

Hölder continuity for two-phase flows in porous media

Li-Ming Yeh^{*,†}

*Department of Applied Mathematics, National Chiao Tung University,
Hsinchu 30050, Taiwan, R.O.C.*

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SUMMARY

This work is to prove the Hölder continuity of the solutions of the degenerate differential equations describing two-phase, incompressible, immiscible flows in porous media. The differential equations allow degeneracy at two end points and the assumption on mild degeneracy is not required in this study. The regularity result is proved by an alternative argument. Uniqueness of the weak solutions of the differential equations is a direct consequence from this Hölder continuity. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: two-phase flow; global pressure; alternative argument

1. INTRODUCTION

Hölder continuity of the solutions of the degenerate differential equations describing two-phase, incompressible, immiscible flows in porous media is concerned. The existence of weak solutions of the differential equations is well-known (see References [1–7] and references therein). However, the regularity of the weak solutions has not been well-established. In this work, we show Hölder continuity of the weak solutions for the differential equations. Uniqueness of the weak solutions is then a direct consequence from this result. If $\Omega \subset \mathbb{R}^N$ ($N = 3$ in reality) is a porous medium, equations for the two-phase flows in porous media in global pressure formulation are (see References [2,4]),

*Correspondence to: Li-Ming Yeh, Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan, R.O.C.

†E-mail: liming@math.nctu.edu.tw

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for $t > 0$,

$$\Phi \partial_t S - \nabla \cdot \left(K \Lambda_w(S) \nabla (P - E_w) - K \frac{\Lambda_w \Lambda_o}{\Lambda} \nabla \Upsilon(S) \right) = 0 \tag{1}$$

$$-\nabla \cdot (K \Lambda(S) \nabla P - K \Lambda_w(S) \nabla E_w - K \Lambda_o(S) \nabla E_o) = 0 \tag{2}$$

Here Φ is porosity, K is absolute permeability field, $S \in [0, 1]$ is water saturation, Λ_i ($i = w, o$) is phase mobility of i -phase and is a nonnegative monotone function of S , Λ ($\equiv \Lambda_w + \Lambda_o$) is the total mobility, P denotes global pressure, E_i ($i = w, o$) is a function depending on density, gravity, and position, and Υ is capillary pressure and is a nonnegative decreasing function of S . In practice, $(\Lambda_w \Lambda_o / \Lambda) \Upsilon'(0) = (\Lambda_w \Lambda_o / \Lambda) \Upsilon'(1) = 0$ [1,2,4]. So Equation (1) is a degenerate parabolic equation with degeneracy at two end points (that is, degeneracy appears at $S = 0, 1$).

Boundary $\partial\Omega$ of the porous medium Ω includes Γ_1 and Γ_2 satisfying $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$. The initial and boundary conditions are given by

$$\left\{ \begin{array}{ll} K(\Lambda_w(S) \nabla (P - E_w) - \frac{\Lambda_w \Lambda_o}{\Lambda} \nabla \Upsilon(S)) \cdot \mathbf{n} = 0 & \text{for } x \in \Gamma_1 \\ K(\Lambda(S) \nabla P - \Lambda_w(S) \nabla E_w - \Lambda_o(S) \nabla E_o) \cdot \mathbf{n} = 0 & \text{for } x \in \Gamma_1 \\ S = S_b & \text{for } x \in \Gamma_2 \\ P = P_b & \text{for } x \in \Gamma_2 \\ S(0, x) = S_{\text{init}}(x) & \text{for } x \in \Omega \end{array} \right. \tag{3}$$

where \mathbf{n} is the unit vector outward normal to Γ_1 .

Regularity results of the weak solutions for porous media problems in nondegenerate case are well-known (see References [2,4,6] and reference therein). Continuity of saturation S for (1)–(3) in interior region of Ω^T ($\equiv (0, T) \times \Omega$) had been shown in Reference [1] if mild degeneracy was assumed at one end point. Hölder continuity of S with degeneracy only at one end point was considered in Reference [5]. In this work, we prove Hölder continuity of S for the case where equations are degenerate at two end points and no mild degeneracy is assumed. Though initial and boundary values of saturation are assumed to be away from the two end points 0 and 1 (see A5 below), the saturation inside the domain Ω^T still can reach 0 and 1. The process of proof is first to derive a uniform Hölder estimate for the solutions of regularized problems of (1)–(3). Then by compactness principle we get Hölder continuity of the solution of (1)–(3). Rest of the paper is organized as follows: Notation and main result are stated in Section 2. In Sections 3–5, we shall derive a uniform Hölder estimate for the solutions of the regularized problems of (1)–(3) by an alternative argument [8]. More precisely, in Section 3 we state some auxiliary results needed in Section 4. Hölder estimate of the solutions for the regularized equations in the interior region is given in Section 4. Hölder estimate of the solutions for the regularized equations on the parabolic boundary can be proved by a similar argument as that for interior region and is sketched in Section 5. Proof of main result is in Section 6.

2. NOTATION AND MAIN RESULT

We shall use the following notation:

$$\left\{ \begin{array}{l}
 \Gamma_i^T \equiv (0, T) \times \Gamma_i, \quad i = 1, 2 \\
 \mathcal{U} \equiv \{ \zeta \in H^1(\Omega) : \zeta|_{\Gamma_2} = 0 \} \\
 \mathcal{V}(\Omega^T) \equiv L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{U}) \\
 \text{dual } X \equiv \text{dual space of } X \\
 L^{q,r}(\Omega^T) \equiv L^r(0, T; L^q(\Omega)) \\
 \alpha \equiv \frac{2}{N} - \frac{2}{\mathbf{k}_1} \quad (\mathbf{k}_1 \text{ is given in A4 below}) \\
 \lambda \equiv \min \left\{ \frac{2}{N+2}, \alpha \right\} \\
 \mathcal{J}(z) \equiv \frac{-\Lambda_w \Lambda_o}{\Lambda} \Upsilon'(z), \quad z \in (0, 1) \\
 \mathcal{R}(z) \equiv \int_0^z \mathcal{J}(\xi) \, d\xi, \quad z \in (0, 1) \\
 \mathcal{X}_D \text{ is a characteristic function of } D
 \end{array} \right. \tag{4}$$

Definition 2.1

Boundary $\partial\Omega$ of the bounded domain Ω belongs to \mathbf{H}^m , $m \geq 1$, if (1) in the neighbourhood $U(x)$ of each boundary point $x \notin \bar{\Gamma}_1 \cap \bar{\Gamma}_2$ there exists a homeomorphic transformation $\tilde{x}(x) = (\tilde{x}_1(x), \tilde{x}_2(x), \dots, \tilde{x}_N(x)) \in C^m$, $|\tilde{d}\tilde{x}/dx| \geq c > 0$ ($\tilde{d}\tilde{x}/dx$ is the Jacobian of the transformation) such that $\tilde{x}_N(\partial\Omega \cap U(x)) = 0, \tilde{x}_N(\Omega \cap U(x)) > 0$, i.e. $\Gamma_i (i = 1, 2)$ can be locally straightened, (2) in the neighbourhood of each point $x \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2$ there exists a transformation $\tilde{x} = \tilde{x}(x)$ with the same properties mapping it at the neighbourhood of the edge of a cube in variable \tilde{x} .

Next we make the following assumptions:

- A1. $\partial\Omega \in \mathbf{H}^1_*$,
- A2. Λ_w (resp. Λ_o) : $[0, 1] \rightarrow [0, 1]$ is continuous and increasing (resp. decreasing), $\Lambda_w(0) = \Lambda_o(1) = 0, \Lambda_w \Lambda_o(z)|_{z \in (0,1)} \neq 0, \inf_{z \in [0,1]} \Lambda(z) > 0$,
- A3. $\Upsilon : (0, 1] \rightarrow \mathfrak{R}^+_0$ is onto, decreasing, and a locally Lipschitz continuous function, and $\inf_{z \in (0,1]} |\Upsilon'(z)| > 0, (\Lambda_w \Lambda_o / \Lambda) \Upsilon'(z) \in L^\infty((0, 1))$,
- A4. $0 < K \in W^{1,\infty}(\Omega), E_w, E_o \in L^\infty(0, T; W^{1,\infty}(\Omega)), P_b \in L^\infty(0, T; W^{1,\mathbf{k}_1}(\Omega)), \mathbf{k}_1 > N$,
- A5. $S_b, S_{\text{init}} \in L^2(0, T; H^1(\Omega)) \cap C^{0,\mathbf{k}_2}(\bar{\Omega}^T), \partial_t \Upsilon(S_b) \in L^1(\Omega^T), S_b, S_{\text{init}} \in (\mathbf{k}_2, 1 - \mathbf{k}_2), S_{\text{init}}|_{\Gamma_2} = S_b|_{\Gamma_2}(t = 0), \Phi \in (\mathbf{k}_3, \mathbf{k}_4)$,
- A6. $\max_{z \in [0,1]} |\Lambda(z) - 1| + |(\mathbf{k}_4/\mathbf{k}_3) - 1| < \mathbf{k}_5$ (\mathbf{k}_5 is small and depends only on Ω, K),
- A7. $\Lambda_w \Lambda_o(z) \propto z|1 - z|\sqrt{\mathcal{J}(z)}$ and $\mathcal{J}(z) \propto z^{\mathbf{m}}|1 - z|^{\mathbf{m}_1}$ for $z \in (0, \vartheta) \cup (1 - \vartheta, 1)$,

where $\mathbf{k}_i (i = 1, \dots, 5), \mathbf{m}, \mathbf{m}_1$ are positive constants, and $\vartheta \in (0, 1/8)$ is a number such that \mathcal{J} is increasing (resp. decreasing) in $(0, \vartheta)$ (resp. $(1 - \vartheta, 1)$).

Some remarks about the assumptions are given below: A1 will be used to derive the regularity of water saturation S and global pressure P around the edge $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$. From A3,

$|\Upsilon'(z)|$ may tend to infinity as $z \rightarrow 0$ or 1 , a property for a capillary pressure function [2,4]. $(\Lambda_w \Lambda_o / \Lambda) \Upsilon'(z) \in L^\infty((0, 1))$ allows equation (1) to be a degenerate parabolic equation, a characteristic of the porous medium equation (PME) [2,4]. A5 means initial and boundary values of the saturation are away from the two end points 0 and 1. However, the saturation inside the domain Ω^T still can reach 0 and 1. The assumption $\max_{z \in [0, 1]} |\Lambda(z) - 1| < \mathbf{k}_5$ in A6 is used in Lemma 3.1 and $|\frac{\mathbf{k}_4}{\mathbf{k}_3} - 1| < \mathbf{k}_5$ is used in Lemma 4.4. The explicit restriction of \mathbf{k}_5 can be found in Reference [2, p. 224, Theorem 4.2] and in the proof of Lemma 4.4. A7 gives restrictions on water (resp. oil) mobility function and capillary pressure around the neighbourhood of 0 (resp. 1). The assumptions on the two end points 0 and 1 are similar, so one can expect water saturation in the neighbourhood of the two end points 0 and 1 has similar properties. These properties are crucial in the regularity proof of saturation in this work. Since we assume \mathbf{m}, \mathbf{m}_1 are positive constants in A7, no mild degeneracy is required. Our main result is

Theorem 2.1

Under A1–A7, saturation S of (1)–(3) is Hölder continuous in $\overline{\Omega}^T$.

Under A1–A7 as well as $\Lambda \in W^{1,\infty}(\mathfrak{R})$ and $P_b, E_w, E_o \in L^\infty(0, T; C^{1,\mathbf{k}}(\overline{\Omega}))$, we have $P \in L^\infty(0, T; W^{1,\infty}(\Omega))$ by Theorem 2.1 and Corollary 8.35 of Reference [9]. Therefore, by Theorem 2.3 [7], we get uniqueness of weak solution of (1)–(3).

3. SOME AUXILIARY LEMMAS

We first derive regularized equations of (1)–(3). Let ε be a small number satisfying $0 < \varepsilon < \mathbf{k}_2/4$. Extend Λ_i ($i = w, o$) constantly and continuously to \mathfrak{R} and define $\tilde{\Lambda}_i^\varepsilon, \tilde{\Lambda}^\varepsilon$ as

$$\tilde{\Lambda}_i^\varepsilon(z) \equiv \Lambda_i \left(0.5 \left(\frac{z - \varepsilon}{0.5 - \varepsilon} \right) \right), \quad \tilde{\Lambda}^\varepsilon(z) \equiv \tilde{\Lambda}_w^\varepsilon(z) + \tilde{\Lambda}_o^\varepsilon(z) \tag{5}$$

By A5, there exist smooth functions $S_{\text{init}}^\varepsilon, S_b^\varepsilon$ such that

$$S_{\text{init}}^\varepsilon, S_b^\varepsilon \in \left(\frac{\mathbf{k}_2}{2}, 1 - \frac{\mathbf{k}_2}{2} \right), \quad S_{\text{init}}^\varepsilon|_{\Gamma_2} = S_b^\varepsilon|_{\Gamma_2}(t = 0) \tag{6}$$

$$\begin{cases} S_{\text{init}}^\varepsilon, S_b^\varepsilon \rightarrow S_{\text{init}}, S_b & \text{in } L^2(0, T; H^1(\Omega)) \cap C^{0,\mathbf{k}_2}(\overline{\Omega}^T) \\ \partial_t \Upsilon(S_b^\varepsilon) \rightarrow \partial_t \Upsilon(S_b) & \text{in } L^1(\Omega^T) \end{cases} \quad \text{as } \varepsilon \rightarrow 0 \tag{7}$$

The regularized problem is: Find $\{S^\varepsilon, P^\varepsilon\}$ satisfying

$$\Phi \partial_t S^\varepsilon \in \text{dual } L^2(0, T; \mathcal{W}) \tag{8}$$

$$\varepsilon \leq S^\varepsilon \leq 1 - \varepsilon \tag{9}$$

$$\mathcal{R}(S^\varepsilon) - \mathcal{R}(S_b^\varepsilon), P^\varepsilon - P_b \in L^2(0, T; \mathcal{U}) \tag{10}$$

$$\int_{\Omega^T} \Phi \partial_t S^\varepsilon \zeta + \int_{\Omega^T} K(\tilde{\Lambda}_w^\varepsilon(S^\varepsilon)) \nabla(P^\varepsilon - E_w) + \nabla \mathcal{R}(S^\varepsilon)) \nabla \zeta = 0 \tag{11}$$

$$\int_{\Omega^T} K \tilde{\Lambda}^\varepsilon(S^\varepsilon) \nabla P^\varepsilon \nabla \zeta - \sum_{i \in \{w, o\}} \int_{\Omega^T} K \tilde{\Lambda}_i^\varepsilon(S^\varepsilon) \nabla E_i \nabla \zeta = 0 \tag{12}$$

$$S^\varepsilon(0, x) = S_{\text{init}}^\varepsilon \tag{13}$$

for any $\zeta, \xi \in L^2(0, T; \mathcal{U})$. It is easy to see that for each fixed ε , (11) is a nondegenerate parabolic equation and (11)–(12) imply, if $S_o^\varepsilon \equiv 1 - S^\varepsilon$,

$$\int_{\Omega^T} \Phi \partial_t S_o^\varepsilon \zeta + \int_{\Omega^T} K(\tilde{\Lambda}_o^\varepsilon(1 - S_o^\varepsilon)) \nabla(P^\varepsilon - E_o) - \nabla \mathcal{R}(1 - S_o^\varepsilon)) \nabla \zeta = 0 \tag{14}$$

By References [5,7,10,11], it is known

Lemma 3.1

Under A1–A6 and (6)–(7), (8)–(13) has a weak solution $\{S^\varepsilon, P^\varepsilon\}$ for each ε . Moreover, $\|P^\varepsilon\|_{L^\infty(0, T; W^{1, k_1}(\Omega))}$ is bounded by a constant which is independent of ε , S^ε is Hölder continuous in $\bar{\Omega}^T$, and

$$\begin{cases} S^\varepsilon \rightarrow S & \text{pointwise and in } L^r(\Omega^T), r < \infty \\ \mathcal{R}(S^\varepsilon), P^\varepsilon \rightarrow \mathcal{R}(S), P & \text{in } L^2(0, T; H^1(\Omega)) \end{cases} \text{ as } \varepsilon \rightarrow 0$$

where $\{S, P\}$ is a weak solution of (1)–(3).

We shall prove that the Hölder norm of S^ε is actually bounded by a constant which is independent of ε . If that is so, then by Lemma 3.1, we obtain S is Hölder continuous and complete the proof of Theorem 2.1. From now on, A1–A7 will be assumed throughout this paper, and ε is fixed and dropped for convenience of presentation. Given any constant $\rho > 0$, define the cube

$$\mathcal{K}_\rho \equiv \left\{ x \in \mathbb{R}^N : \max_{1 \leq i \leq N} |x_i| < \rho \right\} \tag{15}$$

For $\hat{x} \in \mathbb{R}^N$, $\hat{x} + \mathcal{K}_\rho$ denotes the cube of centre \hat{x} . Also, for $\theta > 0$ a given number, define $\mathcal{Q}(\theta, \rho) \equiv (-\theta, 0) \times \mathcal{K}_\rho$. For $(\hat{t}, \hat{x}) \in \mathbb{R}^{N+1}$, let $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)$ be the cylinder congruent to $\mathcal{Q}(\theta, \rho)$; i.e. $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) = (\hat{t} - \theta, \hat{t}) \times \{\hat{x} + \mathcal{K}_\rho\}$. In $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)$, we introduce piecewise smooth cut-off functions $\zeta(t, x)$ and $\tilde{\zeta}(x)$ such that both satisfy

$$\tilde{\zeta} \in [0, 1], \quad |\nabla \tilde{\zeta}| < \infty, \quad \tilde{\zeta}(t, x) = 0 \quad \text{for } x \notin \hat{x} + \mathcal{K}_\rho \tag{16}$$

For $\zeta \in L^1(\Omega^T)$ and $0 < h < T$, introduce the Steklov average ζ_h by

$$\zeta_h(t, x) \equiv \begin{cases} \frac{1}{h} \int_t^{t+h} \zeta(\tau, x) \, d\tau & \text{if } 0 < t \leq T - h \\ 0 & \text{if } t > T - h \end{cases}$$

(11) implies the following equation:

$$\int_{\mathcal{K}_\rho} \Phi \partial_t S_h \varphi + \int_{\mathcal{K}_\rho} K(\nabla \mathcal{R}(S) + \tilde{\Lambda}_w(S) \nabla(P - E_w))_h \nabla \varphi = 0 \tag{17}$$

where \mathcal{K}_ρ is any compact subset of Ω , $0 < t < T - h$, $\varphi \in H_0^1(\mathcal{K}_\rho) \cap L^\infty_{loc}(\Omega)$. Domain Ω^T includes three subregions:

$$\begin{cases} \Omega_1^T \equiv \{(t, x) \in \Omega^T : S(t, x) < \vartheta\} \\ \Omega_2^T \equiv \{(t, x) \in \Omega^T : \frac{\vartheta}{4} < S(t, x) < 1 - \frac{\vartheta}{4}\} \\ \Omega_3^T \equiv \{(t, x) \in \Omega^T : 1 - \vartheta < S(t, x)\} \end{cases} \tag{18}$$

Next we derive energy and logarithmic estimates for the interior region Ω^T . For $(\hat{t}, \hat{x}) \in \Omega^T$ fixed, let θ and ρ be small so that $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega^T$. Define a set $\mathcal{D}_{\mathbf{j}, \rho}^\pm(\tau)$ as, for $\tau \in (\hat{t} - \theta, \hat{t})$ and for every level \mathbf{j} ,

$$\mathcal{D}_{\mathbf{j}, \rho}^\pm(\tau) \equiv \{x \in \hat{x} + \mathcal{K}_\rho : (S - \mathbf{j})_\pm(\tau, x) > 0\} \tag{19}$$

where

$$(S - \mathbf{j})_+ \equiv \begin{cases} S - \mathbf{j} & \text{if } S - \mathbf{j} > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (S - \mathbf{j})_- \equiv \begin{cases} \mathbf{j} - S & \text{if } S - \mathbf{j} < 0 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.2

There is a constant d_1 (independent of $\varepsilon, \theta, \rho, \mathbf{j}$) such that for every cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega_1^T$ and every level \mathbf{j} , we have

$$\begin{aligned} & \sup_{\hat{t} - \theta < t < \hat{t}} \int_{\hat{x} + \mathcal{K}_\rho} (S - \mathbf{j})_+^2 \zeta^2 + \int_{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)} \mathcal{J}(S) |\zeta \nabla (S - \mathbf{j})_+|^2 \\ & \leq d_1 \left(\int_{\hat{x} + \mathcal{K}_\rho} (S - \mathbf{j})_+^2 \zeta^2 (\hat{t} - \theta, x) + \int_{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)} \mathcal{J}(S) (S - \mathbf{j})_+^2 |\nabla \zeta|^2 \right. \\ & \quad \left. + \int_{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)} (S - \mathbf{j})_+^2 \zeta \partial_t \zeta + \int_{\hat{t} - \theta}^{\hat{t}} |\mathcal{D}_{\mathbf{j}, \rho}^+(\tau)|^{(1-2/k_1)} \, d\tau \right) \tag{20} \end{aligned}$$

where ζ is a piecewise smooth cut-off function satisfying (16). Equation (20) also holds if $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)$ is a subset of Ω_2^T or Ω_3^T . But for Ω_3^T case, S in (20) should be replaced by $S_0 \equiv 1 - S$.

Proof

Without loss of generality, let $(\hat{t}, \hat{x}) = (0, 0)$. Take $\varphi = (S_h - \mathbf{j})_+ \zeta^2$ in (17) and integrate the resulting equation over $(-\theta, t)$ for $t \in (-\theta, 0)$ to obtain

$$\begin{aligned} & \int_{-\theta}^t \int_{\mathcal{X}_\rho} \Phi \partial_t S_h \varphi + \int_{-\theta}^t \int_{\mathcal{X}_\rho} (K \mathcal{J}(S) \nabla S)_h \nabla \varphi \\ & + \int_{-\theta}^t \int_{\mathcal{X}_\rho} (K \tilde{\Lambda}_w(S) \nabla (P - E_w))_h \nabla \varphi = 0 \end{aligned} \quad (21)$$

We now estimate each of the terms in (21). First, we integrate by parts in t , let $h \rightarrow 0^+$, and apply Lemma 3.2 of Chapter 1 [8] to see

$$\lim_{h \rightarrow 0^+} \int_{-\theta}^t \int_{\mathcal{X}_\rho} \Phi \partial_t S_h (S_h - \mathbf{j})_+ \zeta^2 = \int_{\mathcal{X}_\rho} \frac{\Phi}{2} (S - \mathbf{j})_+^2 \zeta^2|_{-\theta} - \int_{-\theta}^t \int_{\mathcal{X}_\rho} \frac{\Phi}{2} (S - \mathbf{j})_+^2 \partial_t \zeta^2$$

Next, by Hölder inequality,

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_{-\theta}^t \int_{\mathcal{X}_\rho} (K \mathcal{J}(S) \nabla S)_h \nabla ((S_h - \mathbf{j})_+ \zeta^2) \\ & \geq \frac{1}{2} \int_{-\theta}^t \|\sqrt{K \mathcal{J}(S)} \zeta \nabla (S - \mathbf{j})_+\|_{L^2(\mathcal{X}_\rho)}^2 - c_1 \int_{-\theta}^t \|\sqrt{K \mathcal{J}(S)} (S - \mathbf{j})_+ \nabla \zeta\|_{L^2(\mathcal{X}_\rho)}^2 \end{aligned}$$

Because $\mathcal{Q}(\theta, \rho) \subset \Omega_1^T$, by (5) and A7,

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_{-\theta}^t \int_{\mathcal{X}_\rho} (K \tilde{\Lambda}_w(S) \nabla (P - E_w))_h \nabla ((S_h - \mathbf{j})_+ \zeta^2) \\ & \leq \frac{1}{4} \int_{-\theta}^t \|\sqrt{K \mathcal{J}(S)} \zeta \nabla (S - \mathbf{j})_+\|_{L^2(\mathcal{X}_\rho)}^2 + c_3 \int_{-\theta}^t \|\sqrt{K \mathcal{J}(S)} (S - \mathbf{j})_+ \nabla \zeta\|_{L^2(\mathcal{X}_\rho)}^2 \\ & \quad + c_3 \int_{-\theta}^t \int_{\mathcal{X}_\rho} (|\nabla P|^2 + |\nabla E_w|^2) \mathcal{X}_{\{(S - \mathbf{j})_+ > 0\}} \end{aligned}$$

where $\mathcal{X}_{\{(S - \mathbf{j})_+ > 0\}}$ is the characteristic function (see (4)). Now we combine the above results with Hölder inequality and Lemma 3.1 to obtain (20). If $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega_2^T$, Equation (21) corresponds to a uniform (independent of ε) parabolic equation. So we may repeat above argument to obtain (20). If $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega_3^T$, we use (14) and repeat above argument. Note that Equation (11) around the end point 0 has similar properties as (14) around the other end point 1. Therefore we can easily get the same inequality (20) as $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega_1^T$ case except replacing S in (20) by S_\circ . \square

Let us define a function Ψ in $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)$ as

$$\Psi(H_{\mathbf{j}}, (S - \mathbf{j})_+, \delta) \equiv \ln^+ \left| \frac{H_{\mathbf{j}}}{H_{\mathbf{j}} - (S - \mathbf{j})_+ + \delta} \right| \quad (22)$$

where $H_{\mathbf{j}} \equiv \sup_{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)} (S - \mathbf{j})_+$, $\ln^+ \varphi \equiv \max\{0, \ln \varphi\}$, $0 < \delta < \min\{1, H_{\mathbf{j}}\}$.

Lemma 3.3

There is a constant d_1 (independent of $\varepsilon, \theta, \rho, \mathbf{j}$) such that for every cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega_1^T$ and every level \mathbf{j}

$$\begin{aligned} & \sup_{\hat{t}-\theta < t < \hat{t}} \int_{\hat{x} + \mathcal{K}_\rho} \Phi \Psi^2(H_{\mathbf{j}}, (S - \mathbf{j})_+, \delta)(t, x) \xi^2 \\ & \leq \int_{\hat{x} + \mathcal{K}_\rho} \Phi \Psi^2(H_{\mathbf{j}}, (S - \mathbf{j})_+, \delta)(\hat{t} - \theta, x) \xi^2 + d_1 \left(\int_{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)} \mathcal{J}(S) \Psi(H_{\mathbf{j}}, (S - \mathbf{j})_+, \delta) |\nabla \xi|^2 \right. \\ & \quad \left. + \frac{|1 + \ln^+(H_{\mathbf{j}}/\delta)|}{\delta^2} \int_{\hat{t}-\theta}^{\hat{t}} |\mathcal{D}_{\mathbf{j}, \rho}^+(\tau)|^{(1-2/k_1)} d\tau \right) \end{aligned} \tag{23}$$

where ξ is a piecewise smooth cut-off function satisfying (16). Equation (23) also holds if $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)$ is a subset of Ω_2^T or Ω_3^T . But for Ω_3^T case, S in (23) should be replaced by $S_0 \equiv 1 - S$.

Proof

Let $(\hat{t}, \hat{x}) = (0, 0)$. For convenience, set

$$\psi(x) \equiv \Psi(H_{\mathbf{j}}, (x - \mathbf{j})_+, \delta) = \ln^+ \left| \frac{H_{\mathbf{j}}}{H_{\mathbf{j}} - (x - \mathbf{j})_+ + \delta} \right| \tag{24}$$

By (22) and (24), we have $0 \leq \psi(S_h) \leq \ln^+(H_{\mathbf{j}}/\delta), 0 \leq \psi'(S_h) \leq 1/\delta, 0 < \delta < 1$. Since $(\psi^2)''(S_h) = 2(1 + \psi)(\psi')^2 \in L_{loc}^\infty(\Omega^T)$ by (22), we take $\varphi = \xi^2 (\psi^2)'(S_h)$ in (17) and integrate the resulting equation over $(-\theta, t)$ for $t \in (-\theta, 0)$ to see

$$\begin{aligned} & \int_{-\theta}^t \int_{\mathcal{K}_\rho} \Phi \partial_t S_h (\psi^2)' \xi^2 + \int_{-\theta}^t \int_{\mathcal{K}_\rho} (K \mathcal{J}(S) \nabla S)_h \nabla((\psi^2)' \xi^2) \\ & \quad + \int_{-\theta}^t \int_{\mathcal{K}_\rho} (K \tilde{\Lambda}_w(S) \nabla(P - E_w))_h \nabla((\psi^2)' \xi^2) = 0 \end{aligned} \tag{25}$$

Each term of (25) is estimated similarly as that for (21). First

$$\lim_{h \rightarrow 0^+} \int_{-\theta}^t \int_{\mathcal{K}_\rho} \Phi \partial_t S_h (\psi^2)' \xi^2 = \lim_{h \rightarrow 0^+} \int_{-\theta}^t \int_{\mathcal{K}_\rho} \Phi \partial_t \psi^2 \xi^2 = \int_{\mathcal{K}_\rho} \Phi \psi^2(S) \xi^2(\tau)|_{-\theta}^t$$

Next

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_{-\theta}^t \int_{\mathcal{K}_\rho} (K \mathcal{J}(S) \nabla S)_h \nabla((\psi^2)' \xi^2) = \int_{-\theta}^t \int_{\mathcal{K}_\rho} K \mathcal{J}(S) \nabla S \nabla((\psi^2)' \xi^2) \\ & \geq \int_{-\theta}^t \int_{\mathcal{K}_\rho} K \mathcal{J}(S) |\nabla S|^2 (1 + \psi)(\psi')^2 \xi^2 - c_1 \int_{-\theta}^t \int_{\mathcal{K}_\rho} K \mathcal{J}(S) \psi |\nabla \xi|^2 \end{aligned}$$

Since $\mathcal{Q}(\theta, \rho) \subset \Omega_1^T$, by (5) and A7,

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \int_{-\theta}^t \int_{\mathcal{K}_\rho} (K \tilde{\Lambda}_w(S) \nabla(P - E_w))_h \nabla((\psi^2)' \xi^2) \\ & \leq \frac{1}{4} \int_{-\theta}^t \int_{\mathcal{K}_\rho} K \mathcal{J}(S) |\nabla S|^2 (1 + \psi)(\psi')^2 \xi^2 + \int_{-\theta}^t \int_{\mathcal{K}_\rho} K \mathcal{J}(S) \psi |\nabla \xi|^2 \\ & \quad + c_1 \int_{-\theta}^t \int_{\mathcal{K}_\rho} (|\nabla E_w|^2 + |\nabla P|^2) (1 + \psi)(\psi')^2 \xi^2 \end{aligned}$$

Then, combine above estimates with Hölder inequality and Lemma 3.1 to get (23). If $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega_2^T$, (25) is in uniform (independent of ε) parabolic equation case and we may repeat above argument to obtain (23). If $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega_3^T$, we use (14) and repeat above argument to get (23) except that S in (23) should be replaced by S_\circ . \square

4. HÖLDER ESTIMATE OF REGULARIZED EQUATIONS IN INTERIOR REGION

In this section, we give a Hölder estimate for S in the interior region of Ω^T . Let $\bar{\mathcal{O}} (\subset \Omega^T)$ be a closure of \mathcal{O} and let γ^*, ϖ be two small positive numbers satisfying

$$|\gamma^*|^{\varpi} \leq \min\{1, \mathcal{J}(\vartheta/2), \mathcal{J}(1 - \vartheta/2)\} \tag{26}$$

as well as $(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma) \subset \Omega^T$ for any $(\hat{t}, \hat{x}) \in \bar{\mathcal{O}}$ and $\gamma < \gamma^*$. Define $\mu^+ \equiv \sup_{(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)} S$, $\mu^- \equiv \inf_{(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)} S$, and $\omega \equiv \mu^+ - \mu^-$. By (9),

$$\omega = 0 \quad \text{or} \quad 0 < \omega \leq \vartheta/2 \quad \text{or} \quad \vartheta/2 \leq \omega \leq 1 \tag{27}$$

If $\omega = 0$, S is constant in $(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$. So S is Hölder continuous in $(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$. The other two cases of (27) are discussed in Sections 4.1, 4.2 separately.

4.1. For $0 < \omega \leq \vartheta/2$ case

Since $\omega \leq \vartheta/2$, we have $\mu^+ < \vartheta$ or $\vartheta/4 < \mu^- \leq \mu^+ < 1 - (\vartheta/4)$ or $1 - \vartheta < \mu^-$. By (18),

$$(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma) \subset \begin{cases} \Omega_1^T & \text{if } \mu^+ < \vartheta \\ \Omega_2^T & \text{if } \frac{\vartheta}{4} < \mu^- \leq \mu^+ < 1 - \frac{\vartheta}{4} \\ \Omega_3^T & \text{if } 1 - \vartheta < \mu^- \end{cases} \tag{28}$$

4.1.1. For $\mu^+ < \vartheta$ case. Set M to be a constant satisfying $M/2 \leq \mu^+ \leq M \leq \vartheta$ and define $1/\beta \equiv \mathcal{J}(M)$. Then $\beta \geq \gamma^{-\varpi}$ or $\beta < \gamma^{-\varpi}$. The former (i.e. $\beta \geq \gamma^{-\varpi}$) implies, by A7,

$$\omega \leq c\gamma^{\varpi/m} \tag{29}$$

for a constant c . The latter (i.e. $\beta < \gamma^{-\varpi}$) means $\mathcal{Q}(\beta\gamma^2, \gamma) \subset \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$ and is discussed below. Note $(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma) \subset \Omega_1^T$ here.

4.1.1.1. First alternative

Lemma 4.1

For any $\ell_1 \geq 2$, there is a $v_1 \in (0,1)$ (depending on ℓ_1 and data but independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) so that if $\mathcal{Q}(\beta\gamma^2, \gamma) \subset \mathcal{Q}(\gamma^{2-\sigma}, 2\gamma)$ and

$$\begin{cases} \left| \left\{ (t,x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2, \gamma) : S(t,x) < \mu^- + \frac{\omega}{2^{\ell_1}} \right\} \right| \leq v_1 |\mathcal{Q}(\beta\gamma^2, \gamma)| \\ \left| \frac{\omega}{2^{\ell_1}} \right|^{-2} \beta \gamma^{N\alpha} \leq 1 \end{cases} \tag{30}$$

then $S(t,x) \geq \mu^- + (\omega/2^{\ell_1+4})$ for $(t,x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta|\gamma/2|^2, \gamma/2)$. See (4) for α .

Proof

After translation we assume $(\hat{t}, \hat{x}) = (0, 0)$. Define $\gamma_n \equiv (\gamma/2) + (\gamma/2^{n+1}), n = 0, 1, 2, \dots$. Construct a family of nested cylinders $\mathcal{Q}(\beta\gamma_n^2, \gamma_n)$ and let ζ_n be a piecewise smooth cut-off functions in $\mathcal{Q}(\beta\gamma_n^2, \gamma_n)$ such that

$$\begin{cases} 0 < \zeta_n(t,x) \leq 1 & \text{for } (t,x) \in \mathcal{Q}(\beta\gamma_n^2, \gamma_n) \\ \zeta_n = 1 & \text{in } \mathcal{Q}(\beta\gamma_{n+1}^2, \gamma_{n+1}) \\ \zeta_n = 0 & \text{on the parabolic boundary of } \mathcal{Q}(\beta\gamma_n^2, \gamma_n) \\ |\nabla \zeta_n| \leq c \frac{2^{n+2}}{\gamma} \\ |\Delta \zeta_n| \leq c \left| \frac{2^{(n+2)}}{\gamma} \right|^2 \\ 0 \leq \partial_t \zeta_n \leq \frac{1}{\beta} \left| \frac{2^{(n+1)}}{\gamma} \right|^2 \end{cases} \tag{31}$$

Define in $\mathcal{Q}(\gamma^{2-\sigma}, 2\gamma)$

$$\begin{cases} \phi \equiv \max \left\{ S, \mu^- + \frac{\omega}{2^{\ell_1+4}} \right\} \\ \mathbf{j}_n \equiv \mu^- + \frac{\omega}{2^{\ell_1+1}} + \frac{\omega}{2^{\ell_1+1+n}}, \quad n = 0, 1, 2, \dots \\ \mathcal{F}(S) \equiv - \int_{\mathbf{j}_n}^S \left(\max \left(\xi, \mu^- + \frac{\omega}{2^{\ell_1+4}} \right) - \mathbf{j}_n \right)_- d\xi \end{cases}$$

Then

$$\begin{cases} \mathcal{F}(S) = \frac{1}{2} |(\phi - \mathbf{j}_n)_-|^2 + \left(\frac{7\omega}{2^{\ell_1+4}} + \frac{\omega}{2^{\ell_1+1+n}} \right) \left(S - \mu^- - \frac{\omega}{2^{\ell_1+4}} \right)_- \\ \partial_t \mathcal{F}(S) = -(\phi - \mathbf{j}_n)_- \partial_t S \\ \mathcal{F}(S) \leq c \left| \frac{\omega}{2^{\ell_1}} \right|^2 \\ (\phi - \mathbf{j}_n)_- \leq (S - \mathbf{j}_n)_- \leq \frac{\omega}{2^{\ell_1}} \end{cases} \tag{32}$$

We now take $\zeta = -(\phi - \mathbf{j}_n)_- \zeta_n^2 \mathcal{X}_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)}$ in (11) and estimate each term as follows: By (31)–(32),

$$\begin{aligned} & - \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} \Phi(\phi - \mathbf{j}_n)_- \zeta_n^2 \partial_t S = \int_{\mathcal{K}_{\gamma_n}} \Phi \mathcal{F}(S) \zeta_n^2(0) - \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} \Phi \mathcal{F}(S) \partial_t \zeta_n^2 \\ & \geq \int_{\mathcal{K}_{\gamma_n}} \frac{1}{2} \Phi |(\phi - \mathbf{j}_n)_-|^2 \zeta_n^2(0) - \frac{c}{\beta} \left| \frac{2^n \omega}{\gamma 2^{\ell_1}} \right|^2 \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} \mathcal{X}_{\{(\phi - \mathbf{j}_n)_- > 0\}} \\ & - \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} K \nabla \mathcal{R}(S) \nabla((\phi - \mathbf{j}_n)_- \zeta_n^2) \\ & \geq \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} K \mathcal{J}(\phi) |\nabla(\zeta_n(\phi - \mathbf{j}_n)_-)|^2 \\ & - \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} K(\phi - \mathbf{j}_n)_- \nabla \mathcal{R}(S) \nabla \zeta_n^2 - \frac{c}{\beta} \left| \frac{2^n \omega}{\gamma 2^{\ell_1}} \right|^2 \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} \mathcal{X}_{\{(\phi - \mathbf{j}_n)_- > 0\}} \end{aligned}$$

If $\mu^- \leq \frac{1}{2} \mu^+$, then $\mu^+ \leq 2\omega$. So $\mathcal{J}(M/2^{\ell_1+6}) \leq \mathcal{J}(\mu^+/2^{\ell_1+5}) \leq \mathcal{J}(\phi)$. If $\frac{1}{2} \mu^+ < \mu^-$, then $\mathcal{J}(M/4) \leq \mathcal{J}(\mu^+/2) \leq \mathcal{J}(\phi)$. Therefore, by A7, we have $\mathcal{J}(\phi) \geq (1/\beta) |1/2^{\ell_1+6}|^m$ in $\mathcal{Q}(\beta\gamma_n^2, \gamma_n)$. Note $0 \leq \mathcal{R}(S) - \mathcal{R}(\mu^-) \leq (S - \mu^-) \mathcal{J}(S)$ for $S \in (\mu^-, \mathbf{j}_n)$. By (31)–(32), (28)₁, and A7,

$$\begin{aligned} & \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} K(\phi - \mathbf{j}_n)_- \nabla \mathcal{R}(S) \nabla \zeta_n^2 = \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} K(\phi - \mathbf{j}_n)_- \nabla(\mathcal{R}(S) - \mathcal{R}(\mu^-)) \nabla \zeta_n^2 \\ & = - \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} (\mathcal{R}(S) - \mathcal{R}(\mu^-)) \nabla(K(\phi - \mathbf{j}_n)_- \nabla \zeta_n^2) \\ & \leq \frac{1}{4} \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} K \mathcal{J}(S) |\nabla(\phi - \mathbf{j}_n)_-|^2 \zeta_n^2 + \frac{c}{\beta} \left| \frac{2^n \omega}{\gamma 2^{\ell_1}} \right|^2 \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} \mathcal{X}_{\{(\phi - \mathbf{j}_n)_- > 0\}} \\ & \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} K \tilde{\Lambda}_w(S) \nabla(P - E_w) \nabla((\phi - \mathbf{j}_n)_- \zeta_n^2) \\ & \leq \frac{1}{4} \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} K \mathcal{J}(S) |\nabla(\phi - \mathbf{j}_n)_-|^2 \zeta_n^2 \\ & + \frac{c}{\beta} \left| \frac{2^n \omega}{\gamma 2^{\ell_1}} \right|^2 \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} \mathcal{X}_{\{(\phi - \mathbf{j}_n)_- > 0\}} \, d\tau + d_1 \int_{-\beta\gamma_n^2}^0 |\mathcal{D}_{\mathbf{j}_n, \gamma_n}^-(\tau)|^{(1-2/k_1)} \, d\tau \end{aligned}$$

where $d_1 = d_1(K, \|\nabla P, \nabla E_w\|_{L^\infty(0, T; L^{k_1}(\Omega))})$. Combining above estimates, it is not difficult to see

$$\begin{aligned} & \sup_{-\beta\gamma_n^2 < t < 0} \int_{\mathcal{K}_{\gamma_n}} \zeta_n^2 (\phi - \mathbf{j}_n)_-^2 \, dx + \frac{1}{\beta} \left| \frac{1}{2^{\ell_1+6}} \right|^m \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} |\nabla(\zeta_n(\phi - \mathbf{j}_n)_-)|^2 \\ & \leq \frac{c}{\beta} \left| \frac{2^n \omega}{\gamma 2^{\ell_1}} \right|^2 \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} \mathcal{X}_{\{(\phi - \mathbf{j}_n)_- > 0\}} \, d\tau + d_1 \int_{-\beta\gamma_n^2}^0 |\mathcal{D}_{\mathbf{j}_n, \gamma_n}^-(\tau)|^{(1-2/k_1)} \, d\tau \end{aligned} \tag{33}$$

We claim

$$\lim_{n \rightarrow \infty} \int_{\mathcal{Q}(\beta\gamma_n^2, \gamma_n)} \mathcal{X}_{\{(S-\mathbf{j}_n)_- > 0\}} \, d\tau = 0$$

If so, since $\mathbf{j}_n \searrow \mathbf{j}_\infty \geq \mu^- + (\omega/2^{\ell_1+4})$, this would imply

$$\left| \left\{ (t, x) \in \mathcal{Q} \left(\beta \left| \frac{\gamma}{2} \right|^2, \frac{\gamma}{2} \right) : S(t, x) < \mu^- + \frac{\omega}{2^{\ell_1+4}} \right\} \right| = 0$$

and proves the lemma. Change variable $z = t/\beta$ which transforms $\mathcal{Q}(\beta\gamma_n^2, \gamma_n)$ into $\mathcal{Q}_n \equiv \mathcal{Q}(\gamma_n^2, \gamma_n) = [-\gamma_n^2, 0] \times \mathcal{K}_{\gamma_n}$. Define $\hat{\phi}(z, \cdot) \equiv \phi(z\beta, \cdot)$, $\hat{\zeta}_n(z, \cdot) \equiv \zeta_n(z\beta, \cdot)$, $\mathcal{D}_n(z) \equiv \{x \in \mathcal{K}_{\gamma_n} : \hat{\phi}(z, x) < \mathbf{j}_n\}$ and $|\mathcal{D}_n| \equiv \int_{-\gamma_n^2}^0 |\mathcal{D}_n(z)| \, dz$. Equation (33) can be written as

$$\|\hat{\zeta}_n(\hat{\phi} - \mathbf{j}_n)_-\|_{\mathcal{V}(\mathcal{Q}_n)}^2 \leq c2^{\mathbf{m}(\ell_1+6)} \left(\left| \frac{2^n \omega}{\gamma 2^{\ell_1}} \right|^2 |\mathcal{D}_n| + \beta \int_{-\gamma_n^2}^0 |\mathcal{D}_n(\tau)|^{(1-2/\mathbf{k}_1)} \, d\tau \right) \tag{34}$$

Since $\hat{\zeta}_n(\hat{\phi} - \mathbf{j}_n)_-$ vanishes on the lateral boundary of \mathcal{Q}_n , by Corollary 3.1 of Chapter 1 [8] we have, by (34),

$$\begin{aligned} \frac{|\mathcal{D}_{n+1}|}{2^{2(n+2)}} \left| \frac{\omega}{2^{\ell_1}} \right|^2 &\leq |\mathbf{j}_n - \mathbf{j}_{n+1}|^2 |\mathcal{D}_{n+1}| \leq \|(\hat{\phi} - \mathbf{j}_n)_-\|_{L^2(\mathcal{Q}_{n+1})}^2 \\ &\leq \|\hat{\zeta}_n(\hat{\phi} - \mathbf{j}_n)_-\|_{L^2(\mathcal{Q}_n)}^2 \leq c \|\hat{\zeta}_n(\hat{\phi} - \mathbf{j}_n)_-\|_{\mathcal{V}(\mathcal{Q}_n)}^2 |\mathcal{D}_n|^{2/(N+2)} \\ &\leq c2^{\mathbf{m}(\ell_1+6)} \left(\left| \frac{2^n \omega}{\gamma 2^{\ell_1}} \right|^2 |\mathcal{D}_n|^{(N+4)/(N+2)} + \beta |\mathcal{D}_n|^{2/(N+2)} \int_{-\gamma_n^2}^0 |\mathcal{D}_n(\tau)|^{(1-2/\mathbf{k}_1)} \, d\tau \right) \end{aligned} \tag{35}$$

Define $Y_n \equiv |\mathcal{D}_n|/|\mathcal{Q}_n|$, $Z_n \equiv 1/|\mathcal{K}_{\gamma_n}| (\int_{-\gamma_n^2}^0 |\mathcal{D}_n(\tau)|^{(1-2/\mathbf{k}_1)} \, d\tau)^{1/(1+\alpha)}$, where α is defined in (4). Divide (35) by $|\mathcal{Q}_{n+1}|$ to obtain, by (30)₂,

$$Y_{n+1} \leq c2^{\mathbf{m}(\ell_1+6)} 4^{2n} (Y_n^{1+(2/(N+2))} + Y_n^{2/(N+2)} Z_n^{1+\alpha}), \quad n = 0, 1, 2, \dots \tag{36}$$

Next by the embedding of Proposition 3.3 of Chapter 1 [8]

$$Z_{n+1} (\mathbf{j}_n - \mathbf{j}_{n+1})^2 \leq \frac{\|\hat{\zeta}_n(\hat{\phi} - \mathbf{j}_n)_-\|_{L^{q,r}(\mathcal{Q}_n)}^2}{|\mathcal{K}_{\gamma_{n+1}}|} \leq c\gamma^{-N} \|\hat{\zeta}_n(\hat{\phi} - \mathbf{j}_n)_-\|_{\mathcal{V}(\mathcal{Q}_n)}^2$$

where $r = 2(1 + \alpha) \in [2, \infty)$, $q = r/(1 - 2/\mathbf{k}_1) \in [2, 2N/(N - 2)]$. Therefore

$$Z_{n+1} \leq c2^{\mathbf{m}(\ell_1+6)} 4^{2n} (Y_n + Z_n^{1+\alpha}), \quad n = 0, 1, 2, \dots \tag{37}$$

By (36)–(37) and Lemma 4.2 of Chapter 1 [8], it follows that $\lim_{n \rightarrow \infty} Y_n = \lim_{n \rightarrow \infty} Z_n = 0$, provided

$$Y_0 + Z_0^{1+\alpha} \leq v \equiv |c2^{m(\ell_1+6)}|^{-(1+\alpha)/\lambda} 4^{-2(1+\alpha)/\lambda^2} \tag{38}$$

See (4) for λ . By (30)₁ and $Z_0 \leq Y_0^{2/q}$, if we select v_1 satisfying $v_1 + v_1^{2(1+\alpha)/q} = v$, then (38) holds. So we complete the proof of the lemma. \square

Lemma 4.2

For any $\ell_1 \geq 2$, there exist numbers $v_1 \in (0, 1), \eta_1 \in (1/2, 1)$ and $I_1 \gg 1$ (depending upon ℓ_1 and given data but independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) so that if $\mathcal{Q}(\beta\gamma^2, \gamma) \subset \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$ and

$$\left| \left\{ (t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2, \gamma) : S(t, x) < \mu^- + \frac{\omega}{2^{\ell_1}} \right\} \right| \leq v_1 |\mathcal{Q}(\beta\gamma^2, \gamma)| \tag{39}$$

then either

$$\omega \leq I_1 \gamma^{N\alpha/(2+m)} \tag{40}$$

or $\text{ess osc}_{(\hat{t}, \hat{x}) + \mathcal{Q}(\beta|\gamma/2|^2, \gamma/2)} S \leq \eta_1 \omega$.

Proof

Assume (40) is violated. By A7 and Lemma 4.1,

$$\inf_{(\hat{t}, \hat{x}) + \mathcal{Q}(\beta|\gamma/2|^2, \gamma/2)} S \geq \mu^- + \frac{\omega}{2^{\ell_1+4}} \tag{41}$$

From (41) and definition of μ^+ , we get $\text{ess osc}_{(\hat{t}, \hat{x}) + \mathcal{Q}(\beta|\gamma/2|^2, \gamma/2)} S \leq (1 - (1/2^{\ell_1+4}))\omega$. So the lemma follows with $\eta_1 = 1 - (1/2^{\ell_1+4})$. \square

4.1.1.2. Second alternative. In this subsection, we shall fix ℓ_1 (so v_1 is fixed as well) and assume that (39) of Lemma 4.2 is violated, i.e. for the subcylinder $\mathcal{Q}(\beta\gamma^2, \gamma) \subset \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$

$$\left| \left\{ (t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2, \gamma) : S(t, x) < \mu^- + \frac{\omega}{2^{\ell_1}} \right\} \right| > v_1 |\mathcal{Q}(\beta\gamma^2, \gamma)| \tag{42}$$

Since $\mu^+ - (\omega/2^{\ell_1}) \geq \mu^- + (\omega/2^{\ell_1})$ for all $\ell_1 \geq 2$, (42) implies

$$\left| \left\{ (t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2, \gamma) : S(t, x) \geq \mu^+ - \frac{\omega}{2^{\ell_1}} \right\} \right| \leq (1 - v_1) |\mathcal{Q}(\beta\gamma^2, \gamma)| \tag{43}$$

In view of (43), we will study the behaviour of S near its supremum μ^+ and work with the truncated function $(S - \mathbf{j})_+$ for the levels $\mathbf{j} = \mu^+ - (\omega/2^{\ell_1+n}), n \geq 0$.

Lemma 4.3

Under (43), there is a time level $t^* \in [\hat{t} - \beta\gamma^2, \hat{t} - \beta\gamma^2 v_1/2]$ so that

$$\left| \left\{ x \in \mathcal{X}_\gamma : S(t^*, \hat{x} + x) \geq \mu^+ - \frac{\omega}{2^{\ell_1}} \right\} \right| \leq \left(\frac{1 - v_1}{1 - v_1/2} \right) |\mathcal{X}_\gamma|$$

See Lemma 4.2 for ℓ_1, v_1 and (15) for \mathcal{X}_γ .

Proof

If not, for all $t \in [\hat{t} - \beta\gamma^2, \hat{t} - \beta\gamma^2 v_1/2]$,

$$\left| \left\{ x \in \mathcal{K}_\gamma : S(t, \hat{x} + x) \geq \mu^+ - \frac{\omega}{2^{\ell_1}} \right\} \right| > \left(\frac{1 - v_1}{1 - v_1/2} \right) |\mathcal{K}_\gamma| \tag{44}$$

Equation (44) implies $|\{(t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2, \gamma) : S \geq \mu^+ - (\omega/2^{\ell_1})\}| > (1 - v_1)|\mathcal{Q}(\beta\gamma^2, \gamma)|$, contradicting (43). \square

The above lemma asserts that at some time level t^* , the set where S is close to its supremum occupies only a portion of the cube \mathcal{K}_γ . The next lemma claim that this indeed occurs for all time levels near the top of the cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2, \gamma) \subset \Omega_1^T$.

Lemma 4.4

Under assumptions of Lemma 4.3, there is a positive number \mathbf{n} (depending only on the given data) such that either

$$\left| \frac{\omega}{2^{(\ell_1 + \mathbf{n})}} \right|^{-2} \beta\gamma^{N\alpha} \geq 1 \tag{45}$$

or, for all $t \in [\hat{t} - \beta\gamma^2 v_1/2, \hat{t}]$,

$$\left| \left\{ x \in \mathcal{K}_\gamma : S(t, \hat{x} + x) \geq \mu^+ - \frac{\omega}{2^{\ell_2}} \right\} \right| \leq \left(1 - \left| \frac{v_1}{2} \right|^2 \right) |\mathcal{K}_\gamma|$$

where $\ell_2 \equiv \ell_1 + \mathbf{n}$.

Proof

For convenience, we set $\hat{x} = 0$. Because of (28)₁, we use (23) of Lemma 3.3 written over the box $(t^*, \hat{t}) \times \mathcal{K}_\gamma$ for the function $(S - \mathbf{j})_+$ with level $\mathbf{j} = \mu^+ - (\omega/2^{\ell_1})$. The number δ in (22) is taken to be $\delta = \omega/2^{\ell_1 + \mathbf{n}}$. Thus we have

$$\Psi(t, x) = \ln^+ \left| \frac{H_{\mathbf{j}}}{H_{\mathbf{j}} - (S - (\mu^+ - (\omega/2^{\ell_1})))_+ + (\omega/2^{\ell_1 + \mathbf{n}})} \right| \tag{46}$$

where $H_{\mathbf{j}} = \sup_{(t,0) + \mathcal{Q}(\beta\gamma^2, \gamma)} (S - (\mu^+ - (\omega/2^{\ell_1})))_+$. Note $\Psi = 0$ on $\{S < \mu^+ - (\omega/2^{\ell_1})\}$ and

$$0 \leq \Psi \leq \ln^+ \frac{H_{\mathbf{j}} 2^{\ell_1 + \mathbf{n}}}{\omega} \leq \mathbf{n} \ln 2 \tag{47}$$

The cut-off function $x \rightarrow \zeta(x)$ is taken so that (16) holds, $\zeta = 1$ in the cube $\mathcal{K}_{(1-\sigma)\gamma}$, $\sigma \in (0, 1)$ and $|\nabla \zeta| \leq (\sigma\gamma)^{-1}$. With these choices, Lemma 3.3 gives

$$\begin{aligned} \sup_{t^* \leq t \leq \hat{t}} \int_{\mathcal{K}_{(1-\sigma)\gamma}} \Phi \Psi^2(t, x) \, dx &\leq \int_{\mathcal{K}_\gamma} \Phi \Psi^2(t^*, x) \, dx + d \left(\int_{t^*}^{\hat{t}} \int_{\mathcal{K}_\gamma} \frac{\mathcal{J}(S) \Psi(t, x)}{(\sigma\gamma)^2} \right. \\ &\quad \left. + \left| \frac{\omega}{2^{\ell_1 + \mathbf{n}}} \right|^{-2} \left| 1 + \ln^+ \frac{H_{\mathbf{j}} 2^{\ell_1 + \mathbf{n}}}{\omega} \right| \int_{t^*}^{\hat{t}} |\mathcal{D}_{\mathbf{j}, \gamma}^+(\tau)|^{(1-2/\mathbf{k}_1)} \, d\tau \right) \end{aligned} \tag{48}$$

Using Lemma 4.3, (47), $\hat{t} - t^* \leq \beta\gamma^2$, and (45), we estimate the right-hand side of (48) and get

$$\sup_{t^* \leq t \leq \hat{t}} \int_{\mathcal{K}_{(1-\sigma)\gamma}} \Psi^2(t, x) \, dx \leq \frac{\mathbf{k}_4}{\mathbf{k}_3} |\mathbf{n} \ln 2|^2 \left| \frac{1 - v_1}{1 - v_1/2} \right| |\mathcal{K}_\gamma| + \frac{c}{\sigma^2} \mathbf{n} |\mathcal{K}_\gamma| \tag{49}$$

where c is a constant depending on given data. Left-hand side of (49) is estimated below by integrating over the smaller set $\{x \in \mathcal{K}_{(1-\sigma)\gamma} : S(t, x) \geq \mu^+ - (\omega/2^{\ell_1+\mathbf{n}})\}$. On this set,

$$\Psi^2 \geq \ln^2 \left| \frac{\omega/2^{\ell_1}}{\omega/2^{\ell_1+\mathbf{n}-1}} \right| = |(\mathbf{n} - 1) \ln 2|^2 \tag{50}$$

Equations (49) and (50) imply that, for all $t \in (t^*, \hat{t})$,

$$\left| \left\{ x \in \mathcal{K}_{(1-\sigma)\gamma} : S(t, x) \geq \mu^+ - \frac{\omega}{2^{\ell_1+\mathbf{n}}} \right\} \right| \leq \left(\frac{\mathbf{k}_4}{\mathbf{k}_3} \left| \frac{\mathbf{n}}{\mathbf{n} - 1} \right|^2 \left| \frac{1 - v_1}{1 - v_1/2} \right| + \frac{c}{\mathbf{n}\sigma^2} \right) |\mathcal{K}_\gamma|$$

Also note, for each fixed $t \in (t^*, \hat{t})$,

$$\left| \left\{ x \in \mathcal{K}_\gamma : S \geq \mu^+ - \frac{\omega}{2^{\ell_1+\mathbf{n}}} \right\} \right| \leq \left| \left\{ x \in \mathcal{K}_{(1-\sigma)\gamma} : S \geq \mu^+ - \frac{\omega}{2^{\ell_1+\mathbf{n}}} \right\} \right| + N\sigma |\mathcal{K}_\gamma|$$

By A6 and letting σ small and \mathbf{n} large (depending on the given data), above estimates imply the lemma. □

Next we focus on the cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2 v_1/2, \gamma)$. Introduce following notation:

$$\begin{cases} \beta_1 \equiv v_1 \beta/2 \\ \mathcal{B}_i(t) \equiv \{x \in \hat{x} + \mathcal{K}_\gamma : S(t, x) \geq \mu^+ - \omega/2^i\} \\ \mathcal{G}_i \equiv \{(t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta_1 \gamma^2/2, \gamma) : S(t, x) \geq \mu^+ - \omega/2^i\} \end{cases} \tag{51}$$

The information of Lemma 4.4 will be employed to deduce that the set where S is close to its supremum μ^+ within the cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\beta_1 \gamma^2/2, \gamma)$ can be made arbitrary small.

Lemma 4.5

Under assumptions of Lemma 4.4, for any $v_2 \in (0, 1)$ there is a number ℓ_3 (depending on v_1, v_2, ℓ_1, ℓ_2 and given data but independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) such that if

$$\left| \frac{\omega}{2^{\ell_3}} \right|^{-2} \beta \gamma^{N\alpha} \leq 1 \tag{52}$$

then $|\mathcal{G}_{\ell_3}| \leq v_2 |\mathcal{Q}(\beta_1 \gamma^2/2, \gamma)|$.

Proof

Set $(\hat{t}, \hat{x}) = (0, 0)$. Use (20) of Lemma 3.2 written over box $\mathcal{Q}(\beta_1 \gamma^2, 2\gamma)$ for the functions $(S - \mathbf{j})_+$. The levels \mathbf{j} are given by $\mathbf{j} = \mu^+ - (\omega/2^i)$, where $\ell_2 \leq i \leq \ell_3$ and ℓ_3 is to be chosen. We take a cut-off function ζ such that (16) holds, equals 1 in $\mathcal{Q}((\beta_1/2)\gamma^2, \gamma)$, vanishes on

the parabolic boundary of $\mathcal{Q}(\beta_1\gamma^2, 2\gamma)$, and satisfies $|\nabla\zeta| \leq 1/\gamma, 0 \leq \partial_t\zeta \leq 2(\beta_1\gamma^2)^{-1}$. Neglect the first term on the left-hand side of the energy estimate (20) to obtain

$$\begin{aligned} & \int_{\mathcal{Q}((\beta_1/2)\gamma^2, \gamma)} \mathcal{J}(S) |\nabla(S - \mathbf{j})_+|^2 \\ & \leq d \left(\frac{1}{\beta_1\gamma^2} \int_{\mathcal{Q}(\beta_1\gamma^2, 2\gamma)} (S - \mathbf{j})_+^2 + \int_{-\beta_1\gamma^2}^0 |\mathcal{D}_{\mathbf{j}, 2\gamma}^+(\tau)|^{(1-2/k_1)} d\tau \right) \end{aligned} \tag{53}$$

By (51)₁ and (52), we estimate left- and right-hand sides of (53) to obtain

$$\frac{1}{\beta} \int_{\mathcal{G}_i} |\nabla S|^2 \leq c \int_{\mathcal{Q}((\beta_1/2)\gamma^2, \gamma)} \mathcal{J}(S) |\nabla(S - \mathbf{j})_+|^2 \leq \frac{c}{\gamma^2\beta} \left| \frac{\omega}{2^i} \right|^2 \left| \mathcal{Q} \left(\frac{\beta_1}{2} \gamma^2, \gamma \right) \right| \tag{54}$$

where c above depends on given data only. Lemma 4.4 implies, for $t \in (-\beta_1\gamma^2, 0)$,

$$\left| \left\{ x \in \mathcal{K}_\gamma : S(t, x) < \mu^+ - \frac{\omega}{2^i} \right\} \right| = |\mathcal{K}_\gamma| - |\mathcal{B}_i(t)| \geq \left| \frac{v_1}{2} \right|^2 |\mathcal{K}_\gamma| \tag{55}$$

See (51) for $\mathcal{B}_i(t)$. Next we use (55) and Lemma 2.2 of Chapter 1 [8] applied to the function $S(t, \cdot)$ for all time $-(\beta_1/2)\gamma^2 \leq t \leq 0$ and for the levels $k = \mu^+ - (\omega/2^i)$, $l = \mu^+ - (\omega/2^{i+1})$, $l - k = \omega/2^{i+1}$ to obtain

$$\frac{\omega}{2^{i+1}} |\mathcal{B}_{i+1}(t)| \leq \frac{c}{v_1^2} \frac{\gamma^{N+1}}{|\mathcal{K}_\gamma|} \int_{\mathcal{B}_i(t) \setminus \mathcal{B}_{i+1}(t)} |\nabla S| \tag{56}$$

Integrate (56) over $(-\beta_1/2)\gamma^2, 0$ to get

$$\frac{\omega}{2^{i+1}} |\mathcal{G}_{i+1}| \leq \frac{c\gamma}{v_1^2} \int_{\mathcal{G}_i \setminus \mathcal{G}_{i+1}} |\nabla S| \leq \frac{c\gamma}{v_1^2} \sqrt{|\mathcal{G}_i \setminus \mathcal{G}_{i+1}|} \sqrt{\int_{\mathcal{G}_i} |\nabla S|^2} \tag{57}$$

By (54), (57) gives

$$|\mathcal{G}_{i+1}|^2 \leq \frac{c}{v_1^4} \left| \mathcal{Q} \left(\frac{\beta_1}{2} \gamma^2, \gamma \right) \right| |\mathcal{G}_i \setminus \mathcal{G}_{i+1}| \tag{58}$$

Above inequalities are valid for all $\ell_2 \leq i \leq \ell_3$. We add them for $i = \ell_2, \ell_2 + 1, \dots, \ell_3 - 1$. The right-hand side of (58) can be majorized by a convergent series bounded above by $|\mathcal{Q}((\beta_1/2)\gamma^2, \gamma)|$. Therefore $(\ell_3 - \ell_2) |\mathcal{G}_{\ell_3}|^2 \leq (c/v_1^4) |\mathcal{Q}((\beta_1/2)\gamma^2, \gamma)|^2$. To prove the lemma we divide by $\ell_3 - \ell_2$ and take ℓ_3 so large that $c/v_1^2 \sqrt{\ell_3} - \ell_2 \leq v_2$. \square

Next we show that indeed S is strictly below its supremum μ^+ in a smaller box coaxial with $(\hat{t}, \hat{x}) + \mathcal{Q}(\beta_1\gamma^2/2, \gamma)$ and with the same vertex.

Lemma 4.6

Under assumptions of Lemma 4.4, there is a number ℓ_4 (independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) so that if

$$\left| \frac{\omega}{2^{\ell_4}} \right|^{-2} \beta_1 \gamma^{N\alpha} \leq 1 \tag{59}$$

then $S(t, x) \leq \mu^+ - (\omega/2^{\ell_4+4})$ for $(t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}((\beta_1/2)|\gamma/2|^2, \gamma/2)$.

Proof

Set $(\hat{t}, \hat{x}) = (0, 0)$. We use (20) of Lemma 3.2 written over the boxes $\mathcal{Q}(\beta_1 \gamma_n^2/2, \gamma_n)$ to the function $(S - \mathbf{j}_n)_+$, where

$$\gamma_n = \frac{\gamma}{2} + \frac{\gamma}{2^{n+1}}, \quad \mathbf{j}_n = \mu^+ - \frac{\omega}{2^{\ell_4+1}} - \frac{\omega}{2^{\ell_4+1+n}} \quad \text{for } n = 0, 1, 2, \dots$$

The cut-off functions ζ_n are taken to satisfy

$$\left\{ \begin{array}{ll} 0 < \zeta_n(t, x) \leq 1 & \text{for } (t, x) \in \mathcal{Q}(\beta_1 \gamma_n^2/2, \gamma_n) \\ \zeta_n = 1 & \text{in } \mathcal{Q}(\beta_1 \gamma_{n+1}^2/2, \gamma_{n+1}) \\ \zeta_n = 0 & \text{on the parabolic boundary of } \mathcal{Q}(\beta_1 \gamma_n^2/2, \gamma_n) \\ |\nabla \zeta_n| \leq c \left| \frac{2^{n+2}}{\gamma} \right| \\ 0 \leq \partial_t \zeta_n \leq \frac{1}{\beta_1} \left| \frac{2^{(n+2)}}{\gamma} \right|^2 \end{array} \right. \tag{60}$$

With these choices, (20) of Lemma 3.2 gives

$$\begin{aligned} & \sup_{-\beta_1 \gamma_n^2/2 < t < 0} \int_{\mathcal{K}_{\gamma_n}} \zeta_n^2 (S - \mathbf{j}_n)_+^2 + \frac{1}{\beta_1} \int_{\mathcal{Q}(\beta_1 \gamma_n^2/2, \gamma_n)} |\nabla(\zeta_n (S - \mathbf{j}_n)_+)|^2 \\ & \leq c \left(\left| \frac{2^n \omega}{\gamma 2^{\ell_4}} \right|^2 \frac{1}{\beta_1} \int_{\mathcal{Q}(\beta_1 \gamma_n^2/2, \gamma_n)} \mathcal{X}_{\{(S - \mathbf{j}_n)_+ > 0\}} + \int_{-\beta_1 \gamma_n^2/2}^0 |\mathcal{D}_{\mathbf{j}_n, \gamma_n}^+(\tau)|^{(1 - (2/k_1))} d\tau \right) \end{aligned} \tag{61}$$

Next, in the cylinders $\mathcal{Q}(\beta_1 \gamma_n^2/2, \gamma_n)$ we change variable $z = 2t/\beta_1$ which maps $\mathcal{Q}(\beta_1 \gamma_n^2/2, \gamma_n)$ into $\mathcal{Q}_n \equiv [-\gamma_n^2, 0] \times \mathcal{K}_{\gamma_n}$. Setting $\hat{\phi}(z, \cdot) \equiv S(z\beta_1/2, \cdot)$, $\hat{\zeta}_n(z, \cdot) \equiv \zeta_n(z\beta_1/2, \cdot)$, $\mathcal{D}_n(z) \equiv \{x \in \mathcal{K}_{\gamma_n} : \hat{\phi}(z, x) > \mathbf{j}_n\}$, and $|\mathcal{D}_n| \equiv \int_{-\gamma_n^2}^0 |\mathcal{D}_n(z)| dz$. Inequality (61) can be rewritten as, by (59),

$$\|\hat{\zeta}_n(\hat{\phi} - \mathbf{j}_n)_+\|_{\mathcal{V}(\mathcal{Q}_n)}^2 \leq c \left| \frac{\omega}{2^{\ell_4}} \right|^2 \left(\left| \frac{2^n}{\gamma} \right|^2 |\mathcal{D}_n| + \gamma^{-N\alpha} \int_{-\gamma_n^2}^0 |\mathcal{D}_n(\tau)|^{(1 - (2/k_1))} d\tau \right) \tag{62}$$

Equation (62) and Corollary 3.1 of Chapter 1 [8] give

$$\begin{aligned} & \frac{1}{2^{2(n+2)}} \left| \frac{\omega}{2^{\ell_4}} \right|^2 |\mathcal{D}_{n+1}| \leq |\mathbf{j}_n - \mathbf{j}_{n+1}|^2 |\mathcal{D}_{n+1}| \leq \|(\hat{\phi} - \mathbf{j})_+\|_{L^2(\mathcal{D}_{n+1})}^2 \\ & \leq \|\hat{\xi}_n(\hat{\phi} - \mathbf{j})_+\|_{L^2(\mathcal{D}_n)}^2 \leq c \|\hat{\xi}_n(\hat{\phi} - \mathbf{j})_+\|_{\mathcal{Y}(\mathcal{D}_n)}^2 |\mathcal{D}_n|^{2/(N+2)} \\ & \leq c \left| \frac{\omega}{2^{\ell_4}} \right|^2 \left(\frac{2^{2n}}{\gamma^2} |\mathcal{D}_n|^{1+(2/(N+2))} + \gamma^{-N\alpha} |\mathcal{D}_n|^{2/(N+2)} \int_{-\gamma_n^2}^0 |\mathcal{D}_n(\tau)|^{(1-2/\mathbf{k}_1)} d\tau \right) \end{aligned}$$

Define $Y_n \equiv |\mathcal{D}_n|/|\mathcal{D}_n|$, $Z_n \equiv (1/|\mathcal{X}_{\gamma_n}|)(\int_{-\gamma_n^2}^0 |\mathcal{D}_n(\tau)|^{(1-2/\mathbf{k}_1)} d\tau)^{1/(1+\alpha)}$, we have the recursive inequalities as Lemma 4.1,

$$\begin{cases} Y_{n+1} \leq c4^{2n}(Y_n^{1+(2/(N+2))} + Y_n^{2/(N+2)}Z_n^{1+\alpha}) \\ Z_{n+1} \leq c4^{2n}(Y_n + Z_n^{1+\alpha}) \end{cases} \quad \text{for } n = 0, 1, 2, \dots$$

It follows from these with the aid of Lemma 4.2 of Chapter 1 [8] that Y_n and Z_n tend to zero as $n \rightarrow \infty$, provided $Y_0 + Z_0^{1+\alpha} \leq v \equiv c^{-(1+\alpha)/\lambda} 4^{-2(1+\alpha)/\lambda^2}$. See (4) for λ . Then continuing as the argument of the proof of Lemma 4.1 and using Lemma 4.5, one can choose ℓ_4 and complete the proof of this lemma. □

The results of the second alternative imply

Lemma 4.7

Under assumptions of Lemma 4.3, there are numbers $\eta_2 \in (1/2, 1)$ and $I_2 \gg 1$ (independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) such that either $\omega \leq I_2 \gamma^{N\alpha/(2+\mathbf{m})}$ or $\text{ess osc}_{(\hat{t}, \hat{x}) + \mathcal{Q}((\beta_1/2)|\gamma/2|^2, \gamma/2)} \mathcal{S} \leq \eta_2 \omega$.

We combine Lemmas 4.2, 4.7 with (29) into

Lemma 4.8

If $\gamma \leq \gamma^*$, $0 < \omega \leq \vartheta/2$, and $\mu^+ < \vartheta$, there exist constants

$$\eta \equiv \max\{\eta_1, \eta_2\} \in (1/2, 1) \quad \text{and} \quad I \equiv \max\{I_1, I_2\}$$

that are determined by given data and independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$, such that either $\omega \leq \min\{I\gamma^{\min\{N\alpha/(2+\mathbf{m}), \varpi/\mathbf{m}\}}, \vartheta/2\}$ or $\text{ess osc}_{(\hat{t}, \hat{x}) + \mathcal{Q}((\beta_1/2)|\gamma/2|^2, \gamma/2)} \mathcal{S} \leq \eta\omega$.

4.1.1.3. Hölder estimate. Using notations of Lemma 4.8, we define

$$\begin{cases} M_0 \equiv \mu^+ > \omega \\ \omega_1 \equiv \min \left\{ \max\{\eta\omega, I\gamma^{\min\{N\alpha/(2+\mathbf{m}), \varpi/\mathbf{m}\}}\}, \frac{\vartheta}{2} \right\} \\ M_1 \equiv \max\{\omega_1, \sup_{(\hat{t}, \hat{x}) + \mathcal{Q}((\beta_1/2)|\gamma/2|^2, \gamma/2)} \mathcal{S}\} \end{cases} \quad (63)$$

Lemma 4.9

There is a constant $\mathbf{d} > 1$ (determined by given data) such that $\mathcal{J}(M_0) \leq \mathbf{d} \mathcal{J}(M_1)$.

Proof

If $\omega_1 = \vartheta/2$, then $\mathcal{J}(M_0) \leq \mathbf{d} \mathcal{J}(M_1)$ by A7. If $\omega_1 < \vartheta/2$, then $\omega_1 \geq \eta\omega$. Therefore if $\omega_1 < \vartheta/2$ and $\mu^- \leq \frac{1}{2}\mu^+$, then $M_0 \leq 2\omega \leq 2\omega_1/\eta \leq 2M_1/\eta$. So $\mathcal{J}(M_0) \leq \mathbf{d} \mathcal{J}(M_1)$. If $\omega_1 < \vartheta/2$ and $\mu^- > \frac{1}{2}\mu^+$, then $M_0/2 = \mu^+/2 < \mu^- \leq M_1$. Therefore, $\mathcal{J}(M_0) \leq \mathcal{J}(2M_1) \leq \mathbf{d} \mathcal{J}(M_1)$. \square

Let us estimate from below the length of the cylinder $\mathcal{Q}((\beta_1/2)|\gamma/2|^2, \gamma/2)$ for which the conclusion of Lemma 4.8 holds. We have, by (51),

$$\frac{\beta_1}{2} \left| \frac{\gamma}{2} \right|^2 = \frac{v_1}{4\mathcal{J}(M_0)} \left| \frac{\gamma}{2} \right|^2 \geq \frac{v_1}{4\mathbf{d}\mathcal{J}(M_1)} \left| \frac{\gamma}{2} \right|^2 = \frac{\gamma_1^2}{\mathcal{J}(M_1)}$$

where $\gamma_1 \equiv \sqrt{(v_1/16\mathbf{d})}\gamma \equiv \mathcal{A}\gamma$. So $\mathcal{Q}(\gamma_1^2/\mathcal{J}(M_1), \gamma_1) \subset \mathcal{Q}((\beta_1/2)|\gamma/2|^2, \gamma/2)$.

Lemma 4.10

There are constants $\mathbf{a}, \mathcal{A} \in (0, 1), \eta \in (1/2, 1)$ and I (depending on given data but independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) and it is possible to construct a sequence, for $n = 0$,

$$\omega_0 \equiv \omega, \quad M_0 \equiv \mu^+, \quad \gamma_0 \equiv \gamma, \quad \mathcal{E}_0 \equiv \mathcal{Q}\left(\frac{\gamma_0^2}{\mathcal{J}(M_0)}, \gamma_0\right) \tag{64}$$

and, for $n \in \mathbf{N}$,

$$\omega_n, \hat{\mathcal{E}}_{n-1}, M_n \equiv \max \left\{ \omega_n, \sup_{(\hat{t}, \hat{x}) + \hat{\mathcal{E}}_{n-1}} S \right\}, \quad \gamma_n \equiv \mathcal{A}\gamma_{n-1}, \quad \mathcal{E}_n \equiv \mathcal{Q}\left(\frac{\gamma_n^2}{\mathcal{J}(M_n)}, \gamma_n\right)$$

such that, for all $n \in \mathbf{N}$,

$$\begin{cases} \text{ess osc}_{(\hat{t}, \hat{x}) + \hat{\mathcal{E}}_{n-1}} S \leq \omega_n \equiv \min\{\max\{\eta\omega_{n-1}, I\gamma_{n-1}^{\mathbf{a}}\}, \vartheta/2\} \\ \mathcal{E}_n \subset \hat{\mathcal{E}}_{n-1} \subset \mathcal{E}_{n-1} \end{cases} \tag{65}$$

Proof

This is proved by induction. Let $(\hat{t}, \hat{x}) = (0, 0)$. Take $\mathbf{a} = \min\{N\alpha/(2 + \mathbf{m}), \varpi/\mathbf{m}\}$, \mathcal{A} is the one defined in the remark after Lemma 4.9, and η, I are the ones in Lemma 4.8. For $i = 0$ case, $\omega_0, M_0, \gamma_0, \mathcal{E}_0$ are given in (64). By (63), Lemma 4.9, and remark after Lemma 4.9, we get $\omega_1, \hat{\mathcal{E}}_0, M_1, \gamma_1, \mathcal{E}_1$ and it is easy to see that they satisfy (65). Assume that the sequence is obtained up to $i = n$. We reset $\mu^+ = \sup_{\hat{\mathcal{E}}_{n-1}} S, \mu^- = \inf_{\hat{\mathcal{E}}_{n-1}} S, \omega = \mu^+ - \mu^-$. By (65), $\omega \leq \omega_n$.

(1) If $M_n = \omega_n$ and $\sup_{\hat{\mathcal{E}}_{n-1}} S \leq M_n/2$, we pick

$$\begin{cases} \omega_{n+1} \equiv \min\{\max\{\eta\omega_n, I\gamma_n^{\mathbf{a}}\}, \vartheta/2\} \\ \hat{\mathcal{E}}_n = \mathcal{E}_n \\ M_{n+1} \equiv \max\{\omega_{n+1}, \sup_{\hat{\mathcal{E}}_n} S\} \\ \gamma_{n+1} \equiv \mathcal{A}\gamma_n \\ \mathcal{E}_{n+1} \equiv \mathcal{Q}\left(\frac{\gamma_{n+1}^2}{\mathcal{J}(M_{n+1})}, \gamma_{n+1}\right) \end{cases} \tag{66}$$

Since $\text{ess osc}_{\hat{\mathcal{E}}_n} S \leq \omega_n/2 \leq \eta\omega_n \leq \omega_{n+1}$, (65)₁ holds. Also note if $\omega_{n+1} = \vartheta/2$, then $\mathcal{J}(M_n) \leq \mathcal{J}(M_{n+1})$, and if $\omega_{n+1} < \vartheta/2$, then $\mathcal{J}(M_n) = \mathcal{J}(\omega_n) \leq \mathcal{J}(\omega_{n+1}/\eta) \leq \mathbf{d}\mathcal{J}(M_{n+1})$. Therefore, we have $\mathcal{E}_{n+1} \subset \hat{\mathcal{E}}_n$. So we prove (65)₂.

(2) If $M_n = \omega_n$ and $M_n/2 \leq \sup_{\hat{\mathcal{E}}_{n-1}} S \leq M_n$, we repeat the proofs of Lemmas 4.1–4.8 to see that if $\omega_{n+1} \equiv \min\{\max\{\eta\omega_n, I\gamma_n^{\mathbf{a}}\}, \vartheta/2\}$, $\hat{\mathcal{E}}_n \equiv \mathcal{Q}((v_1/4 \mathcal{J}(M_n))|\gamma_n/2|^2, \gamma_n/2)$ then, by (63),

$$\begin{cases} \text{ess osc}_{\hat{\mathcal{E}}_n} S \leq \min\{\max\{\eta\omega, I\gamma_n^{\mathbf{a}}\}, \vartheta/2\} \leq \omega_{n+1} \\ \hat{\mathcal{E}}_n \subset \mathcal{Q}\left(\frac{\gamma_n^2}{\mathcal{J}(M_n)}, \gamma_n\right) = \mathcal{E}_n \end{cases}$$

Define $M_{n+1}, \gamma_{n+1}, \mathcal{E}_{n+1}$ as (66) and argue as Lemma 4.9 to get $\mathcal{E}_{n+1} \subset \hat{\mathcal{E}}_n$. So we prove (65).

(3) If $M_n = \sup_{\hat{\mathcal{E}}_{n-1}} S$, one repeats the proofs of Lemmas 4.1–4.9 as case (2), and it is easy to get (65). □

Lemma 4.11

There exist constants $\mathbf{a}, \mathcal{A}, \sigma \in (0, 1)$, $\eta \in (1/2, 1)$, and I (depending on given data but independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) such that if $\eta \leq \mathcal{A}^{\mathbf{a}}$, then

$$\text{ess osc}_{(\hat{t}, \hat{x}) + \mathcal{Q}(\rho^2/\mathcal{J}(M_0), \rho)} S \leq c(\eta, \mathbf{a}, \mathcal{A}, I) |\omega_0 + \gamma_0^{\mathbf{a}}| \left| \frac{\rho}{\gamma_0} \right|^\sigma$$

for all cylinders $\mathcal{Q}(\rho^2/\mathcal{J}(M_0), \rho)$ and $0 < \rho \leq \gamma_0^* \leq \gamma_0$ (see (64) for ω_0, M_0, γ_0). Here γ_0^* depends on γ_0, ω_0 , and given data only.

Proof

Let $(\hat{t}, \hat{x}) = (0, 0)$. By (65), $\omega_{n+1} \leq \eta\omega_n + I\gamma_n^{\mathbf{a}}$. By iteration,

$$\omega_n \leq \eta^n \omega_0 + I\gamma_0^{\mathbf{a}} \sum_{i=0}^{n-1} \eta^i \mathcal{A}^{\mathbf{a}(n-i)}$$

Since $\eta \leq \mathcal{A}^{\mathbf{a}}$,

$$\omega_n \leq \eta^n \omega_0 + nI|\gamma_0 \mathcal{A}^{\mathbf{a}}|^{\mathbf{a}} \tag{67}$$

For any $\rho \in (0, \gamma_0 \mathcal{A})$, there exists an integer n_ρ (depending on ρ) such that

$$\gamma_0 \mathcal{A}^{(n_\rho+1)} \leq \rho \leq \gamma_0 \mathcal{A}^{n_\rho} \tag{68}$$

Equation (68) implies

$$\begin{cases} n_\rho \leq \frac{1}{\ln \mathcal{A}} \ln \frac{\rho}{\gamma_0} \leq n_\rho + 1 \\ \eta^{n_\rho} \leq \eta^{-1} \left| \frac{\rho}{\gamma_0} \right|^{\sigma_1} \quad \text{where } \sigma_1 \equiv \frac{\ln \eta}{\ln \mathcal{A}} \end{cases} \tag{69}$$

We take $\gamma_0^* < \gamma_0$ so small that, for $\rho \leq \gamma_0^*$,

$$\begin{cases} \left| \ln \frac{\rho}{\gamma_0} \right| \leq \left| \frac{\gamma_0}{\rho} \right|^{\mathbf{a}/2} \\ \omega_{n_\rho} \leq \omega_0 \end{cases} \tag{70}$$

Here γ_0^* depends on γ_0, ω_0 , and given data. Equations (68), (69)₁, and (70)₁ imply that if $\rho \leq \gamma_0^*$,

$$n_\rho I |\gamma_0 \mathcal{A}^{n_\rho}|^{\mathbf{a}} \leq I \mathcal{A}^{-\mathbf{a}} \frac{\rho^{\mathbf{a}}}{|\ln \mathcal{A}|} \left| \ln \frac{\rho}{\gamma_0} \right| \leq c(\mathbf{a}, \mathcal{A}, I) \gamma_0^{\mathbf{a}/2} \rho^{\mathbf{a}/2} \tag{71}$$

Therefore, (67), (69)₂, and (71) imply $\omega_{n_\rho} \leq c(\eta, \mathbf{a}, \mathcal{A}, I) |\omega_0 + \gamma_0^{\mathbf{a}}| |\rho/\gamma_0|^\sigma$ where $\sigma = \min\{\sigma_1, \mathbf{a}/2\}$. Note $\mathcal{Q}(\rho^2/\mathcal{J}(M_0), \rho) \subset \mathcal{E}_{n_\rho}$ by (68), (70)₂, and the construction of \mathcal{E}_n in Lemma 4.10. So we prove the lemma. \square

4.1.2. For $\vartheta/4 < \mu^- \leq \mu^+ < 1 - \vartheta/4$ and $1 - \vartheta < \mu^-$ cases. For $\vartheta/4 < \mu^- \leq \mu^+ < 1 - \vartheta/4$ case, $(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma) \subset \Omega_2^T$ by (28)₂. Equation (11) corresponds to a uniform (independent of ε) parabolic equation in $(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$. So we may repeat above argument to obtain Lemma 4.11. For $1 - \vartheta < \mu^-$ case, $(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma) \subset \Omega_3^T$ by (28)₃. Since Equation (11) around the end point 0 has similar properties as (14) around the other end point 1 (see remark in Section 2), we use (14) and repeat above argument to get the same conclusion as Lemma 4.11. In summary, Lemma 4.11 holds for $0 < \omega \leq \vartheta/2$ case.

4.2. For $\vartheta/2 \leq \omega \leq 1$ case

In this section, we shall prove that if $\vartheta/2 \leq \omega \leq 1$ and if γ is small enough, lower (resp. upper) bound of saturation S is greater than 0 (resp. less than 1), and the lower and upper bounds are independent of ε . If this is the case, we are in case (28)₂ again and Lemma 4.11 can be shown by following the argument in Section 4.1. Define $\beta^{-1} \equiv \sup_{\mu^- \leq \xi \leq \mu^+} \mathcal{J}(\xi)$. Since $\omega \geq \vartheta/2$, by (26),

$$|\gamma^*|^{\varpi} \leq \min\{\mathcal{J}(\vartheta/2), \mathcal{J}(1 - \vartheta/2)\} \leq \beta^{-1} \leq \sup_{0 \leq \xi \leq 1} \mathcal{J}(\xi)$$

So $\mathcal{Q}(\beta\gamma^2, \gamma) \subset \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$ for all $\gamma \leq \gamma^*$.

Lemma 4.12

Under $\gamma \leq \gamma^*$ and $\omega \geq \vartheta/2$, there is a $v_3 \in (0, 1)$ (depending on given data but independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) such that (1) if

$$\begin{cases} \ell_5 \geq 2, 2^{-\ell_5} \leq \vartheta, \left| \frac{\omega}{2^{\ell_5}} \right|^{-2} \gamma^{N\alpha} \leq 1 \\ \left| \left\{ (t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2, \gamma) : S(t, x) < \frac{\omega}{2^{\ell_5}} \right\} \right| \leq v_3 |\mathcal{Q}(\beta\gamma^2, \gamma)| \end{cases}$$

then $S(t, x) \geq \omega/2^{\ell_5+4}$ a.e. in $(\hat{t}, \hat{x}) + \mathcal{Q}(\beta|\gamma/2|^2, \gamma/2)$, and (2) if

$$\begin{cases} \ell_5 \geq 2, 2^{-\ell_5} \leq \vartheta, \left| \frac{\omega}{2^{\ell_5}} \right|^{-2} \gamma^{N\alpha} \leq 1 \\ \left| \left\{ (t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta\gamma^2, \gamma) : S(t, x) > 1 - \frac{\omega}{2^{\ell_5}} \right\} \right| \leq v_3 |\mathcal{Q}(\beta\gamma^2, \gamma)| \end{cases}$$

then $S(t, x) \leq 1 - (\omega/2^{\ell_5+4})$ a.e. in $(\hat{t}, \hat{x}) + \mathcal{Q}(\beta|\gamma/2|^2, \gamma/2)$.

Proof

Proof is almost same as that of Lemma 4.1. □

Lemma 4.13

If $\tau (\leq T)$, $\ell (\geq 2 + \ell_0 \geq 4) \in \mathbb{N}$, and $\mathbf{k}_2/2^{\ell_0} \leq \vartheta$, solutions of Lemma 3.1 satisfy

$$\sup_{t \leq \tau} |\{x \in \Omega : S(t, x) \leq \mathbf{w} \text{ or } 1 - \mathbf{w} \leq S(t, x)\}| \leq \frac{c_0 |c_0 \tau|^{\ell - \ell_0}}{(\ell - \ell_0)^{(\ell - \ell_0) f_\ell}} \tag{72}$$

where $\lim_{\ell \rightarrow \infty} f_\ell = 1$, $\mathbf{w} \equiv \mathbf{k}_2/2^\ell$ and c_0 is a constant independent of τ, ℓ, ε .

Proof

Define $\mathcal{L}_w, \mathcal{L}_{\zeta, w}, \check{\mathcal{X}}_w$ as

$$\mathcal{L}_w(z) \equiv \begin{cases} 0 & \text{for } 2\mathbf{w} \leq z \\ z - 2\mathbf{w} & \text{for } \mathbf{w} \leq z \leq 2\mathbf{w} \\ -\mathbf{w} & \text{for } z \leq \mathbf{w} \end{cases}$$

$$\mathcal{L}_{\zeta, w}(z) \equiv \begin{cases} 0 & \text{for } \zeta(2\mathbf{w}) \leq z \\ z - \zeta(2\mathbf{w}) & \text{for } \zeta(\mathbf{w}) \leq z \leq \zeta(2\mathbf{w}) \\ \zeta(\mathbf{w}) - \zeta(2\mathbf{w}) & \text{for } z \leq \zeta(\mathbf{w}) \end{cases}$$

$$\check{\mathcal{X}}_w(z) \equiv \begin{cases} 1 & \text{for } \mathbf{w} \leq z \leq 2\mathbf{w} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\zeta(z) \equiv \int_{0.5}^z \frac{\tilde{\Lambda}_w}{\tilde{\Lambda}}(\xi) d\xi, \quad z \in (0, 1) \tag{73}$$

Then $\check{\mathcal{X}}_w(z) = \mathcal{L}'_w(z) = \mathcal{L}'_{\zeta, w}(\zeta(z))$, $(d/dz)\zeta(z) = (\tilde{\Lambda}_w/\tilde{\Lambda})(z)$. By $2\mathbf{w} \leq \mathbf{k}_2/2$ and (10), both $\mathcal{L}_w(S), \mathcal{L}_{\zeta, w}(\zeta(S)) \in L^2(0, T; \mathcal{W})$. Take $\zeta = \mathcal{L}_w(S), \xi = \mathcal{L}_{\zeta, w}(\zeta(S))$ in (11)–(12) to obtain, by A4,

$$\int_{\Omega^r} \Phi \mathcal{L}_w(S) \partial_t S - \int_{\Omega^r} K \Lambda_w(S) \check{\mathcal{X}}_w(S) \nabla \Upsilon(S) \nabla S \leq c_1 \int_{\Omega^r} K \tilde{\Lambda}_w \check{\mathcal{X}}_w(S) |\nabla S| \tag{74}$$

where constant c_1 is independent of \mathbf{w}, ε . If $\int_{\Omega^r} \Phi \mathcal{L}_w(S) \partial_t S \geq 0$, (74) implies

$$\int_{\Omega^r} K \tilde{\Lambda}_w \check{\mathcal{X}}_w(S) |\nabla S| \leq c_2 \sqrt{\int_{\Omega^r} K \frac{\tilde{\Lambda}_w \check{\mathcal{X}}_w}{|d\Upsilon/dS|}(S)} \sqrt{\int_{\Omega^r} K \tilde{\Lambda}_w \check{\mathcal{X}}_w(S) |\nabla S|} \tag{75}$$

where constant c_2 is independent of \mathbf{w}, ε . Equations (74)–(75) imply

$$\int_{\Omega^r} \Phi \mathcal{L}_w(S) \partial_t S \leq c_3 \int_{\Omega^r} K \frac{\tilde{\Lambda}_w \check{\mathcal{X}}_w}{|d\Upsilon/dS|}(S) \tag{76}$$

Define $\mathcal{Z}_w \equiv \int_{2\mathbf{w}}^S \mathcal{L}_w(z) dz$. Equation (76) implies

$$\int_{\Omega^r} \Phi \partial_t \mathcal{Z}_w = \int_{\Omega^r} \Phi \mathcal{L}_w(S) \partial_t S \leq c_3 \int_{\Omega^r} K \frac{\tilde{\Lambda}_w \check{\mathcal{X}}_w}{|d\Upsilon/dS|}(S) \tag{77}$$

Equation (77) and A7 yield that, if $0 \leq t_1 \leq t_2 \leq T$,

$$\int_{t_1}^{t_2} \int_{\Omega} \Phi \partial_t \mathcal{Z}_w \leq c_4 \int_{t_1}^{t_2} \int_{\Omega} \mathcal{Z}_{2\mathbf{w}} \tag{78}$$

where c_4 is independent of $t_1, t_2, \mathbf{w}, \varepsilon$. Define $\mathcal{F}(\mathbf{w}, \tau) \equiv (1/\mathbf{w}^2) \sup_{t \leq \tau} \int_{\Omega} \mathcal{Z}_w(t, \cdot)$. A5 and (78) imply that, for $0 \leq t_1 \leq t_2 \leq T$,

$$\mathcal{F}(\mathbf{w}, t_2) - \mathcal{F}(\mathbf{w}, t_1) \leq c_5(t_2 - t_1) \mathcal{F}(2\mathbf{w}, t_2) \tag{79}$$

where c_5 is independent of $t_1, t_2, \mathbf{w}, \varepsilon$. By induction and (5), one obtains, for $j \in \mathbf{N}, jh \leq T$,

$$\mathcal{F}\left(\frac{\mathbf{k}_2}{2^\ell}, jh\right) \leq (\ell - \ell_0 + 1)^{j-1} |c_5 h|^{\ell - \ell_0} \mathcal{F}\left(\frac{\mathbf{k}_2}{2^{\ell_0}}, jh\right) \tag{80}$$

If $j = (\ell - \ell_0)/\log(\ell - \ell_0)$ and $\tau = jh$ in (80), then

$$\mathcal{F}\left(\frac{\mathbf{k}_2}{2^\ell}, \tau\right) \leq \frac{|c_5 \tau|^{\ell - \ell_0}}{(\ell - \ell_0)^{(\ell - \ell_0) f_\ell}} \mathcal{F}\left(\frac{\mathbf{k}_2}{2^{\ell_0}}, \tau\right) \tag{81}$$

where $f_\ell \rightarrow 1$ as $\ell \rightarrow \infty$. Define $\mathcal{B}(t) \equiv \{x \in \Omega : S(t, x) \leq \mathbf{w} = \mathbf{k}_2/2^\ell\}$. Equation (81) implies

$$\sup_{t \leq \tau} \int \mathcal{X}_{\mathcal{B}(t)} \leq c_6 \mathcal{F}\left(\frac{\mathbf{k}_2}{2^\ell}, \tau\right) \leq \frac{c_6 |c_5 \tau|^{\ell - \ell_0}}{(\ell - \ell_0)^{(\ell - \ell_0) f_\ell}} \mathcal{F}\left(\frac{\mathbf{k}_2}{2^{\ell_0}}, \tau\right)$$

where c_6 is independent of $\tau, \ell, \mathbf{w}, \varepsilon$. So proof for one end point of (72) is completed. Proof of the other end point of (72) can be proceeded in the same manner, so we skip it. \square

By Lemmas 4.12 and 4.13, one can conclude:

Corollary 4.1

There is a ℓ_6 (depending on given data and $\bar{\mathcal{O}}$ but independent of ε) such that if $\ell \geq \ell_6$, $\gamma = |\vartheta/2^{1+\ell}|^{2/N\alpha}$, and $\vartheta/2 \leq \omega$, then

$$\frac{\vartheta}{2^{\ell+5}} \leq S(t, x) \leq 1 - \frac{\vartheta}{2^{\ell+5}}$$

for any $(t, x) \in (\hat{t}, \hat{x}) + \mathcal{Q}(\beta|\gamma/2|^2, \gamma/2)$ and $(\hat{t}, \hat{x}) \in \bar{\mathcal{O}}$.

Similar to the case where $0 < \omega \leq \vartheta/2$ and $\vartheta/4 < \mu^- \leq \mu^+ < 1 - (\vartheta/4)$ hold (i.e. (28)₂ case), Corollary 4.1 implies that Equation (11) corresponds to a uniform (independent of ε) parabolic equation in $(\hat{t}, \hat{x}) + \mathcal{Q}(\beta|\gamma/2|^2, \gamma/2)$ for $(\hat{t}, \hat{x}) \in \bar{\mathcal{O}}$. So we may repeat the argument in Section 4.1 to get the same conclusion as Lemma 4.11 for $\omega \geq \vartheta/2$ case. Combining (27), Lemma 4.11, remark in Section 4.1.2, Corollary 4.1 with a standard covering argument, we get uniformly Hölder continuous for S over any compact subset \mathcal{O} of Ω^T .

5. HÖLDER ESTIMATE OF REGULARIZED EQUATIONS ON PARABOLIC BOUNDARY

In this section, we give a Hölder estimate for S on parabolic boundary of Ω^T . Basically, the estimate can be shown by following the proof for the interior region. Therefore we only give a sketch of the proof. The idea to treat parabolic boundary can also be found in Reference [8]. The boundary consists of Dirichlet boundary (Γ_2^T), Neumann boundary (Γ_1^T), edge ($\bar{\Gamma}_1^T \cap \bar{\Gamma}_2^T$), and initial boundary ($T=0$). They are discussed below:

5.1. Dirichlet boundary, Neumann boundary, and edge

We first derive a result similar to Lemmas 3.2 and 3.3 for Dirichlet boundary. Fix $(\hat{t}, \hat{x}) \in \Gamma_2^T$ and consider the cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)$, where θ and ρ are so small that $\hat{t} - \theta > 0$ and $\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \partial\Omega^T \subset \Gamma_2^T$. Then we modify the interior quantities in (19), (22) as

$$\hat{\mathcal{D}}_{\mathbf{j}, \rho}^{\pm}(\tau) \equiv \{x \in \{\hat{x} + \mathcal{K}_{\rho}\} \cap \Omega : (S - \mathbf{j})_{\pm}(\tau, x) > 0\} \tag{82}$$

$$\Psi(\hat{H}_{\mathbf{j}}^{\pm}, (S - \mathbf{j})_{\pm}, \delta) \equiv \ln^+ \left| \frac{\hat{H}_{\mathbf{j}}^{\pm}}{\hat{H}_{\mathbf{j}}^{\pm} - (S - \mathbf{j})_{\pm} + \delta} \right| \tag{83}$$

where $\hat{H}_{\mathbf{j}}^{\pm} \equiv \sup_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T} (S - \mathbf{j})_{\pm}$ and $0 < \delta < \min\{1, \hat{H}_{\mathbf{j}}^{\pm}\}$. In $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)$, we introduce a piecewise smooth cut-off function $(t, x) \rightarrow \zeta(t, x)$ satisfying (16). We observe that for all $t \in (\hat{t} - \theta, \hat{t}), x \rightarrow \zeta(t, x)$ vanishes on the boundary of $\hat{x} + \mathcal{K}_{\rho}$ and not on the boundary of $\{\hat{x} + \mathcal{K}_{\rho}\} \cap \Omega$. Local energy estimate for S near (\hat{t}, \hat{x}) are obtained by taking $\varphi = \pm(S_h - \mathbf{j})_{\pm} \zeta^2$ in (17), integrating over $(\hat{t} - \theta, \hat{t})$ and letting $h \rightarrow 0^+$. Such a choice of testing function is admissible if for a.e. $t \in (\hat{t} - \theta, \hat{t})$,

$$(S_h(\cdot, t) - \mathbf{j})_{\pm} \zeta^2(t, \cdot) \in H_0^1(\{\hat{x} + \mathcal{K}_{\rho}\} \cap \Omega) \tag{84}$$

Since $x \rightarrow \zeta(t, x)$ vanishing on the boundary of $\{\hat{x} + \mathcal{K}_\rho\}$ and not on the boundary of $\{\hat{x} + \mathcal{K}_\rho\} \cap \Omega$, condition (84) needs to be verified whether for a.e. $t \in (\hat{t} - \theta, \hat{t})$,

$$(S_h - \mathbf{j})_\pm = 0 \text{ in the sense of the trace on boundary of } \{\hat{x} + \mathcal{K}_\rho\} \cap \Omega$$

This can be realized for the function $(S_h - \mathbf{j})_+$ if \mathbf{j} is chosen to satisfy

$$\mathbf{j} \geq \sup_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Gamma_2^T} S_b \tag{85}$$

Analogously, the function $-(S_h - \mathbf{j})_-$ can be taken as testing function in (17) if

$$\mathbf{j} \leq \inf_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Gamma_2^T} S_b \tag{86}$$

With these choices of \mathbf{j} , we may repeat calculation in all analogous to those of Lemma 3.2 and derive energy inequality for S near Γ_2^T . Analogous considerations hold for a version of the logarithmic estimates along the lines of Lemma 3.3. We summarize

Lemma 5.1

There is a constant d (independent of $\varepsilon, \theta, \rho, \mathbf{j}$) such that for every $(\hat{t}, \hat{x}) \in \Gamma_2^T$, every cylinder $\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T \subset \Omega_1^T$ satisfying $\hat{t} - \theta > 0$ and $\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \partial\Omega^T \subset \Gamma_2^T$, and every level \mathbf{j} satisfying

$$\begin{cases} \mathbf{j} \geq \sup_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Gamma_2^T} S_b & \text{for the function } (S_h - \mathbf{j})_+ \\ \mathbf{j} \leq \inf_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Gamma_2^T} S_b & \text{for the function } (S_h - \mathbf{j})_- \end{cases} \tag{87}$$

the following inequalities hold:

$$\begin{aligned} & \sup_{\hat{t} - \theta < t < \hat{t}} \int_{\{\hat{x} + \mathcal{K}_\rho\} \cap \Omega} (S - \mathbf{j})_\pm^2 \zeta^2 + \int_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T} \mathcal{J}(S) |\zeta \nabla (S - \mathbf{j})_\pm|^2 \\ & \leq d \left(\int_{\{\hat{x} + \mathcal{K}_\rho\} \cap \Omega} (S - \mathbf{j})_\pm^2 \zeta^2 (\hat{t} - \theta, x) + \int_{\hat{t} - \theta}^{\hat{t}} |\hat{\mathcal{D}}_{\mathbf{j}, \rho}^\pm(\tau)|^{(1-2/k_1)} d\tau \right. \\ & \quad \left. + \int_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T} (\mathcal{J}(S) (S - \mathbf{j})_\pm^2 |\nabla \zeta|^2 + (S - \mathbf{j})_\pm^2 \zeta \partial_t \zeta) \right) \end{aligned} \tag{88}$$

$$\begin{aligned} & \sup_{\hat{t} - \theta < t < \hat{t}} \int_{\{\hat{x} + \mathcal{K}_\rho\} \cap \Omega} \Psi^2(\hat{H}_{\mathbf{j}}^\pm, (S - \mathbf{j})_\pm, \delta)(t, x) \zeta^2(x) \\ & \leq d \left(\int_{\{\hat{x} + \mathcal{K}_\rho\} \cap \Omega} \Psi^2(\hat{H}_{\mathbf{j}}^\pm, (S - \mathbf{j})_\pm, \delta)(\hat{t} - \theta, x) \zeta^2(x) \right. \\ & \quad \left. + \int_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T} \mathcal{J}(S) \Psi(\hat{H}_{\mathbf{j}}^\pm, (S - \mathbf{j})_\pm, \delta) |\nabla \zeta|^2 \right. \\ & \quad \left. + \frac{1}{\delta^2} \left| 1 + \ln^+ \frac{\hat{H}_{\mathbf{j}}^\pm}{\delta} \right| \int_{\hat{t} - \theta}^{\hat{t}} |\hat{\mathcal{D}}_{\mathbf{j}, \rho}^\pm(\tau)|^{(1-2/k_1)} d\tau \right) \end{aligned} \tag{89}$$

where ζ, ξ are piecewise smooth cut-off functions satisfying (16). Equations (88)–(89) also hold if $\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T$ is a subset of Ω_2^T and Ω_3^T . But for Ω_3^T case, S in (88)–(89) should be replaced by S_0 and proofs of those results are required to use Equation (14).

Now take a cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$, where $\gamma > 0$ is so small that $\hat{t} - \gamma^{2-\varpi} > 0$ and $\{(\hat{t}, \hat{x}) + \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)\} \cap \partial\Omega^T \subset \Gamma_2^T$. Assume $(\hat{t}, \hat{x}) = (0, 0)$. Set $\mu^+ \equiv \sup_{\mathcal{Q}(\gamma^{2-\varpi}, 2\gamma) \cap \Omega^T} S$, $\mu^- \equiv \inf_{\mathcal{Q}(\gamma^{2-\varpi}, 2\gamma) \cap \Omega^T} S$, $\omega \equiv \mu^+ - \mu^-$. The Dirichlet boundary problem is studied by considering the following four cases:

$$\left\{ \begin{array}{l} \min \left\{ \frac{\mathbf{k}_2}{4}, \frac{\vartheta}{4} \right\} \leq \mu^- \leq \mu^+ \leq 1 - \min \left\{ \frac{\mathbf{k}_2}{4}, \frac{\vartheta}{4} \right\} \\ \mu^+ \leq \min \left\{ \frac{\mathbf{k}_2}{2}, \frac{\vartheta}{2} \right\} \\ 1 - \min \left\{ \frac{\mathbf{k}_2}{2}, \frac{\vartheta}{2} \right\} \leq \mu^- \\ \text{Not above three cases} \end{array} \right. \tag{90}$$

For (90)₁ case, we define $\beta^{-1} \equiv \sup_{\mu^- \leq \xi \leq \mu^+} \mathcal{J}(\xi)$ and let $\mathcal{Q}(\beta\gamma^2, \gamma) \subset \mathcal{Q}(\gamma^{2-\varpi}, 2\gamma)$ for all $\gamma \leq \gamma^*$. Set $\mu_b^+ \equiv \sup_{\mathcal{Q}(\beta\gamma^2, \gamma) \cap \Gamma_2^T} S_b$, $\mu_b^- \equiv \inf_{\mathcal{Q}(\beta\gamma^2, \gamma) \cap \Gamma_2^T} S_b$, $\omega_b \equiv \mu_b^+ - \mu_b^-$. If the two inequalities

$$\left\{ \begin{array}{l} \mu^+ - \frac{\omega}{2^{\ell_1}} \leq \mu_b^+, \\ \mu^- + \frac{\omega}{2^{\ell_1}} \geq \mu_b^-, \end{array} \right. \quad \ell_1 \geq 2 \tag{91}$$

are both true, subtracting the second from the first gives $\omega \leq 2\omega_b$. If (91)₁ is violated, then the levels $\mathbf{j} \equiv \mu^+ - (\omega/2^i)$, $i \geq \ell_1$ satisfy (87)₁, and we may derive an energy estimate for $(S - \mathbf{j})_+$. Since $(S - \mathbf{j})_+$ vanishes on $\mathcal{Q}(\beta\gamma^2, \gamma) \cap \partial\Omega^T$, we may extend it to the whole $\mathcal{Q}(\beta\gamma^2, \gamma)$ by setting it to be zero outside Ω^T within the box $\mathcal{Q}(\beta\gamma^2, \gamma)$. Thus conclusion of Lemma 4.4 is satisfied by $(S - \mathbf{j})_+$. We then use Lemma 5.1 and argue as Lemma 4.5 to deduce that for all $v_2 \in (0, 1)$, there are positive numbers I, ℓ_3 that are dependent on given data and independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$ such that either $\omega \leq I\gamma^{N\alpha/2}$ or

$$\left| \left\{ (t, x) \in \mathcal{Q}(\beta\gamma^2/2, \gamma) : S(t, x) > \mu^+ - \frac{\omega}{2^{\ell_3}} \right\} \right| \leq v_2 |\mathcal{Q}(\beta\gamma^2/2, \gamma)|$$

An application of Lemma 4.6 now gives

Lemma 5.2

There exist constants $I > 1$ and $\eta \in (1/2, 1)$, dependent on given data and independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$, such that either $\text{ess osc}_{\mathcal{Q}((\beta/2)|\gamma/2|^{2/\varpi}, \gamma/2)} S \leq \max\{\eta\omega, 2\omega_b\}$ or $\omega \leq I\gamma^{N\alpha/2}$.

If (91)₂ is violated, one can use Equation (14) and repeat the above argument to get the same conclusion as Lemma 5.2, and then argue as Lemmas 4.9–4.11 to show the locally Hölder continuity of S .

For cases (90)_{2,3}, local Hölder continuity can be proved by following the arguments in Section 4.1. If (90)_{1,2,3} are violated, then we first show S is bounded away from the two end points by using the arguments in Section 4.2, and then follow the argument for (90)₁ case to show the local Hölder continuity of S .

Proof of Hölder continuity of S on Neumann boundary Γ_1^T and edge $\bar{\Gamma}_1^T \cap \bar{\Gamma}_2^T$ is a straightforward modification of that for Dirichlet boundary.

5.2. Initial boundary

Again we first derive a result similar to Lemmas 3.2 and 3.3. Fix $(\hat{t}, \hat{x}) \in \Omega^T$ and consider the cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\hat{t}, \rho)$. Therefore $(\hat{t}, \hat{x}) + \mathcal{Q}(\hat{t}, \rho)$ lies on the bottom of the cylindrical domain Ω^T . Consider the cutoff function ξ satisfying (16) and independent of t . Local energy estimates for S near $t=0$ are derived by taking $\varphi = \pm (S_h - \mathbf{j})_{\pm} \xi^2$ in (17), integrating over $(0, t), t \in (0, \hat{t})$, and letting $h \rightarrow 0^+$. The first term in (17) gives

$$\frac{1}{2} \int_{\{\hat{x} + \mathcal{K}_\rho\}} \Phi(S_h - \mathbf{j})_{\pm}^2(t, x) \xi^2 dx - \frac{1}{2} \int_{\{\hat{x} + \mathcal{K}_\rho\}} \Phi(S_h - \mathbf{j})_{\pm}^2(0, x) \xi^2 dx$$

If \mathbf{j} is chosen so that $\mathbf{j} \geq \sup_{\{\hat{x} + \mathcal{K}_\rho\}} S_{\text{init}}$, then we have

$$\frac{1}{2} \int_{\{\hat{x} + \mathcal{K}_\rho\}} \Phi(S_h - \mathbf{j})_+^2(0, x) \xi^2 dx \rightarrow 0 \quad \text{as } h \rightarrow 0^+$$

By (83), it follows that

$$\Psi^2(\hat{H}_{\mathbf{j}}^{\pm}, (S_h - \mathbf{j})_{\pm}, \delta)(t, x) = 0 \quad \text{whenever } (S_h - \mathbf{j})_{\pm} = 0$$

Thus if $\mathbf{j} \geq \sup_{\{\hat{x} + \mathcal{K}_\rho\}} S_{\text{init}}$, then

$$\int_{\{\hat{x} + \mathcal{K}_\rho\}} \Psi^2(\hat{H}_{\mathbf{j}}^+, (S_h - \mathbf{j})_+, \delta)(0, x) \xi^2(x) \rightarrow 0 \quad \text{as } h \rightarrow 0^+$$

Analogous considerations hold for $(S_h - \mathbf{j})_- \xi^2$. We summarize

Lemma 5.3

There is a constant d (independent of $\varepsilon, \theta, \rho, \mathbf{j}$) such that for every cylinder $(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho) \subset \Omega_1^T$ satisfying $\hat{t} - \theta = 0$, and every level \mathbf{j} satisfying

$$\begin{cases} \mathbf{j} \geq \sup_{\{\hat{x} + \mathcal{K}_\rho\}} S_{\text{init}} & \text{for the function } (S_h - \mathbf{j})_+ \\ \mathbf{j} \leq \inf_{\{\hat{x} + \mathcal{K}_\rho\}} S_{\text{init}} & \text{for the function } (S_h - \mathbf{j})_- \end{cases} \quad (92)$$

the following inequalities hold:

$$\begin{aligned} & \sup_{\hat{t} - \theta < t < \hat{t}} \int_{\{\hat{x} + \mathcal{K}_\rho\} \cap \Omega} (S - \mathbf{j})_{\pm}^2 \xi^2 + \int_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T} \mathcal{J}(S) |\xi \nabla (S - \mathbf{j})_{\pm}|^2 \\ & \leq d \left(\int_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T} \mathcal{J}(S) (S - \mathbf{j})_{\pm}^2 |\nabla \xi|^2 + \int_{\hat{t} - \theta}^{\hat{t}} |\hat{\mathcal{D}}_{\mathbf{j}, \rho}^{\pm}(\tau)|^{(1-2/k_1)} d\tau \right) \end{aligned} \quad (93)$$

$$\begin{aligned}
 & \sup_{\hat{t}-\theta < t < \hat{t}} \int_{\{\hat{x} + \mathcal{K}_\rho\} \cap \Omega} \Psi^2(\hat{H}_j^\pm, (S - \mathbf{j})_\pm, \delta)(t, x) \xi^2(x) \\
 & \leq d \left(\int_{\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T} \mathcal{J}(S) \Psi(\hat{H}_j^\pm, (S - \mathbf{j})_\pm, \delta) |\nabla \xi|^2 \right. \\
 & \quad \left. + \frac{1}{\delta^2} \left| 1 + \ln^+ \frac{\hat{H}_j^\pm}{\delta} \right| \int_{\hat{t}-\theta}^{\hat{t}} |\hat{\mathcal{G}}_{j, \rho}^\pm(\tau)|^{(1-2/k_1)} d\tau \right) \tag{94}
 \end{aligned}$$

where ξ is a piecewise smooth cut-off function satisfying (16). Equations (93)–(94) hold if $\{(\hat{t}, \hat{x}) + \mathcal{Q}(\theta, \rho)\} \cap \Omega^T$ is a subset of Ω_2^T and Ω_3^T . But for Ω_3^T case, S in (93)–(94) should be replaced by S_\circ and proofs need to use Equation (14).

Fix $(0, \hat{x}) \in \{0\} \times \Omega$ and construct the cylinder $(0, \hat{x}) + \hat{\mathcal{Q}}(\gamma^{2-\sigma}, 2\gamma) \equiv (0, \hat{x}) + (0, \gamma^{2-\sigma}) \times \mathcal{K}_{2\gamma} \subset \Omega^T$. We assume $\hat{x} = 0$. Set $\mu^+ \equiv \sup_{\hat{\mathcal{Q}}(\gamma^{2-\sigma}, 2\gamma)} S$, $\mu^- \equiv \inf_{\hat{\mathcal{Q}}(\gamma^{2-\sigma}, 2\gamma)} S$, $\omega \equiv \mu^+ - \mu^-$. We consider the following four cases:

$$\begin{cases}
 \min \left\{ \frac{\mathbf{k}_2}{4}, \frac{\vartheta}{4} \right\} \leq \mu^- \leq \mu^+ \leq 1 - \min \left\{ \frac{\mathbf{k}_2}{4}, \frac{\vartheta}{4} \right\} \\
 \mu^+ \leq \min \left\{ \frac{\mathbf{k}_2}{2}, \frac{\vartheta}{2} \right\} \\
 1 - \min \left\{ \frac{\mathbf{k}_2}{2}, \frac{\vartheta}{2} \right\} \leq \mu^- \\
 \text{Not above three cases}
 \end{cases} \tag{95}$$

For (95)₁ case, we define $\beta^{-1} \equiv \sup_{\mu^- \leq \xi \leq \mu^+} \mathcal{J}(\xi)$ and construct the box $\hat{\mathcal{Q}}(\beta\gamma^2, \gamma) \equiv (0, \beta\gamma^2) \times \mathcal{K}_\gamma \subset \hat{\mathcal{Q}}(\gamma^{2-\sigma}, 2\gamma)$. Set $\mu_{\text{init}}^+ \equiv \sup_{\mathcal{K}_\gamma} S_{\text{init}}$, $\mu_{\text{init}}^- \equiv \inf_{\mathcal{K}_\gamma} S_{\text{init}}$, $\omega_{\text{init}} \equiv \mu_{\text{init}}^+ - \mu_{\text{init}}^-$, and consider the two inequalities

$$\begin{cases}
 \mu^+ - \frac{\omega}{2^{\ell_1}} \leq \mu_{\text{init}}^+, \\
 \mu^- + \frac{\omega}{2^{\ell_1}} \geq \mu_{\text{init}}^-,
 \end{cases} \quad \ell_1 \geq 2 \tag{96}$$

If both of (96) hold, subtract the second from the first to obtain $\omega \leq 2\omega_{\text{init}}$. If (96)₁ is violated, then the level $\mathbf{j} = \mu^+ - (\omega/2^i)$, $i \geq \ell_1$ satisfies (92)₁. By Lemma 5.3, we have energy and logarithmic estimates for the truncated functions $(S - \mathbf{j})_+$. Using the logarithmic estimate (94) and proceeding as Lemma 4.4, one can show that for any $v_2 \in (0, 1)$, there are positive numbers ℓ_2, I (depending on given data and independent of ε) such that either

$$\omega \leq I\gamma^{N\alpha/2} \tag{97}$$

or, for all $t \in (0, \beta\gamma^2)$, $|\{x \in \mathcal{K}_\gamma : S(t, x) \geq \mu^+ - (\omega/2^{\ell_2})\}| \leq v_2 |\mathcal{K}_\gamma|$. By the energy inequality (93) and the procedure of Lemma 4.6, we conclude that if (97) does not hold, then it is easy to derive $\text{ess osc}_{\hat{\mathcal{Q}}(\beta|\gamma|^{2/\ell_2}, \gamma/2)} S \leq \eta_2 \omega$. If (96)₂ is violated, we use Equation (14) and still proceed

as above. To summarize, going down from $\hat{\mathcal{Q}}(\gamma^{2-\varpi}, 2\gamma)$ to the smaller box $\hat{\mathcal{Q}}(\beta|\gamma/2|^2, \gamma/2)$, the essential oscillation of S decreases by a factor of η , unless either $\omega \leq 2\omega_{\text{init}}$ or $\omega \leq I\gamma^{N\alpha/2}$.

Lemma 5.4

There exist constants $\eta \in (1/2, 1)$, I (depending on given data and independent of $\hat{t}, \hat{x}, \varepsilon, \mu^+, \mu^-, \gamma$) such that either $\text{ess osc}_{\hat{\mathcal{Q}}(\beta|\gamma/2|^2, \gamma/2)} S \leq \max\{\eta\omega, 2\omega_{\text{init}}\}$ or $\omega \leq I\gamma^{N\alpha/2}$.

Then we argue as Lemmas 4.9–4.11 to show locally Hölder continuity of S .

For cases (95)_{2,3}, one can follow above arguments to prove locally Hölder continuity. If (95)_{1,2,3} are violated, then we first show S is bounded away from the two end points by using the arguments in Section 4.2, and then follow the argument for (95)₁ case to show the local Hölder continuity of S .

The regularity of S on the boundary $\partial\Omega$ at $t=0$ can be proved by A1 and a straightforward modification of above argument, so it is skipped.

6. HÖLDER CONTINUITY IN WHOLE REGION

From results of Sections 4, 5 and standard covering argument, one can see that S^ε are Hölder continuous in $\bar{\Omega}^T$ for all ε . Moreover, their Hölder bounds are independent of ε . By Lemma 3.1, the limit function S in Lemma 3.1 is also Hölder continuous in $\bar{\Omega}^T$. So we prove Theorem 2.1.

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