_0^1(\Omega)]^6$ and $Q \in [H^{\frac{1}{2}}(\partial\Omega)]^3$, the unique solution $\widetilde{U}$ to the following problem

$$\begin{aligned}\mathcal{L}\widetilde{U} &= V \qquad \text{in } \Omega,\\ \mathcal{R}\widetilde{U} &= Q \qquad \text{on } \partial\Omega\end{aligned} \tag{5.3}$$

belongs to $[H^2(\Omega)]^6$, where $\mathcal{R}$ is the displacement boundary operator (cf. $(2.14b')$). This assumption is reasonable since $\mathcal{L}$ is a first-order differential operator.

Theorem 5.2. *Let $U \in \mathcal{V}^s \cap [H^{p+1}(\Omega)]^6$, $U_h^s \in \mathcal{V}_h^s$ be the solutions of* $(2.14a/b')$ *and* (3.7), *respectively. If the regularity assumption of* (5.3) *holds with $Q = \mathbf{0}$,* *then*

$$\|U - U_h^s\|_{0,\Omega} \le Ch^{p+1}\|U\|_{p+1,\Omega}. \tag{5.4}$$

Proof. For $V \in [H_0^1(\Omega)]^6$, let $\widetilde{U} \in [H^2(\Omega)]^6$ be the solution of (5.3) with $Q = \mathbf{0}$. Then,

$$\begin{aligned}
|(\mathcal{L}(U - U_h^s), V)_{0,\Omega}| &= |(\mathcal{L}(U - U_h^s), \mathcal{L}\widetilde{U})_{0,\Omega}| \\
&= |(\mathcal{L}(U - U_h^s), \mathcal{L}(\widetilde{U} - V_h))_{0,\Omega}| \quad \forall\, V_h \in \mathcal{V}_h^s \quad (\text{by } (3.20)) \\
&\le C\|\mathcal{L}(U - U_h^s)\|_{0,\Omega}\|\mathcal{L}(\widetilde{U} - V_h)\|_{0,\Omega} \quad \forall\, V_h \in \mathcal{V}_h^s \\
&\le C\|U - U_h^s\|_{1,\Omega}\|\widetilde{U} - V_h\|_{1,\Omega} \quad \forall\, V_h \in \mathcal{V}_h^s \\
&\le Ch\|U - U_h^s\|_{1,\Omega}\|\widetilde{U}\|_{2,\Omega} \quad (\text{by } (3.8)) \\
&\le Ch\|U - U_h^s\|_{1,\Omega}\|\mathcal{L}\widetilde{U}\|_{1,\Omega} \quad (\text{by } (4.2)) \\
&= Ch\|U - U_h^s\|_{1,\Omega}\|V\|_{1,\Omega}.
\end{aligned}$$

In addition, the L^2-inner product $(\mathcal{L}(U - U_h^s), V)_{0,\Omega}$ defines a bounded linear functional on $[H_0^1(\Omega)]^6$ since

$$|(\mathcal{L}(U - U_h^s), V)_{0,\Omega}| \le \|\mathcal{L}(U - U_h^s)\|_{0,\Omega}\|V\|_{1,\Omega} \qquad \forall\, V \in [H_0^1(\Omega)]^6.$$

It follows that

$$\|\mathcal{L}(U - U_h^s)\|_{-1,\Omega} \le Ch\|U - U_h^s\|_{1,\Omega}. \tag{5.5}$$

Therefore, the proof is completed by combining (5.5), (4.4), and (5.2). ∎

Remark 5.3. *As a special case of Theorem 5.2, we prove a conjecture made in [20] in which the SLSFEM is applied to the 2D stress-pressure-velocity Stokes equations with the displacement boundary conditions (2.14b′), and numerical experiments contained therein predict that the rate of convergence in the L^2-norm is optimal.* ∎

5.2. *ERROR ESTIMATES FOR THE WLSFEM*

Following the techniques developed in [36, pp. 352-356], we shall first present the optimal L^2-estimates and then the optimal H^1-estimates for the WLSFEM.

Similar to the proof of part (v) in Theorem 3.2, we note that, for any $W \in [H^{p+1}(\Omega)]^6$, $p \ge 0$ integer, there exists $W_h \in \mathcal{V}_h^w$ such that

$$\|W - W_h\|_w \le Ch^p\|W\|_{p+1,\Omega}, \tag{5.6}$$

where C is a positive constant independent of h.

Theorem 5.4. *Let $U \in \mathcal{V}^w \cap [H^{p+1}(\Omega)]^6$, $U_h^w \in \mathcal{V}_h^w$ be the solutions of (2.14a/b′) and (3.15), respectively. Assume that the regularity assumption of (5.3) holds, then*

$$\|U - U_h^w\|_{0,\Omega} \le Ch^{p+1}\|U\|_{p+1,\Omega}. \tag{5.7}$$

Proof. For $V \in [H_0^1(\Omega)]^6$, let $\widetilde{U} \in [H^2(\Omega)]^6$ be the solution of (5.3) with $Q = \mathbf{0}$. Then,

$$
\begin{aligned}
&|(\mathcal{L}(U-U_h^w),V)_{0,\Omega}| \\
&\quad = |(\mathcal{L}(U-U_h^w),\mathcal{L}\widetilde{U})_{0,\Omega}| \\
&\quad = |\mathcal{B}^w(U-U_h^w,\widetilde{U})| \\
&\quad = |\mathcal{B}^w(U-U_h^w,\widetilde{U}-V_h)| \quad \forall\, V_h \in \mathcal{V}_h^w \quad \text{(by (3.24))} \\
&\quad \le \left\{\mathcal{B}^w(U-U_h^w,U-U_h^w)\right\}^{\frac{1}{2}}\left\{\mathcal{B}^w(\widetilde{U}-V_h,\widetilde{U}-V_h)\right\}^{\frac{1}{2}} \quad \forall\, V_h \in \mathcal{V}_h^w \\
&\quad \le Ch\left\{\mathcal{B}^w(U-U_h^w,U-U_h^w)\right\}^{\frac{1}{2}}\|\widetilde{U}\|_{2,\Omega} \quad \text{(by (5.6))} \\
&\quad \le Chh^p\|U\|_{p+1,\Omega}\|\mathcal{L}\widetilde{U}\|_{1,\Omega} \quad \text{(by (3.26), (4.5))} \\
&\quad = Ch^{p+1}\|U\|_{p+1,\Omega}\|V\|_{1,\Omega}.
\end{aligned}
$$

It follows that

$$\|\mathcal{L}(U-U_h^w)\|_{-1,\Omega} \le Ch^{p+1}\|U\|_{p+1,\Omega}. \tag{5.8}$$

On the other hand, take $V = \mathbf{0} \in [H_0^1(\Omega)]^6$ in (5.3), then for any $Q \in [H^{\frac{1}{2}}(\partial\Omega)]^3$,

$$
\begin{aligned}
&|h^{-1}(\mathcal{R}(U-U_h^w),Q)_{0,\partial\Omega}| \\
&\quad = |h^{-1}(\mathcal{R}(U-U_h^w),\mathcal{R}\widetilde{U})_{0,\partial\Omega}| \\
&\quad = |\mathcal{B}^w(U-U_h^w,\widetilde{U})| \\
&\quad = |\mathcal{B}^w(U-U_h^w,\widetilde{U}-V_h)| \quad \forall\, V_h \in \mathcal{V}_h^w \quad \text{(by (3.24))} \\
&\quad \le \left\{\mathcal{B}^w(U-U_h^w,U-U_h^w)\right\}^{\frac{1}{2}}\left\{\mathcal{B}^w(\widetilde{U}-V_h,\widetilde{U}-V_h)\right\}^{\frac{1}{2}} \quad \forall\, V_h \in \mathcal{V}_h^w \\
&\quad \le C\left\{\mathcal{B}^w(U-U_h^w,U-U_h^w)\right\}^{\frac{1}{2}}\|\widetilde{U}\|_{1,\Omega} \quad \text{(by (5.6))} \\
&\quad \le Ch^p\|U\|_{p+1,\Omega}\|\mathcal{R}\widetilde{U}\|_{\frac{1}{2},\partial\Omega} \quad \text{(by (3.26), (4.6))} \\
&\quad = Ch^p\|U\|_{p+1,\Omega}\|Q\|_{\frac{1}{2},\partial\Omega}.
\end{aligned}
$$

Thus, for any $Q \in [H^{\frac{1}{2}}(\partial\Omega)]^3$ we have

$$|(\mathcal{R}(U-U_h^w),Q)_{0,\partial\Omega}| \le Ch^{p+1}\|U\|_{p+1,\Omega}\|Q\|_{\frac{1}{2},\partial\Omega},$$

and so

$$\|\mathcal{R}(U-U_h^w)\|_{-\frac{1}{2},\partial\Omega} \le Ch^{p+1}\|U\|_{p+1,\Omega}. \tag{5.9}$$

Therefore, the proof is completed by combining (4.7), (5.8), and (5.9). ∎

Note that in the proof of Theorem 5.4, we utilize the estimates (3.26) to circumvent the use of the optimal H^1-estimates which have not been yet established. In order to give the optimal H^1-estimates, we need to define the following Gauss projection [36]:

$$\mathcal{G}_h:\ \mathcal{V}^w \to \mathcal{V}_h^w, \qquad \mathcal{G}_h W \equiv W_h^w, \tag{5.10}$$

where W_h^w is the solution of the discretized problem (3.15) corresponding to problem (2.14a/b') with suitable data function F such that its unique exact solution is W. Since problem (3.15) is uniquely solvable, the Gauss mapping $\mathcal{G}_h$ is well-defined, and we have

$$\mathcal{G}_h V_h = V_h \qquad \forall\, V_h \in \mathcal{V}_h^w. \tag{5.11}$$

Taking $p = 0$ in (5.7), we get

$$\begin{aligned}\|\mathcal{G}_h U\|_{0,\Omega} &= \|U_h^w\|_{0,\Omega} \\ &\le \|U\|_{0,\Omega} + \|U - U_h^w\|_{0,\Omega} \\ &\le \|U\|_{0,\Omega} + Ch\|U\|_{1,\Omega}.\end{aligned}$$

Thus, we can conclude that, for any $V \in \mathcal{V}^w$,

$$\|\mathcal{G}_h V\|_{0,\Omega} \le \|V\|_{0,\Omega} + Ch\|V\|_{1,\Omega}. \tag{5.12}$$

We also need the following inverse assumption on the finite element space $\mathcal{V}_h^w$. There exists a constant $C > 0$ independent of h such that

$$\|V_h\|_{1,\Omega} \le Ch^{-1}\|V_h\|_{0,\Omega} \qquad \forall\, V_h \in \mathcal{V}_h^w. \tag{5.13}$$

The inverse assumption is commonly used in many least squares finite element analyses [3, 36]. More precisely, if the regular family $\{\mathcal{T}_h\}$ of triangulations of $\overline{\Omega}$ associated with the finite element space $\mathcal{V}_h^w$ is quasi-uniform [22, 31], i.e., there exists a positive constant C independent of h such that

$$h \le C\ \mathrm{diam}(\Omega_i^h), \qquad \forall\, \Omega_i^h \in \mathcal{T}_h, \mathcal{T}_h \in \{\mathcal{T}_h\},$$

then (5.13) is satisfied.

The optimal order of convergence for the WLSFEM in the H^1-norm is thus concluded.

Theorem 5.5. *Let $U \in \mathcal{V}^w \cap [H^{p+1}(\Omega)]^6$, $U_h^w \in \mathcal{V}_h^w$ be the solutions of* (2.14a/b') *and* (3.15), *respectively. Suppose that the regularity assumption of* (5.3) *and the inverse assumption* (5.13) *hold, then*

$$\|U - U_h^w\|_{1,\Omega} \le Ch^p\|U\|_{p+1,\Omega}. \tag{5.14}$$

Proof. By (5.11), (5.13), (5.12), and the approximation property (3.16), we have

$$\begin{aligned}\|U - U_h^w\|_{1,\Omega} &\le \|U - V_h\|_{1,\Omega} + \|U_h^w - V_h\|_{1,\Omega}\ \forall\, V_h \in \mathcal{V}_h^w \\ &= \|U - V_h\|_{1,\Omega} + \|\mathcal{G}_h(U - V_h)\|_{1,\Omega}\ \forall\, V_h \in \mathcal{V}_h^w \\ &\le \|U - V_h\|_{1,\Omega} + Ch^{-1}\|\mathcal{G}_h(U - V_h)\|_{0,\Omega}\ \forall\, V_h \in \mathcal{V}_h^w\end{aligned}$$

Table 6.1. The SLSFE approximations with $E = 2.5$ and $\nu = 0.25$ ($\epsilon = 2.0$)

1/h	$\|\mathbf{e}\|_{0,\Omega}$	RelErr	conv. rate	$\|\mathbf{e}\|_s$	RelErr	conv. rate
2	0.92226	3.05125e-1	–	7.25450	4.64876e-1	–
4	0.26879	8.89285e-2	1.78	3.58707	2.29863e-1	1.02
8	0.07245	2.39709e-2	1.89	1.80001	1.15347e-1	0.99
16	0.01861	6.15548e-3	1.96	0.90161	5.77763e-2	1.00
32	0.00469	1.55220e-3	1.99	0.45105	2.89037e-2	1.00

Table 6.2. Rates of convergence in the $\|\cdot\|_s$-norm with $E = 2.5$ and small ϵ

	ν=0.49	ν=0.499	ν=0.4999	ν=0.49999	ν=0.499999
1/h	$\epsilon \simeq$4.1e-2	$\epsilon \simeq$4.0e-3	$\epsilon \simeq$4.0e-4	$\epsilon \simeq$4.0e-5	$\epsilon \simeq$4.0e-6
2	–	–	–	–	–
4	0.97	0.96	0.96	0.96	0.96
8	0.98	0.98	0.98	0.98	0.98
16	0.99	0.99	0.99	0.99	0.99
32	1.00	1.00	1.00	1.00	1.00

Table 6.3. Rates of convergence in the L^2-norm with $E = 2.5$ and small ϵ

	ν=0.49	ν=0.499	ν=0.4999	ν=0.49999	ν=0.499999
1/h	$\epsilon \simeq$4.1e-2	$\epsilon \simeq$4.0e-3	$\epsilon \simeq$4.0e-4	$\epsilon \simeq$4.0e-5	$\epsilon \simeq$4.0e-6
2	–	–	–	–	–
4	1.77	1.75	1.75	1.75	1.75
8	1.89	1.88	1.88	1.88	1.88
16	1.96	1.95	1.95	1.95	1.95
32	1.99	1.98	1.98	1.98	1.98

$$\leq \|U - V_h\|_{1,\Omega} + Ch^{-1}\Big\{\|U - V_h\|_{0,\Omega} + Ch\|U - V_h\|_{1,\Omega}\Big\} \;\forall\, V_h \in \mathcal{V}_h^w$$
$$\leq Ch^p\|U\|_{p+1,\Omega}.$$

∎

6. NUMERICAL EXPERIMENTS

We shall give a simple example which will be solved by using the SLSFEM (3.7). Consider the stress-pressure-displacement elasticity equations (2.14a) equipped with the homogeneous displacement boundary conditions (2.14b'). Taking $\Omega = (0,1) \times (0,1)$ and choosing

$$f_1 = 2\mu\pi^2\Big\{(\frac{3}{2} + \frac{1}{\epsilon})\sin(\pi x)\sin(\pi y) - (\frac{1}{2} + \frac{1}{\epsilon})\cos(\pi x)\cos(\pi y)\Big\},$$
$$f_2 = 2\mu\pi^2\Big\{(\frac{3}{2} + \frac{1}{\epsilon})\sin(\pi x)\sin(\pi y) - (\frac{1}{2} + \frac{1}{\epsilon})\cos(\pi x)\cos(\pi y)\Big\},$$

the exact solution is then given by

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ p \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \pi\cos(\pi x)\sin(\pi y) \\ \pi\sin(\pi x)\cos(\pi y) \\ \pi\cos(\pi x)\sin(\pi y) \\ -\frac{\pi}{\epsilon}\Big(\cos(\pi x)\sin(\pi y) + \sin(\pi x)\cos(\pi y)\Big) \\ \sin(\pi x)\sin(\pi y) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix}.$$

To simplify the numerical implementation, we shall assume that the square domain Ω is uniformly partitioned into a set of $1/h^2$ square subdomains Ω_i^h with side-length h. Piecewise bilinear finite elements are used to approximate all components of the exact solution. For the case of Poisson's ratio $\nu = 0.25$ ($\epsilon = 2.0$) and Young's modulus $E = 2.5$, the results are collected in Table 6.1, where **e** denotes the exact error $U - U_h^s$ and RelErr denotes the relative error. Since the H^1-norm is equivalent to the $\|\cdot\|_s$-norm for the standard least squares case, Table 6.1 exhibits that the SLSFEM achieves optimal converegnce both in the L^2-norm and in the H^1-norm for all the components.

The influence by the nonnegative parameter ϵ for the behavior of convergence is also examined. It is interesting to note that the SLSFEM for the elasticity problem in the new formulation does not exhibit any significant numerical locking, i.e., the results of Tables 6.2 and Table 6.3 do not deteriorate as $\epsilon \to 0^+$ (i.e., $\nu \to 0.5^-$). The locking phenomenon is of major concern for the standard and mixed finite element methods [2, 6, 7]. With the numerical evidence shown in Table 6.2 and Table 6.3, a theoretical verification about possible improvement in regard to the locking problem by the present methods appears to be promising.

ACKNOWLEDGMENTS

The authors would like to thank Professor C. L. Chang (Department of Mathematics, Cleveland State University, Cleveland, Ohio, U.S.A.) for helpful discussions when he visited the Department of Applied Mathematics of the National Chiao Tung University, Taiwan, in the summer of 1996.

REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, *Comm. Pure Appl. Math.* **17** (1964) 35-92.

[2] D. N. Arnold, Discretization by finite elements of a model parameter dependent problem, *Numer. Math.* **37** (1981) 405-421.

[3] A. K. Aziz, R. B. Kellogg, and A. B. Stephens, Least squares methods for elliptic systems, *Math. Comp.* **44** (1985) 53-70.

[4] A. K. Aziz and J.-L. Liu, A weighted least squares method for the backward-forward heat equation, *SIAM J. Numer. Anal.* **28** (1991) 156-167.

[5] A. K. Aziz and A. Werschulz, On the numerical solution of Helmholtz's equation by the finite element method, *SIAM J. Numer. Anal.* **17** (1980) 681-686.

[6] I. Babuška and M. Suri, On locking and robustness in the finite element method, *SIAM J. Numer. Anal.* **29** (1992) 1261-1293.

[7] I. Babuška and M. Suri, Locking effects in the finite element approximation of elasticity problems, *Numer. Math.* **62** (1992) 439-463.

[8] P. B. Bochev and M. D. Gunzburger, Analysis of least squares finite element methods for the Stokes equations, *Math. Comp.* **63** (1994) 479-506.

[9] J. H. Bramble and J. E. Pasciak, Least-squares methods for Stokes equations based on a discrete minus one inner product, preprint, June 1995.

[10] J. H. Bramble and A. H. Schatz, Rayleigh-Ritz-Galerkin methods for Dirichlet's problem using subspaces without boundary conditions, *Comm. Pure Appl. Math.* **23** (1970) 653-675.

[11] J. H. Bramble and V. Thomée, Semidiscrete-least squares methods for a parabolic boundary value problem, *Math. Comp.* **26** (1972) 633-648.

[12] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods* (Springer-Verlag, New York, 1991).

[13] Z. Cai, R. Lazarov, T. A. Manteuffel, and S. F. McCormick, First-order system least squares for second-order partial differential equations: Part I, *SIAM J. Numer. Anal.* **31** (1994) 1785-1799.

[14] C. L. Chang, A mixed finite element method for the Stokes problem: an acceleration-pressure formulation, *Appl. Math. Comput.* **36** (1990) 135-146.

[15] C. L. Chang, Finite element approximation for grad-div type systems in the plane, *SIAM J. Numer. Anal.* **29** (1992) 452-461.

[16] C. L. Chang, Least-squares finite elements for second-order boundary value problems with optimal rates of convergence, *Appl. Math. Comput.* **76** (1996) 267-284.

[17] C. L. Chang and B.-N. Jiang, An error analysis of least squares finite element method of velocity-pressure-vorticity formulation for Stokes problem, *Comput. Methods Appl. Mech. Engrg.* **84** (1990) 247-255.

[18] C. L. Chang and J. J. Nelson, Least-squares finite element method for the Stokes problem with zero residual of mass conservation, preprint, May 1995, to appear in *SIAM J. Numer. Anal.*.

[19] C. L. Chang and S.-Y. Yang, Piecewise linear approximations to the Stokes problem with velocity boundary conditions, preprint, November 1996. Submitted to *SIAM J. Numer. Anal.*.

[20] C. L. Chang, S.-Y. Yang, and C.-H. Hsu, A least-squares finite element method for incompressible flow in stress-velocity-pressure version, *Comput. Methods Appl. Mech. Engrg.* **128** (1995) 1-9.

[21] T.-F. Chen, On least squares approximations to compressible flow problems, *Numerical Methods for PDEs*, **12** (1986) 207-228.

[22] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, Amsterdam, 1978).

[23] A. S. Dikanskij, Conjugate problems of elliptic differential and pseudo differential boundary value problems in a bounded domain, *Math. USSR Sbornik* **20** (1973) 67-83.

[24] G. J. Fix, M. D. Gunzburger, and R. A. Nicolaides, On finite element methods of the least squares type, *Computer Math. Applic.* **5** (1979) 87-98.

[25] G. J. Fix and M. E. Rose, A comparative study of finite element and finite difference methods for Cauchy-Riemann type equations, *SIAM J. Numer. Anal.* **22** (1985) 250-261.

[26] L. P. Franca and T. J. R. Hughes, Two classes of mixed finite element methods, *Comput. Methods Appl. Mech. Engrg.* **69** (1988) 89-129.

[27] L. P. Franca, T. J. R. Hughes, A. F. D. Loula, and I. Miranda, A new family of stable elements for nearly incompressible elasticity based on a mixed Petrov-Galerkin finite element formulation, *Numer. Math.* **53** (1988) 123-141.

[28] L. P. Franca and R. Stenberg, Error analysis of some Galerkin least squares methods for the elasticity equations, *SIAM J. Numer. Anal.* **28** (1991) 1680-1697.

[29] B.-N. Jiang, A least-squares finite element method for incompressible Navier-Stokes problem, *Internat. J. Numer. Meth. Fluids* **14** (1992) 843-859.

[30] B.-N. Jiang and C. L. Chang, Least-squares finite elements for the Stokes problem, *Comput. Methods Appl. Mech. Engrg.* **78** (1990) 297-311.

[31] C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method* (Cambridge University Press, 1990).

[32] J. L. Lions and E. Magenes, *Nonhomogeneous Elliptic Boundary Value Problems and Applications, Vol. I* (Springer-Verlag, Berlin, 1972).

[33] A. I. Pehlivanov, G. F. Carey, and R. D. Lazarov, Least-squares mixed finite elements for second-order elliptic problems, *SIAM J. Numer. Anal.* **31** (1994) 1368-1377.

[34] J. Pitkäranta and R. Stenberg, Analysis of some mixed finite element methods for plane elasticity equations, *Math. Comp.* **41** (1983) 399-423.

[35] R. Stenberg, On the construction of optimal mixed finite element methods for the linear elasticity problem, *Numer. Math.* **48** (1986) 447-462.

[36] W. L. Wendland, *Elliptic Systems in the Plane* (Pitman, London, 1979).