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Cubic planar hamiltonian graphs of various types

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Abstract

Let U be the set of cubic planar hamiltonian graphs, A the set of graphs G in U such that $G - v$ is hamiltonian for any vertex v of G, B the set of graphs G in U such that $G - e$ is hamiltonian for any edge e of G, and C the set of graphs G in U such that there is a hamiltonian path between any two different vertices of G. With the inclusion and/or exclusion of the sets A, B, and C, U is divided into eight subsets. In this paper, we prove that there is an infinite number of graphs in each of the eight subsets.
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1. Definitions and notations

In this paper, for the graph definitions and notations we follow [2]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set of* G. Two vertices u and v are *adjacent* if $(u, v) \in E$. We use $N(x)$ to denote the set of vertices in G that are adjacent to x. Sometimes, we use $N_G(x)$ to emphasize that the underlying graph is G. A *path* P is represented by $\langle v_0, v_1, v_2, \ldots, v_k\rangle$. We use P^{-1} to denote $\langle v_k, v_{k-1}, \ldots, v_1, v_0\rangle$. A path is a *hamiltonian path* if its vertices are distinct and span V. A *cycle* is a path with at least three vertices such that its first vertex is the same as the last vertex. A cycle is a *hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G.

The architecture of an interconnection network is usually represented by a graph. There are a lot of mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum for all conditions. One has to design a suitable network according to the requirements of their properties. The hamiltonian property is one of the major requirements in designing the topology of networks. Fault tolerance is also desirable in massive parallel systems that have a relatively high probability of failure.

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Since vertex faults and edge faults may happen when a network is used, it is practically meaningful to consider faulty networks. A graph G is *k-hamiltonian* if $G - F$ is hamiltonian for any $F \subseteq V \cup E$ and $|F| = k$. It is easy to see that the degree of any vertex in a k-hamiltonian graph is at least $k + 2$. A k-hamiltonian graph is *optimal* if it contains the least number of edges among all k-hamiltonian graphs with the same number of vertices. Thus, an n-regular graph G is *optimal fault-tolerant hamiltonian* if it is $(n - 2)$-hamiltonian. There are some interconnection networks proved to be optimal fault-tolerant hamiltonian, such as twisted-cubes, crossed-cubes, pancake graphs, etc. [6–13].

In these paper, the concept of fault-tolerant hamiltonian connected graphs is introduced. A graph G is *k-hamiltonian connected* if $G - F$ is hamiltonian connected for any $F \subseteq V \cup E$ and $|F| = k$. A k-hamiltonian connected graph is *optimal* if it contains the least number of edges among all k-hamiltonian connected graphs having the same number of vertices.

Suppose that G is a k-hamiltonian connected graph with at least $k + 4$ vertices, then the degree of any vertex of G is at least $k + 3$. For otherwise, let $x \in V(G)$ be any vertex of degree at most $k + 2$ and $N(x) = \{x_1, x_2, \ldots, x_r\}$ with $r \leqslant k + 2$. Let $F = \{x_i \mid 3 \leqslant i \leqslant r\}$, then $|F| \leqslant k$ and $G - F$ has at least four vertices. Let P be any path between x_1 and x_2. Obviously, either $x \notin P$ or $P = \langle x_1, x, x_2 \rangle$. Thus, $G - F$ is not hamiltonian connected. Thus, an n-regular graph G is *optimal fault-tolerant hamiltonian connected* if it is $(n - 3)$-hamiltonian connected.

We observed the aforementioned interconnection networks are recursively constructed. Based on the recursive structure, we use induction to prove that such networks are not only optimal fault-tolerant hamiltonian but also optimal fault-tolerant hamiltonian connected. However, we cannot prove such networks are optimal fault-tolerant hamiltonian without the optimal fault-tolerant hamiltonian connected property. For this reason, we are wondering if any optimal k-hamiltonian graph is optimal $(k - 1)$-hamiltonian connected. In this paper, we will concentrate on the special case with $k = 1$.

A hamiltonian graph G is 1-*vertex hamiltonian* if $G - v$ is hamiltonian for any $v \in V(G)$. (A non-hamiltonian graph such that $G - v$ is hamiltonian for any $v \in V(G)$ is called a *hypohamiltonian* graph. There are numerous studies on hypohamiltonian graphs. Readers can refer to [5] for a survey of hypohamiltonian graphs.) A graph G is 1-*edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E(G)$. Obviously, any 1-edge hamiltonian graph is hamiltonian. A graph $G = (V, E)$ is 1-hamiltonian if it is 1-vertex hamiltonian and 1-edge hamiltonian. A 1-hamiltonian graph G is *optimal* if it contains the least number of edges among all 1-hamiltonian graphs with the same number of vertices. It is proved in [3,4] that the degree of any vertex in any optimal 1-hamiltonian graph is at least 3. Moreover, all optimal 1-hamiltonian graphs with an even number of vertices are *cubic*, i.e., the degree of any vertex is 3.

Let U be the set of cubic planar hamiltonian graphs, A the set of hyper hamiltonian graphs in U, B the set of 1-edge hamiltonian graphs in U, and C the set of hamiltonian connected graphs in U. With the inclusion and/or exclusion of the sets A, B, and C, the set U is divided into eight subsets, namely, $\bar{A} \cap \bar{B} \cap \bar{C}$, $\bar{A} \cap \bar{B} \cap C$, $\bar{A} \cap B \cap C$, $A \cap \bar{B} \cap C$, $A \cap \bar{B} \cap \bar{C}$, $\bar{A} \cap B \cap \bar{C}$, $A \cap B \cap \bar{C}$, and $A \cap B \cap C$. We will prove that there is an infinite number of elements in each of the eight subsets.

2. Preliminaries

We will use a graph operator, 3-*join*, to construct graphs in each of the eight subsets of U. Let G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. Let $x \in V(G_1)$ with $\deg_{G_1}(x) = 3$ and $y \in V(G_2)$ with $\deg_{G_2}(y) = 3$. Let $N(x) = \{x_1, x_2, x_3\}$ be an ordered set of the neighbors of x and $N(y) = \{y_1, y_2, y_3\}$ be an oredered set of the neighbors of y. 3-*join* of G_1 and G_2 at x and y, denoted by $J(G_1, N(x); G_2, N(y))$, is the graph with $V(J(G_1, N(x); G_2, N(y)) = (V(G_1) - \{x\}) \cup (V(G_2) - \{y\})$ and $E(J(G_1, N(x); G_2, N(y)) = (E(G_1) - \{(x, x_i) \mid 1 \leqslant i \leqslant 3\}) \cup (E(G_2) - \{(y, y_i) \mid 1 \leqslant i \leqslant 3\}) \cup \{(x_i, y_i) \mid 1 \leqslant i \leqslant 3\}$. A graph is called a 3-*join* of G_1 and G_2 if $G = J(G_1, N(x); G_2, N(y))$ for some vertices $x \in V(G_1)$ and $y \in V(G_2)$with $\deg_{G_1}(x) = \deg_{G_2}(y) = 3$. We note that a different ordering of $N(x)$ and $N(y)$ generates a different 3-join of G_1 and G_2 at x and y.

In this section, we always assume that G_1 is a graph with a vertex x of degree 3 and G_2 is a graph with a vertex y of degree 3. Let $G = J(G_1, N(x); G_2, N(y))$, where $N(x) = \{x_1, x_2, x_3\}$ and $N(y) = \{y_1, y_2, y_3\}$. Depending on the hamiltonian properties of G_1 and G_2, i.e., whether they are 1-vertex hamiltonian, 1-edge hamiltonian and/or hamiltonian connected, and some local properties at x and y, we may have various hamiltonian properties of G, as stated in the following lemmas.

Lemma 1. *G is* 1-*edge hamiltonian if and only if both G_1 and G_2 are* 1-*edge hamiltonian.*

Proof. Assume that G is 1-edge hamiltonian. We claim that both G_1 and G_2 are 1-edge hamiltonian. By symmetry, it is sufficient to prove that G_1 is 1-edge hamiltonian.

Let e be any edge of G_1. Then e is either incident with x or not. Suppose that e is not incident with x. Then e is an edge of G. Since G is 1-edge hamiltonian, there exists a hamiltonian cycle C of $G-e$. Because there are exactly three edges of G joining $V(G_1)-\{x\}$ to $V(G_2)-\{y\}$, C can be written as $\langle x_i, P, x_j, y_j, Q, y_i, x_i\rangle$ for some $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$, where P is a hamiltonian path of $G_1 - x$ joining x_i to x_j. Thus, $\langle x, x_i, P, x_j, x\rangle$ forms a hamiltonian cycle of $G_1 - e$. Suppose that e is an edge incident with x. Without loss of generality, we may assume that $e=(x, x_1)$. Since G is 1-edge hamiltonian, there exists a hamiltonian cycle C of $G-(x_1, y_1)$. Because there are exactly three edges joining $V(G_1)-\{x\}$ to $V(G_2)-\{y\}$, C can be written as $\langle x_2, P, x_3, y_3, Q, y_2, x_2\rangle$, where P is a hamiltonian path of $G_1 - x$. Thus, $\langle x, x_2, P, x_3, x\rangle$ forms a hamiltonian cycle of $G_1-(x, x_1)$. Then G_1 is 1-edge hamiltonian.

Suppose that both G_1 and G_2 are 1-edge hamiltonian. Let e be any edge of G. Suppose that $e \notin \{(x_i, y_i) \mid i=1, 2, 3\}$. Then e is either in $E(G_1)$ or in $E(G_2)$. Without loss of generality, we assume that e is in $E(G_1)$. Since G_1 is 1-edge hamiltonian, there exists a hamiltonian cycle C_1 in $G_1 - e$. Obviously, C_1 can be written as $\langle x, x_i, P, x_j, x\rangle$ for some $i, j \in \{1, 2, 3\}$ with $i \neq j$. Let k be the only element in $\{1, 2, 3\}-\{i, j\}$. Since G_2 is 1-edge hamiltonian, there exists a hamiltonian cycle C_2 of $G_2-(y, y_k)$. Obviously, C_2 can be written as $\langle y, y_j, Q, y_i, y\rangle$. Obviously, $\langle x_i, P, x_j, y_j, Q, y_i, x_i\rangle$ forms a hamiltonian cycle of $G-e$.

Suppose that $e \in \{(x_i, y_i) \mid i=1, 2, 3\}$. Without loss of generality, we assume that $e=(x_1, y_1)$. Since G_1 is 1-edge hamiltonian, there exists a hamiltonian cycle C_1 in $G_1-(x, x_1)$. Obviously, C_1 can be written as $\langle x, x_2, P, x_3, x\rangle$. Since G_2 is 1-edge hamiltonian, there exists a hamiltonian cycle C_2 in $G_2-(y, y_1)$. Obviously, C_2 can be written as $\langle y, y_3, Q, y_2, y\rangle$. Obviously, $\langle x_2, P, x_3, y_3, Q, y_2, x_2\rangle$ forms a hamiltonian cycle of $G-e$. Hence, G is 1-edge hamiltonian. □

We say that a vertex x is *good* in a graph G if $\deg_G(x)=3$ and $G-e$ is hamiltonian for any edge e incident with x. We use $\mathrm{Good}(G)$ to denote the set of good vertices in G.

Obviously, $\mathrm{Good}(G)=V(G)$ if G is a 3-regular 1-edge hamiltonian graph.

Lemma 2. *Assume that both G_1 and G_2 are* 1-*vertex hamiltonian graphs. Then G is* 1-*vertex hamiltonian if and only if x is good in G_1 and y is good in G_2. Moreover*, $\mathrm{Good}(G)=(\mathrm{Good}(G_1)\cup \mathrm{Good}(G_2))-\{x, y\}$ *if x is good in G_1 and y is good in G_2.*

Proof. We prove this lemma through the following steps.

(1) Suppose that G is 1-vertex hamiltonian. We claim that x is good in G_1 and y is good in G_2. By symmetry, it is sufficient to prove that x is good in G_1. Let e be any edge incident with x. Without loss of generality, we assume that $e=(x, x_1)$. Since G is 1-vertex hamiltonian, there exists a hamiltonian cycle C in $G-y_1$. Obviously, C can be written as $\langle x_2, P, x_3, y_3, Q, y_2, x_2\rangle$. Thus, $\langle x, x_2, P, x_3, x\rangle$ forms a hamiltonian cycle of $G_1-(x, x_1)$. Hence, x is good in G_1.

(2) Suppose that x is good in G_1 and y is good in G_2. We claim that G is hamiltonian. Since x is good in G_1, there exists a hamiltonian cycle C_1 in $G_1-(x, x_1)$. Obviously, C_1 can be written as $\langle x, x_2, P, x_3, x\rangle$. Similarly, there exists a hamiltonian cycle C_2 in $G_2-(y, y_1)$. Obviously, C_2 can be written as $\langle y, y_3, Q, y_2, y\rangle$. Thus, $\langle x_2, P, x_3, y_3, Q, y_2, x_2\rangle$ forms a hamiltonian cycle of G.

(3) We prove that G is 1-vertex hamiltonian. Let u be any vertex of G. Then u is either in $(V(G_1)-\{x\})$ or in $(V(G_2)-\{y\})$. By symmetry, we assume that $u \in V(G_1)-\{x\}$.

Suppose that $u \in N(x)$. Without loss of generality, we assume that $u=x_1$. Since G_1 is 1-vertex hamiltonian, there exists a hamiltonian cycle C_1 in G_1-u. Obviously, C_1 can be written as $\langle x, x_2, P, x_3, x\rangle$. Since y is good in G_2, there exists a hamiltonian cycle C_2 in $G_2-(y, y_1)$. Obviously, C_2 can be written as $\langle y, y_3, Q, y_2, y\rangle$. Thus, $\langle x_2, P, x_3, y_3, Q, y_2, x_2\rangle$ forms a hamiltonian cycle of $G-u$.

Suppose that $u \notin N(x)$. Since G_1 is 1-vertex hamiltonian, there exists a hamiltonian cycle C_1 in G_1-u. Obviously, C_1 can be written as $\langle x, x_i, P, x_j, x\rangle$ for some $i, j \in \{1, 2, 3\}$ with $i \neq j$. Let k be the only element in $\{1, 2, 3\}-\{i, j\}$. Since y is good in G_2, there exists a hamiltonian cycle C_2 in $G_2-(y, y_k)$. Obviously, C_2 can be written as $\langle y, y_j, Q, y_i, y\rangle$. Thus, $\langle x_i, P, x_j, y_j, Q, y_i, x_i\rangle$ forms a hamiltonian cycle of $G-u$.

Hence, G is 1-vertex hamiltonian.

(4) We claim that $\mathrm{Good}(G) \subseteq (\mathrm{Good}(G_1)\cup \mathrm{Good}(G_2))-\{x, y\}$. Let u be any good vertex in G. Obviously, $\deg_G(u)=3$ and $u \in (V(G_1)-\{x\}) \cup (V(G_2)-\{y\})$. Without loss of generality, we assume that $u \in V(G_1)-\{x\}$. Obviously, $\deg_{G_1}(u)=3$. Let $e=(u, v)$ be any edge of G_1 incident with u.

Suppose that $u \in N_{G_1}(x)$. Without loss of generality, we assume that $u = x_1$. Suppose that $v = x$. Since u is good in G, there exists a hamiltonian cycle C in $G - (x_1, y_1)$. Obviously, C can be written as $\langle u, P, x_i, y_i, Q, y_j, x_j, R, u\rangle$ where $i, j \in \{2, 3\}$ with $i \neq j$. Obviously, $\langle u, P, x_i, x, x_j, R, u\rangle$ forms a hamiltonian cycle of $G_1 - (u, v)$. Suppose that $v \neq x$. Since u is good in G, there exists a hamiltonian cycle C in $G - (u, v)$. Obviously, C can be written as $\langle u, y_1, Q, y_i, x_i, R, u\rangle$, where $i \in \{2, 3\}$. Obviously, $\langle u, x, x_i, R, u\rangle$ forms a hamiltonian cycle of $G_1 - (u, v)$. Thus, u is good in G_1.

Suppose that $u \notin N_{G_1}(x)$. Since u is good in G, there exists a hamiltonian cycle C in $G - (u, v)$. Obviously, C can be written as $\langle u, P, x_i, y_i, Q, y_j, x_j, R, u\rangle$, where $i, j \in \{1, 2, 3\}$ with $i \neq j$. Obviously, $\langle u, P, x_i, x, x_j, R, u\rangle$ forms a hamiltonian cycle of $G_1 - (u, v)$. Thus, u is good in G_1.

Therefore, $\text{Good}(G) \subseteq (\text{Good}(G_1) \cup \text{Good}(G_2)) - \{x, y\}$.

(5) We claim that $\text{Good}(G_1) - \{x\} \subseteq \text{Good}(G)$. Suppose that u is good in G_1 with $u \neq x$. Obviously, $\deg_G(u) = 3$. Let $e = (u, v)$ be any edge of G incident with u.

Suppose that $u = x_i$ and $e = (x_i, y_i)$ for some i with $i \in \{1, 2, 3\}$. Without loss of generality, we assume that $i = 1$. Since u is good in G_1, there exists a hamiltonian cycle C_1 in $G_1 - (x_1, x)$. Obviously, C_1 can be written as $\langle x, x_2, P, x_3, x\rangle$. Since y is good in G_2, there exists a hamiltonian cycle C_2 in $G_2 - (y, y_1)$. Obviously, C_2 can be written as $\langle y, y_3, Q, y_2, y\rangle$. Obviously, $\langle x_2, P, x_3, y_3, Q, y_2, x_2\rangle$ forms a hamiltonian cycle in $G - e$.

Suppose that $u \neq x_i$ or $e \neq (x_i, y_i)$ for any i with $i \in \{1, 2, 3\}$. Since u is good in G_1, there exists a hamiltonian cycle C_1 in $G_1 - e$. Obviously, C_1 can be written as $\langle u, P_1, x_i, x, x_j, P_2, u\rangle$ where $i, j \in \{1, 2, 3\}$ with $i \neq j$. (Note that the length of P_t is 0 if $u = x_t$ for $t \in \{1, 2\}$.) Let k be the only element in $\{1, 2, 3\} - \{i, j\}$. Since y is good in G_2, there exists a hamiltonian cycle C_2 in $G_2 - (y, y_k)$. Thus, C_2 can be written as $\langle y, y_i, Q, y_j, y\rangle$. Obviously, $\langle u, P_1, x_i, y_i, Q, y_j, x_j, P_2, u\rangle$ forms a hamiltonian cycle in $G - e$.

Therefore, $\text{Good}(G_1) - \{x\} \subseteq \text{Good}(G)$.

(6) Similar to Step 5, we have $\text{Good}(G_2) - \{y\} \subseteq \text{Good}(G)$.

(7) Combining Steps 4–6, $\text{Good}(G) = (\text{Good}(G_1) \cup \text{Good}(G_2)) - \{x, y\}$. □

Lemma 3. *Let a and b be two distinct vertices of G_1 such that $a \neq x$ and $b \neq x$. Let $y \in \text{Good}(G_2)$. Then there exists a hamiltonian path of G_1 joining a to b if and only if there exists a hamiltonian path of G joining a to b.*

Proof. Assume that there exists a hamiltonian path P_1 of G_1 joining a to b. We can write P_1 as $\langle a, P_1^1, x_i, x, x_j, P_1^2, b\rangle$ with $\{i, j\} \subset \{1, 2, 3\}$ and $i \neq j$. Note that the length of P_1^1 and the length of P_1^2 could be 0. Let k be the only element in $\{1, 2, 3\} - \{i, j\}$. Since y is good in G_2, there exists a hamiltonian cycle C_2 in $G_2 - (y, y_k)$. We can write C_2 as $\langle y, y_i, P_2, y_j, y\rangle$. Then $\langle a, P_1^1, x_i, y_i, P_2, y_j, x_j, P_1^2, b\rangle$ forms a hamiltonian path of G joining a to b.

Assume that there exists a hamiltonian path P of G joining a to b. Since there are exactly three edges, namely (x_1, y_1), (x_2, y_2) and (x_3, y_3), between $V(G_1) - \{x\}$ and $V(G_2) - \{y\}$ in G, P can be written as $\langle a, P_1, x_i, y_i, P_2, y_j, x_j, P_3, b\rangle$ for some $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$. Note that the length of P_1 and the length of P_3 could be 0. Thus, $\langle a, P_1, x_i, x, x_j, P_3, b\rangle$ forms a hamiltonian path in G_1 joining a to b. □

A vertex x is *nice* in a graph G if it is good in G with the following property: *let $N(x) = \{x_1, x_2, x_3\}$ be the neighborhood of x in G. For any $i \in \{1, 2, 3\}$, there exists a hamiltonian path of $G - (x, x_i)$ joining u to x_i for any vertex u of G with $u \notin \{x, x_i\}$.*

We use $\text{Nice}(G)$ to denote the set of nice vertices in G.

Lemma 4. *Assume that both G_1 and G_2 are hamiltonian connected graphs, $x \in \text{Nice}(G_1)$, and $y \in \text{Nice}(G_2)$. Then G is hamiltonian connected.*

Proof. To prove that G is hamiltonian connected, we want to show that there exists a hamiltonian path joining a to b for any $a, b \in V(G)$ with $a \neq b$. By symmetry, we only need to consider the following cases.

Case 1: $a, b \in V(G_1)$. Since G_1 is hamiltonian connected, there exists a hamiltonian path $\langle a, P_1, x_i, x, x_j, P_2, b\rangle$ in G_1. Let k be the only element in $\{1, 2, 3\} - \{i, j\}$. Since y is good, there exists a hamiltonian cycle $\langle y, y_i, Q, y_j, y\rangle$ in $G_2 - (y, y_k)$. Then $\langle a, P_1, x_i, y_i, Q, y_j, x_j, P_2, b\rangle$ is a hamiltonian path in G.

Case 2: $a \in V(G_1) - N(x)$ and $b \in V(G_2) - N(y)$. Since x is nice in G_1, $G_1 - (x, x_1)$ contains a hamiltonian path P_1 joining a to x_1. Without loss of generality, P_1 can be written as $\langle a, P_1^1, x_2, x, x_3, P_1^2, x_1\rangle$. Since y is nice in

G_2, $G_2 - (y, y_2)$ contains a hamiltonian path P_2 joining b to y_2. P_2 can be written as $\langle y_2, P_2^1, y_j, y, y_k, P_2^2, b\rangle$ with $\{j, k\} = \{1, 3\}$. If $j = 1$, then $k = 3$. Thus, $\langle a, P_1^1, x_2, y_2, P_2^1, y_1, x_1, (P_1^2)^{-1}, x_3, y_3, P_2^2, b\rangle$ is a hamiltonian path in G that joins a to b. If $j = 3$, then $k = 1$. Thus, $\langle a, P_1^1, x_2, y_2, P_2^1, y_3, x_3, P_1^2, x_1, y_1, P_2^2, b\rangle$ is a hamiltonian path in G that joins a to b.

Case 3: $a \in N(x)$ and $b \in V(G_2)$. Without loss of generality, we assume that $a = x_1$. Since x is nice in G_1, there exists a hamiltonian path P_1 of $G_1 - (x, x_2)$ joining a to x_2. Write P_1 as $\langle a, x, x_3, Q_1, x_2\rangle$. Since y is nice in G_2, there exists a hamiltonian path P_2 of $G_2 - (y, y_1)$ joining y_1 to b. P_2 can be written as $\langle y_1, Q_2^1, y_2, y, y_3, Q_2^2, b\rangle$ or $\langle y_1, Q_2^1, y_3, y, y_2, Q_2^2, b\rangle$. If $P_2 = \langle y_1, Q_2^1, y_2, y, y_3, Q_2^2, b\rangle$, then $\langle a, y_1, Q_2^1, y_2, x_2, Q_1^{-1}, x_3, y_3, Q_2^2, b\rangle$ is a hamiltonian path joining a to b in G. If $P_2 = \langle y_1, Q_2^1, y_3, y, y_2, Q_2^2, b\rangle$, then $\langle a, y_1, Q_2^1, y_3, x_3, Q_1, x_2, y_2, Q_2^2, b\rangle$ is a hamiltonian path joining a to b in G. □

Lemma 5. *Assume that G_1 is a hamiltonian connected graph with a nice vertex x. Let K_4 be the complete graph defined on $\{y, y_1, y_2, y_3\}$ and $G = J(G_1, N(x); K_4, N(y))$. Then G is hamiltonian connected. Moreover,* $\{y_1, y_2, y_3\} \subseteq \mathrm{Nice}(G)$.

Proof. Obviously, K_4 is hamiltonian connected, and it is easy to check that $\mathrm{Nice}(K_4) = V(K_4)$. Using Lemma 4, G is hamiltonian connected. Now, we show that $\{y_1, y_2, y_3\} \subseteq \mathrm{Nice}(G)$. Using symmetry, we only need to verify that y_1 is nice in G. It is obvious that $\deg_G(y_1) = 3$.

We first claim that y_1 is good in G. Thus, we show that $G - (y_1, z)$ is hamiltonian for any $z \in \{x_1, y_2, y_3\}$. Since x is good in G_1, there exists a hamiltonian cycle C_1^i of $G_1 - (x, x_i)$ for any $i \in \{1, 2, 3\}$. We may write C_1^i as $\langle x, x_j, P_i, x_k, x\rangle$ with $\{i, j, k\} = \{1, 2, 3\}$. Let $C^i = \langle x_j, P_i, x_k, y_k, y_i, y_j, x_j\rangle$. Obviously, C^1 is a hamiltonian cycle in $G - (y_1, x_1)$; C^2 is a hamiltonian cycle in $G - (y_1, y_3)$; and C^3 is a hamiltonian cycle in $G - (y_1, y_2)$. Hence, y_1 is good in G.

Let b be any element in $N(y_1) = \{x_1, y_2, y_3\}$. To show that y_1 is nice in G, we need to find a hamiltonian path of $G - (y_1, b)$ that joins a to b for any vertex a of G with $a \notin \{y_1, b\}$.

Case i: $a \in V(G_1) - \{x, x_1\}$ and $b = x_1$. Since x is nice in G_1, there exists a hamiltonian path P_1 of $G_1 - (x, x_1)$ joining a to x_1. We can write P_1 as $\langle a, P_1^1, x_k, x, x_j, P_1^2, x_1\rangle$ where $\{k, j\} = \{2, 3\}$. Obviously, $\langle a, P_1^1, x_k, y_k, y_1, y_j, x_j, P_1^2, x_1\rangle$ is a hamiltonian path joining a to x_1 in $G - (x_1, y_1)$.

Case ii: $a \in V(G_1) - \{x, x_1\}$ and $b = y_k$ for some $k \in \{2, 3\}$. Since x is nice in G_1, there exists a hamiltonian path P_1 joining a to x_k in $G_1 - (x, x_k)$. Write P_1 as $\langle a, P_1^1, x_i, x, x_j, P_1^2, x_k\rangle$ where $\{i, j, k\} = \{1, 2, 3\}$. Obviously, $\langle a, P_1^1, x_i, y_i, y_j, x_j, P_1^2, x_k, y_k\rangle$ is a hamiltonian path of $G - (y_1, y_k)$joining a to y_k.

Case iii: $a \in \{y_2, y_3\}$ and $b = x_1$. Since x is good in G_1, there exists a hamiltonian cycle C_1 in $G_1 - (x, x_2)$. We can write C_1 as $\langle x, x_3, P_1, x_1, x\rangle$. Obviously, $\langle y_2, y_1, y_3, x_3, P_1, x_1\rangle$ forms a hamiltonian path of $G - (y_1, x_1)$ joining y_2 to x_1. Since x is good in G_1, there exists a hamiltonian cycle C_2 in $G_1 - (x, x_3)$. We can write C_2 as $\langle x, x_2, P_2, x_1, x\rangle$. Obviously, $\langle y_3, y_1, y_2, x_2, P_2, x_1\rangle$ is a hamiltonian path of $G - (y_1, x_1)$ joining y_3 to x_1.

Case iv: $a = x_1$ and $b \in \{y_2, y_3\}$. Let $b = y_k$ for some $k \in \{2, 3\}$ and j be the only index in $\{2, 3\} - \{k\}$. Since x is a nice vertex in G_1, there exists a hamiltonian path P that joins x_1 to x_k in $G_1 - (x, x_k)$. We can write P as $\langle x_1, x, x_j, Q, x_k\rangle$. Obviously, $\langle x_1, y_1, y_j, x_j, Q, x_k, y_k\rangle$ forms a hamiltonian path of $G - (y_1, y_k)$ joining x_1 to y_k.

Case v: $\{a, b\} = \{y_2, y_3\}$. Without loss of generality, let $a = y_2$ and $b = y_3$. Since x is good in G_1, there exists a hamiltonian cycle C in $G_1 - (x, x_2)$. We can write C as $\langle x, x_1, R, x_3, x\rangle$. Obviously, $\langle y_2, y_1, x_1, R, x_3, y_3\rangle$ is a hamiltonian path of $G - (y_1, y_3)$ joining y_2 to y_3. □

Lemma 6. *Assume that $y \in \mathrm{Good}(G_2)$. Let $a \in V(G_1)$ such that $a \neq x$. Then $G_1 - a$ is hamiltonian if and only if $G - a$ is hamiltonian.*

Proof. Assume that there exists a hamiltonian cycle C_1 in $G_1 - a$. We can write C_1 as $\langle x, x_i, P, x_j, x\rangle$ with $i, j \in \{1, 2, 3\}$ and $i \neq j$. Let k be the only element in $\{1, 2, 3\} - \{i, j\}$. Since y is good in G_2, there exists a hamiltonian cycle C_2 in $G_2 - (y, y_k)$. We can write C_2 as $\langle y, y_j, Q, y_i, y\rangle$. Obviously, $\langle x_i, P, x_j, y_j, Q, y_i, x_i\rangle$ forms a hamiltonian cycle in $G - a$.

Assume that there exists a hamiltonian cycle C of $G - a$. Since there are exactly three edges between $V(G_1) - \{x\}$ and $V(G_2) - \{y\}$ in G, C can be written as $\langle x_i, P, x_j, y_j, Q, y_i, x_i\rangle$ for some $i, j \in \{1, 2, 3\}$ with $i \neq j$. Obviously, $\langle x, x_i, P, x_j, x\rangle$ forms a hamiltonian cycle in $G_1 - a$. □

Lemma 7. *Let $a \in V(G_1)$ such that $a \neq x$. Let K_4 be the complete graph defined on $\{y, y_1, y_2, y_3\}$ and $G = J(G_1, N(x); K_4, N(y))$. Then $G - y_i$ is hamiltonian if and only if $G_1 - (x, x_i)$ is hamiltonian. Moreover, $G - a$ is hamiltonian if and only if $G_1 - a$ is hamiltonian.*

Proof. Assume that there exists a hamiltonian cycle C in $G - y_i$. We can write C as $\langle x_j, y_j, y_k, x_k, P, x_j\rangle$ where $\{i, j, k\} = \{1, 2, 3\}$. Obviously, $\langle x_j, x, x_k, P, x_j\rangle$ forms a hamiltonian cycle of $G_1 - (x, x_i)$.

Assume that there exists a hamiltonian cycle C_1 in $G_1 - (x, x_i)$. We can write C_1 as $\langle x_j, x, x_k, P, x_j\rangle$ where $\{i, j, k\} = \{1, 2, 3\}$. Obviously, $\langle x_j, y_j, y_k, x_k, P, x_j\rangle$ forms a hamiltonian cycle of $G - y_i$.

Note that $\text{Good}(K_4) = V(K_4)$. Using Lemma 6, $G_1 - a$ is hamiltonian if and only if $G - a$ is hamiltonian. □

Lemma 8. *Assume that G_1 is a hamiltonian graph and G_2 is a 1-edge hamiltonian graph. Then G is hamiltonian.*

Proof. Since G_1 is hamiltonian, G_1 has a hamiltonian cycle C_1. Write C_1 as $\langle x_i, P_1, x_j, x, x_i\rangle$, where $i, j \in \{1, 2, 3\}$ and $i \neq j$. Let k be the only element in $\{1, 2, 3\} - \{i, j\}$. Since G_2 is 1-edge hamiltonian, $G_2 - (y, y_k)$ has a hamiltonian cycle C_2. Obviously, we can write C_2 as $\langle y_i, y, y_j, P_2, y_i\rangle$. Thus, G has a hamiltonian cycle $\langle x_i, P_1, x_j, y_j, P_2, y_i, x_i\rangle$. □

Let $G = (V, E)$ be a graph with a vertex x of degree 3. *The* 3*-vertex expansion of* G, $\text{Ex}(G, x)$, is the graph $J(G, N(x); K_4, N(y))$, where $y \in V(K_4)$. Any vertex in $\text{Ex}(G, x) - V(G)$ is called an *expanded vertex* of G at x. Obviously, $\text{Ex}(G, x)$ is cubic if G is cubic, $\text{Ex}(G, x)$ is planar if G is planar, and $\text{Ex}(G, x)$ is connected if G is connected.

3. Various types of cubic planar hamiltonian graphs

Let U be the set of cubic planar hamiltonian graphs, A the set of 1-vertex hamiltonian graphs in U, B the set of 1-edge hamiltonian graphs in U, and C the set of hamiltonian connected graphs in U. With the inclusion and/or exclusion of the sets A, B, and C, the set U is divided into eight subsets. In this section, we prove that there is an infinite number of elements in each of the eight subsets.

3.1. $A \cap B \cap C$

Obviously, K_4 is the smallest cubic planar hamiltonian graph. It is easy to check that K_4 is a graph in $A \cap B \cap C$. Moreover, $\text{Nice}(K_4) = V(K_4)$. Let x_1 be any vertex of K_4. Using Lemmas 1, 2, and 5, $\text{Ex}(K_4, x_1)$ is a graph in $A \cap B \cap C$. Moreover, with Lemma 5, any expanded vertex of K_4 at x_1 is nice. Now, we recursively define a sequence of graphs as follows: let $G_1 = K_4$ and $G_2 = \text{Ex}(K_4, x_1)$. Suppose that we have defined $G_1, G_2, \ldots, G_i$ with $i \geqslant 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} a $\text{Ex}(G_i, x_i)$. Recursively applying Lemmas 1, 2, and 5, $G_i \in A \cap B \cap C$ for every $i \geqslant 1$. Hence, we have the following theorem.

Theorem 1. *There is an infinite number of planar* 1*-vertex hamiltonian*, 1*-edge hamiltonian*, *and hamiltonian connected graphs.*

3.2. $\bar{A} \cap B \cap \bar{C}$

Let Q_3 be the three-dimensional hypercube shown in Fig. 1(a). Obviously, Q_3 is planar. It is easy to check that Q_3 is 1-edge hamiltonian. Since Q_3 is a bipartite graph, there is no cycle of length 7. Hence, $Q_3 - x$ is not hamiltonian for any vertex x in Q_3. Thus, Q_3 is not 1-vertex hamiltonian. Since there are four vertices in each partite set, there is no hamiltonian path joining any two vertices of the same partite set. Thus, Q_3 is not hamiltonian connected. Therefore, Q_3 is a graph in $\bar{A} \cap B \cap \bar{C}$. Let x_1 be any vertex in Q_3. Using Lemma 1, $\text{Ex}(Q_3, x_1)$ is 1-edge hamiltonian. Using Lemma 7, $\text{Ex}(Q_3, x_1)$ is not 1-vertex hamiltonian. Using Lemma 3, $\text{Ex}(Q_3, x_1)$ is not hamiltonian connected. Thus, $\text{Ex}(Q_3, x_1)$ is a graph in $\bar{A} \cap B \cap \bar{C}$. Now, we recursively define a sequence of graphs as follows: let $G_1 = Q_3$ and $G_2 = \text{Ex}(Q_3, x_1)$. Suppose that we have defined $G_1, G_2, \ldots, G_i$ with $i \geqslant 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $\text{Ex}(G_i, x_i)$. Recursively applying Lemmas 1, 7, and 3, $G_i \in \bar{A} \cap B \cap \bar{C}$ for every $i \geqslant 1$. Hence, we have the following theorem.

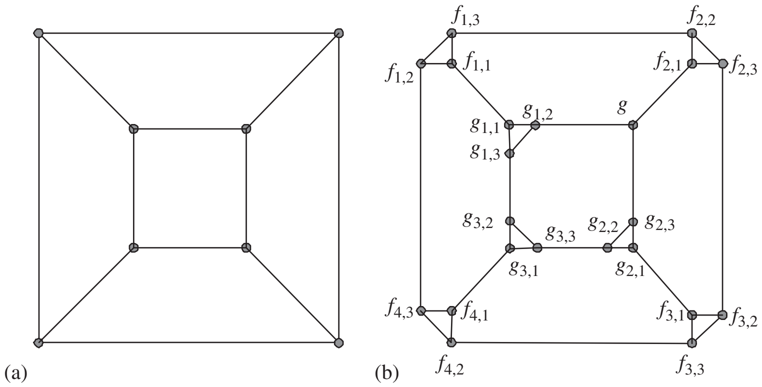

Fig. 1. The graphs (a) Q_3 and (b) Q.

Theorem 2. *There is an infinite number of planar graphs that are* 1-*edge hamiltonian*, *but neither* 1-*vertex hamiltonian nor hamiltonian connected.*

3.3. $\bar{A} \cap B \cap C$

Let Q be the graph in Fig. 1(b). Obviously, Q is obtained from the graph Q_3 by a sequence of 3-vertex expansions. From Section 3.2, $Q_3 \in \bar{A} \cap B \cap \bar{C}$. Using Lemma 1, Q is 1-edge hamiltonian. Using Lemma 7, $Q - g$ is not hamiltonian. Hence, Q is not 1-vertex hamiltonian. By brute force, we can check that Q is hamiltonian connected. (See Fact 1 in Appendix.) Thus, $Q \in \bar{A} \cap B \cap C$.

Theorem 3. *There is an infinite number of planar graphs that are* 1-*edge hamiltonian and hamiltonian connected but not* 1-*vertex hamiltonian.*

Proof. Let $g_{3,1}$ be the vertex of Q shown in Fig. 1(b). By brute force, we can check that $g_{3,1} \in \mathrm{Nice}(Q)$. (See Fact 2 in Appendix.) Let $Y = \mathrm{Ex}(Q, g_{3,1})$. Using Lemmas 1, 5, and 7, Y is a graph in $\bar{A} \cap B \cap C$. Moreover, with Lemma 5, any expanded vertex of Q at $g_{3,1}$ is nice. Now, we recursively define a sequence of graphs as follows: let $G_1 = Q$, $x_1 = g_{3,1}$ and $G_2 = \mathrm{Ex}(Q, x_1)$. Suppose that we have defined $G_1, G_2, \ldots, G_i$ with $i \geqslant 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $\mathrm{Ex}(G_i, x_i)$. Recursively applying Lemmas 1, 5, and 7, $G_i \in \bar{A} \cap B \cap C$ for every $i \geqslant 1$. $\square$

3.4. $A \cap B \cap \bar{C}$

Let M be the graph in Fig. 2(a) and M_0 the graph in Fig. 2(b). Obviously, M is obtained from M_0 by a sequence of 3-vertex expansions.

Lemma 9. *The graph M is* 1-*edge hamiltonian and* 1-*vertex hamiltonian.*

Proof. We first check that M_0 is 1-edge hamiltonian. By symmetry, we only need to prove that $M_0 - e$ is hamiltonian for any $e \in \{(s_1, s_2), (s_1, r_1), (r_1, q_2)\}$. The corresponding cycles are listed below:

$M_0 - (s_1, s_2)$	$\langle s_1, r_1, q_1, p_1, p_2, q_2, r_2, s_2, s_3, r_3, q_3, p_3, p_4, q_4, r_4, s_4, s_1 \rangle$
$M_0 - (s_1, r_1)$	$\langle s_1, s_2, r_2, q_3, p_3, p_4, p_1, p_2, q_2, r_1, q_1, r_4, q_4, r_3, s_3, s_4, s_1 \rangle$
$M_0 - (r_1, q_2)$	$\langle s_1, s_2, s_3, s_4, r_4, q_4, r_3, q_3, r_2, q_2, p_2, p_3, p_4, p_1, q_1, r_1, s_1 \rangle$

Note that M is obtained from M_0 by a sequence of 3-vertex expansions. Recursively applying Lemma 1, M is 1-edge hamiltonian. Applying Lemma 7, $M - v$ is hamiltonian if $v \in \{p_{i,j} \mid 1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant 3\} \cup \{s_{i,j} \mid 1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant 3\}$.

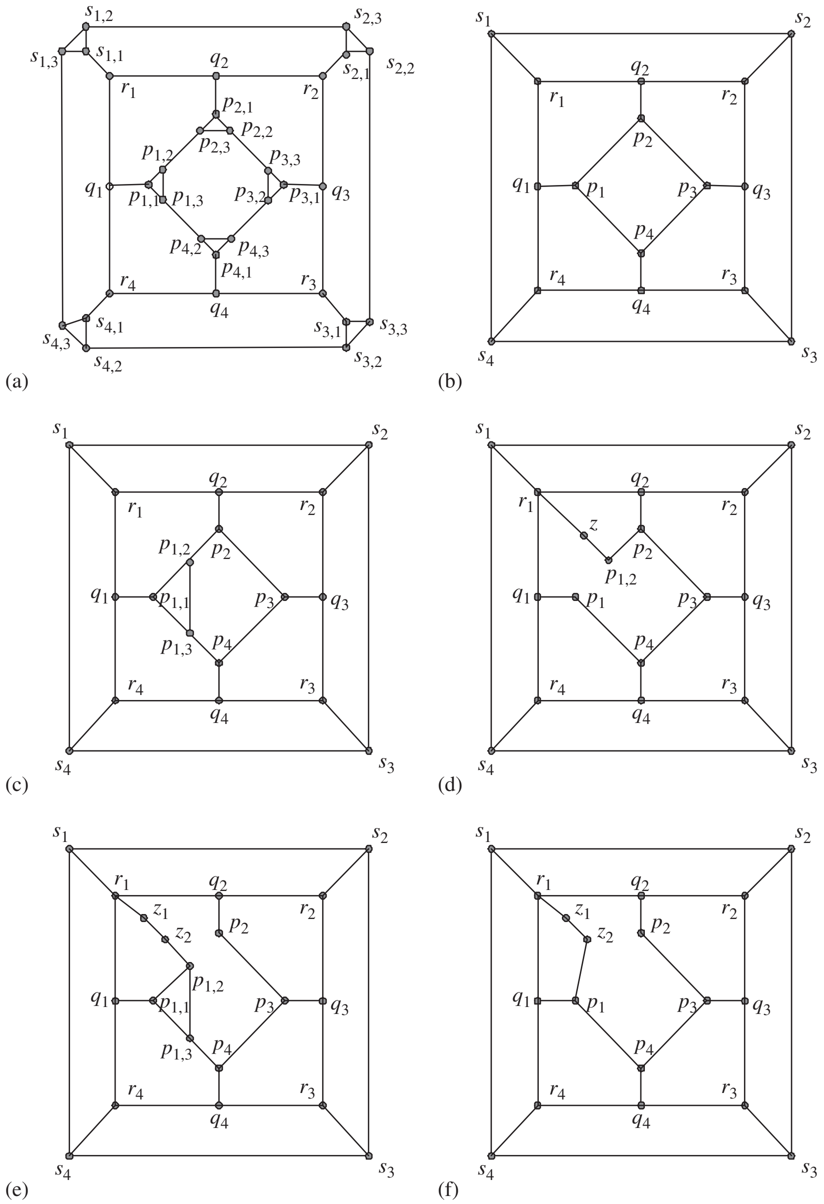

Fig. 2. Graphs (a) M; (b) M_0; (c) M_1; (d) M_2; (e) M_3; and (f) M_4.

To prove that M is 1-vertex hamiltonian, we only need to check that $M - r_1$ and $M - q_1$ are hamiltonian by the symmetric property of M. Using Lemma 7, it suffices to show that $M_0 - r_1$ and $M_0 - q_1$ are hamiltonian. Obviously, $\langle q_1, p_1, p_2, q_2, r_2, s_2, s_1, s_4, s_3, r_3, q_3, p_3, p_4, q_4, r_4, q_1\rangle$ is a hamiltonian cycle of $M_0 - r_1$, and $\langle r_1, q_2, r_2, s_2, s_3, r_3, q_3, p_3, p_2, p_1, p_4, q_4, r_4, s_4, s_1, r_1\rangle$ is a hamiltonian cycle of $M_0 - q_1$. Hence, M is 1-vertex hamiltonian. □

Lemma 10. *There is no hamiltonian path of M joining $p_{1,2}$ to r_1. Hence, M is not hamiltonian connected.*

Proof. Let M_1 be the graph shown in Fig. 2(c). Obviously, M is obtained from M_1 by a sequence of 3-vertex expansions. With Lemma 3, it suffices to show that there is no hamiltonian path in M_1 joining $p_{1,2}$ to r_1.

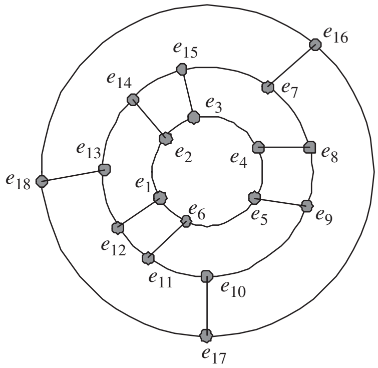

Fig. 3. The graph Eye(2).

We prove it by contradiction. Suppose there exists a hamiltonian path P in M_1 joining $p_{1,2}$ to r_1. Let $P = \langle v_1 = p_{1,2}, v_2, \ldots, v_{18}\rangle$. Obviously, v_2 is p_2, $p_{1,1}$, or $p_{1,3}$.

Case 1: $v_2 = p_2$. Then neither $(p_{1,1}, p_{1,2})$ nor $(p_{1,2}, p_{1,3})$ is in P. However, both $(p_{1,1}, p_{1,3})$ and $(p_{1,3}, p_4)$ are in P. Hence, the graph M_2 in Fig. 2(d) is hamiltonian. Let H be any hamiltonian cycle of M_2. Since M_2 is a planar graph, M_2 and H satisfy the *Grinberg condition* [1]. The Grinberg condition can also be found in some standard textbooks, say [14, p. 302], and is stated below.

If G is a planar graph having a hamiltonian cycle C, and G has f_i' faces of length i inside C and f_i'' faces of length i outside C, then $\sum_i (i-2)(f_i' - f_i'') = 0$.

Thus, $2(f_4' - f_4'') + 3(f_5' - f_5'') + 6(f_8' - f_8'') = 0$, where f_i' is the number of faces of length i inside H and f_i'' is the number of faces of length i outside H for $i = 4, 5, 8$. Obviously, $2(f_4' - f_4'') = 0 \pmod 3$. Since $|f_4' - f_4''| = 1$, the equation cannot hold and we arrive at a contradiction.

Case 2: v_2 is either $p_{1,1}$ or $p_{1,3}$. Then the graph M_3 shown in Fig. 2(e) is hamiltonian. Using Lemma 3, the graph M_4 in Fig. 2(f) is hamiltonian. It is easy to see that M_4 is isomorphic to M_2. Thus, M_4 is not hamiltonian. Again, we get a contradiction. □

Theorem 4. *There is an infinite number of planar graphs that are* 1*-edge hamiltonian and* 1*-vertex hamiltonian but not hamiltonian connected.*

Proof. Let $Y = \mathrm{Ex}(M, q_1)$. Using Lemmas 9 and 10, $M \in A \cap B \cap \bar{C}$. Using Lemmas 1–3, $Y \in A \cap B \cap \bar{C}$. Let $G_1 = M$, $x_1 = q_1$, and $G_2 = \mathrm{Ex}(M, q_1)$. Suppose that we have defined $G_1, G_2, \ldots, G_i$ with $i \geqslant 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $\mathrm{Ex}(G_i, x_i)$. Recursively applying Lemmas 1–3, $G_i \in A \cap B \cap \bar{C}$ for every $i \geqslant 1$. □

3.5. $A \cap \bar{B} \cap C$

Let Eye(2) be the graph in Fig. 3. In [13], it is proved that Eye(2) is 1-vertex hamiltonian but not 1-edge hamiltonian. More precisely, Eye(2) $- e$ is not hamiltonian for any edge in $\{(e_1, e_2), (e_3, e_4), (e_5, e_6)\}$. By brute force, we can check that Eye(2) is hamiltonian connected. (See Fact 3 in Appendix.) Thus, Eye(2) is a graph in $A \cap \bar{B} \cap C$. Let e_{16} be the vertex of Eye(2) shown in Fig. 3. By brute force, we can check that e_{16} is a nice vertex of Eye(2). (See Fact 4 in Appendix.) Let $Y = \mathrm{Ex}(\mathrm{Eye}(2), e_{16})$. Using Lemmas 1, 2, and 5, Y is a graph in $A \cap \bar{B} \cap C$. Using Lemma 5, any expanded vertex of Eye(2) at e_{16} is nice in Y. We recursively define a sequence of graphs as follows: let $G_1 = \mathrm{Eye}(2)$ and $G_2 = \mathrm{Ex}(\mathrm{Eye}(2), e_{16})$. Suppose that we have defined $G_1, G_2, \ldots, G_i$ with $i \geqslant 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $\mathrm{Ex}(G_i, x_i)$. Recursively applying Lemmas 1, 2, and 5, $G_i \in A \cap \bar{B} \cap C$ for every $i \geqslant 1$. Hence, we have the following theorem.

Theorem 5. *There is an infinite number of planar graphs that are* 1*-vertex hamiltonian and hamiltonian connected but not* 1*-edge hamiltonian.*

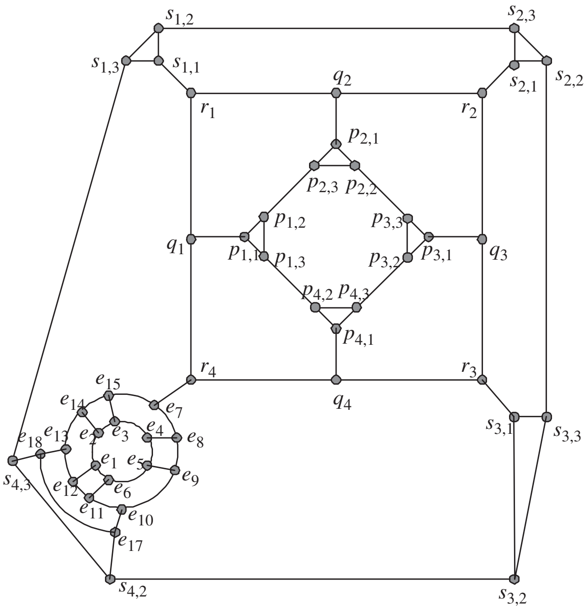

Fig. 4. The graph N.

3.6. $A \cap \bar{B} \cap \bar{C}$

Let $N = J(\text{Eye}(2), N(e_{16}); M, N(s_{4,1}))$ be the graph in Fig. 4. Obviously, N is a cubic planar graph. Since M is 1-edge hamiltonian, $s_{4,1} \in \text{Good}(M)$. From Section 3.5, we know that $e_{16} \in \text{Good}(\text{Eye}(2))$. Using Lemma 2, N is 1-vertex hamiltonian. Again, we know that $\text{Eye}(2) \in A \cap \bar{B} \cap C$. Using Lemma 1, N is not 1-edge hamiltonian. Using Lemma 10, there is no hamiltonian path of M joining $p_{1,2}$ to r_1. Using Lemma 3, there is no hamiltonian path of N joining $p_{1,2}$ to r_1. Therefore, N is not hamiltonian connected. Thus, N is a graph in $A \cap \bar{B} \cap \bar{C}$.

Theorem 6. *There is an infinite number of planar graphs that are* 1-*vertex hamiltonian but neither* 1-*edge hamiltonian nor hamiltonian connected.*

Proof. Let x be the vertex $p_{3,1}$ of N shown in Fig. 4. Since M is 1-edge hamiltonian, $x \in \text{Good}(M)$. Using Lemma 2, $x \in \text{Good}(N)$. Let $Y = \text{Ex}(N, x)$. Using Lemmas 1–3, Y is a graph in $A \cap \bar{B} \cap \bar{C}$. With Lemma 2, any expanded vertex of N at x is good. We recursively define a sequence of graphs as follows: let $G_1 = N$ and $G_2 = \text{Ex}(N, x)$. Suppose that we have define $G_1, G_2, \ldots, G_i$ with $i \geqslant 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $\text{Ex}(G_i, x_i)$. Recursively applying Lemmas 1–3, $G_i \in A \cap \bar{B} \cap \bar{C}$ for every $i \geqslant 1$. □

3.7. $\bar{A} \cap \bar{B} \cap C$

Let R be the graph $J(\text{Eye}(2), N(e_{16}); Q, N(g_{3,1}))$ shown in Fig. 5. Obviously, R is a cubic planar graph. In Section 3.3, we know that $Q \in \bar{A} \cap B \cap C$, $Q - g$ is not hamiltonian, and $g_{3,1}$ is nice in Q. From Section 3.5, we know that $\text{Eye}(2) \in A \cap \bar{B} \cap C$ and $e_{16} \in \text{Nice}(\text{Eye}(2))$. Using Lemma 1, R is not 1-edge hamiltonian. Using Lemma 6, $R - g$ is not hamiltonian. Hence, R is not 1-vertex hamiltonian. Using Lemma 4, R is hamiltonian connected. Thus, R is a graph in $\bar{A} \cap \bar{B} \cap C$.

Theorem 7. *There is an infinite number of planar graphs that are hamiltonian connected but neither* 1-*vertex hamiltonian nor* 1-*edge hamiltonian.*

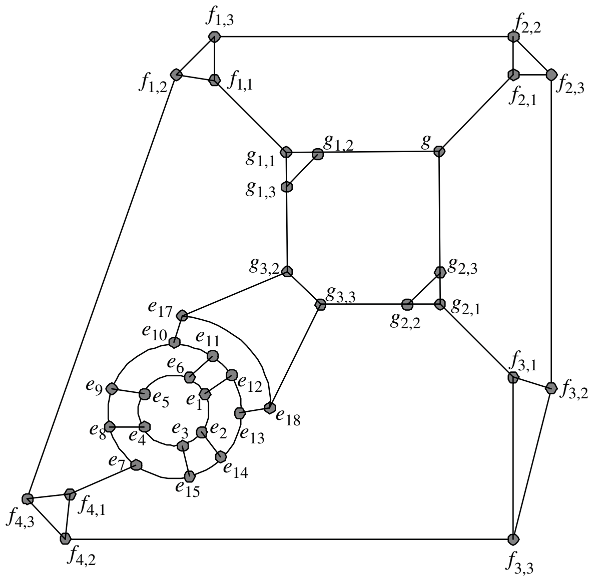

Fig. 5. The graph R.

Proof. Let x be the vertex e_{17} in Eye(2) shown in Fig. 3. By brute force, $x \in \text{Nice}(R)$. (See Fact 5 in Appendix.) Let $Y = \text{Ex}(R, x)$. Using Lemmas 1, 5, and 7, Y is a graph in $\bar{A} \cap \bar{B} \cap C$, and any expanded vertex of R at x is nice. We recursively define a sequence of graphs as follows: let $G_1 = R$ and $G_2 = \text{Ex}(R, x)$. Suppose that we have defined $G_1, G_2, \ldots, G_i$ with $i \geqslant 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $\text{Ex}(G_i, x_i)$. Recursively applying Lemmas 1, 5, and 7, $G_i \in \bar{A} \cap \bar{B} \cap C$ for every $i \geqslant 1$. □

3.8. $\bar{A} \cap \bar{B} \cap \bar{C}$

Let $Z = J(\text{Eye}(2), N(e_{16}); Q_3, N(f_4))$ shown in Fig. 6. Obviously, Z is a connected planar cubic graph. From Sections 3.2 and 3.5, we know that $Q_3 \in \bar{A} \cap B \cap \bar{C}$ and $\text{Eye}(2) \in A \cap \bar{B} \cap C$. Moreover, $Q_3 - f_2$ is not hamiltonian and there is no hamiltonian path of Q_3 joining f_2 to g_2. Using Lemma 1, Z is not 1-edge hamiltonian. From Section 3.5, we know that e_{16} is a good vertex of Eye(2). Using Lemma 6, $Z - f_2$ is not hamiltonian. Hence, Z is not 1-vertex hamiltonian. Using Lemma 3, there is no hamiltonian path of Z joining f_2 to g_2. Therefore, Z is not hamiltonian connected. Thus, Z is a graph in $\bar{A} \cap \bar{B} \cap \bar{C}$.

Theorem 8. *There is an infinite number of hamiltonian planar graphs that are not hamiltonian connected, not* 1-*vertex hamiltonian, and not* 1-*edge hamiltonian.*

Proof. Let x be the vertex g_1 in Z shown in Fig. 6. Let $Y = \text{Ex}(Z, x)$. Using Lemma 8, Y is hamiltonian. Using Lemmas 1, 3, and 7, Y is a graph in $U - (A \cap B \cap C)$. We recursively define a sequence of graphs as follows: let $G_1 = Z$ and $G_2 = \text{Ex}(Z, x)$. Suppose that we have defined $G_1, G_2, \ldots, G_i$ with $i \geqslant 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $\text{Ex}(G_i, x_i)$. Recursively applying Lemmas 1, 3, and 7, $G_i \in U - (A \cap B \cap C)$ for every $i \geqslant 1$. □

Acknowledgements

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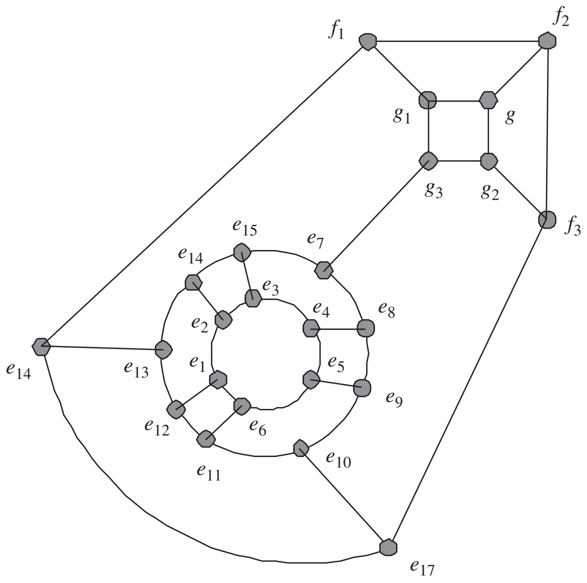

Fig. 6. The graph Z.

Appendix

Fact 1. *Q is hamiltonian connected.*

Proof. We relabel the vertices of Q as in Fig. 7. The corresponding hamiltonian paths between any two vertices x and y are listed below:

⟨1, 10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 12, 8, 7, 19, 13, 16, 5, 3, 20, 2⟩
⟨1, 10, 21, 6, 22, 16, 5, 13, 19, 8, 7, 17, 15, 9, 14, 18, 4, 11, 12, 2, 20, 3⟩
⟨1, 10, 21, 6, 22, 16, 13, 5, 3, 20, 2, 11, 12, 8, 19, 7, 17, 15, 9, 14, 18, 4⟩
⟨1, 10, 21, 6, 22, 16, 13, 19, 8, 7, 17, 15, 9, 14, 18, 4, 11, 12, 2, 20, 3, 5⟩
⟨1, 10, 21, 18, 4, 14, 9, 15, 17, 7, 19, 8, 12, 11, 2, 20, 3, 5, 13, 16, 22, 6⟩
⟨1, 10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 12, 2, 20, 3, 5, 16, 13, 19, 8, 7⟩
⟨1, 10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 12, 2, 20, 3, 5, 16, 13, 19, 7, 8⟩
⟨1, 10, 21, 6, 22, 15, 17, 7, 8, 19, 13, 16, 5, 3, 20, 2, 12, 11, 4, 18, 14, 9⟩
⟨1, 3, 20, 2, 11, 12, 8, 7, 19, 13, 5, 16, 22, 15, 17, 9, 14, 4, 18, 21, 6, 10⟩
⟨1, 10, 21, 6, 22, 16, 13, 5, 3, 20, 2, 12, 8, 19, 7, 17, 15, 9, 14, 18, 4, 11⟩
⟨1, 10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 2, 20, 3, 5, 16, 13, 19, 7, 8, 12⟩
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⟨1, 10, 21, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5, 3, 20, 2, 12, 11, 4, 18, 14⟩
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⟨1, 10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 2, 12, 8, 7, 19, 13, 16, 5, 3, 20⟩
⟨1, 10, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5, 3, 20, 2, 12, 11, 4, 14, 18, 21⟩
⟨1, 10, 6, 21, 18, 4, 14, 9, 15, 17, 7, 19, 8, 12, 11, 2, 20, 3, 5, 13, 16, 22⟩

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⟨9, 15, 17, 7, 19, 8, 12, 2, 11, 4, 14, 18, 21, 6, 22, 16, 13, 5, 3, 20, 1, 10⟩
⟨9, 15, 17, 7, 19, 8, 12, 2, 20, 1, 3, 5, 13, 16, 22, 6, 10, 21, 18, 14, 4, 11⟩
⟨9, 17, 15, 22, 6, 10, 21, 18, 14, 4, 11, 2, 20, 1, 3, 5, 16, 13, 19, 7, 8, 12⟩
⟨9, 17, 15, 22, 16, 5, 3, 20, 1, 10, 6, 21, 18, 14, 4, 11, 2, 12, 8, 7, 19, 13⟩
⟨9, 15, 17, 7, 19, 8, 12, 11, 2, 20, 1, 3, 5, 13, 16, 22, 6, 10, 21, 18, 4, 14⟩
⟨9, 17, 7, 19, 8, 12, 2, 11, 4, 14, 18, 21, 6, 10, 1, 20, 3, 5, 13, 16, 22, 15⟩
⟨9, 15, 17, 7, 8, 19, 13, 5, 3, 1, 20, 2, 12, 11, 4, 14, 18, 21, 10, 6, 22, 16⟩
⟨9, 15, 22, 16, 13, 5, 3, 20, 1, 10, 6, 21, 18, 14, 4, 11, 2, 12, 8, 19, 7, 17⟩
⟨9, 15, 17, 7, 8, 19, 13, 5, 16, 22, 6, 21, 10, 1, 3, 20, 2, 12, 11, 4, 14, 18⟩
⟨9, 17, 15, 22, 16, 13, 5, 3, 20, 1, 10, 6, 21, 18, 14, 4, 11, 2, 12, 8, 7, 19⟩
⟨9, 15, 17, 7, 19, 8, 12, 2, 11, 4, 14, 18, 21, 10, 6, 22, 16, 13, 5, 3, 1, 20⟩
⟨9, 15, 17, 7, 8, 19, 13, 5, 16, 22, 6, 10, 1, 3, 20, 2, 12, 11, 4, 14, 18, 21⟩
⟨9, 15, 17, 7, 19, 8, 12, 2, 11, 4, 14, 18, 21, 6, 10, 1, 20, 3, 5, 13, 16, 22⟩
⟨10, 21, 6, 22, 16, 13, 5, 3, 1, 20, 2, 12, 8, 19, 7, 17, 15, 9, 14, 18, 4, 11⟩
⟨10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 2, 20, 1, 3, 5, 16, 13, 19, 7, 8, 12⟩
⟨10, 21, 6, 22, 16, 5, 3, 1, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19, 13⟩
⟨10, 21, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5, 3, 1, 20, 2, 12, 11, 4, 18, 14⟩
⟨10, 6, 21, 18, 4, 14, 9, 17, 7, 19, 8, 12, 11, 2, 20, 1, 3, 5, 13, 16, 22, 15⟩
⟨10, 6, 21, 18, 14, 4, 11, 12, 2, 20, 1, 3, 5, 13, 19, 8, 7, 17, 9, 15, 22, 16⟩
⟨10, 21, 6, 22, 15, 9, 14, 18, 4, 11, 12, 2, 20, 1, 3, 5, 16, 13, 19, 8, 7, 17⟩
⟨10, 21, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5, 3, 1, 20, 2, 12, 11, 4, 14, 18⟩
⟨10, 21, 6, 22, 16, 13, 5, 3, 1, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19⟩
⟨10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 2, 12, 8, 7, 19, 13, 16, 5, 3, 1, 20⟩
⟨10, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5, 3, 1, 20, 2, 12, 11, 4, 14, 18, 21⟩
⟨10, 6, 21, 18, 4, 14, 9, 15, 17, 7, 19, 8, 12, 11, 2, 20, 1, 3, 5, 13, 16, 22⟩
⟨11, 4, 14, 18, 21, 10, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5, 3, 1, 20, 2, 12⟩
⟨11, 4, 14, 18, 21, 6, 10, 1, 3, 20, 2, 12, 8, 19, 7, 17, 9, 15, 22, 16, 5, 13⟩
⟨11, 4, 18, 21, 6, 10, 1, 3, 20, 2, 12, 8, 7, 19, 13, 5, 16, 22, 15, 17, 9, 14⟩
⟨11, 4, 14, 18, 21, 10, 6, 22, 16, 13, 5, 3, 1, 20, 2, 12, 8, 19, 7, 17, 9, 15⟩
⟨11, 4, 14, 18, 21, 10, 6, 22, 15, 9, 17, 7, 19, 8, 12, 2, 20, 1, 3, 5, 13, 16⟩
⟨11, 4, 14, 18, 21, 6, 10, 1, 3, 20, 2, 12, 8, 7, 19, 13, 5, 16, 22, 15, 9, 17⟩
⟨11, 4, 14, 9, 15, 17, 7, 19, 8, 12, 2, 20, 1, 3, 5, 13, 16, 22, 6, 10, 21, 18⟩
⟨11, 4, 14, 18, 21, 6, 10, 1, 3, 20, 2, 12, 8, 7, 17, 9, 15, 22, 16, 5, 13, 19⟩
⟨11, 4, 14, 18, 21, 6, 10, 1, 3, 5, 13, 16, 22, 15, 9, 17, 7, 19, 8, 12, 2, 20⟩
⟨11, 4, 18, 14, 9, 15, 17, 7, 19, 8, 12, 2, 20, 1, 3, 5, 13, 16, 22, 6, 10, 21⟩
⟨11, 12, 2, 20, 1, 3, 5, 16, 13, 19, 8, 7, 17, 15, 9, 14, 4, 18, 21, 10, 6, 22⟩

$\langle 12, 8, 19, 7, 17, 15, 9, 14, 18, 4, 11, 2, 20, 3, 1, 10, 21, 6, 22, 16, 5, 13\rangle$
$\langle 12, 8, 7, 19, 13, 16, 5, 3, 1, 20, 2, 11, 4, 18, 21, 10, 6, 22, 15, 17, 9, 14\rangle$
$\langle 12, 8, 7, 19, 13, 5, 16, 22, 6, 21, 10, 1, 3, 20, 2, 11, 4, 18, 14, 9, 17, 15\rangle$
$\langle 12, 8, 19, 7, 17, 9, 15, 22, 6, 10, 21, 18, 14, 4, 11, 2, 20, 1, 3, 5, 13, 16\rangle$
$\langle 12, 8, 7, 19, 13, 16, 5, 3, 1, 20, 2, 11, 4, 14, 18, 21, 10, 6, 22, 15, 9, 17\rangle$
$\langle 12, 8, 7, 19, 13, 16, 5, 3, 1, 20, 2, 11, 4, 14, 9, 17, 15, 22, 6, 10, 21, 18\rangle$
$\langle 12, 8, 7, 17, 15, 9, 14, 18, 4, 11, 2, 20, 3, 1, 10, 21, 6, 22, 16, 5, 13, 19\rangle$
$\langle 12, 8, 7, 19, 13, 16, 5, 3, 1, 10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 2, 20\rangle$
$\langle 12, 8, 7, 19, 13, 16, 5, 3, 1, 20, 2, 11, 4, 18, 14, 9, 17, 15, 22, 6, 10, 21\rangle$
$\langle 12, 11, 2, 20, 1, 3, 5, 16, 13, 19, 8, 7, 17, 15, 9, 14, 4, 18, 21, 10, 6, 22\rangle$
$\langle 13, 19, 7, 8, 12, 2, 11, 4, 18, 21, 6, 10, 1, 20, 3, 5, 16, 22, 15, 17, 9, 14\rangle$
$\langle 13, 19, 7, 8, 12, 11, 2, 20, 1, 3, 5, 16, 22, 6, 10, 21, 18, 4, 14, 9, 17, 15\rangle$
$\langle 13, 19, 7, 8, 12, 2, 11, 4, 18, 14, 9, 17, 15, 22, 6, 21, 10, 1, 20, 3, 5, 16\rangle$
$\langle 13, 19, 7, 8, 12, 11, 2, 20, 1, 3, 5, 16, 22, 6, 10, 21, 18, 4, 14, 9, 15, 17\rangle$
$\langle 13, 19, 7, 8, 12, 2, 11, 4, 14, 9, 17, 15, 22, 16, 5, 3, 20, 1, 10, 6, 21, 18\rangle$
$\langle 13, 16, 5, 3, 1, 20, 2, 12, 11, 4, 14, 18, 21, 10, 6, 22, 15, 9, 17, 7, 8, 19\rangle$
$\langle 13, 19, 8, 7, 17, 9, 15, 22, 16, 5, 3, 1, 10, 6, 21, 18, 14, 4, 11, 12, 2, 20\rangle$
$\langle 13, 19, 7, 8, 12, 2, 11, 4, 18, 14, 9, 17, 15, 22, 16, 5, 3, 20, 1, 10, 6, 21\rangle$
$\langle 13, 16, 5, 3, 1, 20, 2, 11, 12, 8, 19, 7, 17, 15, 9, 14, 4, 18, 21, 10, 6, 22\rangle$
$\langle 14, 4, 18, 21, 10, 6, 22, 16, 13, 5, 3, 1, 20, 2, 11, 12, 8, 19, 7, 17, 9, 15\rangle$
$\langle 14, 4, 18, 21, 10, 6, 22, 15, 9, 17, 7, 19, 8, 12, 11, 2, 20, 1, 3, 5, 13, 16\rangle$
$\langle 14, 4, 18, 21, 6, 10, 1, 3, 20, 2, 11, 12, 8, 7, 19, 13, 5, 16, 22, 15, 9, 17\rangle$
$\langle 14, 4, 11, 12, 2, 20, 1, 3, 5, 16, 13, 19, 8, 7, 17, 9, 15, 22, 6, 10, 21, 18\rangle$
$\langle 14, 4, 18, 21, 6, 10, 1, 3, 20, 2, 11, 12, 8, 7, 17, 9, 15, 22, 16, 5, 13, 19\rangle$
$\langle 14, 4, 18, 21, 6, 10, 1, 3, 5, 13, 16, 22, 15, 9, 17, 7, 19, 8, 12, 11, 2, 20\rangle$
$\langle 14, 9, 15, 17, 7, 8, 19, 13, 5, 16, 22, 6, 10, 1, 3, 20, 2, 12, 11, 4, 18, 21\rangle$
$\langle 14, 9, 15, 17, 7, 19, 8, 12, 2, 11, 4, 18, 21, 6, 10, 1, 20, 3, 5, 13, 16, 22\rangle$
$\langle 15, 17, 9, 14, 18, 4, 11, 2, 12, 8, 7, 19, 13, 5, 3, 20, 1, 10, 21, 6, 22, 16\rangle$
$\langle 15, 9, 14, 4, 18, 21, 10, 6, 22, 16, 13, 5, 3, 1, 20, 2, 11, 12, 8, 19, 7, 17\rangle$
$\langle 15, 9, 17, 7, 8, 19, 13, 5, 16, 22, 6, 21, 10, 1, 3, 20, 2, 12, 11, 4, 14, 18\rangle$
$\langle 15, 17, 9, 14, 4, 18, 21, 10, 6, 22, 16, 13, 5, 3, 1, 20, 2, 11, 12, 8, 7, 19\rangle$
$\langle 15, 17, 9, 14, 18, 4, 11, 2, 12, 8, 7, 19, 13, 5, 16, 22, 6, 21, 10, 1, 3, 20\rangle$
$\langle 15, 9, 17, 7, 8, 19, 13, 5, 16, 22, 6, 10, 1, 3, 20, 2, 12, 11, 4, 14, 18, 21\rangle$
$\langle 15, 17, 9, 14, 4, 18, 21, 6, 10, 1, 3, 20, 2, 11, 12, 8, 7, 19, 13, 5, 16, 22\rangle$
$\langle 16, 13, 5, 3, 20, 1, 10, 21, 6, 22, 15, 9, 14, 18, 4, 11, 2, 12, 8, 19, 7, 17\rangle$
$\langle 16, 13, 5, 3, 20, 1, 10, 21, 6, 22, 15, 9, 17, 7, 19, 8, 12, 2, 11, 4, 14, 18\rangle$
$\langle 16, 13, 5, 3, 1, 20, 2, 12, 11, 4, 14, 18, 21, 10, 6, 22, 15, 9, 17, 7, 8, 19\rangle$
$\langle 16, 5, 13, 19, 7, 8, 12, 2, 11, 4, 18, 14, 9, 17, 15, 22, 6, 21, 10, 1, 3, 20\rangle$
$\langle 16, 13, 5, 3, 20, 1, 10, 6, 22, 15, 9, 17, 7, 19, 8, 12, 2, 11, 4, 14, 18, 21\rangle$
$\langle 16, 13, 5, 3, 1, 20, 2, 11, 12, 8, 19, 7, 17, 15, 9, 14, 4, 18, 21, 10, 6, 22\rangle$
$\langle 17, 9, 15, 22, 6, 21, 10, 1, 20, 3, 5, 16, 13, 19, 7, 8, 12, 2, 11, 4, 14, 18\rangle$
$\langle 17, 15, 9, 14, 4, 18, 21, 10, 6, 22, 16, 13, 5, 3, 1, 20, 2, 11, 12, 8, 7, 19\rangle$
$\langle 17, 15, 9, 14, 18, 4, 11, 2, 12, 8, 7, 19, 13, 5, 16, 22, 6, 21, 10, 1, 3, 20\rangle$
$\langle 17, 9, 15, 22, 6, 10, 1, 20, 3, 5, 16, 13, 19, 7, 8, 12, 2, 11, 4, 14, 18, 21\rangle$
$\langle 17, 15, 9, 14, 4, 18, 21, 6, 10, 1, 3, 20, 2, 11, 12, 8, 7, 19, 13, 5, 16, 22\rangle$
$\langle 18, 4, 14, 9, 15, 17, 7, 8, 12, 11, 2, 20, 3, 1, 10, 21, 6, 22, 16, 5, 13, 19\rangle$
$\langle 18, 4, 14, 9, 17, 15, 22, 6, 21, 10, 1, 3, 5, 16, 13, 19, 7, 8, 12, 11, 2, 20\rangle$
$\langle 18, 4, 14, 9, 15, 17, 7, 19, 8, 12, 11, 2, 20, 1, 3, 5, 13, 16, 22, 6, 10, 21\rangle$
$\langle 18, 21, 6, 10, 1, 3, 20, 2, 12, 11, 4, 14, 9, 15, 17, 7, 8, 19, 13, 5, 16, 22\rangle$

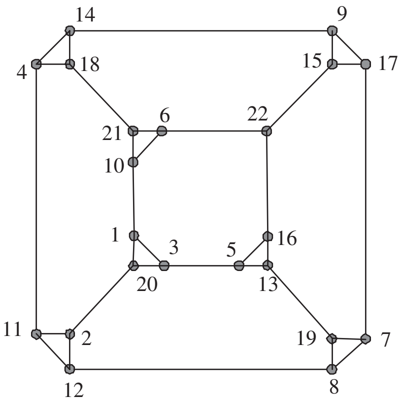

Fig. 7. Relabelling for graph Q.

⟨19, 13, 5, 16, 22, 15, 9, 17, 7, 8, 12, 2, 11, 4, 14, 18, 21, 6, 10, 1, 3, 20⟩
⟨19, 13, 16, 5, 3, 20, 1, 10, 6, 22, 15, 9, 17, 7, 8, 12, 2, 11, 4, 14, 18, 21⟩
⟨19, 13, 16, 5, 3, 1, 20, 2, 11, 12, 8, 7, 17, 15, 9, 14, 4, 18, 21, 10, 6, 22⟩
⟨20, 1, 3, 5, 16, 13, 19, 7, 8, 12, 2, 11, 4, 18, 14, 9, 17, 15, 22, 6, 10, 21⟩
⟨20, 2, 11, 12, 8, 7, 19, 13, 16, 5, 3, 1, 10, 6, 21, 18, 4, 14, 9, 17, 15, 22⟩
⟨21, 6, 10, 1, 3, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19, 13, 5, 16, 22⟩

□

Fact 2. $g_{3,1} \in \mathrm{Nice}(Q)$.

Proof. With the relabelling of vertices, it is equivalent to showing that vertex 20 is nice. Since Q is 1-edge hamiltonian, vertex 20 is good. In the following, we list all hamiltonian paths of Q that joins u to x_i in $Q - (20, x_i)$ for any $u \in V(Q) - \{20, x_i\}$ and $x_i \in N(20)$.

$Q - (1, 20)$	⟨1, 10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 12, 8, 7, 19, 13, 16, 5, 3, 20, 2⟩
	⟨1, 10, 21, 6, 22, 16, 5, 13, 19, 8, 7, 17, 15, 9, 14, 18, 4, 11, 12, 2, 20, 3⟩
	⟨1, 10, 21, 6, 22, 16, 13, 5, 3, 20, 2, 11, 12, 8, 19, 7, 17, 15, 9, 14, 18, 4⟩
	⟨1, 10, 21, 6, 22, 16, 13, 19, 8, 7, 17, 15, 9, 14, 18, 4, 11, 12, 2, 20, 3, 5⟩
	⟨1, 10, 21, 18, 14, 4, 11, 12, 2, 20, 3, 5, 16, 13, 19, 8, 7, 17, 9, 15, 22, 6⟩
	⟨1, 10, 21, 6, 22, 15, 17, 9, 14, 18, 4, 11, 12, 2, 20, 3, 5, 16, 13, 19, 8, 7⟩
	⟨1, 10, 21, 6, 22, 16, 13, 5, 3, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 19, 8⟩
	⟨1, 10, 21, 6, 22, 15, 17, 7, 8, 19, 13, 16, 5, 3, 20, 2, 12, 11, 4, 18, 14, 9⟩
	⟨1, 3, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19, 13, 5, 16, 22, 6, 21, 10⟩
	⟨1, 10, 21, 6, 22, 16, 13, 5, 3, 20, 2, 12, 8, 19, 7, 17, 15, 9, 14, 18, 4, 11⟩
	⟨1, 10, 21, 6, 22, 16, 13, 5, 3, 20, 2, 11, 4, 18, 14, 9, 15, 17, 7, 19, 8, 12⟩
	⟨1, 10, 21, 6, 22, 16, 5, 3, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19, 13⟩
	⟨1, 10, 21, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5, 3, 20, 2, 12, 11, 4, 18, 14⟩
	⟨1, 10, 6, 21, 18, 4, 14, 9, 17, 7, 19, 8, 12, 11, 2, 20, 3, 5, 13, 16, 22, 15⟩
	⟨1, 10, 6, 21, 18, 14, 4, 11, 12, 2, 20, 3, 5, 13, 19, 8, 7, 17, 9, 15, 22, 16⟩
	⟨1, 10, 21, 6, 22, 15, 9, 14, 18, 4, 11, 12, 2, 20, 3, 5, 16, 13, 19, 8, 7, 17⟩
	⟨1, 10, 21, 6, 22, 16, 13, 5, 3, 20, 2, 11, 12, 8, 19, 7, 17, 15, 9, 14, 4, 18⟩
	⟨1, 10, 21, 6, 22, 16, 13, 5, 3, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19⟩
	⟨1, 10, 6, 22, 16, 13, 5, 3, 20, 2, 11, 12, 8, 19, 7, 17, 15, 9, 14, 4, 18, 21⟩
	⟨1, 10, 6, 21, 18, 14, 4, 11, 12, 2, 20, 3, 5, 16, 13, 19, 8, 7, 17, 9, 15, 22⟩

$Q-(2,20)$	⟨2, 11, 12, 8, 19, 7, 17, 15, 9, 14, 4, 18, 21, 10, 6, 22, 16, 13, 5, 3, 20, 1⟩
	⟨2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19, 13, 5, 16, 22, 6, 21, 10, 1, 20, 3⟩
	⟨2, 11, 12, 8, 19, 7, 17, 9, 15, 22, 16, 13, 5, 3, 20, 1, 10, 6, 21, 18, 14, 4⟩
	⟨2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19, 13, 16, 22, 6, 21, 10, 1, 20, 3, 5⟩
	⟨2, 12, 11, 4, 14, 18, 21, 10, 1, 20, 3, 5, 16, 13, 19, 8, 7, 17, 9, 15, 22, 6⟩
	⟨2, 12, 11, 4, 18, 14, 9, 17, 15, 22, 6, 21, 10, 1, 20, 3, 5, 16, 13, 19, 8, 7⟩
	⟨2, 12, 11, 4, 14, 18, 21, 6, 10, 1, 20, 3, 5, 13, 16, 22, 15, 9, 17, 7, 19, 8⟩
	⟨2, 11, 12, 8, 19, 7, 17, 15, 22, 16, 13, 5, 3, 20, 1, 10, 6, 21, 18, 4, 14, 9⟩
	⟨2, 12, 11, 4, 14, 18, 21, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5, 3, 20, 1, 10⟩
	⟨2, 12, 8, 19, 7, 17, 9, 15, 22, 16, 13, 5, 3, 20, 1, 10, 6, 21, 18, 14, 4, 11⟩
	⟨2, 11, 4, 14, 18, 21, 6, 10, 1, 20, 3, 5, 13, 16, 22, 15, 9, 17, 7, 19, 8, 12⟩
	⟨2, 12, 11, 4, 14, 18, 21, 6, 10, 1, 20, 3, 5, 16, 22, 15, 9, 17, 7, 8, 19, 13⟩
	⟨2, 11, 12, 8, 19, 7, 17, 9, 15, 22, 16, 13, 5, 3, 20, 1, 10, 6, 21, 18, 4, 14⟩
	⟨2, 12, 11, 4, 18, 14, 9, 17, 7, 8, 19, 13, 16, 5, 3, 20, 1, 10, 21, 6, 22, 15⟩
	⟨2, 12, 11, 4, 14, 18, 21, 6, 10, 1, 20, 3, 5, 13, 19, 8, 7, 17, 9, 15, 22, 16⟩
	⟨2, 12, 11, 4, 18, 14, 9, 15, 22, 6, 21, 10, 1, 20, 3, 5, 16, 13, 19, 8, 7, 17⟩
	⟨2, 11, 12, 8, 7, 19, 13, 16, 5, 3, 20, 1, 10, 21, 6, 22, 15, 17, 9, 14, 4, 18⟩
	⟨2, 12, 11, 4, 14, 18, 21, 6, 10, 1, 20, 3, 5, 13, 16, 22, 15, 9, 17, 7, 8, 19⟩
	⟨2, 11, 12, 8, 7, 19, 13, 16, 5, 3, 20, 1, 10, 6, 22, 15, 17, 9, 14, 4, 18, 21⟩
	⟨2, 12, 11, 4, 14, 18, 21, 6, 10, 1, 20, 3, 5, 16, 13, 19, 8, 7, 17, 9, 15, 22⟩
$Q-(3,20)$	⟨3, 5, 16, 13, 19, 8, 7, 17, 9, 15, 22, 6, 10, 21, 18, 14, 4, 11, 12, 2, 20, 1⟩
	⟨3, 5, 16, 13, 19, 7, 8, 12, 11, 4, 18, 14, 9, 17, 15, 22, 6, 21, 10, 1, 20, 2⟩
	⟨3, 5, 13, 16, 22, 15, 9, 17, 7, 19, 8, 12, 11, 2, 20, 1, 10, 6, 21, 18, 14, 4⟩
	⟨3, 1, 20, 2, 12, 11, 4, 14, 18, 21, 10, 6, 22, 15, 9, 17, 7, 8, 19, 13, 16, 5⟩
	⟨3, 1, 20, 2, 11, 12, 8, 7, 19, 13, 5, 16, 22, 15, 17, 9, 14, 4, 18, 21, 10, 6⟩
	⟨3, 1, 20, 2, 11, 12, 8, 19, 13, 5, 16, 22, 6, 10, 21, 18, 4, 14, 9, 15, 17, 7⟩
	⟨3, 5, 13, 16, 22, 6, 21, 10, 1, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 19, 8⟩
	⟨3, 1, 20, 2, 12, 11, 4, 14, 18, 21, 10, 6, 22, 16, 5, 13, 19, 8, 7, 17, 15, 9⟩
	⟨3, 1, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19, 13, 5, 16, 22, 6, 21, 10⟩
	⟨3, 5, 13, 16, 22, 15, 9, 17, 7, 19, 8, 12, 2, 20, 1, 10, 6, 21, 18, 14, 4, 11⟩
	⟨3, 5, 13, 16, 22, 6, 21, 10, 1, 20, 2, 11, 4, 18, 14, 9, 15, 17, 7, 19, 8, 12⟩
	⟨3, 1, 20, 2, 11, 12, 8, 19, 7, 17, 15, 9, 14, 4, 18, 21, 10, 6, 22, 16, 5, 13⟩
	⟨3, 1, 20, 2, 12, 11, 4, 18, 21, 10, 6, 22, 16, 5, 13, 19, 8, 7, 17, 15, 9, 14⟩
	⟨3, 1, 20, 2, 12, 11, 4, 14, 18, 21, 10, 6, 22, 16, 5, 13, 19, 8, 7, 17, 9, 15⟩
	⟨3, 1, 20, 2, 12, 11, 4, 14, 18, 21, 10, 6, 22, 15, 9, 17, 7, 8, 19, 13, 5, 16⟩
	⟨3, 1, 20, 2, 11, 12, 8, 7, 19, 13, 5, 16, 22, 6, 10, 21, 18, 4, 14, 9, 15, 17⟩
	⟨3, 1, 20, 2, 12, 11, 4, 14, 9, 15, 17, 7, 8, 19, 13, 5, 16, 22, 6, 10, 21, 18⟩
	⟨3, 1, 20, 2, 11, 12, 8, 7, 17, 15, 9, 14, 4, 18, 21, 10, 6, 22, 16, 5, 13, 19⟩
	⟨3, 1, 20, 2, 12, 11, 4, 18, 14, 9, 15, 17, 7, 8, 19, 13, 5, 16, 22, 6, 10, 21⟩
	⟨3, 5, 16, 13, 19, 8, 7, 17, 15, 9, 14, 18, 4, 11, 12, 2, 20, 1, 10, 21, 6, 22⟩

Thus, $g_{3,1} \in \text{Nice}(Q)$. □

Fact 3. Eye(2) *is hamiltonian connected.*

Proof. We relabel the vertices of Eye(2) as in Fig. 8. Using the symmetric property of Eye(2), we only need to find a hamiltonian path joining a to b for any $a \in \{1, 2, 3, 4\}$ and $b \in V(\text{Eye}(2))$ with $a \neq b$. The corresponding hamiltonian paths are listed below:

⟨1, 16, 5, 7, 10, 12, 17, 14, 13, 15, 4, 8, 11, 9, 18, 6, 3, 2⟩	⟨1, 2, 13, 15, 4, 8, 11, 9, 10, 12, 17, 14, 7, 5, 16, 18, 6, 3⟩
⟨1, 2, 3, 6, 8, 11, 9, 18, 16, 5, 7, 10, 12, 17, 14, 13, 15, 4⟩	⟨1, 2, 13, 15, 4, 3, 6, 8, 11, 12, 17, 14, 7, 10, 9, 18, 16, 5⟩
⟨1, 2, 3, 4, 8, 11, 9, 10, 12, 17, 15, 13, 14, 7, 5, 16, 18, 6⟩	⟨1, 2, 3, 4, 15, 13, 14, 17, 12, 10, 9, 11, 8, 6, 18, 16, 5, 7⟩
⟨1, 2, 3, 6, 18, 16, 5, 7, 10, 9, 11, 12, 17, 14, 13, 15, 4, 8⟩	⟨1, 2, 3, 6, 18, 16, 5, 7, 10, 12, 17, 14, 13, 15, 4, 8, 11, 9⟩
⟨1, 2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 11, 9, 18, 16, 5, 7, 10⟩	⟨1, 2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 10, 7, 5, 16, 18, 9, 11⟩
⟨1, 2, 3, 6, 18, 16, 5, 7, 10, 9, 11, 8, 4, 15, 13, 14, 17, 12⟩	⟨1, 2, 3, 6, 18, 16, 5, 7, 14, 17, 12, 10, 9, 11, 8, 4, 15, 13⟩
⟨1, 2, 3, 4, 8, 6, 18, 16, 5, 7, 10, 9, 11, 12, 17, 15, 13, 14⟩	⟨1, 2, 3, 4, 8, 6, 18, 16, 5, 7, 10, 9, 11, 12, 17, 14, 13, 15⟩
⟨1, 2, 3, 6, 18, 9, 11, 8, 4, 15, 13, 14, 17, 12, 10, 7, 5, 16⟩	⟨1, 2, 3, 6, 18, 16, 5, 7, 14, 13, 15, 4, 8, 11, 9, 10, 12, 17⟩
⟨1, 2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 11, 9, 10, 7, 5, 16, 18⟩	
⟨2, 13, 15, 4, 8, 11, 9, 10, 12, 17, 14, 7, 5, 1, 16, 18, 6, 3⟩	⟨2, 3, 6, 8, 11, 9, 18, 16, 1, 5, 7, 10, 12, 17, 14, 13, 15, 4⟩
⟨2, 13, 15, 4, 3, 6, 8, 11, 12, 17, 14, 7, 10, 9, 18, 16, 1, 5⟩	⟨2, 3, 4, 8, 11, 9, 10, 12, 17, 15, 13, 14, 7, 5, 1, 16, 18, 6⟩
⟨2, 3, 4, 15, 13, 14, 17, 12, 10, 9, 11, 8, 6, 18, 16, 1, 5, 7⟩	⟨2, 3, 6, 18, 16, 1, 5, 7, 10, 9, 11, 12, 17, 14, 13, 15, 4, 8⟩
⟨2, 3, 6, 18, 16, 1, 5, 7, 10, 12, 17, 14, 13, 15, 4, 8, 11, 9⟩	⟨2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 11, 9, 18, 16, 1, 5, 7, 10⟩
⟨2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 10, 7, 5, 1, 16, 18, 9, 11⟩	⟨2, 3, 6, 18, 16, 1, 5, 7, 10, 9, 11, 8, 4, 15, 13, 14, 17, 12⟩
⟨2, 3, 6, 18, 16, 1, 5, 7, 14, 17, 12, 10, 9, 11, 8, 4, 15, 13⟩	⟨2, 3, 4, 8, 6, 18, 16, 1, 5, 7, 10, 9, 11, 12, 17, 15, 13, 14⟩
⟨2, 3, 4, 8, 6, 18, 16, 1, 5, 7, 10, 9, 11, 12, 17, 14, 13, 15⟩	⟨2, 3, 6, 18, 9, 11, 8, 4, 15, 13, 14, 17, 12, 10, 7, 5, 1, 16⟩
⟨2, 3, 6, 18, 16, 1, 5, 7, 14, 13, 15, 4, 8, 11, 9, 10, 12, 17⟩	⟨2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 11, 9, 10, 7, 5, 1, 16, 18⟩
⟨3, 6, 8, 11, 12, 17, 14, 7, 10, 9, 18, 16, 5, 1, 2, 13, 15, 4⟩	⟨3, 6, 8, 4, 15, 17, 14, 13, 2, 1, 16, 18, 9, 11, 12, 10, 7, 5⟩
⟨3, 4, 15, 17, 12, 10, 7, 14, 13, 2, 1, 5, 16, 18, 9, 11, 8, 6⟩	⟨3, 6, 8, 4, 15, 17, 14, 13, 2, 1, 5, 16, 18, 9, 11, 12, 10, 7⟩
⟨3, 6, 18, 9, 11, 12, 10, 7, 5, 16, 1, 2, 13, 14, 17, 15, 4, 8⟩	⟨3, 6, 18, 16, 5, 1, 2, 13, 15, 4, 8, 11, 12, 17, 14, 7, 10, 9⟩
⟨3, 6, 8, 4, 15, 17, 12, 11, 9, 18, 16, 5, 1, 2, 13, 14, 7, 10⟩	⟨3, 6, 8, 4, 15, 17, 12, 10, 7, 14, 13, 2, 1, 5, 16, 18, 9, 11⟩
⟨3, 6, 18, 9, 10, 7, 5, 16, 1, 2, 13, 14, 17, 15, 4, 8, 11, 12⟩	⟨3, 6, 8, 4, 15, 17, 14, 7, 10, 12, 11, 9, 18, 16, 5, 1, 2, 13⟩
⟨3, 6, 18, 9, 10, 7, 5, 16, 1, 2, 13, 15, 4, 8, 11, 12, 17, 14⟩	⟨3, 6, 18, 9, 10, 7, 5, 16, 1, 2, 13, 14, 17, 12, 11, 8, 4, 15⟩
⟨3, 6, 8, 4, 15, 17, 14, 13, 2, 1, 5, 7, 10, 12, 11, 9, 18, 16⟩	⟨3, 6, 8, 4, 15, 13, 2, 1, 5, 16, 18, 9, 11, 12, 10, 7, 14, 17⟩
⟨3, 6, 8, 4, 15, 17, 12, 11, 9, 10, 7, 14, 13, 2, 1, 5, 16, 18⟩	
⟨4, 8, 11, 12, 17, 15, 13, 14, 7, 10, 9, 18, 6, 3, 2, 1, 16, 5⟩	⟨4, 8, 11, 12, 10, 9, 18, 16, 1, 5, 7, 14, 17, 15, 13, 2, 3, 6⟩
⟨4, 15, 17, 14, 13, 2, 3, 6, 8, 11, 12, 10, 9, 18, 16, 1, 5, 7⟩	⟨4, 15, 13, 14, 17, 12, 10, 7, 5, 16, 1, 2, 3, 6, 18, 9, 11, 8⟩
⟨4, 15, 17, 14, 13, 2, 3, 6, 8, 11, 12, 10, 7, 5, 1, 16, 18, 9⟩	⟨4, 8, 11, 12, 17, 15, 13, 14, 7, 5, 16, 1, 2, 3, 6, 18, 9, 10⟩
⟨4, 8, 6, 3, 2, 13, 15, 17, 14, 7, 5, 1, 16, 18, 9, 10, 12, 11⟩	⟨4, 15, 17, 14, 13, 2, 3, 6, 8, 11, 9, 18, 16, 1, 5, 7, 10, 12⟩
⟨4, 8, 11, 12, 10, 9, 18, 6, 3, 2, 1, 16, 5, 7, 14, 17, 15, 13⟩	⟨4, 8, 11, 9, 10, 12, 17, 15, 13, 2, 3, 6, 18, 16, 1, 5, 7, 14⟩
⟨4, 8, 11, 9, 10, 12, 17, 14, 7, 5, 1, 16, 18, 6, 3, 2, 13, 15⟩	⟨4, 8, 11, 12, 10, 9, 18, 6, 3, 2, 13, 15, 17, 14, 7, 5, 1, 16⟩
⟨4, 8, 11, 12, 10, 9, 18, 6, 3, 2, 1, 16, 5, 7, 14, 13, 15, 17⟩	⟨4, 8, 11, 9, 10, 12, 17, 15, 13, 14, 7, 5, 16, 1, 2, 3, 6, 18⟩

□

Fact 4. $e_{16} \in \text{Nice}(\text{Eye}(2))$.

Proof. With the relabelling of vertices, it is equivalent to showing vertex 1 is nice. We first check that vertex 1 is good. The hamiltonian cycles of $\text{Eye}(2) - (1, x_i)$ with $x_i \in N(1)$ are listed below:

Eye(2) − (1, 2)	⟨1, 5, 7, 14, 13, 2, 3, 4, 15, 17, 12, 10, 9, 11, 8, 6, 18, 16, 1⟩
Eye(2) − (1, 5)	⟨1, 16, 5, 7, 10, 12, 17, 14, 13, 15, 4, 8, 11, 9, 18, 6, 3, 2, 1⟩
Eye(2) − (1, 16)	⟨1, 5, 16, 18, 9, 10, 7, 14, 17, 12, 11, 8, 6, 3, 4, 15, 13, 2, 1⟩

Thus, vertex 1 is good in Eye(2). In the following, we list all hamiltonian paths of Eye(2) that joins u to x_i in $\text{Eye}(2) - (1, x_i)$ for any $u \in V(\text{Eye}(2)) - \{1, x_i\}$ and $x_i \in N(1)$.

Eye(2) − (16, 1)	⟨16, 18, 6, 3, 4, 8, 11, 9, 10, 12, 17, 15, 13, 14, 7, 5, 1, 2⟩
	⟨16, 18, 9, 11, 12, 10, 7, 5, 1, 2, 13, 14, 17, 15, 4, 8, 6, 3⟩
	⟨16, 5, 1, 2, 3, 6, 18, 9, 10, 7, 14, 13, 15, 17, 12, 11, 8, 4⟩
	⟨16, 18, 9, 10, 7, 14, 17, 12, 11, 8, 6, 3, 4, 15, 13, 2, 1, 5⟩
	⟨16, 18, 9, 10, 7, 5, 1, 2, 3, 4, 15, 13, 14, 17, 12, 11, 8, 6⟩
	⟨16, 18, 6, 8, 11, 9, 10, 12, 17, 14, 13, 15, 4, 3, 2, 1, 5, 7⟩
	⟨16, 5, 1, 2, 13, 14, 7, 10, 9, 18, 6, 3, 4, 15, 17, 12, 11, 8⟩
	⟨16, 18, 6, 8, 11, 12, 17, 14, 13, 15, 4, 3, 2, 1, 5, 7, 10, 9⟩
	⟨16, 18, 9, 11, 8, 6, 3, 4, 15, 13, 2, 1, 5, 7, 14, 17, 12, 10⟩
	⟨16, 5, 1, 2, 3, 4, 8, 6, 18, 9, 10, 7, 14, 13, 15, 17, 12, 11⟩
	⟨16, 18, 9, 11, 8, 6, 3, 4, 15, 17, 14, 13, 2, 1, 5, 7, 10, 12⟩
	⟨16, 5, 1, 2, 3, 4, 8, 6, 18, 9, 11, 12, 10, 7, 14, 17, 15, 13⟩
	⟨16, 18, 6, 8, 4, 3, 2, 1, 5, 7, 10, 9, 11, 12, 17, 15, 13, 14⟩
	⟨16, 5, 1, 2, 13, 14, 7, 10, 9, 18, 6, 3, 4, 8, 11, 12, 17, 15⟩
	⟨16, 18, 6, 3, 2, 1, 5, 7, 14, 13, 15, 4, 8, 11, 9, 10, 12, 17⟩
	⟨16, 5, 1, 2, 13, 15, 4, 3, 6, 8, 11, 12, 17, 14, 7, 10, 9, 18⟩
Eye(2) − (2, 1)	⟨2, 13, 15, 4, 8, 11, 9, 10, 12, 17, 14, 7, 5, 1, 16, 18, 6, 3⟩
	⟨2, 3, 6, 8, 11, 9, 18, 16, 1, 5, 7, 10, 12, 17, 14, 13, 15, 4⟩
	⟨2, 13, 15, 4, 3, 6, 8, 11, 12, 17, 14, 7, 10, 9, 18, 16, 1, 5⟩
	⟨2, 3, 4, 8, 11, 9, 10, 12, 17, 15, 13, 14, 7, 5, 1, 16, 18, 6⟩
	⟨2, 3, 4, 15, 13, 14, 17, 12, 10, 9, 11, 8, 6, 18, 16, 1, 5, 7⟩
	⟨2, 3, 6, 18, 16, 1, 5, 7, 10, 9, 11, 12, 17, 14, 13, 15, 4, 8⟩
	⟨2, 3, 6, 18, 16, 1, 5, 7, 10, 12, 17, 14, 13, 15, 4, 8, 11, 9⟩
	⟨2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 11, 9, 18, 16, 1, 5, 7, 10⟩
	⟨2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 10, 7, 5, 1, 16, 18, 9, 11⟩
	⟨2, 3, 6, 18, 16, 1, 5, 7, 10, 9, 11, 8, 4, 15, 13, 14, 17, 12⟩
	⟨2, 3, 6, 18, 16, 1, 5, 7, 14, 17, 12, 10, 9, 11, 8, 4, 15, 13⟩
	⟨2, 3, 4, 8, 6, 18, 16, 1, 5, 7, 10, 9, 11, 12, 17, 15, 13, 14⟩
	⟨2, 3, 4, 8, 6, 18, 16, 1, 5, 7, 10, 9, 11, 12, 17, 14, 13, 15⟩
	⟨2, 3, 6, 18, 9, 11, 8, 4, 15, 13, 14, 17, 12, 10, 7, 5, 1, 16⟩
	⟨2, 3, 6, 18, 16, 1, 5, 7, 14, 13, 15, 4, 8, 11, 9, 10, 12, 17⟩
	⟨2, 3, 6, 8, 4, 15, 13, 14, 17, 12, 11, 9, 10, 7, 5, 1, 16, 18⟩
Eye(2) − (5, 1)	⟨5, 7, 14, 13, 15, 17, 12, 10, 9, 11, 8, 4, 3, 6, 18, 16, 1, 2⟩
	⟨5, 7, 10, 9, 11, 12, 17, 14, 13, 15, 4, 8, 6, 18, 16, 1, 2, 3⟩
	⟨5, 7, 14, 13, 15, 17, 12, 10, 9, 11, 8, 6, 18, 16, 1, 2, 3, 4⟩
	⟨5, 16, 1, 2, 3, 4, 8, 11, 12, 17, 15, 13, 14, 7, 10, 9, 18, 6⟩
	⟨5, 16, 1, 2, 3, 6, 18, 9, 11, 8, 4, 15, 13, 14, 17, 12, 10, 7⟩
	⟨5, 16, 1, 2, 13, 15, 17, 14, 7, 10, 12, 11, 9, 18, 6, 3, 4, 8⟩
	⟨5, 16, 1, 2, 13, 15, 17, 14, 7, 10, 12, 11, 8, 4, 3, 6, 18, 9⟩
	⟨5, 16, 1, 2, 3, 4, 8, 6, 18, 9, 11, 12, 17, 15, 13, 14, 7, 10⟩
	⟨5, 16, 1, 2, 3, 4, 8, 6, 18, 9, 10, 7, 14, 13, 15, 17, 12, 11⟩
	⟨5, 16, 1, 2, 13, 15, 17, 14, 7, 10, 9, 18, 6, 3, 4, 8, 11, 12⟩
	⟨5, 16, 1, 2, 3, 6, 18, 9, 10, 7, 14, 17, 12, 11, 8, 4, 15, 13⟩
	⟨5, 16, 1, 2, 13, 15, 17, 12, 11, 8, 4, 3, 6, 18, 9, 10, 7, 14⟩
	⟨5, 16, 1, 2, 13, 14, 7, 10, 9, 18, 6, 3, 4, 8, 11, 12, 17, 15⟩
	⟨5, 7, 10, 12, 17, 14, 13, 15, 4, 8, 11, 9, 18, 6, 3, 2, 1, 16⟩
	⟨5, 16, 1, 2, 3, 6, 18, 9, 10, 7, 14, 13, 15, 4, 8, 11, 12, 17⟩
	⟨5, 16, 1, 2, 13, 15, 4, 3, 6, 8, 11, 12, 17, 14, 7, 10, 9, 18⟩

Thus, e_{16} is a nice vertex of Eye(2). □

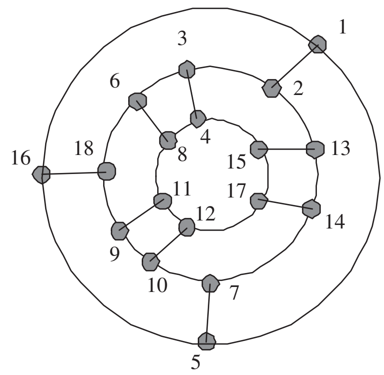

Fig. 8. Relabelling for graph Eye(2).

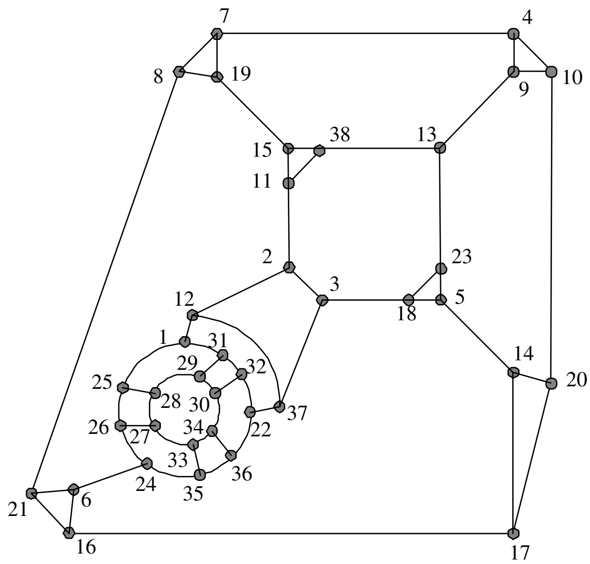

Fig. 9. Relabelling for graph R.

Fact 5. $e_{17} \in \text{Nice}(R)$.

Proof. We relabel the vertices of R as in Fig. 9. It is equivalent to showing that vertex 12 is a nice vertex. We first check that vertex 12 is good. The hamiltonian cycles of $R-(12, x_i)$ with $x_i \in N(12)$ are listed below:

$R-(12,1)$	$\langle 1, 31, 32, 22, 37, 12, 2, 3, 18, 23, 5, 14, 17, 20, 10, 4, 9, 13, 38, 11, 15, 19, 7, 8, 21, 16, 6, 24, 26, 27, 33, 35, 36, 34, 30, 29, 28, 25, 1\rangle$
$R-(12,2)$	$\langle 1, 12, 37, 3, 2, 11, 15, 38, 13, 23, 18, 5, 14, 17, 20, 10, 9, 4, 7, 19, 8, 21, 16, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1\rangle$
$R-(12,37)$	$\langle 1, 12, 2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 22, 36, 34, 30, 32, 31, 29, 28, 27, 33, 35, 24, 26, 25, 1\rangle$

Thus, vertex 12 is good in R. In the following, we list all hamiltonian paths that join u to x_i in $R-(12, x_i)$ for any $u \in V(R)-\{12, x_i\}$ and $x_i \in N(12)$.

$R-(12,1)$	⟨1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 37, 12, 2⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 7, 19, 15, 11, 38, 13, 9, 10, 4⟩
	⟨1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 7, 19, 15, 38, 11, 2, 12, 37, 3, 18, 23, 13, 9, 4, 10, 20, 17, 14, 5⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 17, 20, 10, 4, 9, 13, 38, 11, 15, 19, 7, 8, 21, 16, 6⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 15, 11, 38, 13, 9, 10, 4, 7⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 7, 4, 9, 10, 20, 14, 17, 16, 6, 21, 8⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 7, 8, 21, 6, 16, 17, 14, 20, 10, 4, 9⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 7, 19, 15, 11, 38, 13, 9, 4, 10⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 15, 11⟩
	⟨1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 2, 11, 38, 15, 19, 8, 7, 4, 10, 9, 13⟩
	⟨1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 7, 19, 15, 38, 11, 2, 12, 37, 3, 18, 5, 23, 13, 9, 4, 10, 20, 17, 14⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 11, 15⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 17, 20, 10, 4, 9, 13, 38, 11, 15, 19, 7, 8, 21, 6, 16⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 20, 10, 4, 9, 13, 38, 11, 15, 19, 7, 8, 21, 6, 16, 17⟩
	⟨1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 20, 14, 5, 23, 13, 9, 10, 4, 7, 8, 19, 15, 38, 11, 2, 12, 37, 3, 18⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 7, 4, 10, 9, 13, 38, 11, 15, 19⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 17, 16, 6, 21, 8, 7, 19, 15, 11, 38, 13, 9, 4, 10, 20⟩
	⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 6, 21⟩
	⟨1, 25, 28, 29, 31, 32, 30, 34, 36, 35, 33, 27, 26, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22⟩
	⟨1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 20, 14, 5, 18, 3, 37, 12, 2, 11, 38, 15, 19, 8, 7, 4, 10, 9, 13, 23⟩
	⟨1, 25, 26, 27, 28, 29, 31, 32, 30, 34, 33, 35, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 17, 20, 10, 4, 9, 13, 38, 11, 15, 19, 7, 8, 21, 16, 6, 24⟩
	⟨1, 31, 32, 22, 37, 12, 2, 3, 18, 23, 5, 14, 17, 20, 10, 4, 9, 13, 38, 11, 15, 19, 7, 8, 21, 16, 6, 24, 26, 27, 33, 35, 36, 34, 30, 29, 28, 25⟩

$R-(12,1)$	⟨1, 25, 28, 27, 33, 35, 36, 34, 30, 29, 31, 32, 22, 37, 12, 2, 3, 18, 23, 5, 14, 17, 20, 10, 4, 9, 13, 38, 11, 15, 19, 7, 8, 21, 16, 6, 24, 26⟩ ⟨1, 25, 26, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 36, 35, 33, 34, 30, 32, 31, 29, 28, 27⟩ ⟨1, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 36, 34, 30, 32, 31, 29, 28⟩ ⟨1, 25, 28, 27, 26, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 36, 35, 33, 34, 30, 32, 31, 29⟩ ⟨1, 25, 26, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 32, 31, 29, 28, 27, 33, 35, 36, 34, 30⟩ ⟨1, 25, 26, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 32, 30, 34, 36, 35, 33, 27, 28, 29, 31⟩ ⟨1, 25, 28, 27, 26, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 36, 35, 33, 34, 30, 29, 31, 32⟩ ⟨1, 25, 26, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 23, 5, 14, 17, 20, 10, 4, 9, 13, 38, 11, 15, 19, 7, 8, 21, 16, 6, 24, 35, 33⟩ ⟨1, 25, 26, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 36, 35, 33, 27, 28, 29, 31, 32, 30, 34⟩ ⟨1, 25, 26, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 36, 34, 30, 32, 31, 29, 28, 27, 33, 35⟩ ⟨1, 25, 26, 27, 28, 29, 31, 32, 30, 34, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37, 22, 36⟩ ⟨1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 5, 18, 3, 2, 12, 37⟩ ⟨1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22, 37, 12, 2, 3, 18, 5, 23, 13, 9, 4, 10, 20, 14, 17, 16, 6, 21, 8, 7, 19, 15, 11, 38⟩

$R-(12,2)$	⟨2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6, 24, 26, 25, 28, 27, 33, 35, 36, 34, 30, 29, 31, 32, 22, 37, 12, 1⟩ ⟨2, 11, 15, 38, 13, 23, 18, 5, 14, 17, 20, 10, 9, 4, 7, 19, 8, 21, 16, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 37, 3⟩ ⟨2, 11, 15, 38, 13, 9, 10, 20, 17, 14, 5, 23, 18, 3, 37, 12, 1, 31, 32, 22, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7, 4⟩ ⟨2, 11, 15, 38, 13, 23, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7, 4, 9, 10, 20, 17, 14, 5⟩ ⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6⟩ ⟨2, 11, 15, 38, 13, 9, 4, 10, 20, 17, 14, 5, 23, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7⟩ ⟨2, 11, 15, 38, 13, 23, 5, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 19, 8⟩ ⟨2, 11, 38, 15, 19, 8, 7, 4, 10, 20, 14, 17, 16, 21, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 37, 3, 18, 5, 23, 13, 9⟩ ⟨2, 11, 15, 38, 13, 9, 4, 7, 19, 8, 21, 16, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 37, 3, 18, 23, 5, 14, 17, 20, 10⟩ ⟨2, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 20, 14, 5, 18, 23, 13, 9, 10, 4, 7, 8, 19, 15, 38, 11⟩ ⟨2, 11, 38, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 37, 3, 18, 5, 23, 13⟩

$R-(12,2)$	⟨2, 11, 15, 38, 13, 23, 5, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7, 4, 9, 10, 20, 17, 14⟩
	⟨2, 11, 38, 13, 23, 5, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15⟩
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	⟨2, 11, 15, 38, 13, 23, 5, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7, 4, 9, 10, 20, 14, 17⟩
	⟨2, 11, 15, 38, 13, 23, 5, 14, 17, 20, 10, 9, 4, 7, 19, 8, 21, 16, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 37, 3, 18⟩
	⟨2, 11, 15, 38, 13, 23, 5, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19⟩
	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 16, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 37, 3, 18, 23, 5, 14, 17, 20⟩
	⟨2, 11, 15, 38, 13, 23, 5, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 17, 14, 20, 10, 9, 4, 7, 19, 8, 21⟩
	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22⟩
	⟨2, 11, 38, 15, 19, 7, 8, 21, 16, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 37, 3, 18, 5, 14, 17, 20, 10, 4, 9, 13, 23⟩
	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24⟩
	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 27, 33, 35, 24, 26, 25⟩
	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 25, 28, 27, 33, 34, 30, 29, 31, 32, 22, 36, 35, 24, 26⟩
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	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 25, 26, 24, 35, 36, 22, 32, 31, 29, 30, 34, 33, 27, 28⟩
	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 31, 32, 22, 36, 35, 24, 26, 25, 28, 27, 33, 34, 30, 29⟩
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	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 25, 26, 24, 35, 33, 27, 28, 29, 30, 34, 36, 22, 32, 31⟩
	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 31, 29, 30, 34, 33, 27, 28, 25, 26, 24, 35, 36, 22, 32⟩
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	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 31, 29, 30, 32, 22, 36, 35, 24, 26, 25, 28, 27, 33, 34⟩
	⟨2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3, 37, 12, 1, 31, 29, 30, 32, 22, 36, 34, 33, 27, 28, 25, 26, 24, 35⟩
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	⟨2, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 37, 3, 18, 5, 23, 13, 38⟩

$R-(12,37)$
⟨37, 22, 36, 34, 30, 32, 31, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 7, 19, 15, 11, 38, 13, 9, 4, 10, 20, 17, 14, 5, 23, 18, 3, 2, 12, 1⟩
⟨37, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 2⟩
⟨37, 22, 36, 34, 30, 32, 31, 29, 28, 27, 33, 35, 24, 26, 25, 1, 12, 2, 11, 15, 38, 13, 9, 10, 4, 7, 19, 8, 21, 6, 16, 17, 20, 14, 5, 23, 18, 3⟩
⟨37, 3, 18, 5, 23, 13, 9, 10, 20, 14, 17, 16, 21, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 2, 11, 38, 15, 19, 8, 7, 4⟩
⟨37, 3, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19, 15, 11, 38, 13, 23, 18, 5⟩
⟨37, 22, 36, 34, 30, 32, 31, 29, 28, 27, 33, 35, 24, 26, 25, 1, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6⟩
⟨37, 22, 36, 34, 30, 32, 31, 29, 28, 27, 33, 35, 24, 26, 25, 1, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 21, 6, 16, 17, 14, 20, 10, 9, 4, 7⟩
⟨37, 22, 36, 34, 30, 32, 31, 29, 28, 27, 33, 35, 24, 26, 25, 1, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 7, 4, 9, 10, 20, 14, 17, 16, 6, 21, 8⟩
⟨37, 22, 36, 34, 30, 32, 31, 29, 28, 27, 33, 35, 24, 26, 25, 1, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 7, 8, 21, 6, 16, 17, 14, 20, 10, 4, 9⟩
⟨37, 3, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 20, 14, 5, 18, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10⟩
⟨37, 3, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7, 4, 9, 10, 20, 17, 14, 5, 18, 23, 13, 38, 15, 11⟩
⟨37, 3, 18, 23, 5, 14, 20, 17, 16, 21, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 2, 11, 38, 15, 19, 8, 7, 4, 10, 9, 13⟩
⟨37, 3, 18, 5, 23, 13, 38, 15, 11, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7, 4, 9, 10, 20, 17, 14⟩
⟨37, 3, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7, 4, 9, 10, 20, 17, 14, 5, 18, 23, 13, 38, 11, 15⟩
⟨37, 3, 18, 5, 23, 13, 38, 15, 11, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 8, 19, 7, 4, 9, 10, 20, 14, 17, 16⟩
⟨37, 3, 18, 5, 23, 13, 38, 15, 11, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 19, 7, 4, 9, 10, 20, 14, 17⟩
⟨37, 3, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 7, 19, 15, 11, 38, 13, 9, 4, 10, 20, 17, 14, 5, 23, 18⟩
⟨37, 3, 18, 5, 23, 13, 38, 15, 11, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20, 10, 9, 4, 7, 8, 19⟩
⟨37, 3, 18, 5, 23, 13, 9, 10, 4, 7, 8, 19, 15, 38, 11, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 21, 16, 17, 14, 20⟩
⟨37, 3, 18, 5, 23, 13, 9, 4, 10, 20, 14, 17, 16, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36, 22, 32, 31, 1, 12, 2, 11, 38, 15, 19, 7, 8, 21⟩
⟨37, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 15, 11, 2, 12, 1, 25, 26, 24, 35, 33, 27, 28, 29, 31, 32, 30, 34, 36, 22⟩
⟨37, 3, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 7, 19, 15, 11, 38, 13, 9, 4, 10, 20, 17, 14, 5, 18, 23⟩
⟨37, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 15, 11, 2, 12, 1, 31, 32, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24⟩

$R-(12,37)$
$\langle 37, 22, 36, 34, 33, 35, 24, 6, 16, 21, 8, 7, 19, 15, 11, 38, 13, 9, 4, 10, 20, 17, 14, 5, 23, 18, 3, 2, 12, 1, 31, 32, 30, 29, 28, 27, 26, 25\rangle$
$\langle 37, 22, 32, 30, 29, 31, 1, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6, 24, 35, 36, 34, 33, 27, 28, 25, 26\rangle$
$\langle 37, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 15, 11, 2, 12, 1, 31, 29, 30, 32, 22, 36, 34, 33, 35, 24, 26, 25, 28, 27\rangle$
$\langle 37, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 15, 11, 2, 12, 1, 25, 26, 24, 35, 36, 22, 32, 31, 29, 30, 34, 33, 27, 28\rangle$
$\langle 37, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 15, 11, 2, 12, 1, 31, 32, 22, 36, 35, 24, 26, 25, 28, 27, 33, 34, 30, 29\rangle$
$\langle 37, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 15, 11, 2, 12, 1, 25, 26, 24, 35, 36, 22, 32, 31, 29, 28, 27, 33, 34, 30\rangle$
$\langle 37, 22, 36, 34, 33, 35, 24, 6, 16, 21, 8, 7, 19, 15, 11, 38, 13, 9, 4, 10, 20, 17, 14, 5, 23, 18, 3, 2, 12, 1, 25, 26, 27, 28, 29, 30, 32, 31\rangle$
$\langle 37, 22, 36, 34, 30, 29, 28, 25, 26, 27, 33, 35, 24, 6, 16, 21, 8, 7, 19, 15, 11, 38, 13, 9, 4, 10, 20, 17, 14, 5, 23, 18, 3, 2, 12, 1, 31, 32\rangle$
$\langle 37, 22, 32, 31, 1, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6, 24, 35, 36, 34, 30, 29, 28, 25, 26, 27, 33\rangle$
$\langle 37, 3, 18, 23, 5, 14, 20, 17, 16, 6, 21, 8, 19, 7, 4, 10, 9, 13, 38, 15, 11, 2, 12, 1, 31, 29, 30, 32, 22, 36, 35, 24, 26, 25, 28, 27, 33, 34\rangle$
$\langle 37, 22, 36, 34, 30, 32, 31, 29, 28, 25, 1, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6, 24, 26, 27, 33, 35\rangle$
$\langle 37, 22, 32, 31, 1, 12, 2, 3, 18, 5, 23, 13, 38, 11, 15, 19, 8, 7, 4, 9, 10, 20, 14, 17, 16, 21, 6, 24, 35, 33, 27, 26, 25, 28, 29, 30, 34, 36\rangle$
$\langle 37, 22, 36, 34, 30, 32, 31, 29, 28, 27, 33, 35, 24, 26, 25, 1, 12, 2, 3, 18, 5, 23, 13, 9, 4, 10, 20, 14, 17, 16, 6, 21, 8, 7, 19, 15, 11, 38\rangle$

Thus, e_{17} is a nice vertex in R. □

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