AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **47**, 072704 (2006); doi: 10.1063/1.2218674
View online: http://dx.doi.org/10.1063/1.2218674
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/47/7?ver=pdfcov
Published by the AIP Publishing

Eigenvalue problems and their application to the wavelet method of chaotic control

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(Received 5 January 2006; accepted 5 June 2006; published online 31 July 2006)

Controlling chaos via wavelet transform was recently proposed by Wei, Zhan, and Lai [Phys. Rev. Lett. **89**, 284103 (2002)]. It was reported there that by modifying a tiny fraction of the wavelet subspace of a coupling matrix, the transverse stability of the synchronous manifold of a coupled chaotic system could be dramatically enhanced. The stability of chaotic synchronization is actually controlled by the second largest eigenvalue $\lambda_1(\alpha,\beta)$ of the (wavelet) transformed coupling matrix $C(\alpha,\beta)$ for each α and β. Here β is a mixed boundary constant and α is a scalar factor. In particular, $\beta=1$ (respectively, 0) gives the nearest neighbor coupling with periodic (respectively, Neumann) boundary conditions. The first, rigorous work to understand the eigenvalues of $C(\alpha,1)$ was provided by Shieh *et al.* [J. Math. Phys. (to be published)]. The purpose of this paper is twofold. First, we apply a different approach to obtain the explicit formulas for the eigenvalues of $C(\alpha,1)$ and $C(\alpha,0)$. This, in turn, yields some new information concerning $\lambda_1(\alpha,1)$. Second, we shed some light on the question whether the wavelet method works for general coupling schemes. In particular, we show that the wavelet method is also good for the nearest neighbor coupling with Neumann boundary conditions. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2218674]

I. INTRODUCTION

Chaotic synchronization (Refs. 1, 8, 12–14, and references cited therein) is a fundamental phenomenon in physical systems with dissipation. It was first observed in Ref. 8 for identical master-slave Lorenz equations. This phenomenon was later observed in many different fields—physics, electrical engineering, biology, laser systems, etc. Experimental observations show that chaotic subsystems in a lattice manifest synchronized chaotic behavior in time provided they are coupled with a dissipative coupling and its coupling strength is greater than some critical value. Specifically, let there be N nodes (oscillators). Assume $\mathbf{u}_i$ is the m-dimensional vector of dynamical variables of the ith node. Let the isolated (uncoupling) dynamics be $\dot{\mathbf{u}}_i=f(\mathbf{u}_i)$ for each node. We assume that $\mathbf{u}_i$ has a chaotic dynamics in the sense that its largest Lyapunov exponent is positive. Let h: $\mathbb{R}^m\to\mathbb{R}^m$ be an arbitrary function describing the coupling within the components of each node. Thus, the dynamics of the ith node are

$$\dot{\mathbf{u}}_i = f(\mathbf{u}_i) + \epsilon\sum_{j=1}^{N} a_{ij}h(\mathbf{u}_j), \quad i = 1,2, \ldots ,N, \tag{1.1a}$$

where ϵ is a coupling strength. Here sum $\Sigma_{j=1}^{N}a_{ij}=0$. Let $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_N)^T$, $F(\mathbf{u})=(f(\mathbf{u}_1),f(\mathbf{u}_2),\ldots,f(\mathbf{u}_N))^T$, $H(\mathbf{u})=(h(\mathbf{u}_1),h(\mathbf{u}_2),\ldots,h(\mathbf{u}_N))^T$, and $A=(a_{ij})$. We may write (1.1a) as

$$\dot{\mathbf{u}} = F(\mathbf{u}) + \epsilon A \times H(\mathbf{u}). \tag{1.1b}$$

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Here $\times$ is the direct product of two matrices B and C defined as follows. Let $B=(b_{ij})_{k_1\times k_2}$ be a $k_1\times k_2$ matrix and $C=(C_{ij})_{k_2\times k_3}$ be a $k_2\times k_3$ block matrix, where each of C_{ij}, $1\leqslant i\leqslant k_2$, $1\leqslant j\leqslant k_3$, is a $k_4\times k_5$ matrix. Then

$$B\times C=\left(\sum_{l=1}^{k_2} b_{il}C_{lj}\right)_{k_1\times k_3}.$$

Many coupling schemes are covered by Eq. (1.1b). For example, if the Lorenz system is used and the coupling is through its three components x, y, and z, then the function h is just the matrix

$$I_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}. \tag{1.2}$$

The choice of A will provide the connectivity of nodes. For instance, the nearest neighbor coupling with mixed boundary conditions is given as follows:

$$A=A(\beta)=\begin{pmatrix}-1-\beta&1&0&\cdots&\cdots&\beta\\1&-2&1&\cdots&\cdots&0\\0&1&-2&1&\cdots&0\\\vdots&\ddots&\ddots&\ddots&&\vdots\\\vdots&&\ddots&\ddots&\ddots&\vdots\\\beta&0&\cdots&\cdots&1&-1-\beta\end{pmatrix}_{N\times N}. \tag{1.3}$$

Note that $\beta=1$ corresponds to periodic boundary conditions, while $\beta=0$ is associated with Neumann boundary conditions. The synchronous manifold of the chaotic system (1.1) can be studied by setting $\mathbf{u}_1(t)=\mathbf{u}_2(t)=\cdots=\mathbf{u}_N(t)=\mathbf{s}(t)$. Here the chaotic solution $\mathbf{s}(t)$ satisfies the single oscillator equation $\mathrm{d}\mathbf{s}/\mathrm{d}t=f[\mathbf{s}(t)]$. The stability property of the synchronous manifold can then be studied in the space of difference variables $\delta\mathbf{u}_i(t)=\mathbf{u}_i(t)-\mathbf{s}(t)$, which are governed by[7,10]

$$\frac{\mathrm{d}\,\delta\mathbf{u}}{\mathrm{d}t}=(I_N\times DF+\epsilon A\times DH)\,\delta\mathbf{u}, \tag{1.4a}$$

where $DH=\mathrm{d}H(\mathbf{u})/\mathrm{d}\mathbf{u}$, and $\delta\mathbf{u}=(\delta\mathbf{u}_1,\delta\mathbf{u}_2,\ldots,\delta\mathbf{u}_N)^T$. When H is just a matrix E, $DH=E$. The first term in (1.4a) is block diagonal. The second term can be treated by diagonalizing A. The transformation which does this does not affect the first term, since it acts only on the identity matrix I_N. This leaves us with a block diagonalized variational equation with each block having the form[7]

$$\delta\dot{\mathbf{u}}_i=[DF+\epsilon\lambda_i DH]\,\delta\mathbf{u}_i, \tag{1.4b}$$

where λ_i is an eigenvalue of A, $i=0,1,\cdots,N-1$. The Jacobian functions DF and DH are the same for each block, since they are evaluated on the synchronized state. It then follows from (1.4b) that the largest eigenvalue λ_0 of A being equal to 0 governs the motion on the synchronized manifold, and all of other eigenvalues $\lambda_i(i\neq 0)$ control the transverse stability[9] of the chaotic synchronous state. The stability condition is then given by $L_{\max}+\epsilon\lambda_1\leqslant 0$, where $L_{\max}>0$ is the largest Lyapunov exponent of a single chaotic oscillator. As a consequence, the second largest eigenvalue λ_1 is dominant in controlling the stability of chaotic synchronization, and the critical strength ϵ_c can be determined in term of λ_1,

$$\epsilon_c=\frac{I_{\max}}{-\lambda_1}. \tag{1.4c}$$

Note that the eigenvalues of $A=A(1)$ are given by $\lambda_i=-4\sin^2(\pi i/N)$, $i=0,1,\ldots,N-1$. In general, a larger number of nodes gives a smaller nonzero eigenvalue λ_1 in magnitude, and, hence, a larger ϵ_c. As a consequence, controlling chaos is apparently of great interest and importance.[4–7,11,12] In Ref. 11, a new efficient strategy for controlling nonlinear dynamics was presented. To be self-contained, we briefly describe such procedures. Let

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}_{n\times n}, \tag{1.5a}$$

be a matrix with the dimension of each block matrix A_{kl} being $2^i \times 2^i$. By an i-scale wavelet operator W,[2,11] the matrix A is transformed into $W(A)$ of the form

$$W(A) = \begin{pmatrix} \tilde{A}_{11} & \cdots & \tilde{A}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{n1} & \cdots & \tilde{A}_{nn} \end{pmatrix}_{n\times n}, \tag{1.5b}$$

where each entry of $\tilde{A}_{kl}$ is the average of entries of A_{kl}, $1 \leq k, l \leq n$.

For a given matrix, the above wavelet transform allows a perfect reconstruction (inverse wavelet transform), by which there is nothing to gain: $A = W^{-1}(W(A))$. In Ref. 11, a simple operator O_k is introduced to attain a desirable coupling matrix. That is,

$$C = W^{-1}(O_k(W(A))) = A + (k-1)W(A) =: A + \alpha W(A), \tag{1.5c}$$

where O_k is the multiplication of a scalar factor K on each block matrix $\tilde{A}_{kl}$. After such reconstruction, the critical strength ϵ_c is again determined in terms of the second largest eigenvalue of C. A numerical simulation of a coupled system of 512 Lorenz oscillators in Ref. 11 shows that with $h = I_3$ and $A = A(1)$, the critical coupling strength ϵ_c decreases linearly with respect to the increase of α up to a critical value α_c. The smallest ϵ_c is about 6, which is about 10^3 times smaller than the original critical coupling strength, indicating the efficiency of the proposed approach.

To verify this phenomenon mathematically, we first consider the coupling matrix $A = A(\beta)$, as given in (1.3). Let $n = N/2^i \in \mathbb{N}$, where i is a fixed positive integer. We then write A into an $n \times n$ block matrix of the form

$$A = A(\beta) = \begin{pmatrix} A_1(\beta) & A_2(1) & 0 & . & . & 0 & A_2^T(\beta) \\ A_2^T(1) & A_1(1) & A_2(1) & . & . & 0 & 0 \\ 0 & . & . & . & & & . \\ . & & . & . & . & & . \\ . & & & . & . & . & 0 \\ 0 & 0 & . & . & A_2^T(1) & A_1(1) & A_2(1) \\ A_2(\beta) & 0 & . & . & 0 & A_2^T(1) & \bar{A}_1(\beta) \end{pmatrix}_{n\times n}, \tag{1.6a}$$

where

$$A_1(\beta) = \begin{pmatrix} -1-\beta & 1 & & & & \\ 1 & -2 & 1 & & \mathbf{0} & \\ & . & . & . & & \\ & & . & . & . & \\ & \mathbf{0} & & . & . & 1 \\ & & & & 1 & -2 \end{pmatrix}_{2^i\times 2^i},$$

$$\bar{A}_1(\beta)=\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & \mathbf{0} & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & \cdot & 1 \\ & & & & 1 & -1-\beta \end{pmatrix}_{2^i\times 2^i}, \tag{1.6b}$$

and

$$A_2(\beta)=\begin{pmatrix} 0 & . & . & . & 0 \\ . & & & & . \\ . & & & & . \\ 0 & & & & 0 \\ \beta & 0 & . & . & 0 \end{pmatrix}_{2^i\times 2^i}. \tag{1.6c}$$

Then the newly transformed coupling matrix $C=C(\alpha,\beta)$ can be written as

$$C(\alpha,\beta)$$

$$=\begin{pmatrix} A_1(\beta)+\tilde{A}_1(\beta) & A_2(1)+\tilde{A}_2(1) & 0 & \cdots & 0 & A_2^T(\beta)+\tilde{A}_2^T(\beta) \\ A_2^T(1)+\tilde{A}_2^T(1) & A_1(1)+\tilde{A}_1(1) & A_2(1)+\tilde{A}_2(1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_2^T(1)+\tilde{A}_2^T(1) & A_1(1)+\tilde{A}_1(1) & A_2(1)+\tilde{A}_2(1) \\ A_2(\beta)+\tilde{A}_2(\beta) & 0 & \cdots & 0 & A_2^T(1)+\tilde{A}_2^T(1) & \tilde{A}_1(\beta)+\tilde{\bar{A}}_1(\beta) \end{pmatrix}$$

$$=:\begin{pmatrix} C_1(\beta) & C_2(1) & 0 & \cdots & 0 & C_2^T(\beta) \\ C_2^T(1) & C_1(1) & C_2(1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & C_2^T(1) & C_1(1) & C_2(1) \\ C_2(\beta) & 0 & \cdots & 0 & C_2^T(1) & \bar{C}_1(\beta) \end{pmatrix}. \tag{1.7}$$

Here for any matrix B of dimension $2^i\times 2^i$, the kl entry $(\tilde{B})_{kl}$ of $\tilde{B}$ is defined to be

$$(\tilde{B})_{kl}=\frac{\alpha}{2^{2i}}\sum_{l=1}^{2^i}\sum_{k=1}^{2^i}(B)_{kl}.$$

Here α is a scalar factor. The matrix $C(\alpha,\beta)$ carries a new relationship among the coupled oscillators, which might not be as simple as the original matrix A. Nevertheless, the stability of the synchronous states can be determined by matrix $C(\alpha,\beta)$, whose eigenvalues $\lambda_i(\alpha,\beta)$ ($i=0,1,2,\cdots,N-1$) determine the synchronous stability of the coupled chaotic system. The following theorem of Shieh *et al.*[9] showed, indeed, the dramatic reduction in the critical coupling strength can be achieved with the periodic boundary conditions. We summarize their main results in the following.

Theorem 1.1: *Let $N\times N$, $N=8k$, $k\in N$, be the dimension of the matrix $C(\alpha,1)$. Let the dimension of each block matrix in $C(\alpha,1)$ be $2^i\times 2^i$. Then the following assertions hold.*

(i) $\rho_i:=2\cos(\pi/2^i)-2$ *is an eigenvalue of* $C(\alpha,1)$.
(ii) *The second eigenvalue* $\lambda_1(\alpha,1)$ *of* $C(\alpha,1)$ *is decreasing in* α. *Moreover,* $\lambda_1(\alpha,1)=\rho_i$ *when-*

ever $\alpha \geqslant -2^i \rho_i / 4 \sin^2(2^i \pi / N)$.

Note that $C(\alpha,1)$ is a block circulant matrix (see e.g., Ref. 3). A classical result of a block circulant matrix states that its eigenvalues exactly consist of those of certain linear combinations of its block matrices (see, e.g., Theorem 5.6.4 of Ref. 3). The proof of Theorem 1.1 was then reduced to working on the eigenvalues of those linear combinations of block matrices of $C(\alpha,1)$. Note that $C(\alpha,\beta)$, $\beta \neq 1$ are not block circulant matrix. The objective of the present work is to present another approach to study the eigenvalues of $C(\alpha,\beta)$. Specifically, we use this new method to study two coupling schemes, the nearest neighbor coupling with periodic boundary conditions and the nearest neighbor coupling with Neumann boundary conditions. To simplify our calculation, we consider only the case $i=1$. In both coupling schemes, we are able to obtain, respectively, exact form of eigenvalues $\lambda_m^{\pm}(\alpha,\beta)$ of its corresponding matrix $C(\alpha,\beta)$, see (2.16) and (3.9). Here $\beta=0$ or 1. For each α and β, let $\lambda_1(\alpha,\beta)$ be the second largest eigenvalue of $C(\alpha,\beta)$. We prove that for N being a multiple of 4, then

$$\lambda_1(\alpha,1) = \begin{cases} \lambda_1^{+}(\alpha,1), & 0 \leqslant \alpha \leqslant \dfrac{1}{\sin^2 \dfrac{z}{n}} \\ \lambda_{n/2}^{+}(\alpha,1) = -2, & \alpha \geqslant \dfrac{1}{\sin^2 \dfrac{\pi}{n}}. \end{cases}$$

Let $N=2n$ be an even number which is not multiple of 4. We show that $\lambda_1(\alpha,1)=\lambda_{[n/2]}^{+}(\alpha,1)$ for α sufficiently large, where $[n/2]$=the largest positive integer that is less than or equal to $n/2$. Moreover, we prove that for such N that $\lambda_1(\alpha,1) < -2$, whenever $\alpha > 1/\sin^2(\pi/n)$. With those results above, we get considerably more information than those obtained in Ref. 9. Among other, such result suggests that if the number N of oscillators is even but not a multiple of 4, then the wavelet method works even better. Specifically, it is better in the sense that the corresponding second largest eigenvalue $\lambda_1(\alpha,1)$ is further away from 0, and, hence, gives even smaller critical length. Our second main result is concerned with $\lambda_1(\alpha,0)$ of $C(\alpha,0)$, which corresponds to the nearest neighbor coupling with Neumann boundary conditions. We show that for all even number N its second largest eigenvalue $\lambda_1(\alpha,0)$ for each α behaves like its periodic counterpart for which its corresponding N is a multiple of 4.

II. PERIODIC BOUNDARY CONDITIONS

Here, we consider the nearest neighbor coupling with periodic boundary conditions. The resulting coupling matrix $A(1)$ is given as in (1.6). Let the dimension of $A_1(1)$, $A_2(1)$, and $\bar{A}_1(1)$ be 2×2. Then

$$A_1(1) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = \bar{A}_1(1), \quad A_2(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.1a}$$

$$\tilde{A}_1(1) = \alpha \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & -\dfrac{1}{2} \end{pmatrix} = \bar{\tilde{A}}_1(1), \quad \tilde{A}_2(1) = \alpha \begin{pmatrix} \dfrac{1}{4} & \dfrac{1}{4} \\ \dfrac{1}{4} & \dfrac{1}{4} \end{pmatrix}. \tag{2.1b}$$

Then $C_i(1)=A_i(1)+\tilde{A}_i(1)$, i=1,2, $\bar{C}_1(1)=\bar{A}_1(1)+\bar{\tilde{A}}_1(1)$. Thus,

$$C_1(1)=\begin{pmatrix}-\frac{1}{2}(4+\alpha) & \frac{1}{2}(2-\alpha)\\ \frac{1}{2}(2-\alpha) & -\frac{1}{2}(4+\alpha)\end{pmatrix}=\bar{C}_1(1),\quad C_2(1)=\begin{pmatrix}\frac{\alpha}{4} & \frac{\alpha}{4}\\ \frac{1}{4}(4+\alpha) & \frac{\alpha}{4}\end{pmatrix}. \tag{2.1c}$$

We begin by identifying some trivial eigenvalues of $C(\alpha,1)$.

Proposition 2.1: For each α, 0 *and* -4 *are eigenvalues of* $C(\alpha,1)$. *If, in addition*, $n/2(>1)$ *is a positive integer, then* -2 *is also an eigenvalue of* $C(\alpha,1)$ *for any* α.

Proof: Let $C(\alpha,1)+4I=(\mathbf{c}_1,\mathbf{c}_2,\ldots,\mathbf{c}_N)$, where $\mathbf{c}_i$, $1\le i\le N$, are column vectors. Then $\Sigma_{j=1}^{N}(-1)^{j+1}\mathbf{c}_j=0$. Thus -4 is an eigenvalue of $C(\alpha,1)$ for each $\alpha>0$. Let $C(\alpha,1)+2I=(\mathbf{c}_1,\mathbf{c}_2,\ldots,\mathbf{c}_N)$. If $N=2n(>4)$ is a multiple of four, then $\Sigma_{j=1}^{N}\delta(j)\mathbf{c}_j=0$, where

$$\delta(j)=\begin{cases}1 & \textit{if } j=4k \textit{ or } 4k+1 \textit{ for some } k\\ -1 & \textit{if } j=4k+2 \textit{ or } 4k+3 \textit{ for some } k.\end{cases}$$

Thus, -2 is an eigenvalue of $C(\alpha,1)$ for each α with such N. □

Writing the corresponding eigenvalue problem $C(\alpha,1)\mathbf{b}=\lambda\mathbf{b}$, where $\mathbf{b}=(\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_n)^T$ and $\mathbf{b}_i\in\mathbb{C}^2$, in block component form, we have

$$C_2^T(1)\mathbf{b}_{i-1}+C_1(1)\mathbf{b}_i+C_2(1)\mathbf{b}_{i+1}=\lambda\mathbf{b}_i,\quad 1\le i\le n. \tag{2.2a}$$

Periodic boundary conditions would yield that

$$C_2^T(1)\mathbf{b}_0+C_1(1)\mathbf{b}_1+C_2(1)\mathbf{b}_2=\lambda\mathbf{b}_1=C_1(1)\mathbf{b}_1+C_2(1)\mathbf{b}_2+C_2^T(1)\mathbf{b}_n$$

and

$$C_2^T(1)\mathbf{b}_{n-1}+C_1(1)\mathbf{b}_n+C_2(1)\mathbf{b}_{n+1}=\lambda\mathbf{b}_n=C_2(1)\mathbf{b}_1+C_2^T(1)\mathbf{b}_{n-1}+\bar{C}_1(1)\mathbf{b}_n,$$

or, equivalently,

$$\mathbf{b}_0=\mathbf{b}_n, \tag{2.2b}$$

$$\mathbf{b}_1=\mathbf{b}_{n+1}. \tag{2.2c}$$

To study the block difference equation (2.2), we first seek to find the solution $\mathbf{b}_i$ of the form

$$\mathbf{b}_i=\delta^i\begin{pmatrix}1\\ \nu\end{pmatrix}. \tag{2.3}$$

Substituting (2.3) into (2.2a), we get

$$[C_2^T(1)+\delta(C_1(1)-\lambda I)+\delta^2C_2(1)]\begin{pmatrix}1\\ \nu\end{pmatrix}=0. \tag{2.4}$$

To have a nontrivial solution $\binom{1}{\nu}$ to Eq. (2.4), we need to have

$$\det[C_2^T(1)+\delta(C_1(1)-\lambda I)+\delta^2C_2(1)]=0, \tag{2.5a}$$

or, equivalently,

$$\alpha\delta^4+(4\alpha+4+2\alpha\lambda)\delta^3-(8+10\alpha+16\lambda+4\alpha\lambda+4\lambda^2)\delta^2+(4\alpha+4+2\alpha\lambda)\delta+\alpha=0. \tag{2.5b}$$

Equation (2.5b) is to be called the characteristic equation of the block difference equation (2.2a). To study the property of Eq. (2.5b), we need the following proposition.

Proposition 2.2: Let D_1, D_2, and D_3 *be* 2×2 *matrices. Suppose* $D_1=D_3^T$ *and* $D_2=D_2^T$. *Let* x_1,

x_2, x_3, and x_4 be roots of $\det[D_1+xD_2+x^2D_3]=0$, *where $x\in\mathbb{C}$. Then we may renumber the subscripts if necessary so that*

$$x_1x_2 = 1 = x_3x_4. \tag{2.6a}$$

If, in addition, diagonal elements of D_1 and D_2, respectively, are both equal, then

$$y_1y_2 = 1 = y_3y_4. \tag{2.6b}$$

Here $\binom{1}{y_i}$, $i=1,2,3,4$, *are vectors satisfying*

$$[D_1 + x_iD_2 + x_i^2D_3]\begin{pmatrix}1\\ y_i\end{pmatrix} = 0. \tag{2.6c}$$

Proof: If D_1, D_2, and D_3 are as assumed, then

$$\det[D_1 + xD_2 + x^2D_3] = ax^4 + bx^3 + cx^2 + bx + a \tag{2.7}$$

for some constants $a\neq 0$, b, and c. Letting $y=x+1/x$, then (2.7) can be written as $\alpha y^2+\beta y+\gamma$, where α, β, and γ depend on the constants a, b, and c. Thus $\det[D_1+xD_2+x^2D_3]=0$ is equivalent to $x^2-\lambda_\pm x+1=0$, where $\lambda_\pm$ are the roots $a_1y^2+b_1y+c_1=0$. Consequently, $x_1x_2=1=x_3x_4$.

Letting $D_1=\begin{pmatrix}a_1 & b_1\\ c_1 & a_1\end{pmatrix}=D_3^T$ and $D_2=\begin{pmatrix}a_2 & b_2\\ b_2 & a_2\end{pmatrix}$, we write (2.6c) in component form,

$$(a_1 + y_ib_1) + (a_2 + y_ib_2)x_i + (a_1 + y_ic_1)x_i^2 = 0, \quad i = 1,2,3,4, \tag{2.8a}$$

$$(c_1 + y_ia_1) + (b_2 + y_ia_2)x_i + (b_1 + y_ia_1)x_i^2 = 0, \quad i = 1,2,3,4, \tag{2.8b}$$

For $i=1$, (2.8a) is equal to

$$(a_1 + y_1b_1) + (a_2 + y_1b_2)\frac{1}{x_2} + (a_1 + y_1c_1)\frac{1}{x_2^2} = 0$$

or

$$(a_1 + y_1c_1) + (a_2 + y_1b_2)x_2 + (a_1 + y_1b_1)x_2^2 = 0$$

or

$$\left(c_1 + \frac{1}{y_1}a_1\right) + \left(b_2 + \frac{1}{y_1}a_2\right)x_2 + \left(b_1 + \frac{1}{y_1}a_1\right)x_2^2 = 0. \tag{2.8c}$$

Using Eqs. (2.8c) and (2.8b) with $i=2$, and the uniqueness of y_i, $i=1,2,3,4$, we conclude that $y_1y_2=1$. Similarly, $y_3y_4=1$. We just complete the proof of the proposition. □

We are now in a position to further study Eq. (2.5). We assume, momentarily, that Eq. (2.5) has four distinct roots δ_1, δ_2, δ_3, and δ_4. The general solutions to (2.2a) can then be written as

$$\mathbf{b}_i = c_1\delta_1^i\begin{pmatrix}1\\ \nu_1\end{pmatrix} + c_2\delta_2^i\begin{pmatrix}1\\ \nu_2\end{pmatrix} + c_3\delta_3^i\begin{pmatrix}1\\ \nu_3\end{pmatrix} + c_4\delta_4^i\begin{pmatrix}1\\ \nu_4\end{pmatrix}. \tag{2.9}$$

Here ν_i, $i=1,2,3,4$, are some constants depending on δ_i.

Applying (2.9) to boundary conditions (2.2b) and (2.2c), we get

$$c_1(\delta_1^n - 1)\begin{pmatrix}1\\ \nu_1\end{pmatrix} + c_2(\delta_2^n - 1)\begin{pmatrix}1\\ \nu_2\end{pmatrix} + c_3(\delta_3^n - 1)\begin{pmatrix}1\\ \nu_3\end{pmatrix} + c_4(\delta_4^n - 1)\begin{pmatrix}1\\ \nu_4\end{pmatrix} = 0 \tag{2.10a}$$

and

$$c_1\delta_1(\delta_1^n-1)\begin{pmatrix}1\\ \nu_1\end{pmatrix}+c_2\delta_2(\delta_2^n-1)\begin{pmatrix}1\\ \nu_2\end{pmatrix}+c_3\delta_3(\delta_3^n-1)\begin{pmatrix}1\\ \nu_3\end{pmatrix}+c_4\delta_4(\delta_4^n-1)\begin{pmatrix}1\\ \nu_4\end{pmatrix}=0. \tag{2.10b}$$

Writing (2.10) in matrix form, we have

$$\begin{pmatrix}1 & 1 & 1 & 1\\ \nu_1 & \nu_2 & \nu_3 & \nu_4\\ \delta_1 & \delta_2 & \delta_3 & \delta_4\\ \nu_1\delta_1 & \nu_2\delta_2 & \nu_3\delta_3 & \nu_4\delta_4\end{pmatrix}\operatorname{diag}(\delta_1^n-1,\delta_2^n-1,\delta_3^n-1,\delta_4^n-1)\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix}=0. \tag{2.11}$$

Now if, $\operatorname{diag}(\delta_1^n-1,\delta_2^n-1,\delta_3^n-1,\delta_4^n-1)$ is singular, then Eq. (2.9) has nontrivial solutions c_i, $i=1,2,3,4$. Note that $\operatorname{diag}(\delta_1^n-1,\delta_2^n-1,\delta_3^n-1,\delta_4^n-1)$ is singular if and only if δ_i, $i=1,2,3,4$, satisfy

$$\delta^n=1 \tag{2.12}$$

and (2.5b). To solve the system of equations (2.12) and (2.5b), we first note that

$$\delta_m=e^{i2m\pi/n},\quad 0\leqslant m\leqslant n-1, \tag{2.13}$$

are roots of Eq. (2.12). Substituting (2.13) and (2.5b), we get that the imaginary part of the resulting equation is

$$\left[-4\sin\frac{4m\pi}{n}\right]\lambda^2+\left[2\alpha\sin\frac{6m\pi}{n}-(4\alpha+16)\sin\frac{4m\pi}{n}+2\alpha\sin\frac{2m\pi}{n}\right]\lambda$$
$$+\left[\alpha\sin\frac{8m\pi}{n}+4(1+\alpha)\sin\frac{6m\pi}{n}-(8+10\alpha)\sin\frac{4m\pi}{n}+4(1+\alpha)\sin\frac{2m\pi}{n}\right]=0. \tag{2.14}$$

Before we proceed to compute the real part of the resulting equation, we need the following lemma.

Lemma 2.1: Let a, b, and c be any complex number, then

$$\cos 2\theta(\sin 4\theta+a\sin 3\theta+b\sin 2\theta+a\sin\theta)=\sin 2\theta(\cos 4\theta+a\cos 3\theta+b\cos 2\theta+a\cos\theta+1). \tag{2.15}$$

Since the proof of the lemma is straightforward, we will skip it.

Using (2.14) and (2.15), we see immediately that the real part of (2.5b) with $\delta=e^{i2m\pi/n}$ is a constant multiple $\sin/\cos(4m\pi/n)/(4m\pi/n)$ of its imaginary part. We next show that (2.14) is indeed the characteristic equation of the matrix $C(\alpha,1)$.

Theorem 2.1: *Let $N\times N$, $N=2k$, $k\in N$, be the dimension of the matrix $C(\alpha,1)$. Let dimension of each block matrix in $C(\alpha,1)$ be 2×2. Then the eigenvalues $\lambda_m^{\pm}(\alpha,1)$ of $C(\alpha,1)$ are of the following form:*

$$\lambda_m^{\pm}(\alpha,1)=\frac{1}{2}\left(\alpha\cos\frac{2m\pi}{n}-\alpha-4\right)\pm\frac{1}{2}\left[\left(\alpha\cos\frac{2m\pi}{n}-\alpha-4\right)^2+4\left(\alpha\cos^2\frac{2m\pi}{n}\right.\right.$$
$$\left.\left.+2(\alpha+1)\cos\frac{2m\pi}{n}-2-3\alpha\right)\right]^{1/2}=:\check{\lambda}_m(\alpha,1)\pm\hat{\lambda}_m(\alpha,1),\quad m=0,1,\ldots,n-1. \tag{2.16}$$

Proof: Solving (2.14), we get (2.16). Using Proposition 2.2, we see that if $\delta=1$ or -1 is a root of Eq. (2.5b), then the multiplicity of $\delta=1$ or -1 is both two. Thus, we have only proved the following. (i) If $n/2$ is not a positive integer, then for each α, $\lambda_m^{\pm}(\alpha,1)$, $m=1,2,\ldots,n-1$, are eigenvalues of $C(\alpha,1)$. (ii) If $n/2$ is a positive integer, then for each α, $\lambda_m^{\pm}(\alpha,1)$, $m=1,2,\ldots,n/2-1,n/2+1,\ldots,n-1$, are eigenvalues of $C(\alpha,1)$. To complete the proof of the theorem, it remains to show that for each α, $\lambda_0^{\pm}(\alpha,1)(=0,-4)$ are eigenvalues of $C(\alpha,1)$ for each α and that if, additionally, $n/2>1$ is a positive integer, then for each α, $\lambda_{n/2}^{\pm}(\alpha,1)(=-2,-2-2\alpha)$ are also eigenvalues of $C(\alpha,1)$. Using Proposition 2.1, we only need to show that $-2-2\alpha=(\lambda_{n/2}^{-}(\alpha,1))$ is an eigenvalue of $C(\alpha,1)$ for fixed α. To this end, we see that

$$\textit{trace of } C(\alpha,1)=-n(\alpha+4). \tag{2.17}$$

Let $N=2n>4$ be a multiple of four, then

$$\lambda_{n/2}^{+}(\alpha,1)+\left(\sum_{j=1,j\neq n/2}^{n}\lambda_j^{\pm}(\alpha,1)\right)+\lambda_0^{\pm}(\alpha,1)=-2-(n-2)(\alpha+4)-4. \tag{2.18}$$

Using (2.17) and (2.18), we have that the remaining eigenvalue of $C(\alpha,1)$ for each α is $-2-2\alpha$, which is equal to $\lambda_{n/2}^{-}(\alpha,1)$. We thus complete the proof of the theorem. □

Proposition 2.3: For all $\alpha>0$, *we have that* $\hat{\lambda}_m(\alpha,1)>0$, $\check{\lambda}_m(\alpha,1)<0$ *and* $\lambda_m^{\pm}(\alpha,1)\leqslant 0$.

Proof: Obviously, $\check{\lambda}_m(\alpha,1)<0$. Now, letting $t=\cos(2m\pi/n)$, we have that

$$4(\hat{\lambda}_m(\alpha,1))^2=(t-1)^2\alpha^2+4(t^2-1)\alpha+8(1+t)=((t-1)\alpha+2(t+1))^2+4(1-t^2)>0$$

for any $\alpha>0$. Thus $\hat{\lambda}_m(\alpha,1)>0$. To prove the last assertion of the proposition, we note, via (2.16), that

$$0>4\left(\alpha\cos^2\frac{2m\pi}{n}+2(\alpha+1)\cos\frac{2m\pi}{n}-2-3\alpha\right)=: l.$$

Thus,

$$2\lambda_m^{\pm}(\alpha,1)=2\check{\lambda}_m(\alpha,1)\pm(4\check{\lambda}_m^2(\alpha,1)+l)^{1/2}\leqslant 0.$$

We just complete the proof of the proposition. □

Proposition 2.4: If $n/2$ *is not a positive integer, then the eigencurves* $\lambda_m^{\pm}(\alpha,1)$, $m=1,2,\ldots,n-1$, *are strictly decreasing in* $\alpha\in(0,\infty)$. *If* $n/2(>1)$ *is a positive integer, then* $\lambda_m^{\pm}(\alpha,1)$, $m=1,2,\ldots,n/2-1,n/2+1,\ldots,n-1$, *and* $\lambda_{n/2}^{-}(\alpha,1)$ *are strictly decreasing in* $\alpha\in(0,\infty)$.

Proof: Letting $t=\cos(2m\pi/n)$, we write (2.16) as

$$\begin{aligned}\lambda_m^{\pm}(\alpha,1)&=\tfrac{1}{2}\{\alpha(t-1)-4\pm[(t-1)^2\alpha^2+4(t^2-1)\alpha+8(1+t)]^{1/2}\}\\&=:\tfrac{1}{2}\{\alpha(t-1)-4\pm(t_\alpha)^{1/2}\}=:\lambda_t^{\pm}(\alpha).\end{aligned} \tag{2.19}$$

Then

$$2\frac{\mathrm{d}\lambda_m^{\pm}(\alpha,1)}{\mathrm{d}\alpha}=(t-1)\left(1\pm\frac{(t-1)\alpha+2(t+1)}{\sqrt{t_\alpha}}\right).$$

A direct computation would yield that

$$t_\alpha\geqslant((t-1)\alpha+2(t+1))^2.$$

Thus, $\mathrm{d}\lambda_m^{\pm}(\alpha,1)/\mathrm{d}\alpha\leqslant 0$. The equality holds only if $t=1$ or $t=-1$ for λ_m^{+}. □

Proposition 2.5: (i) In the $\alpha-\lambda$ *plane,* $\lambda_t^{\lambda}(\alpha,1)$ *interest with* $\lambda=-2+k$ *at* $\alpha_{t,k}$, *where*

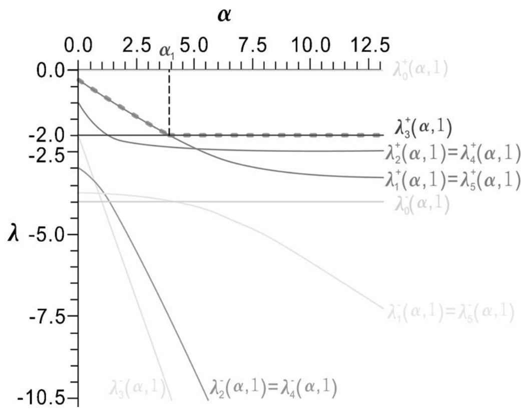

FIG. 1. The curves $\lambda_m^{\pm}(\alpha,1)$ with $N=2n=12$ are provided. As predicted in Theorem 2.2–(i), $\lambda(\alpha,1)$ turns flat after α_1.

$$\alpha_{t,k} = \frac{2(1+t)-k^2}{(1-t)(1+t+k)}. \tag{2.20}$$

(ii) For $-1 \leqslant t < 1$, $\lim_{\alpha\to\infty}\lambda_t^+(\alpha,1) = -(t+3)$.

Proof: Solving equation $-2+k=\lambda_t^+(\alpha,1)$, we easily get that $\alpha_{t,k}$ are as asserted. Rewriting $\lambda_t^+(\alpha,1)$ as

$$\lambda_t^+(\alpha,1) = \frac{-2\alpha(t-1)(t+3)+4(1-t)}{\alpha(t-1)-4-\sqrt{t_\alpha}},$$

we see that $\lim_{\alpha\to\infty}\lambda_t^+(\alpha,1) = -(t+3)$ for $-1 \leqslant t < 1$. □

Theorem 2.2: *Let N be any positive even integer. The dimension of each block matrix in* $C(\alpha,1)$ *is* 2×2. *Then (i) suppose N is a multiple of four and* $N>4$. *For each* $\alpha>0$, *let* $\lambda(\alpha,1)$ *be the second largest eigenvalue of* $C(\alpha,1)$. *Then* $\lambda(\alpha,1)=\lambda_1^+(\alpha,1)$, *for* $0 \leqslant \alpha \leqslant 1/\sin^2(\pi/n) \coloneqq \alpha_1$; *and* $\lambda(\alpha,1)=\lambda_{n/2}^+(\alpha,1)=-2$ *for all* $\alpha\in[\alpha_1,\infty)$. *See Fig. 1.*

(ii) Suppose N is not a multiple of four. Then there exists a $\tilde{\alpha}_c$ *such that* $\lambda(\alpha,1)=\lambda_{[n/2]}^+(\alpha)$ *for all* $\alpha \geqslant \tilde{\alpha}_c$. *Here* $[n/2]$ = *the largest positive integer that is less than or equal to* $n/2$. *Moreover*, $\lambda(\alpha,1) < -2$ *whenever* $\alpha>\alpha_1$. *See Fig. 2.*

Proof: For $\alpha_{t,k}$ to be positive, we must have

$$2(1+t) > k^2. \tag{2.21}$$

Now,

$$(1-t)^2(1+t+k)^2\frac{d\alpha_{t,k}}{dt} = 2(t+1)^2 - k^3 + 4k - 2tk^2 > (1+t)k^2 - k^3 + 4k - 2tk^2$$
$$= -k(k^2+(t-1)k-4) = -k(k-t_+)(k-t_-),$$

where $t_\pm = (1-t\pm\sqrt{16+(1-t)^2})/2$. Note that we have used (2.21) to justify the above inequality. Moreover $t_-<0$ and $t_+\geqslant 2$. Thus, $d\alpha_{t,k}/dt>0$ whenever $\lambda=-2+k$, $0\leqslant k<2$, and $\lambda=\lambda_t^+(\alpha,1)$ have the intersections intersect at the positive $\alpha_{t,k}$. Upon using Proposition 2.4, we conclude that for $0\leqslant m\leqslant n-1$, the portion of the graphs of $\lambda_m^+(\alpha,1)$ lying above the line $\lambda=-2$ do not intersect each other. Thus, $\lambda(\alpha,1)$ is as asserted.

By Proposition 2.5(ii), we have that

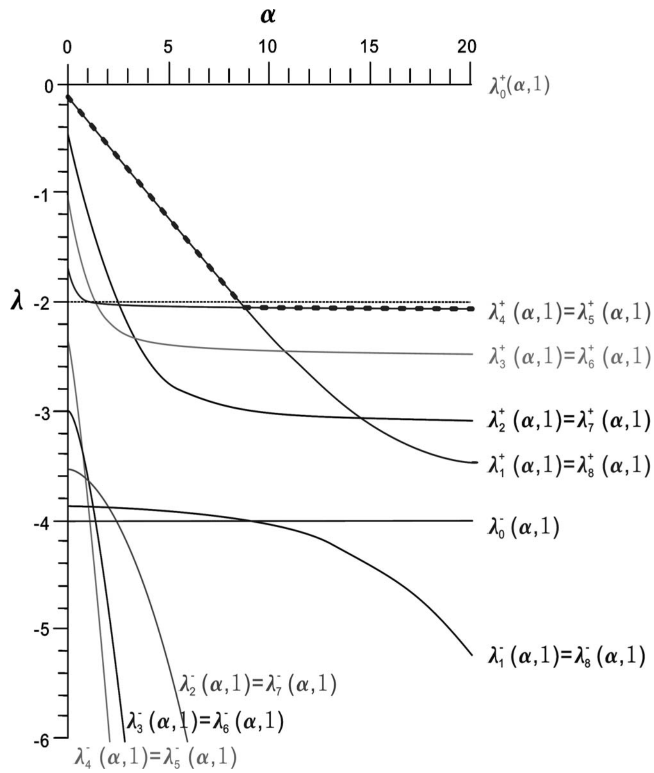

FIG. 2. The curves $\lambda_m^{\pm}(\alpha,1)$ with $N=2n=18$ are provided. As predicted in Theorem 2.2–(ii), $\lambda(\alpha,1)$ lies below -2 eventually.

$$\lim_{\alpha\to\infty} \lambda_m^+(\alpha,1) = -\left(\cos\frac{2m\pi}{n}+3\right) =: \lambda_m^\infty = \lambda_t^\infty .$$

Then $\lambda_m^\infty, 0<m\leqslant n-1$, have a maximum at $m=[n/2]$. Thus, there exists a $\tilde{\alpha}_c$ such that $\lambda(\alpha,1) = \lambda_{[n/2]}^+(\alpha,1)$ for all $\alpha \geqslant \tilde{\alpha}_c$. The last assertion of the theorem follows from Proposition 2.5-(i) and Proposition 2.1. □

Remark 2.1: (i) Since $\lambda_t^+(\alpha,1)$ is increasing in t and λ_t^∞ is decreasing in t, the eigencurves $\lambda_m^+(\alpha,1)$, $0<m\leqslant[n/2]$ must be crossing each other.

(ii) The first column in Table I contains the values of $\lambda_m^{\pm}(1,1)$, $m=0,1,\dots,5$, while the second column contains the eigenvalues of $C(1,1)$ obtained by using MATHEMATICA. As indicated, the $C(1,1)$ and $C(5,1)$ obtained by both methods are identical. The values $\lambda_m^{\pm}(3,1)$, $m=0,1,\dots,8$, in the first and third columns of Table II are computed by MAPLE, while those in the second and fourth columns are computed by MATLAB. Some discrepancies between the values in the respective columns occur due to the round-off errors.

(iii) Figure 1 illustrates the graph of $\lambda_m^{\pm}(\alpha,1)$, $m=0,1,\dots,5$, with $n=6$. The dotted part of the curve is $\lambda(\alpha,1)$. Figure 2 gives the same information with $n=9$.

(iv) We conclude, via the last assertion of Theorem 2.1, that the wavelet approach works even better when N is an even number but not a multiple of four. Indeed, in such case, it synchronizes faster when α is chosen to be the critical value $\tilde{\alpha}_c$.

III. NEUMANN BOUNDARY CONDITIONS

Here, we consider the nearest neighbor coupling with Neumann boundary conditions. The resulting coupling matrix A is then $A(0)$, given as in (1.6a). With $i=1$, we have

TABLE I. The first and third columns contain the values computed by using formulas $\lambda_m^{\pm}(\alpha,1)$ as given in (2.16). The values in the second and fourth columns are eigenvalues of $C(\alpha,1)$ obtained by using MATHEMATICA.

$n=6$			
$\lambda_m^{\pm}(1,1)$	Eigenvalues of $C(1,1)$	$\lambda_m^{\pm}(5,1)$	Eigenvalues of $C(5,1)$
$\lambda_0^+(1,1)=0$	0	$\lambda_0^+(5,1)=0$	0
$\lambda_1^+(1,1)=-\frac{9}{4}+\frac{1}{4}\sqrt{37}$	$-\frac{9}{4}+\frac{1}{4}\sqrt{37}$	$\lambda_1^+(5,1)=-\frac{13}{4}+\frac{1}{4}\sqrt{13}$	$-\frac{13}{4}+\frac{1}{4}\sqrt{13}$
$\lambda_2^+(1,1)=-\frac{11}{4}+\frac{1}{4}\sqrt{13}$	$-\frac{11}{4}+\frac{1}{4}\sqrt{13}$	$\lambda_2^+(5,1)=-\frac{23}{4}+\frac{1}{4}\sqrt{181}$	$-\frac{23}{4}+\frac{1}{4}\sqrt{181}$
$\lambda_3^+(1,1)=-2$	-2	$\lambda_3^+(5,1)=-2$	-2
$\lambda_2^+(1,1)=-\frac{11}{4}+\frac{1}{4}\sqrt{13}$	$-\frac{11}{4}+\frac{1}{4}\sqrt{13}$	$\lambda_4^+(5,1)=-\frac{23}{4}+\frac{1}{4}\sqrt{181}$	$-\frac{23}{4}+\frac{1}{4}\sqrt{181}$
$\lambda_2^+(1,1)=-\frac{11}{4}+\frac{1}{4}\sqrt{13}$	$-\frac{11}{4}+\frac{1}{4}\sqrt{13}$	$\lambda_5^+(5,1)=-\frac{13}{4}+\frac{1}{4}\sqrt{13}$	$-\frac{13}{4}+\frac{1}{4}\sqrt{13}$
$\lambda_0^-(1,1)=-4$	-4	$\lambda_0^-(5,1)=-4$	-4
$\lambda_1^-(1,1)=-\frac{9}{4}-\frac{1}{4}\sqrt{37}$	$-\frac{9}{4}-\frac{1}{4}\sqrt{37}$	$\lambda_1^-(5,1)=-\frac{13}{4}-\frac{1}{4}\sqrt{13}$	$-\frac{13}{4}-\frac{1}{4}\sqrt{13}$
$\lambda_2^-(1,1)=-\frac{11}{4}-\frac{1}{4}\sqrt{13}$	$-\frac{11}{4}-\frac{1}{4}\sqrt{13}$	$\lambda_2^-(5,1)=-\frac{23}{4}-\frac{1}{4}\sqrt{181}$	$-\frac{23}{4}-\frac{1}{4}\sqrt{181}$
$\lambda_3^-(1,1)=-4$	-4	$\lambda_3^-(5,1)=-12$	-12
$\lambda_4^-(1,1)=-\frac{11}{4}-\frac{1}{4}\sqrt{13}$	$-\frac{11}{4}-\frac{1}{4}\sqrt{13}$	$\lambda_4^-(5,1)=-\frac{23}{4}-\frac{1}{4}\sqrt{181}$	$-\frac{23}{4}-\frac{1}{4}\sqrt{181}$
$\lambda_5^-(1,1)=-\frac{11}{4}-\frac{1}{4}\sqrt{13}$	$-\frac{11}{4}-\frac{1}{4}\sqrt{13}$	$\lambda_5^-(5,1)=-\frac{13}{4}-\frac{1}{4}\sqrt{13}$	$-\frac{13}{4}-\frac{1}{4}\sqrt{13}$

TABLE II. The first and third columns contain the values computed by using formulas $\lambda_m^{\pm}(\alpha,1)$ as given in (2.16). The values in the second and fourth columns are eigenvalues of $C(\alpha,1)$ obtained by using MATHEMATICA.

$n=9$			
$\lambda_m^{\pm}(3,1)$	Eigenvalues of $C(3,1)$	$\lambda_m^{\pm}(10,1)$	Eigenvalues of $C(10,1)$
$\lambda_0^+(3,1)=0$	0	$\lambda_0^+(10,1)=0$	0
$\lambda_1^+(3,1)\approx-0.7967$	-0.7967	$\lambda_1^+(10,1)\approx-2.2938$	-2.2930
$\lambda_2^+(3,1)\approx-2.2524$	-2.2525	$\lambda_2^+(10,1)\approx-3.0135$	-3.0140
$\lambda_3^+(3,1)\approx-2.2975$	-2.2974	$\lambda_3^+(10,1)\approx-2.4465$	-2.4466
$\lambda_4^+(3,1)\approx-2.0399$	-2.0399	$\lambda_4^+(10,1)\approx-2.0535$	-2.0542
$\lambda_5^+(3,1)\approx-2.0399$	-2.0399	$\lambda_5^+(10,1)\approx-2.0535$	-2.0542
$\lambda_6^+(3,1)\approx-2.2975$	-2.2974	$\lambda_6^+(10,1)\approx-2.4465$	-2.4466
$\lambda_7^+(3,1)\approx-2.2524$	-2.2525	$\lambda_7^+(10,1)\approx-3.0135$	-3.0140
$\lambda_8^+(3,1)\approx-0.7967$	-0.7967	$\lambda_8^+(10,1)\approx-2.2938$	-2.2930
$\lambda_0^-(3,1)=-4$	-4	$\lambda_0^-(10,1)=-4$	-4
$\lambda_1^-(3,1)\approx-3.9051$	-3.9052	$\lambda_1^-(10,1)\approx-4.0458$	-4.0465
$\lambda_2^-(3,1)\approx-4.2268$	-4.2265	$\lambda_2^-(10,1)\approx-9.2505$	-9.2495
$\lambda_3^-(3,1)\approx-6.2025$	-6.2026	$\lambda_3^-(10,1)\approx-16.5534$	-16.5534
$\lambda_4^-(3,1)\approx-7.7791$	-7.7792	$\lambda_4^-(10,1)\approx-21.3427$	-21.3427
$\lambda_5^-(3,1)\approx-7.7791$	-7.7792	$\lambda_5^-(10,1)\approx-21.3427$	-21.3427
$\lambda_6^-(3,1)\approx-6.2025$	-6.2026	$\lambda_6^-(10,1)\approx-16.5534$	-16.5534
$\lambda_7^-(3,1)\approx-4.2268$	-4.2265	$\lambda_7^-(10,1)\approx-9.2505$	-9.2495
$\lambda_8^-(3,1)\approx-3.9051$	-3.9052	$\lambda_8^-(10,1)\approx-4.0458$	-4.0465

$$A_1(0)=\begin{pmatrix}-1 & 1\\ 1 & -2\end{pmatrix},\quad \bar{A}_1(0)=\begin{pmatrix}-2 & 1\\ 1 & -1\end{pmatrix},\quad A_2(1)=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

$$A_1(1)=\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix},\quad A_2(0)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\quad \tilde{A}_2(0)=\alpha\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

$$\tilde{A}_1(0)=\alpha\begin{pmatrix}-\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4}\end{pmatrix}=\tilde{\tilde{A}}_1(0),\quad \tilde{A}_2(1)=\alpha\begin{pmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4}\end{pmatrix},$$

and

$$\tilde{A}_1(1)=\alpha\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{pmatrix} \tag{3.1}$$

A direct calculation would yield that

$$C_2(0)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

$$C_1(0)=\begin{pmatrix}-\frac{1}{4}(4+\alpha) & \frac{1}{4}(4-\alpha)\\ \frac{1}{4}(4-\alpha) & -\frac{1}{4}(8+\alpha)\end{pmatrix},\quad C_2(1)=\begin{pmatrix}\frac{\alpha}{4} & \frac{\alpha}{4}\\ \frac{1}{4}(\alpha+4) & \frac{\alpha}{4}\end{pmatrix},$$

$$C_1(1)=\begin{pmatrix}-\frac{1}{2}(4+\alpha) & \frac{1}{2}(2-\alpha)\\ \frac{1}{2}(2-\alpha) & -\frac{1}{2}(4+\alpha)\end{pmatrix},\quad \bar{C}_1(0)=\begin{pmatrix}-\frac{1}{4}(8+\alpha) & \frac{1}{4}(4-\alpha)\\ \frac{1}{4}(4-\alpha) & -\frac{1}{4}(4+\alpha)\end{pmatrix}. \tag{3.2}$$

As in the case of periodic boundary conditions, the eigenvalue problem $C(\alpha,0)\mathbf{b}=\lambda\mathbf{b}$, where $\mathbf{b}=(\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_n)^T$, $\mathbf{b}_i\in\mathbb{C}^2$, can be formed as block difference equation

$$C_2^T(1)\mathbf{b}_{i-1}+C_1(1)\mathbf{b}_i+C_2(1)\mathbf{b}_{i+1}=\lambda\mathbf{b}_i,\quad 1\leqslant i\leqslant n. \tag{3.3}$$

With Neumann boundary conditions, $\mathbf{b}_0$, and $\mathbf{b}_{n+1}$ must satisfy

$$C_1(0)\mathbf{b}_1+C_2(1)\mathbf{b}_2=\lambda\mathbf{b}_1=C_2^T(1)\mathbf{b}_0+C_1(1)\mathbf{b}_1+C_2(1)\mathbf{b}_2 \tag{3.4a}$$

and

$$C_2^T(1)\mathbf{b}_{n-1}+\bar{C}_1(0)\mathbf{b}_n=\lambda\mathbf{b}_n=C_2^T(1)\mathbf{b}_{n-1}+C_1(1)\mathbf{b}_n+C_2(1)\mathbf{b}_{n+1}. \tag{3.4b}$$

Solving (3.4a) and (3.4b), respectively, we get

$$\mathbf{b}_0=(C_2^T(1))^{-1}(C_1(0)-C_1(1))\mathbf{b}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathbf{b}_1 \tag{3.5a}$$

and

TABLE III. The first and third columns contain the values computed by using formulas $\lambda_m^{\pm}(\alpha,0)$ as given in (3.9). The values in the second and fourth columns are eigenvalues of $C(\alpha,1)$ obtained by using MATHEMATICA.

$n=3$			
$\lambda_m^{\pm}(2,0)$	Eigenvalues of $C(2,0)$	$\lambda_m^{\pm}(5,0)$	Eigenvalues of $C(5,0)$
$\lambda_0^+(2,0)=0$	0	$\lambda_0^+(5,0)=0$	0
$\lambda_1^+(2,0)=-\frac{5}{2}+\frac{1}{2}\sqrt{7}$	$-\frac{5}{2}+\frac{1}{2}\sqrt{7}$	$\lambda_1^+(5,0)=-\frac{13}{4}+\frac{1}{4}\sqrt{13}$	$-\frac{13}{4}+\frac{1}{4}\sqrt{13}$
$\lambda_2^+(2,0)=-\frac{7}{2}+\frac{1}{2}\sqrt{7}$	$-\frac{7}{2}+\frac{1}{2}\sqrt{7}$	$\lambda_2^+(5,0)=-\frac{23}{4}+\frac{1}{4}\sqrt{181}$	$-\frac{23}{4}+\frac{1}{4}\sqrt{181}$
$\lambda_3^+(2,0)=-2$	-2	$\lambda_3^+(5,0)=-2$	-2
$\lambda_1^-(2,0)=-\frac{5}{2}-\frac{1}{2}\sqrt{7}$	$-\frac{5}{2}-\frac{1}{2}\sqrt{7}$	$\lambda_1^-(5,0)=-\frac{13}{4}-\frac{1}{4}\sqrt{13}$	$-\frac{13}{4}-\frac{1}{4}\sqrt{13}$
$\lambda_2^-(2,0)=-\frac{7}{2}-\frac{1}{2}\sqrt{7}$	$-\frac{7}{2}-\frac{1}{2}\sqrt{7}$	$\lambda_2^-(5,0)=-\frac{23}{4}-\frac{1}{4}\sqrt{181}$	$-\frac{23}{4}-\frac{1}{4}\sqrt{181}$

$$\mathbf{b}_{n+1} = C_2(1)^{-1}(\bar{C}_1(0) - C_1(1))\mathbf{b}_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{b}_n. \tag{3.5b}$$

We then see that the characteristic equation of the block difference equation (3.3) is

$$\det[C_2^T(1) + \delta(C_1(1) - \lambda I) + \delta^2 C_2] = 0. \tag{3.6a}$$

Here δ is such that $\mathbf{b}_i = \delta^i\binom{1}{\nu}$, where ν is a constant depending on δ. Expanding the determinant in (3.6a), we get

$$\alpha\delta^4 + 2(2\alpha + 2 + \lambda\alpha)\delta^3 - 2(4 + 5\alpha + 2(\alpha + 4)\lambda + 2\lambda^2)\delta^2 + 2(2\alpha + 2 + \lambda\alpha)\delta + \alpha = 0. \tag{3.6b}$$

We assume, momentarily, that Eq. (3.6b) has four distinct roots δ_1, δ_2, δ_3, and δ_4. The general solutions to (3.3) can then be written as

$$\mathbf{b}_i = \sum_{j=1}^{4} c_j \delta_j^i \begin{pmatrix} 1 \\ \nu_j \end{pmatrix}. \tag{3.7}$$

Substituting (3.7) into boundary conditions (3.5), we get

$$\begin{pmatrix} \delta_1\nu_1 - 1 & \delta_2\nu_2 - 1 & \delta_3\nu_3 - 1 & \delta_4\nu_4 - 1 \\ \delta_1 - \nu_1 & \delta_2 - \nu_2 & \delta_3 - \nu_3 & \delta_4 - \nu_4 \\ \delta_1^n(\delta_1\nu_1 - 1) & \delta_2^n(\delta_2\nu_2 - 1) & \delta_3^n(\delta_3\nu_3 - 1) & \delta_4^n(\delta_4\nu_4 - 1) \\ \delta_1^n(\delta_1 - \nu_1) & \delta_2^n(\delta_2 - \nu_2) & \delta_3^n(\delta_3 - \nu_3) & \delta_4^n(\delta_4 - \nu_4) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} =: D\mathbf{c} = 0, \tag{3.8}$$

where $\mathbf{c} = (c_1, c_2, c_3, c_4)^T$. We are now in a position to simplify $\det D$,

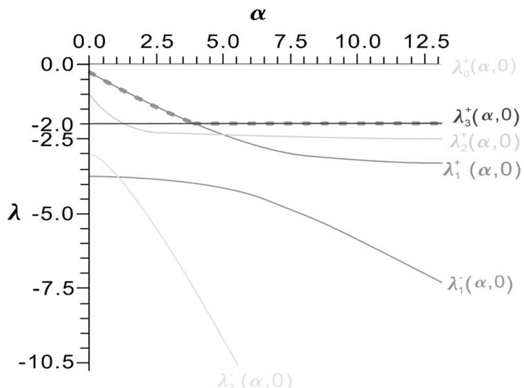

FIG. 3. The curves $\lambda_m^{\pm}(\alpha,0)$ with $N=2n=6$ are provided. As predicted in Theorem 3.2, $\lambda(\alpha,0)$ turns flat after $\bar{\alpha}_1$.

$$\det D=(\delta_2\nu_2)(\delta_4\nu_4)\begin{vmatrix}\delta_1\nu_1-1 & 1-\delta_1\nu_1 & \delta_3\nu_3-1 & 1-\delta_3\nu_3\\ \delta_1-\nu_1 & \nu_1-\delta_1 & \delta_3-\nu_3 & \nu_3-\delta_3\\ \delta_1^n(\delta_1\nu_1-1) & \delta_2^n(1-\delta_1\nu_1) & \delta_3^n(\delta_3\nu_3-1) & \delta_4^n(1-\delta_3\nu_3)\\ \delta_1^n(\delta_1-\nu_1) & \delta_2^n(\delta_1-\nu_1) & \delta_3^n(\delta_3-\nu_3) & \delta_4^n(\delta_3-\nu_3)\end{vmatrix}$$

$$=(\delta_2\nu_2)(\delta_4\nu_4)(\delta_1^n-\delta_2^n)(\delta_3^n-\delta_4^n)\begin{vmatrix}0 & 1-\delta_1\nu_1 & 0 & 1-\delta_3\nu_3\\ 0 & \nu_1-\delta_1 & 0 & \nu_3-\delta_3\\ \delta_1\nu_1-1 & \delta_2^n(1-\delta_1\nu_1) & \delta_3\nu_3-1 & \delta_4^n(1-\delta_3\nu_3)\\ \delta_1-\nu_1 & \delta_2^n(\nu_1-\delta_1) & \delta_3-\nu_3 & \delta_4^n(\nu_3-\delta_3)\end{vmatrix}$$

$$=(\delta_2\nu_2)(\delta_4\nu_4)(\delta_1^n-\delta_2^n)(\delta_3^n-\delta_4^n)\Bigg\{[(\delta_1\nu_1-1)(\nu_3-\delta_3)+(\delta_1-\nu_1)(\delta_3\nu_3-1)]$$
$$\times\begin{vmatrix}1-\delta_1\nu_1 & 1-\delta_3\nu_3\\ \nu_1-\delta_1 & \nu_3-\delta_3\end{vmatrix}\Bigg\}.$$

Therefore, $\det D$ being equal to zero amounts to $\delta_i^{2n}=1$ for $i=1,2,3,4$.

To get the characteristic equation of $C(\alpha,0)$, we need to solve $\delta^{2n}=1$ and Eq. (3.6b). This leads to the following theorem.

Theorem 3.1: *Let N be any positive even integer. The dimension of each block matrix in $C(\alpha,0)$ is 2×2. Let $\lambda_m^{\pm}(\alpha,0)$ be defined as follows:*

$$\lambda_m^{\pm}(\alpha,0)=\frac{1}{2}\left(\alpha\cos\frac{m\pi}{n}-\alpha-4\right)\pm\frac{1}{2}\left[\left(\alpha\cos\frac{m\pi}{n}-\alpha-4\right)^2+4\left(\alpha\cos^2\frac{m\pi}{n}+2(\alpha+1)\times\cos\frac{m\pi}{n}-2-3\alpha\right)\right]^{1/2}. \qquad (3.9)$$

Then $\lambda_m^{\pm}(\alpha,0)$, $m=1,2,\ldots,n-1$, $\lambda_0^{+}(\alpha,0)=0$ and $\lambda_n^{+}(\alpha,0)=-2$ are eigenvalues of $C(\alpha,0)$ for each $\alpha>0$.

Proof: Substituting $\delta=e^{im\pi/n}$, $0\leqslant m\leqslant n-1$, into (3.6b), we get (3.9). Clearly, if $\delta\neq 1$ or -1, or equivalently, $\cos(m\pi/n)\neq 1$ or -1, then $\lambda_m^{\pm}(\alpha,0)$, $m=1,2,\ldots,n-1$, are eigencurves of $C(\alpha,0)$. Since $0=\lambda_0^{+}(\alpha,0)$ is an eigenvalue of $C(\alpha,0)$ for all α, we only need to show that $\lambda_n^{+}(\alpha,0)$ is, indeed, the eigenvalue of $C(\alpha,0)$ for each α. To this end, we see that $\mathrm{trace}(C(\alpha,0))=-(n-2)(\alpha+4)-6-\alpha$. However, $\lambda_0^{+}(\alpha,0)+\Sigma_{j=1}^{n-1}\lambda_j^{\pm}(\alpha,0)=-(n-1)(\alpha+4)=:k$. Thus, $\mathrm{trace}(C(\alpha,0))-k=-2=\lambda_n^{+}(\alpha,0)$. We just complete the proof of the theorem. □

Remark 3.1: (i) Letting $t=\cos(m\pi/n)$, $\lambda_m^{\pm}(\alpha,0)=\lambda_t^{\pm}(\alpha,0)$ and treating t as a real parameter, we see that for fixed $\alpha>0$, the eigenvalues of C with periodic boundary conditions and Neumann boundary conditions, respectively, lie on the curve $\lambda_t^{\pm}(\alpha,0)$ in $t-\lambda$ plane.

(ii) Note that $\lambda_m^{\pm}(\alpha,0)=\lambda_{2n-m}^{\pm}(\alpha,0)$.

Theorem 3.2: *For each* α, *let* $\lambda(\alpha,0)$ *be the second largest eigenvalue of* $C(\alpha,0)$. *Then* $\lambda(\alpha,0)=\lambda_1^{+}(\alpha,0)$, *for* $0\leqslant\alpha\leqslant 1/\sin^2(\pi/2n)=:\bar{\alpha}_1$; *and* $\lambda(\alpha,0)=\lambda_n^{+}(\alpha,0)=-2$ *for all* $\alpha\in[\bar{\alpha}_1,\infty)$.

We skip the proof of theorem due to its similarity with that of Theorem 2.1 (ii).

Remark 3.2: Table III and Fig. 3 illustrate, again, the accuracy of our theorems.

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