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Extended Vogan diagrams ☆

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Abstract

An extended Vogan diagram is an extended Dynkin diagram with a diagram involution, such that the vertices fixed by the involution can be painted or unpainted. Every extended Vogan diagram represents an almost compact real form of some affine Kac–Moody Lie algebra. Two diagrams may represent isomorphic algebras, and in this case we say that the diagrams are equivalent. In this paper, we classify the equivalence classes of extended Vogan diagrams, and provide a complete list of all diagrams within each class. It gives a combinatorial classification of the isomorphic classes of almost compact real forms of the affine Kac–Moody Lie algebras.

Keywords: Extended Vogan diagram; Almost compact real form; Kac–Moody Lie algebra

1. Introduction

A Vogan diagram is a Dynkin diagram with a diagram involution, such that the vertices fixed by the involution are either painted or unpainted. This terminology first appeared in [7], and the Vogan diagrams represent the real forms of the complex simple Lie algebras. Similarly, given a complex affine Kac–Moody Lie algebra, we can represent it with

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a diagram, known as the extended Dynkin diagram [6, Chapter 4]. We define the extended Vogan diagrams as above, namely with an involution whose fixed points are painted or unpainted. The equivalence classes of extended Vogan diagrams correspond to the isomorphic classes of almost compact real forms [1,2]. In this paper, we classify all the equivalence classes of extended Vogan diagrams, and give a complete list of all the diagrams within each equivalence class. Consequently, this gives a combinatorial classification of the almost compact real forms of affine Kac–Moody Lie algebras, which is parallel to the algebraic classification given in [3].

Let $\mathfrak{g}_{\mathbf{R}}$ be a real form of a complex affine Kac–Moody Lie algebra $\mathfrak{g}$. Fix an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbf{R}} \otimes \mathbf{C}$, and let the Galois group $\Gamma = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ act on $\mathfrak{g}$. We identify $\mathfrak{g}_{\mathbf{R}}$ with the fixed points of Γ. We say that $\mathfrak{g}_{\mathbf{R}}$ is almost compact if the nontrivial element of Γ transforms a Borel subalgebra of $\mathfrak{g}$ to the Borel subalgebra of the opposite sign. Suppose that $\mathfrak{g}_{\mathbf{R}}$ is an almost compact real form. By choosing a maximally compact Cartan subalgebra of $\mathfrak{g}_{\mathbf{R}}$ which is stable under a Cartan involution, we can represent $\mathfrak{g}_{\mathbf{R}}$ by an extended Vogan diagram [1, Section 3].

In what follows, we recall the equivalence relation [1, (3.7.1)] on the extended Vogan diagrams v. It considers v whose edges may be single, double with one arrow, triple with one arrow. Namely we tentatively ignore $A_1^{(1)}$ (contains double edge with two arrows) and $A_2^{(2)}$ (contains quadruple edge), and treat them separately later. If i is a painted vertex in v, let F_i be the algorithm which reverses the colors of all the vertices j adjacent to i, except when j is a longer root joint to i by a double edge. Namely, define the neighborhood of vertex i by

$$N(i) = \{\text{vertices adjacent to } i\}, \tag{1.1}$$

excluding i itself. Then $F_i(v)$ is the diagram given by

$$\begin{aligned} F_i\colon\ & \text{Reverse the colors of all } j \in N(i), \text{ except when } j \text{ is a longer root joint to } i \\ & \text{by a double edge or when } j \text{ is not fixed by the involution.} \end{aligned} \tag{1.2}$$

The operation F_i corresponds to the reflection which sends the simple root i to $-i$. So v and $F_i(v)$ represent isomorphic Lie algebras. We say that two extended Vogan diagrams v and w are *equivalent* if there is a sequence of operations $v = v_0 \to v_1 \to \cdots \to v_k = w$, where each $v_a \to v_{a+1}$ is either some algorithm F_i as given in (1.2), or a diagram automorphism. This definition is justified by the following theorem.

Theorem 1.1 *(Batra). Every extended Vogan diagram represents an almost compact real form of an affine Kac–Moody Lie algebra. Two extended Vogan diagrams are equivalent if and only if their corresponding algebras are isomorphic.*

Proof. The first statement follows from [2, Theorem 5.2], and the second statement follows from [1, Theorem 5.2]. □

By this theorem, the equivalence classes of extended Vogan diagrams correspond to the isomorphic classes of almost compact real forms of affine Kac–Moody Lie algebras. It allows us to use the diagrams to study the isomorphism of algebras. Nonequivalent diagrams of $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_4^{(1)}$ are shown in [1]. In this paper, we give a complete list of all the distinct equivalence classes, as well as all the diagrams within each class.

It is convenient to represent an equivalence class with a diagram with minimum number of painted vertices. So the following theorem will be useful.

Theorem 1.2 *(Borel and de Siebenthal). Every equivalence class of extended Vogan diagrams has a representative with at most two vertices painted.*

Proof. The Borel and de Siebenthal theorem [4] says that every real form of a complex simple Lie algebra can be represented by a Vogan diagram with at most one painted vertex. In [5], we verify this theorem by using algorithms (1.2) and diagram automorphisms to explicitly reduce every painting on a Dynkin diagram D to another painting with at most one painted vertex.

Consider an extended Dynkin diagram given by a Dynkin diagram D, an extra vertex p, and some extra edges joint to p. Since a painting on D is equivalent to another one with at most one painted vertex, together with p, we obtain a painting with at most two painted vertices. □

This theorem does not help to judge whether two diagrams are equivalent, or how to reduce a diagram to another one with at most two painted vertices. For instance two diagrams, both with two painted vertices, could be nonequivalent to each other.

Clearly a diagram with trivial involution and no painted vertex is not equivalent to any other diagram. So once and for all, we ignore such diagrams. In Tables 1 and 2 below, we apply Theorem 1.2 and represent each equivalence class by a diagram with one or two painted vertices. Tables 1 and 2 handle the diagrams with trivial and nontrivial involutions, respectively. The tables give a complete list of all the diagrams within each equivalence class. We shall label the vertices, so that an extended Vogan diagram is denoted by

$$(i_1, \ldots, i_k) \quad \text{or} \quad (\theta; i_1, \ldots, i_k), \quad i_1 < i_2 < \cdots < i_k. \tag{1.3}$$

Here $(i_1, \ldots, i_k)$ has trivial diagram involution and vertices $i_1, \ldots, i_k$ painted; while $(\theta; i_1, \ldots, i_k)$ has diagram involution θ and vertices $i_1, \ldots, i_k$ painted. We also write $(\theta; \emptyset)$ for the diagram with involution θ and no painted vertex.

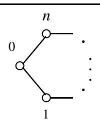
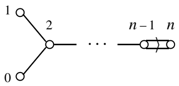
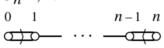
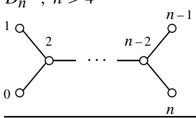
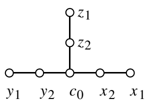
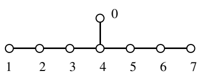
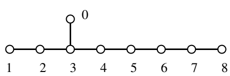
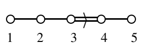
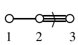
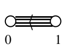
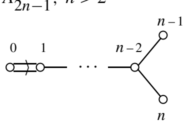
In what follows, we explain the notations ϕ, c, B, M, ξ used in Table 1.

The notation ϕ shall be used very often. Given a diagram $(i_1, \ldots, i_k)$ where the painted vertices are ordered by $i_1 < i_2 < \cdots < i_k$, we define

$$\phi(i_1, \ldots, i_k) = i_k - i_{k-1} + \cdots + (-1)^{k-1} i_1 = \sum_{p=1}^{k} (-1)^{k-p} i_p. \tag{1.4}$$

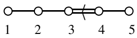
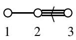
For a vertex i of a given diagram v, let $c(i)$ denote the color of i in v, which can be painted or unpainted.

Table 1
Trivial diagram involution

Extended Dynkin diagram	Representative diagram	Equivalent diagrams
$A_1^{(1)}$ 	(0)	(1).
	(0, 1)	
$A_n^{(1)},\ n>1$ 	$(0,N),\ 1\leqslant N\leqslant\frac{n+1}{2}$	$(i_1,\dots,i_k)$, k is even and $\phi=N,\ n-N$.
	(0)	$(i_1,\dots,i_k)$, k is odd.
$B_n^{(1)},\ n>2$ 	(1)	$c(0)\neq c(1)$.
	$(N),\ N\geqslant 2$	$c(0)=c(1)$ and $\phi=N$.
	(0, 1)	$(0,1,2)$, $(k,k-1),\ k\geqslant 3$.
$C_n^{(1)},\ n>1$ 	(0)	$c(0)\neq c(n)$.
	$(N),\ N\leqslant\frac{n}{2}$	$c(0)=c(n)=\circ$ and $\phi=N,\ n-N$.
	$(0,n)$	$c(0)=c(n)=\bullet$.
$D_n^{(1)},\ n>4$ 	(0)	$c(0)\neq c(1),\ c(n-1)=c(n)$ or $c(0)=c(1),\ c(n-1)\neq c(n)$.
	$(0,n)$	$c(0)\neq c(1)$ and $c(n-1)\neq c(n)$.
	(0, 1)	$(0,1,2)$, $(n-2,n-1,n)$, $(k-1,k),\ 3\leqslant k\leqslant n-2$.
	$(N),\ 2\leqslant N\leqslant\frac{n}{2}$	$v\neq(0,1,2)$; $c(0)=c(1)$ and $c(n-1)=c(n)$, $\phi=N,\ n-N$.
$E_6^{(1)}$ 	(x_1)	M is odd.
	(x_2)	M is even and B is odd.
	(x_1,y_1)	M, B are even.
$E_7^{(1)}$ 	(1)	ϕ is odd.
	(2)	ϕ and ξ are even.
	(0)	$\phi=0,4$ and ξ is odd.
	(1, 7)	$\phi=2,6$ and ξ is odd.
$E_8^{(1)}$ 	(1)	(5), $(0,N)$, $N=4,8$.
	(7)	(2), (3), $(0,6)$.
	(8)	(0), (4), (6), $(0,N)$, $N=1,2,3,5,7$.
$F_4^{(1)}$ 	(1)	$(1\leqslant i_1,\dots,i_a\leqslant 3, 4\leqslant i_{a+1},\dots)$, $\phi(i_1,\dots,i_a)$ is odd.
	(2)	$(1\leqslant i_1,\dots,i_a\leqslant 3, 4\leqslant i_{a+1},\dots)$, $\phi(i_1,\dots,i_a)$ is even $(\neq 0)$.
	(4)	$(1\leqslant i_1,\dots,i_a\leqslant 3, 4\leqslant i_{a+1},\dots)$, $\phi(i_1,\dots,i_a)=0$.
$G_2^{(1)}$ 	(1)	ϕ is odd.
	(2)	ϕ is even.
$A_2^{(2)}$ 	(0)	
	(1)	(0, 1).
$A_{2n}^{(2)},\ n>1$ 	(0)	$(i_1,\dots,i_k),\ i_1=0$.
	$(N),\ 1\leqslant N\leqslant n$	$(i_1,\dots,i_k),\ i_1\neq 0$ and $\phi=N$.
$A_{2n-1}^{(2)},\ n>2$ 	(0)	$c(0)=\bullet$ and $c(n-1)=c(n)$.
	(n)	$c(0)=\circ$ and $c(n-1)\neq c(n)$.
	$(N),\ 1\leqslant N\leqslant\frac{n}{2}$	$\phi=N,\ n-N$ and $c(0)=\circ,\ c(n-1)=c(n)$.
	$(0,n)$	$c(0)=\bullet$ and $c(n-1)\neq c(n)$.

(continued on next page)

Table 1 (*continued*)

Extended Dynkin diagram	Representative diagram	Equivalent diagrams
$D_{n+1}^{(2)}$, $n>1$ 	$(0,N)$, $1\leqslant N\leqslant n$	$(i_1,\ldots,i_k)$, $\phi=N$ and k is even.
	(N), $0\leqslant N\leqslant \frac{n}{2}$	$(i_1,\ldots,i_k)$, $\phi=N$, $n-N$ and k is odd.
$E_6^{(2)}$ 	(1)	$(i_1,\ldots,i_k)$, $i_k\leqslant 3$ and ϕ is odd.
	(2)	$(i_1,\ldots,i_k)$, $i_k\leqslant 3$ and ϕ is even.
	(4)	$(i_1,\ldots,i_k)$, $i_k\geqslant 4$ and ϕ is even.
	(5)	$(i_1,\ldots,i_k)$, $i_k\geqslant 4$ and ϕ is odd.
$D_4^{(3)}$ 	(1)	ϕ is odd.
	(2)	ϕ is even.

For a Vogan diagram v in $E_6^{(1)}$, let

$$B(v)=\text{number of branches which contain painted vertices in } v,$$

and

$$M(v)=\text{number of painted odd vertices in } v.$$

So $0\leqslant M(v)\leqslant 4$. More explanations for $B(v)$ and $M(v)$ are given in (3.1) and (3.2).

For a Vogan diagram v in $E_7^{(1)}$, we write

$$v=(s,i_1,\ldots,i_a,i_{a+1},\ldots,i_k), \tag{1.5}$$

where $1\leqslant i_1<\cdots<i_a\leqslant 4<i_{a+1}<\cdots<i_k\leqslant 7$, and $s\subset\{0\}$. In this case, let

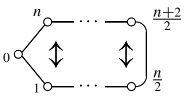
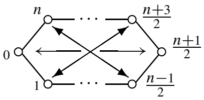
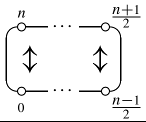
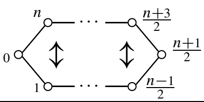
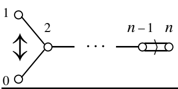
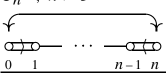
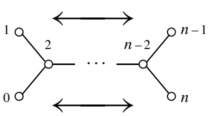
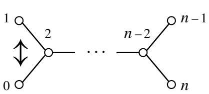
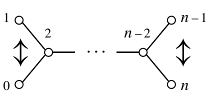
$$\xi=\begin{cases}\sum_{p=1}^{a}(-1)^{a-p}i_p, & \text{if the vertex 0 is unpainted,}\\ \sum_{p=1}^{a}(-1)^{a-p}i_p+1, & \text{if the vertex 0 is painted.}\end{cases} \tag{1.6}$$

For example, let $v=(0,1,3,4,7)$ be a diagram for $E_7^{(1)}$. Then $\phi(v)=7-4+3-1+0=5$ and $\xi=4-3+1-0+1=3$ (the last $+1$ in the above equation is due to the vertex 0 being painted).

In Table 2, there are several cases where equivalent diagrams can be obtained by replacing θ with other σ via diagram automorphisms. For instance, consider the first diagram which deals with $A_n^{(1)}$, n even. Here θ fixes 0 and $\theta(i)=n+1-i$. If we rotate the indices by one unit, we obtain σ which fixes n and $\sigma(i)=n-1-i$. But clearly the diagrams resulting from θ and σ can be identified. So we exclude such diagrams because they are obvious (but require messy notations). The same happen for other $A_n^{(1)}$, $D_n^{(1)}$ (replacing $0\leftrightarrow 1$ with $n-1\leftrightarrow n$) and $E_6^{(1)}$ (permuting x_i, y_i, z_i).

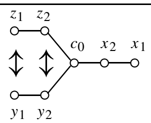
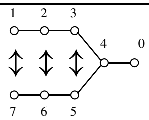
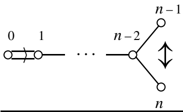
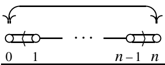
The classification in Tables 1 and 2 is consistent with the classification of the almost compact real forms of affine Kac–Moody Lie algebras in [3, pp. 487–494]. For example, the equivalences classes for $A_1^{(1)}$ given in [3, p. 487] are $\tau_0\tau_1$, τ_0 and ρ. And the corresponding classes are represented by (0, 1), (0) in Table 1 and $(\theta;\emptyset)$ in Table 2.

Table 2
Nontrivial diagram involution

Extended Dynkin diagram with nontrivial θ	Representative diagram	Equivalent diagrams
$A_n^{(1)}$, n even	$(\theta; \emptyset)$	
	$(\theta; 0)$	
$A_n^{(1)}$, n odd 	$(\theta; \emptyset)$	
$A_n^{(1)}$, n odd 	$(\theta; \emptyset)$	
$A_n^{(1)}$, n odd	$(\theta; \emptyset)$	
	$(\theta; 0)$	$(\theta; \frac{n+1}{2})$.
	$(\theta; 0, \frac{n+1}{2})$	
$B_n^{(1)}$, $n > 4$	$(\theta; \emptyset)$	
	$(\theta; N)$ $N = 2, 3, \ldots, n$	$(\theta; v)$, $\phi = N$.
$C_n^{(1)}$, $n > 3$	$(\theta; \emptyset)$	
	$(\theta; \frac{n}{2})$, n even	
$D_n^{(1)}$, $n > 4$	$(\theta; \emptyset)$	
	$(\theta; \frac{n}{2})$, n even	
$D_n^{(1)}$, $n > 4$	$(\theta; \emptyset)$	
	$(\theta; N)$, $2 \leqslant N \leqslant \frac{n+1}{2}$ or $N = n$	$(\theta; v)$, $\phi = N$.
$D_n^{(1)}$, $n > 4$	$(\theta; \emptyset)$	
	$(\theta; N)$, $2 \leqslant N \leqslant \frac{n}{2}$	$(\theta; v)$, $\phi = N$.

(*continued on next page*)

Table 2 (*continued*)

Extended Dynkin diagram with nontrivial θ	Representative diagram	Equivalent diagrams
$E_6^{(1)}$ 	$(\theta; \emptyset)$	
	$(\theta; x_1)$	
	$(\theta; x_2)$	
$E_7^{(1)}$ 	$(\theta; \emptyset)$	
	$(\theta; 0)$	
$A_{2n-1}^{(2)}$, $n > 2$	$(\theta; \emptyset)$	
	$(\theta; N)$, $0 \leqslant N \leqslant \frac{n-1}{2}$	$(\theta; v)$, $\phi = N$.
$D_{n+1}^{(2)}$, $n > 1$	$(\theta; \emptyset)$	
	$(\theta; \frac{n}{2})$, n even	

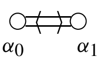
Our arguments are divided into the following sections. In Section 2, we consider the classical nontwisted diagrams for $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$. In Section 3, we consider the exceptional nontwisted diagrams for $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$. In Section 4, we consider the twisted diagrams for $A_n^{(2)}$, $D_n^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$. There are two propositions for $E_7^{(1)}$ and $E_8^{(1)}$ which treat the Dynkin diagrams purely from a graph theoretic viewpoint. Their arguments are lengthy and less relevant, so we place them in Appendix A to keep the rest of the paper fluent.

2. Classical nontwisted diagrams

We consider the extended Vogan diagrams for $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$. Nonequivalent diagrams of $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_4^{(1)}$ are given in [1]. In this section, we show that the diagrams in [1] (as well as general $D_n^{(1)}$) exhaust all the equivalence classes, and describe the other diagrams which are equivalent to each of them.

2.1. $A_1^{(1)}$

We start with $A_1^{(1)}$. Recall that the operation F_i in (1.2) does not cover the cases $A_1^{(1)}$ and $A_2^{(2)}$. We now treat $A_1^{(1)}$, leaving $A_2^{(2)}$ for Section 4 later. Let



be the diagram for $A_1^{(1)}$.

Proposition 2.1. *There are three mutually nonequivalent nontrivial diagrams of $A_1^{(1)}$ given by* $\{\alpha_0$ *painted alone*$\}$, $\{\alpha_0, \alpha_1$ *painted*$\}$ *and* $\{$*involution* $\alpha_0 \leftrightarrow \alpha_1\}$.

Proof. Recall that the Cartan matrix of $A_1^{(1)}$ is $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$, and the positive roots are [6, p. 93]

$$\Delta^+ = \big\{(k-1)\alpha_0 + k\alpha_1, k\alpha_0 + (k-1)\alpha_1, k\alpha_0 + k\alpha_1, \text{ where } k = 1, 2, \ldots\big\}.$$

It implies that

$$\alpha_0, \alpha_0 + \alpha_1, 2\alpha_0 + \alpha_1 \in \Delta^+. \tag{2.1}$$

Suppose now that α_0 is painted. The operation F_{α_0} corresponds to the effect on the diagram due to the Weyl reflection r_{α_0} which sends α_0 to $-\alpha_0$. By [6, p. 86],

$$r_{\alpha_0}(\alpha_1) = \alpha_1 - (-2)\alpha_0 = 2\alpha_0 + \alpha_1. \tag{2.2}$$

The coefficient -2 in the above equation comes from the Cartan matrix.

Let $c(\cdot)$ denotes "the color of," which could be painted or unpainted. The almost compact real form determines the colors of all real roots, though only the colors of simple roots are indicated on the extended Vogan diagrams. Suppose that we regard the two colors as the two element group with "unpainted" being the identity. Then whenever $i, j, i+j$ are roots, they satisfy $c(i) + c(j) = c(i+j)$. For example, the sum of two painted roots is unpainted, and so on.

Since α_0 is painted, by (2.1), $c(\alpha_0 + \alpha_1) \neq c(\alpha_1)$, and also $c(2\alpha_0 + \alpha_1) \neq c(\alpha_0 + \alpha_1)$. So $c(2\alpha_0 + \alpha_1) = c(\alpha_1)$. Together with (2.2), we conclude that F_{α_0} does not change the color of α_1. By symmetry of the diagram, clearly the diagram with α_0 painted alone is equivalent to the one with α_1 painted alone. The proposition follows. □

By the above proposition, we have proved the information for $A_1^{(1)}$ in Tables 1 and 2. Let X be a type of complex simple or affine Kac–Moody Lie algebra. Let $V(X)$ and $V(\theta; X)$ respectively denote the diagrams with trivial diagram involution (with at least one painted vertex) and with diagram involution θ. We write

$$(i_1, \ldots, i_k) \in V(X) \quad \text{and} \quad (\theta; i_1, \ldots, i_k) \in V(\theta; X), \quad i_1 < i_2 < \cdots < i_k,$$

where $i_1, \ldots, i_k$ are the painted vertices. Here we label the vertices as in Tables 1 and 2. Define the function ϕ on $V(X)$ by (1.4).

2.2. $A_n^{(1)}$, $n > 1$

The diagram of $A_n^{(1)}$ is a loop with vertices $0, 1, \ldots, n$ in this order.

Proposition 2.2. *Let* $v = (i_1, \ldots, i_k) \in V(A_n^{(1)})$. *Then*

$$v \sim \begin{cases} (0, N) \sim (0, n+1-N) & \text{if } k \text{ is even and } \phi(v) = N \text{ or } n+1-N; \\ (0) & \text{if } k \text{ is odd.} \end{cases}$$

Proof. By Theorem 1.2, v is equivalent to another diagram with at most two painted vertices. Further, each F_i preserves the parity of the number of painted vertices in v. So if v has odd number of painted vertices, it is equivalent to a diagram with one painted vertex. By rotating the diagram of $A_n^{(1)}$, it is clear that all the diagrams with one painted vertex are equivalent to one another.

Next we consider the case where v has even number of painted vertices. For the proof of this proposition (only), we modify the requirement for the notation $(i_1, \ldots, i_k)$ in (1.3) by allowing $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$. In this notation, a vertex appears odd number of times if and only if it is painted. So for instance $(1, 1, 2)$ and $(2, 2, 2)$ refer to the same diagram with vertex 2 painted. Observe that ϕ of (1.4) remains well defined in this convention. As we shall see, it allows us to express F_i easily. Note that k is even.

For $i_r \neq 0, n$,

$$\begin{aligned} \phi \cdot F_{i_r}(v) &= \phi(i_1, \ldots, i_{r-1}, i_r - 1, i_r, i_r + 1, i_{r+1}, \ldots, i_k) \\ &= \phi(i_{r+1}, \ldots, i_k) + (-1)^{k-r}\big((i_r + 1) - i_r + (i_r - 1) - \phi(i_1, \ldots, i_{r-1})\big) \\ &= \phi(v). \end{aligned} \tag{2.3}$$

If we can apply F_n to $v = (i_1, \ldots, i_k)$, then $i_k = n$ and so

$$\phi \cdot F_n(v) = \phi(0, i_1, \ldots, i_{k-1}, n - 1, n) = n + 1 - \phi(v). \tag{2.4}$$

If we can apply F_0 to $v = (i_1, \ldots, i_k)$, then $i_1 = 0$ and so

$$\phi \cdot F_0(v) = \phi(0, 1, i_2, \ldots, i_k, n) = n + 1 - \phi(v). \tag{2.5}$$

The last equation uses the fact that k is even.

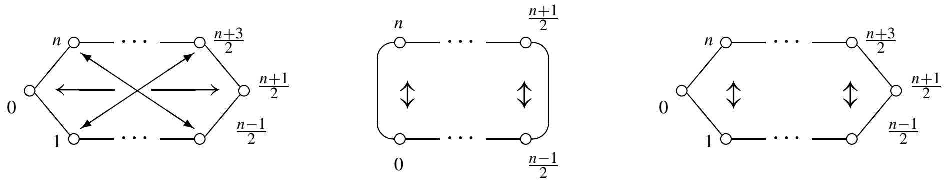
We conclude from (2.3)–(2.5) that

$$v \sim w \quad \Leftrightarrow \quad \begin{cases} \phi(v) = \phi(w) \text{ or} \\ \phi(v) = n + 1 - \phi(w). \end{cases} \tag{2.6}$$

By Theorem 1.2, v is equivalent to some diagram with two painted vertices i and j. By (2.6), $|j - i|$ is either $\phi(v)$ or $n + 1 - \phi(v)$. But both cases represent equivalent diagrams, via diagram automorphisms. For instance the diagrams $v = (1, 3)$ and $w = (1, n)$ are equivalent, with $\phi(w) = n + 1 - \phi(v)$. This proves the proposition. □

As explained in the proof, the diagrams with odd number of painted vertices form an equivalence class. Proposition 2.2, together with (2.6), show that two Vogan diagrams v and w with even number of painted vertices are equivalent if and only if $\phi(v) = \phi(w)$ or $\phi(v) = n + 1 - \phi(w)$. This leads to all the information for $V(A_n^{(1)})$ in Table 1.

We next consider $V(\theta; A_n^{(1)})$. If n is even (i.e. odd number of vertices), then up to diagram automorphisms, θ has only one possibility $1 \leftrightarrow n, 2 \leftrightarrow n - 1, \ldots$ where 0 is fixed by θ. So there are two equivalence classes, given by vertex 0 being painted or unpainted. If n is odd (i.e. even number of vertices), then up to diagram automorphisms, there are three cases for θ:



(a) (b) (c)

In cases (a) and (b), θ has no fixed point. In case (c), θ has fixed points 0 and $\frac{n+1}{2}$. So case (c) has three equivalence classes represented by $(\theta; \emptyset)$, $(\theta; 0)$ and $(\theta; 0, \frac{n+1}{2})$. This completes the discussion for $V(\theta; A_n^{(1)})$ in Table 2.

In our labeling for $X = A, B, C, D$, if we omit vertex 0 and its adjacent edges in $X_n^{(1)}$, then we obtain the Dynkin diagram for X_n. This idea allows us to apply the results of [5] in the following manner. Suppose that S is a collection of extended Vogan diagrams, and S is closed under each F_i. To study S, we shall often omit one or two vertices (especially vertices 0 and n) from each diagram in S, and denote the resulting Vogan diagrams by T. The bijection $\pi : S \to T$ is an isomorphism in the sense that $F_i \cdot \pi(v) = \pi \cdot F_i(v)$ and $\phi(v) = \phi \cdot \pi(v)$ for all $v \in S$. In this way, we can apply the results of [5] on T to S. We first recall some results of [5].

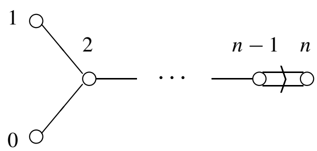
Proposition 2.3.

(a) *In A_n and B_n, $(i_1, \ldots, i_k) \sim (\sum_{p=1}^{k} (-1)^{k-p} i_p)$.*
(b) *In C_n, if $i_k = n$, then $(i_1, \ldots, i_k) \sim (n)$.*
(c) *In D_n, $(i_1, \ldots, i_k, n-1) \sim (n-1)$, $(i_1, \ldots, i_k \leqslant n-2) \sim (\sum_{p=1}^{k} (-1)^{k-p} i_p)$ and $(i_1, \ldots, i_k, n-1, n) \sim (1 + \sum_{p=1}^{k} (-1)^{k-p} i_p)$.*

Proof. Part (a) follows from [5, Proposition 2.3], part (b) follows from [5, Proposition 2.4], and part (c) follows from [5, Proposition 2.5]. $\square$

2.3. $B_n^{(1)}$, $n > 2$

Given a Vogan diagram, recall that $c(i)$ denote the color of vertex i in that diagram. The vertices of $B_n^{(1)}$ are labeled as follows:



Proposition 2.4. *Let $v \in V(B_n^{(1)})$. Then*

$$v \sim \begin{cases} (\phi(v)) & \text{if } c(0) = c(1) \text{ and } \phi(v) \neq 1; & \text{(a)} \\ (0, 1) & \text{if } c(0) = c(1) \text{ and } \phi(v) = 1; & \text{(b)} \\ (1) & \text{if } c(0) \neq c(1). & \text{(c)} \end{cases}$$

Proof. We prove parts (a) and (b) simultaneously. Let $S \subset V(B_n^{(1)})$ be the diagrams with vertices 0, 1 having the same color. It is preserved by all the F_i. By ignoring vertex 0, we obtain an isomorphism $\pi : S \to V(B_n)$. Recall from [5] that in B_n, the distinct equivalence classes are represented by $(1), (2), \ldots, (n)$, where $v \sim (\phi(v))$. We conclude that in S, the equivalence classes are $(0,1), (2), (3), \ldots, (n)$, with

$$v \sim \begin{cases} (\phi(v)) & \text{if } \phi(v) > 1, \\ (0,1) & \text{if } \phi(v) = 1. \end{cases}$$

We next consider part (c), where vertices 0 and 1 have opposite colors. If we ignore vertex n and think of the diagram as in $V(D_n)$, then Proposition 2.3(c) says that the colors of $2, 3, \ldots, n-1$ are irrelevant. Namely all the diagrams in $\{v \in V(D_n);\ c(0) \neq c(1)\}$ are equivalent to one another. In particular if we let vertex $n-1$ be painted and apply F_{n-1}, then the color of vertex n is irrelevant too. We conclude that all the diagrams in part (c) are equivalent to one another. This completes the proof. □

By Proposition 2.4, to prove all the information for $V(B_n^{(1)})$ in Table 1, it remains only to show that the diagrams $(0,1), (1), (2), \ldots, (n)$ are not equivalent to one another. The diagram (1) is obvious, because the colors of vertices 0 and 1 remain different under all the F_i. For the other diagrams, we use the function ϕ of (1.4). The computation similar to (2.3) shows that $\phi \cdot F_i(v) = \phi(v)$. Since the values of ϕ on $(0,1), (2), (3), \ldots, (n)$ are different, they are not equivalent to one another. This proves all the cases for $V(B_n^{(1)})$ in Table 1.

The only possible nontrivial diagram involution for $B_n^{(1)}$ is given by $0 \leftrightarrow 1$, fixing the other vertices. In this case the arguments are similar to Proposition 2.4(a), and the equivalence classes are represented by diagrams with only one vertex painted from $2, 3, \ldots, n$, respectively.

2.4. $C_n^{(1)}$, $n > 1$

The vertices of $C_n^{(1)}$, $n > 1$, are labeled as follows:

Proposition 2.5. *Let $v \in V(C_n^{(1)})$. Then*

$$v \sim \begin{cases} (\phi(v)) \sim (n - \phi(v)) & \textit{if } 0, n \textit{ are unpainted}; & \text{(a)} \\ (0) \sim (n) & \textit{if exactly one of } 0, n \textit{ is painted}; & \text{(b)} \\ (0, n) & \textit{if } 0, n \textit{ are painted}. & \text{(c)} \end{cases}$$

Proof. We first consider part (a), namely the diagrams v with vertices $0, n$ unpainted. Since $0, n$ are long, they remain unpainted under any F_i. So by ignoring vertices 0 and n, such diagrams are isomorphic to A_{n-1}. By Proposition 2.3(a), $v \sim (\phi(v)) \sim (n - \phi(v))$. This proves part (a).

Next we consider part (b), where exactly one of $0, n$ is painted. Without loss of generality, let S be the diagrams with 0 unpainted and n painted. Once again the colors of 0 and n remain unchanged under any F_i. Let

$$T = \{v \in V(C_n);\ \text{vertex } n \text{ of } v \text{ is painted}\}. \tag{2.7}$$

By ignoring vertex 0, we obtain an isomorphism between S and T. By Proposition 2.3(b), the diagrams in T are all equivalent to (n). Therefore, the diagrams in S are all equivalent to (n). By symmetry of the diagram, $(0) \sim (n)$ in $V(C_n^{(1)})$.

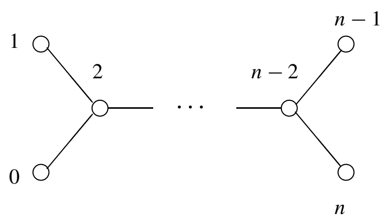
The argument for part (c) is similar to part (b). Namely by ignoring the painted vertex 0, the diagrams with $0, n$ painted can be identified with T of (2.7). By applying Proposition 2.3(b) again, it follows that the diagrams in part (c) are all equivalent to $(0, n)$. The proof follows. □

By Proposition 2.5, to prove the information for $V(C_n^{(1)})$ in Table 1, we only have to show that the diagrams (0), $(0, n)$ and $\{(N);\ 1 \leqslant N \leqslant \frac{n}{2}\}$ are not equivalent to one another. The colors of vertices 0 and n remain the same under all the F_i, so (0) and $(0, n)$ are not equivalent to the other diagrams in this list. As for $\{(N);\ 1 \leqslant N \leqslant \frac{n}{2}\}$, apply the function ϕ of (1.4) to them. Similar to the computation in (2.3), if v is a diagram with vertices 0 and n unpainted, then $\phi \cdot F_i(v)$ equals $\phi(v)$ or $n - \phi(v)$ for all $i = 1, \ldots, n-1$. So the diagrams in $\{(N);\ 1 \leqslant N \leqslant \frac{n}{2}\}$ are not equivalent to one another. This proves the information for $V(C_n^{(1)})$ in Table 1.

In $C_n^{(1)}$, the only nontrivial diagram involution is the reflection $0 \leftrightarrow n$, $1 \leftrightarrow n-1, \ldots.$ If n is odd (i.e. even number of vertices), then the involution has no fixed point and so all vertices remain unpainted. If n is even (i.e. odd number of vertices), then the involution has exactly one fixed point at vertex $\frac{n}{2}$. In this case there are two equivalence classes, given by $\frac{n}{2}$ painted or unpainted.

2.5. $D_n^{(1)}$, $n > 4$

As before, $c(i)$ denotes the color of vertex i. The vertices of $D_n^{(1)}$ are labeled as follows:



Proposition 2.6. *Let $v \in V(D_n^{(1)})$. Then*

$$v \sim \begin{cases} (\phi(v)) \sim (n - \phi(v)) & \text{if } c(0) = c(1),\ c(n-1) = c(n),\ \phi(v) \neq 1; \\ (0, 1) & \text{if } c(0) = c(1),\ c(n-1) = c(n),\ \phi(v) = 1; \end{cases} \quad \text{(a)}$$
$$\begin{cases} (0) & \text{if } c(0) = c(1),\ c(n-1) \neq c(n) \ (\text{or vice versa}); & \text{(b)} \\ (0, n) & \text{if } c(0) \neq c(1),\ c(n-1) \neq c(n). & \text{(c)} \end{cases}$$

Proof. We first prove part (a). Let S denote the diagrams v in which $c(0) = c(1)$ and $c(n-1) = c(n)$. By ignoring vertices 0 and n, we see that S is isomorphic to $V(A_{n-1})$. By Proposition 2.3(a),

$$v \sim \begin{cases} (\phi(v)) \sim (n - \phi(v)) & \text{if } \phi(v) \neq 1, \\ (0, 1) & \text{if } \phi(v) = 1. \end{cases}$$

By symmetry of the diagram, $(0, 1) \sim (n-1, n)$. This proves (a).

We next prove part (b). Without loss of generality, we may consider the Vogan diagrams S in which $c(0) = c(1)$ and $c(n-1) \neq c(n)$. So S is closed under each F_i. Let $T \subset V(D_n)$ be the diagrams where $c(n-1) \neq c(n)$. By ignoring vertex 0, we obtain an isomorphism $S \to T$. By Proposition 2.3(c), the diagrams in T are all equivalent to $(1) \in V(D_n)$. Therefore all the diagrams in S are equivalent to (1). By diagram automorphisms, they are also equivalent to (0), $(n-1)$ and (n). This completes the proof for (b).

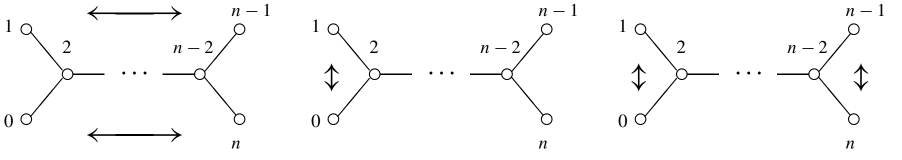
Next we prove (c). Let S be the diagrams with $c(0) \neq c(1)$ and $c(n-1) \neq c(n)$. Then S is closed under each F_i. The argument for Proposition 2.3(c) can be used to show that each $v \in S$ is equivalent to some $w \in S$ whose vertices $2, 3, \ldots, n-2$ are unpainted. For instance if $v = (0, 3, 4, n)$, we may perform F_3, F_2, F_1 and obtain $w = (1, n)$. We conclude that all the diagrams in S are equivalent to $(1, n)$. By diagram automorphisms, they are also equivalent to $(0, n-1)$, $(0, n)$ and $(1, n-1)$. This proves (c). $\square$

We now prove the information for $V(D_n^{(1)})$ in Table 1. If v and w belong to different parts of Proposition 2.6(a), (b) and (c) (for example if vertices 0 and 1 have the same color in v but different colors in w), then they are inequivalent. Since each of parts (b) and (c) consists of a single equivalence class, it suffices to show that in (a), the diagrams in

$$\{(0, 1)\} \cup \left\{(N);\ 2 \leqslant N \leqslant \frac{n}{2}\right\} \tag{2.8}$$

are mutually not equivalent. We modify ϕ of (1.4) by ignoring vertex n, so for instance $\phi(4, 6, n) = 6 - 4 = 2$. By a computation similar to (2.3), $\phi \cdot F_i(v) = \phi(v)$ or $n - \phi(v)$. If v and w are distinct diagrams chosen from (2.8), then $\phi(w)$ is neither $\phi(v)$ nor $n - \phi(v)$. So the diagrams in (2.8) are mutually not equivalent. This proves all the information for $V(D_n^{(1)})$ in Table 1.

Next we consider $V(\theta; D_n^{(1)})$ in Table 2. Up to diagram automorphisms, there are three cases for θ,



(a) (b) (c)

In (a), if n is odd, then there is no fixed point, so all vertices are unpainted. If n is even, there is one fixed point $\frac{n}{2}$, so there are two classes represented by $(\theta; \emptyset)$ and $(\theta; \frac{n}{2})$.

In (b), the diagram obtained by ignoring vertices 0 and 1 is simply D_{n-1}, so the distinct equivalence classes are represented by $\{(\theta; N);\ 2 \leqslant N \leqslant \frac{n+1}{2}\} \cup \{(\theta; n)\}$ [5].

In (c), the diagram obtained by ignoring vertices $0, 1, n-1, n$ is A_{n-3}, so the distinct equivalence classes are represented by $\{(\theta; N);\ 2 \leqslant N \leqslant \frac{n}{2}\}$ [5].

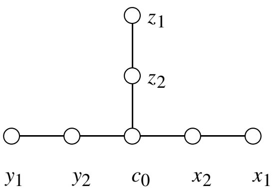
3. Exceptional nontwisted diagrams

In this section, we study the extended Vogan diagrams for $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$. Observe that if σ is a diagram automorphism, then $\sigma \cdot F_i = F_{\sigma(i)} \cdot \sigma$. So given a sequence of mixed F_i and σ_j, we can move the σ_j over the F_i and gather them. This proves the following proposition.

Proposition 3.1. *If diagrams v and w are equivalent, then there exist some $F_{i_1}, \ldots, F_{i_k}$ and diagram automorphisms $\sigma_{j_1}, \ldots, \sigma_{j_l}$ such that $\sigma_{j_l} \cdot \ldots \cdot \sigma_{j_1} \cdot F_{i_k} \cdot \ldots \cdot F_{i_1}(v) = w$.*

3.1. $E_6^{(1)}$

Label the vertices of $E_6^{(1)}$ as follows:



Given a Vogan diagram v, let

$$B(v) = \text{number of branches which contain painted vertices in } v. \tag{3.1}$$

In this definition we ignore vertex c_0, except that $B = 1$ if c_0 is the only painted vertex. For example, $B(c_0) = B(c_0, x_1, x_2) = 1$, while $B(x_1, y_1) = 2$.

We say that a vertex is odd or even depending on whether there are odd or even number of edges joined to it. Given a Vogan diagram v, let

$$M(v) = \text{number of painted odd vertices in } v. \tag{3.2}$$

So $0 \leqslant M(v) \leqslant 4$. For example, $M(c_0, x_1, x_2) = 2$, due to the odd vertices c_0 and x_1.

Proposition 3.2. *There are three equivalence classes of $V(E_6^{(1)})$, namely*

$$Z_1 = \{M(v) \text{ is odd}\},$$
$$Z_2 = \{M(v) \text{ is even and } B(v) \text{ is odd}\},$$
$$Z_3 = \{M(v) \text{ is even and } B(v) \text{ is even}\}.$$

Proof. Observe that in $E_6^{(1)}$, any vertex i has even number of adjacent odd vertices. Therefore F_i preserves the parity of M. We conclude that Z_1 and $Z_2 \cup Z_3$ are both unions of equivalence classes.

Direct manipulation with the various F_i shows that Z_1 is indeed one equivalence class by itself. We next check that each of Z_2 and Z_3 is preserved by the various F_i. Recall that c_0 is the central vertex. Clearly B is preserved by all the F_i except possibly F_{c_0}. So we only need to consider $F_{c_0}(v)$ for diagrams v which contain c_0.

First, consider Z_2. Here $M(v)$ is even and $B(v) = 1, 3$. If $B(v) = 1$, then up to diagram automorphisms, either $v = (c_0, x_1)$ or $v = (c_0, x_1, x_2)$. In either case $B(F_{c_0}(v)) = 3$. If $B(v) = 3$, then up to diagram automorphisms v has three possibilities, namely (c_0, x_1, y_2, z_2), $(c_0, x_1, x_2, y_2, z_2)$ and (c_0, x_1, y_1, z_1, s), where $s \subset \{x_2, y_2, z_2\}$. In the first two possibilities $B(F_{c_0}(v)) = 1$, and in the third $B(F_{c_0}(v)) = 3$. We conclude that Z_2 is preserved by all the F_i.

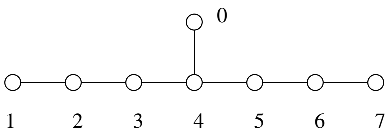
In Z_3, $M(v) = B(v) = 2$; and in particular if c_0 is painted, then $v = (c_0, x_1, y_2)$ or $v = (c_0, x_1, x_2, y_2)$ up to diagram automorphisms. It follows that $B(F_{c_0}(v)) = 2$.

We conclude that each of Z_2 and Z_3 is preserved by all the F_i and so is a union of equivalence classes. Direct manipulation with the various F_i shows that each of them is indeed one equivalence class. The proposition is proved. $\square$

The above proposition proves the information for $V(E_6^{(1)})$ in Table 1. The case with nontrivial diagram involution θ is easy. Up to diagram automorphisms, θ is given by $\{y_1 \leftrightarrow z_1$ and $y_2 \leftrightarrow z_2\}$. The fixed points of θ are x_1, x_2 and c_0. From A_3, we know that there are three equivalence classes. They are represented by $(\theta; \emptyset)$, $(\theta; x_1)$ and $(\theta; x_2)$.

3.2. $E_7^{(1)}$

Label the vertices of $E_7^{(1)}$ as follows:



If v, w are equivalent diagrams,

$$\text{a switching sequence } \langle i_1, \ldots, i_k \rangle \tag{3.3}$$

for (v, w) is a sequence of $F_{i_1}, \ldots, F_{i_k}$ such that $F_{i_k} \cdot \ldots \cdot F_{i_1}(v) = w$. For example $\langle 1, 2, 3 \rangle$ is a switching sequence for $((1), (3, 4))$. There is only one nontrivial diagram automorphism on $V(E_7^{(1)})$ given by the reflection $r(i_1, \ldots, i_k) = (8 - i_k, \ldots, 8 - i_1)$. So by Proposition 3.1, if $v, w \in V(E_7^{(1)})$ are equivalent, then either $s(v) = w$ or $r \cdot s(v) = w$, where s is a switching sequence. For a switching sequence s, let

$$t_i = \text{number of times entry } i \text{ appears in } s. \tag{3.4}$$

Lemma 3.3. (*For diagrams with single edges only.*) *Let s be a switching sequence for (v, w). Then vertex i has the same color in v and w if and only if $\sum_{j\in N(i)} t_j$ is even.*

Here $N(i)$ is the neighborhood of vertex i as defined in (1.1). The lemma is obvious and we omit the proof. Lemma 3.3 will be useful when proving inequivalence of some diagrams. Recall that ϕ and ξ are as defined in (1.4) and (1.6).

Lemma 3.4. *Each F_i preserves the parities of ϕ and ξ.*

Proof. As in (1.5), write $v \in V(E_7^{(1)})$ in the form $v = (s, i_1, \ldots, i_a, i_{a+1}, \ldots, i_k)$, where $1 \leqslant i_1 < \cdots < i_a \leqslant 4 < i_{a+1} < \cdots < i_k \leqslant 7$, and $s \subset \{0\}$. First we show that each F_i preserves the parity of ϕ. By using arguments similar to (2.3) and (2.4), it is clear that this is true for $1 \leqslant i \leqslant 7$. It remains to show that F_0 also preserves the parity of ϕ. Suppose that 0 is painted. Since

$$\begin{aligned}\phi \cdot F_0(v) &= \phi(0, i_1, \ldots, i_a, 4, i_{a+1}, \ldots, i_k)\\ &= \sum_{r=a+1}^{k} (-1)^{k-r}(i_r) + (-1)^{k-a}\left(4 - \sum_{p=1}^{a}(-1)^{a-p} i_p\right)\\ &= \phi(v) + (-1)^{k-a}\left(4 - 2\sum_{p=1}^{a}(-1)^{a-p} i_p\right),\end{aligned}$$

F_0 preserves the parity of ϕ.

Next we show that each F_i preserves the parity of ξ. For $i \neq 4$, F_i preserves the value $\sum_{p=1}^{a}(-1)^{a-p} i_p$ and the color of vertex 0, and hence preserves the parity of ξ. Since F_4 changes the colors of vertices 3 and 0, it follows that F_4 also preserves the parity of ξ. This proves the lemma. □

We shall show that $V(E_7^{(1)})$ consists of the following four equivalence classes,

$$\begin{aligned}Z_1 &= \{\phi \text{ is odd}\},\\ Z_2 &= \{\phi \text{ and } \xi \text{ are even}\},\\ Z_3 &= \{\phi = 0, 4 \text{ and } \xi \text{ is odd}\},\\ Z_4 &= \{\phi = 2, 6 \text{ and } \xi \text{ is odd}\}.\end{aligned} \tag{3.5}$$

This will be proved using the next two propositions.

Proposition 3.5. *Let $v \in V(E_7^{(1)})$. Then*

(a) $v \in Z_1 \Rightarrow v \sim (1)$,
(b) $v \in Z_2 \Rightarrow v \sim (2)$,
(c) $v \in Z_3 \Rightarrow v \sim (0)$,
(d) $v \in Z_4 \Rightarrow v \sim (1, 7)$.

Proof. By Theorem 1.2, v is equivalent to a diagram of the form (k) or $(0,k)$.

We first prove part (a). It suffices to show that the diagrams in $\{(k),(0,k);\ k \text{ is odd}\}$ are mutually equivalent. Further, by symmetry of the diagram, it suffices to show that $(0,1) \sim (7) \sim (0,3) \sim (5)$. We do this by giving their switching sequences:

$$\begin{aligned} &(0,1) \sim (7) \quad \text{by } \langle 1,2,3,4,5,6,7\rangle, \\ &(7) \sim (0,3) \quad \text{by } \langle 7,6,5,4,0\rangle, \\ &(0,3) \sim (5) \quad \text{by } \langle 3,2,1,4,3,2,5,4,3,0,4,5\rangle. \end{aligned}$$

This proves part (a) as claimed.

For $v \in Z_2$, $v \sim (k)$ where k is even. Since $(2) \sim (6)$ by symmetry of the diagram and $(2) \sim (4)$ by $\langle 2,3,4,0,1,2,3,4\rangle$, we have $(2) \sim (4) \sim (6)$. This proves part (b).

The only possibilities of the form (k) and $(0,k)$ are (0), $(0,4)$ in Z_3 and $(0,2)$, $(0,6)$ in Z_4. Clearly each pair of above diagrams are equivalent and since $(0,6) \sim (1,7)$ by $\langle 6,5,4,3,2,1\rangle$, this proves parts (c) and (d). Hence we complete the proof. $\square$

Proposition 3.6. *Each Z_i of* (3.5) *is a union of equivalence classes.*

Proof. The diagram reflection preserves the parity of ϕ. So together with Lemma 3.4, we have

$$\{\phi \text{ is even}\} \nsim \{\phi \text{ is odd}\}.$$

It follows that Z_1 and $Z_2 \cup Z_3 \cup Z_4$ are unions of equivalence classes.

By Theorem 1.2 and Lemma 3.4, for any $v \in Z_i$, there exists an even $k \neq 0$ such that

$$\begin{aligned} v \in Z_2 \quad &\Rightarrow \quad v \sim (k), \\ v \in Z_3 \cup Z_4 \quad &\Rightarrow \quad v \sim (0,k). \end{aligned} \tag{3.6}$$

We claim that

$$k_1, k_2 \in \{2,4,6\} \quad \Rightarrow \quad (k_1) \nsim (0,k_2). \tag{3.7}$$

Suppose otherwise, namely $(k_1) \sim (0,k_2)$ for some $k_1, k_2 \in \{2,4,6\}$. By Proposition 3.1, there exists a switching sequence s and the reflection r such that

$$s(k_1) = (0,k_2) \tag{3.8}$$

or $r \cdot s(k_1) = (0,k_2)$. In the latter case, we may replace k_2 by $8-k_2$ to eliminate r and again obtain (3.8). Apply Lemma 3.3 to (3.8), we see that t_4 is odd because vertex 0 changes color, then t_2 is odd because vertex 3 does not change color. This is a contradiction because vertex 1 does not change color. This proves (3.7) as claimed.

In (3.7), $\xi(k_1)$ and $\xi(0,k_2)$ have different parities. So by (3.6) and (3.7), it follows that Z_2 and $Z_3 \cup Z_4$ are both unions of equivalence classes. To complete the proof of the proposition, it remains to prove that Z_3 and Z_4 are both unions of equivalence classes. By

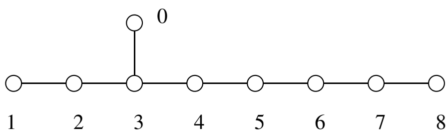
Proposition 3.5(c), (d), this will follow from $(0) \nsim (1,7)$. Unfortunately, and surprisingly, its argument is much harder than other inequivalences, so we accept it for now, leaving $(0) \nsim (1,7)$ for Appendix A. So Z_3, Z_4 are both equivalence classes. This completes the proof. □

It follows from Propositions 3.5 and 3.6 that $V(E_7^{(1)})$ consists of the four equivalence classes given in (3.5).

For $E_7^{(1)}$, the nontrivial involution θ fixes vertices 0 and 4, with $i \leftrightarrow 8-i$. Regarding vertices 0 and 4 as type A_2, there are two distinct classes represented by $(\theta; \emptyset)$ and $(\theta; 0)$.

3.3. $E_8^{(1)}$

Next we study the equivalence classes of $E_8^{(1)}$. Label the vertices of $E_8^{(1)}$ as follows:



Here the only diagram involution is the trivial one. We shall show that there are three nontrivial equivalence classes represented by

$$(1),\ (7),\ (8). \tag{3.9}$$

Proposition 3.7. *The three diagrams* $(1), (7), (8) \in V(E_8^{(1)})$ *are mutually not equivalent.*

Proof. To prove this, we assume that a switching sequence exists between two diagrams and derive a contradiction. By Lemma 3.3, we can prove inequivalence for most of them:

Suppose that there is a switching sequence for $((1),(8))$. The colors of vertices 0, 4, 6 are unchanged. So t_3 is even, which implies that t_5 is even, which implies that t_7 is even. This is a contradiction because vertex 8 changes colors. The same arguments show that there is no switching sequence for $((7),(8))$. So we have

$$(1) \nsim (8), \qquad (7) \nsim (8).$$

Unfortunately, the remaining arguments for $(1) \nsim (7)$ require a lengthy proposition, and we leave it for Appendix A. For the time being, we accept the fact that $(1) \nsim (7)$. □

In the next three propositions, we shall show that every other diagram in $V(E_8^{(1)})$ is equivalent to one of (3.9).

Lemma 3.8.

(a) *For* $q \geqslant 4$ *and* $p = 2, 3$, *we get* $(p,q) \sim (0, p-1, q-1)$ *and* $(0,p,q) \sim (p-1, q-1)$.
(b) *For* $q \geqslant 4$, $(1,q) \sim (0, q-1)$ *and* $(0,1,q) \sim (q-1)$.

Proof. This follows directly from [5, Lemma 3.1]. Note that vertex 0 is denoted by the notation $*$ in [5]. $\square$

The next proposition simplifies a diagram of the form (α) or $(0,\alpha)$ to (3.9).

Proposition 3.9. *The equivalence classes of* (α) *and* $(0,\alpha)$ *in* $E_8^{(1)}$ *are given by*

(a) $(1) \sim (5) \sim (0,4) \sim (0,8)$,
(b) $(7) \sim (2) \sim (3) \sim (0,6)$,
(c) $(8) \sim (0) \sim (4) \sim (6) \sim (0,N)$, *for* $N = 1,2,3,5,7$.

Proof. We prove this proposition by Lemma 3.8 and switching sequences. For part (a),

$$\begin{aligned}
\mathbf{(5)} &\sim (0,1,6) && \text{by Lemma 3.8(b)}\\
&\sim (1,5) && \text{by } \langle 0,1,3,4,5\rangle\\
&\sim \mathbf{(0,4)} && \text{by Lemma 3.8(b)}\\
&\sim \mathbf{(1)} && \text{by } \langle 0,3,2,1\rangle\\
&\sim \mathbf{(0,8)} && \text{by } \langle 1,2,3,4,5,6,7,8\rangle.
\end{aligned}$$

For part (b),

$$\begin{aligned}
\mathbf{(0,6)} &\sim (1,7) && \text{by Lemma 3.8(b)}\\
&\sim (0,2,8) && \text{by Lemma 3.8(a)}\\
&\sim \mathbf{(7)} && \text{by } \langle 0,3,4,5,6,7\rangle\\
&\sim (0,1,8) && \text{by Lemma 3.8(b)}\\
&\sim \mathbf{(2)} && \text{by } \langle 8,7,6,5,4,3,2\rangle\\
&\sim (0,1,4) && \text{by } \langle 2,3,0\rangle\\
&\sim \mathbf{(3)} && \text{by Lemma 3.8(b).}
\end{aligned}$$

For part (c),

$$\begin{aligned}
\mathbf{(0,5)} &\sim (1,6) && \text{by Lemma 3.8(b)}\\
&\sim (0,2,7) && \text{by Lemma 3.8(a)}\\
&\sim \mathbf{(6)} && \text{by } \langle 0,3,4,5,6\rangle\\
&\sim (0,1,7) && \text{by Lemma 3.8(b)}\\
&\sim (2,8) && \text{by Lemma 3.8(a)}\\
&\sim \mathbf{(0)} && \text{by } \langle 8,7,6,5,4,3,0\rangle\\
&\sim \mathbf{(0,3)} && \text{by } \langle 0\rangle\\
&\sim (1,4) && \text{by Lemma 3.8(b)}\\
&\sim (0,2,5) && \text{by Lemma 3.8(a)}\\
&\sim \mathbf{(4)} && \text{by } \langle 0,3,4\rangle
\end{aligned}$$

$$\begin{aligned}
&\sim (0,1,5) && \text{by Lemma 3.8(b)}\\
&\sim (2,6) && \text{by Lemma 3.8(a)}\\
&\sim \mathbf{(0,7)} && \text{by } \langle 6,5,4,3,0\rangle\\
&\sim (1,8) && \text{by Lemma 3.8(b)}\\
&\sim \mathbf{(0,2)} && \text{by } \langle 8,7,6,5,4,3,2\rangle\\
&\sim \mathbf{(0,1)} && \text{by } \langle 0,2,1\rangle\\
&\sim \mathbf{(8)} && \text{by } \langle 1,2,3,4,5,6,7,8\rangle. \qquad \square
\end{aligned}$$

By the above proposition, the only remaining problem is to simplify a diagram to (α) or $(0,\alpha)$. We divide the diagrams into the following two cases:

$$\begin{array}{ll}
\text{(a)} & \begin{cases} (s,2,4), (s,1,3,4), (s,2,3,8), (s,1,2,8), (s,1,8), \\ (s,3,4 \leqslant j_1,\dots,j_l) \text{ where } \phi(j_1,\dots,j_l) = 5, \end{cases}\\
\text{(b)} & \text{diagrams which do not belong to (a).}
\end{array} \tag{3.10}$$

In (3.10)(a), $s \subset \{0\}$ depending on whether vertex 0 is painted. The reason for this division is that we shall apply [5, Proposition 3.2] which is not valid for the special cases (3.10)(a).

Proposition 3.10. *The diagrams* (s,v) *in* (3.10)(a) *are equivalent to* $(t,7)$*, where* $t \subset \{0\}$*. In the first row of* (3.10)(a)*,* $s \neq t$*. In the second row of* (3.10)(a)*,* $s = t$*.*

Proof. Observe that the diagrams in the first and the second row of (3.10)(a) are equivalent to $(s,2,4)$ and $(s,3,5)$, respectively. By [5, Proposition 3.5], $(s,2,4) \sim (t,3,5) \sim (t,7)$ where $s \neq t$. This completes the proof. $\square$

By Propositions 3.9 and 3.10, we have solved the diagrams in (3.10)(a).

We now consider (3.10)(b). Denote a diagram v by

$$v = (s, i_1, \dots, i_a, i_{a+1}, \dots, i_k),$$

where $1 \leqslant i_1 < \cdots < i_a \leqslant 3 < i_{a+1} < \cdots < i_k \leqslant 8$ and $s \subset \{0\}$. Let I, J be defined by

$$I = \sum_{p=1}^{a} (-1)^{a-p} i_p \quad \text{and} \quad J = \sum_{p=a+1}^{k} (-1)^{k-p} i_p \tag{3.11}$$

and let

$$\alpha = \begin{cases} J - I & \text{if } J > 4 \text{ or } J = 4,\ k-a = 1, \\ 9 - J - I & \text{if } J < 4 \text{ or } J = 4,\ k-a \neq 1. \end{cases} \tag{3.12}$$

In fact, there are only two cases for $J = 4$: $v = (i_1,\dots,i_a,4)$ (with $k-a=1$) and $v = (i_1,\dots,i_a,4,8)$ (with $k-a=2$). Using I and α defined above, the next proposition simplifies v to (α) or $(0,\alpha)$.

Proposition 3.11. *Let $v = (s, i_1, \dots, i_k)$ be in* (3.10)(b). *Then $v \sim (t, \alpha)$, where $s = t$ if I is even, and $s \neq t$ if I is odd.*

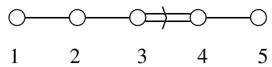
Proof. The proof follows from [5, Proposition 3.2]. □

By Propositions 3.9 and 3.11, we have solved the diagrams in (3.10)(b).

Propositions 3.7, 3.9, 3.10, 3.11 explain all the information for $V(E_8^{(1)})$ in Table 1 as follows. Proposition 3.7 shows that there are at least three distinct classes, represented by (1), (7), (8). The remaining propositions explain how any $v \in V(E_8^{(1)})$ is equivalent to one of them. Namely, if v belongs to (3.10)(a), we apply Propositions 3.9 and 3.10. And if v belongs to (3.10)(b), we apply Propositions 3.9 and 3.11.

3.4. $F_4^{(1)}$

Label the vertices of $F_4^{(1)}$ as follows:



The only diagram involution is the trivial one. In the next proposition, we write a typical $v \in V(F_4^{(1)})$ as

$$v = (v_1, v_2), \quad v_1 \subset \{1, 2, 3\}, \quad v_2 \subset \{4, 5\}.$$

So v_1 is a Vogan diagram of A_3.

Proposition 3.12. *Let $v, w \in V(F_4^{(1)})$ both contain painted vertices. Then $v \sim w$ if and only if $v_1 \sim w_1$.*

Proof. In what follows, "F_4" refers to the algorithm (1.2) on vertex 4, rather than the diagram of type F_4. Since there is no risk of confusion, we do not create extra notation to distinguish them. Since $(4) \sim (4, 5) \sim (5)$, and since F_4, F_5 do not change the colors of vertices 1, 2 and 3, the proposition obviously holds when $v_1 = \emptyset$ or $w_1 = \emptyset$. Therefore, in what follows, we may assume that $v_1 \neq \emptyset$ and $w_1 \neq \emptyset$. By applying F_4 and F_5, obviously

$$(v_1, 4) \sim (v_1, 4, 5) \sim (v_1, 5), \tag{3.13}$$

for any v_1. So we may assume that $v_2 = (5)$.

We first claim that

$$(v_1, v_2) \sim (v_1, \emptyset). \tag{3.14}$$

That is equivalent to prove that

$$(v_1, 5) \sim (v_1, \emptyset). \tag{3.15}$$

Since $v_1 \neq \emptyset$, there exists a sequence s of operations involving F_1 and F_2 such that vertex 3 is painted in sv_1 (if 3 is already painted in v_1, we may take $s = 1$). Hence

$$(v_1, 5) \sim (sv_1, 5). \tag{3.16}$$

By applying F_3, F_4, F_3 in that order,

$$(sv_1, 5) \sim (sv_1, \emptyset). \tag{3.17}$$

Apply s^{-1} to $(sv_1, \emptyset)$, we get

$$(sv_1, \emptyset) \sim (v_1, \emptyset). \tag{3.18}$$

Then (3.16), (3.17) and (3.18) lead to (3.15). This proves (3.14) as claimed.

We are now ready to prove the proposition. Suppose that $v_1 \sim w_1$. So there is a sequence t of operations involving F_1, F_2, F_3, such that $t(v_1) = w_1$. Since F_3 may affect the color of vertex 4, we have

$$t(v_1, v_2) = (w_1, x),$$

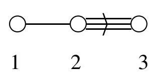
for some x. By (3.14), $(w_1, x) \sim (w_1, \emptyset) \sim (w_1, w_2)$. It follows that $v \sim w$.

Conversely, suppose that $v \sim w$. Let r be a sequence of F_i such that $r(v) = w$. Let r_1 be the subsequence of r obtained by removing all the F_4 and F_5 in r. Then $r_1(v_1) = w_1$, so $v_1 \sim w_1$. This completes the proof of the proposition. □

The equivalence classes for $v_1 \in V(A_3)$ and $v_2 \in V(A_2)$ are well known. So Proposition 3.12 proves the information for $V(F_4^{(1)})$ in Table 1.

3.5. $G_2^{(1)}$

Since the effect of F_i on the Vogan diagrams of $G_2^{(1)}$ is the same as that of A_3, the equivalence classes of $G_2^{(1)}$ is that of A_3 given in [5, Table 1]. That is, if we label the vertices of $G_2^{(1)}$ by



then there are two equivalence classes in $V(G_2^{(1)})$, namely

$$(1) \sim (1,2) \sim (2,3) \sim (3) \quad \text{and} \quad (2) \sim (1,3) \sim (1,2,3).$$

Since the diagram is not symmetric, there is no nontrivial diagram involution.

4. Twisted diagrams

In this section, we study the extended Vogan diagrams for $A_n^{(2)}$, $D_{n+1}^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$. Here the only possible nontrivial θ are in $A_{2n-1}^{(2)}$ and $D_{n+1}^{(2)}$. We separate $A_n^{(2)}$ into three cases $n=2$, even $n>2$, odd $n>3$.

4.1. $A_2^{(2)}$

Label the vertices of $A_2^{(2)}$ as follows:

α_0 α_1

As mentioned before, type $A_2^{(2)}$ is not covered in (1.2). We now treat it separately.

Proposition 4.1. *There are two inequivalent nontrivial diagrams of $A_2^{(2)}$ given by*

$$\{\alpha_0 \textit{ painted alone}\} \quad \textit{and} \quad \{\alpha_1 \textit{ painted alone}\}.$$

Proof. Similar to Proposition 2.1, we want to consider the effects of the Weyl reflections $r_{\alpha_0}, r_{\alpha_1}$ on the diagram. Recall that the Cartan matrix of $A_2^{(2)}$ is $\left(\begin{smallmatrix} 2 & -4 \\ -1 & 2 \end{smallmatrix}\right)$, and the positive roots are [6, p. 94]

$$\begin{aligned} \Delta^+ = \big\{&4k\alpha_0 + (2k-1)\alpha_1, 4(k-1)\alpha_0 + (2k-1)\alpha_1, (2k-1)\alpha_0 + k\alpha_1, \\ &(2k-1)\alpha_0 + (k-1)\alpha_1, 2k\alpha_0 + k\alpha_1;\ \text{where } k = 1, 2, \ldots\big\}. \end{aligned}$$

It implies that

$$\alpha_1, \alpha_0 + \alpha_1, 2\alpha_0 + \alpha_1, 3\alpha_0 + \alpha_1, 4\alpha_0 + \alpha_1 \in \Delta^+. \tag{4.1}$$

Suppose that α_0 is painted. We claim that F_{α_0} does not change the color of α_1. Since the upper right entry of the Cartan matrix is -4, by [6, p. 86],

$$r_{\alpha_0}(\alpha_1) = \alpha_1 - (-4)\alpha_0 = 4\alpha_0 + \alpha_1. \tag{4.2}$$

Let $c(\cdot)$ denote "the color of," as in Proposition 2.1. Since α_0 is painted, by (4.1), $c(k\alpha_0+\alpha_1) \neq c((k+1)\alpha_0+\alpha_1)$ for all $k = 0, 1, 2, 3$. Hence $c(\alpha_1) = c(4\alpha_0+\alpha_1)$. By (4.2), we conclude that F_{α_0} does not change the color of α_1, as claimed.

Next, suppose that α_1 is painted. We claim that F_{α_1} reverses the color of α_0. Since the lower left entry of the Cartan matrix is -1, by [6, p. 86],

$$r_{\alpha_1}(\alpha_0) = \alpha_0 - (-1)\alpha_0 = \alpha_0 + \alpha_1.$$

Since α_1 is painted, $c(\alpha_0) \neq c(\alpha_0 + \alpha_1)$. So F_{α_1} reverses the color of α_0 as claimed.

We conclude that there are two nontrivial equivalence classes, represented by {α_0 painted alone} and {α_1 painted alone}. Note that {α_0 and α_1 painted} is equivalent to {α_1 painted alone}, via F_{α_1}. This proves the proposition. □

4.2. $A_{2n}^{(2)}$, $n > 1$

Next we consider Vogan diagrams of $A_{2n}^{(2)}$, $n > 1$. Label the vertices as follows:

Throughout this section, ϕ denotes the function defined in (1.4).

Proposition 4.2. *Let* $v \in V(A_{2n}^{(2)})$. *Then*

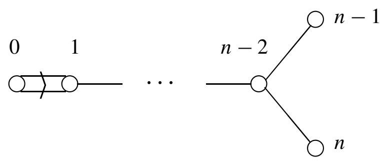
(a) $v \sim (0)$, *if the vertex* 0 *is painted in* v,
(b) *if the vertex* 0 *is not painted in* v, *then* $v \sim (\phi(v))$, $1 \leqslant \phi(v) \leqslant n$.

Proof. Since the vertex 0 represents the longest root, part (a) follows from Proposition 2.3(b). Next suppose that the vertex 0 is not painted in v. Since vertex 0 remains unpainted under any F_i, we can ignore it and regard the remaining diagram as a diagram of B_n. Hence (b) follows from Proposition 2.3(a). This completes the proof. □

Similar to the argument in (2.3), $F_1, \ldots, F_n$ preserve ϕ. Therefore, the diagrams $\{(N);\ 1 \leqslant N \leqslant n\}$ in Proposition 4.2(b) are mutually not equivalent. This proves the information for $V(A_{2n}^{(2)})$ in Table 1.

4.3. $A_{2n-1}^{(2)}$, $n > 2$

We label the vertices of $A_{2n-1}^{(2)}$ as follows:



We shall show that $V(A_{2n-1}^{(2)})$ consists of the following four equivalence classes,

$$\begin{aligned}
Z_1 &= \{c(n-1) = c(n) \text{ and } 0 \text{ is painted}\},\\
Z_2 &= \{c(n-1) \neq c(n) \text{ and } 0 \text{ is painted}\},\\
Z_3 &= \{c(n-1) = c(n) \text{ and } 0 \text{ is unpainted}\},\\
Z_4 &= \{c(n-1) \neq c(n) \text{ and } 0 \text{ is unpainted}\}.
\end{aligned}$$

Proposition 4.3. *If $v \in Z_i$, $w \in Z_j$ and $i \neq j$, then $v \nsim w$.*

Proof. Notice that F_i preserves the color of the long root 0. Moreover, if a diagram v satisfies $c(n-1) = c(n)$ or $c(n-1) \neq c(n)$, then the same property is satisfied by all the diagrams equivalent to v. □

Proposition 4.4. *Let $v \in V(A^{(2)}_{2n-1})$. Then*

(a) $v \in Z_1 \Rightarrow v \sim (0)$,
(b) $v \in Z_2 \Rightarrow v \sim (0, n)$,
(c) $v \in Z_3 \Rightarrow v \sim (\phi(v)) \sim (n - \phi(v))$,
(d) $v \in Z_4 \Rightarrow v \sim (n)$.

Proof. Consider parts (a) and (b), where 0 is painted in v. By Theorem 1.2, v is equivalent to a diagram w with at most two painted vertices. In part (a), $w \in Z_1$ by Proposition 4.3, so $w = (0)$ or $w = (0, k)$ for some $k \leqslant n-2$. Using the arguments in [5, Proposition 2.4(b)], we see that $(0) \sim (0, k)$ for $k \leqslant n-2$. This proves (a). In part (b), $w \in Z_2$ by Proposition 4.3, so $w = (0, n-1)$ or $w = (0, n)$. This proves (b).

Next we prove (c), (d) simultaneously. Since the vertex 0 is long, the color of 0 does not change under any F_i. So we can ignore vertex 0 and its adjacent edges and regard it as a diagram of D_n. Hence (c), (d) follow from Proposition 2.3(c) and we are done. □

By Proposition 4.3, each Z_i is a union of equivalence classes. Proposition 4.4 says that in addition, each of Z_1, Z_2 and Z_4 is an equivalence class. In Z_3, we see that $\phi \cdot F_i(v)$ equals $\phi(v)$ or $n - \phi(v)$, so the distinct equivalence classes in Z_3 are represented by $\{(N);\ 1 \leqslant N \leqslant \frac{n}{2}\}$. This proves all the information for $V(A^{(2)}_{2n-1})$ in Table 1.

For $A^{(2)}_{2n-1}$, the only nontrivial involution is given by $\theta(n-1) = n$. Regarding vertices $0, \dots, n-2$ as type C_{n-1}, there are $\frac{n+3}{2}$ distinct classes represented by $(\theta; \emptyset)$ and $(\theta; N)$, $0 \leqslant N \leqslant \frac{n-1}{2}$.

4.4. $D^{(2)}_{n+1}$, $n > 1$

Label the vertices of $D^{(2)}_{n+1}$ as follows:

Proposition 4.5. *Let $v = (i_1, \dots, i_k) \in V(D^{(2)}_{n+1})$, $n > 1$. Then*

$$v \sim \begin{cases} (\phi(v)) \sim (n - \phi(v)) & \text{if } k \text{ is odd;} \\ (0, \phi(v)) & \text{if } k \text{ is even.} \end{cases}$$

Furthermore, $\{(0), (n)\}$ form an equivalence class.

Proof. We first claim that $\phi \cdot F_i(v) = \phi(v)$ for all i. By the same argument as in (2.3), we have $\phi \cdot F_i(v) = \phi(v)$ for $i \neq 0, n$. Since 0 and n are short, F_0 and F_n do not change the colors of their neighborhoods. So $\phi \cdot F_i(v) = \phi(v)$ for all $i = 0, \ldots, n$ as claimed.

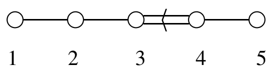
By Theorem 1.2, v is equivalent to some diagram w with one or two painted vertices. Further, each F_i preserves the parity of the number of painted vertices in v. So if v has odd (respectively even) number of painted vertices, then w has one (respectively two) painted vertex. Also, $\phi(v) = \phi(w)$. The equivalence of $(\phi(v))$ and $(n - \phi(v))$ follows from the symmetry of the diagram. And the last statement is obvious since vertices 0 and n are short. So we complete the proof. □

Regarding the subdiagram with $1, \ldots, n-1$ as type A_{n-1}, it follows that the diagrams in $\{(N);\ 0 \leqslant N \leqslant \frac{n}{2}\} \cup \{(0, N);\ 1 \leqslant N \leqslant n\}$ are mutually not equivalent. So together with Proposition 4.5, this proves the information for $V(D^{(2)}_{n+1})$ in Table 1.

In $D^{(2)}_{n+1}$, the only nontrivial diagram involution is the reflection $0 \leftrightarrow n$, $1 \leftrightarrow n-1, \ldots.$ If n is odd (i.e. even number of vertices), then the involution has no fixed point and so all vertices remain unpainted. If n is even (i.e. odd number of vertices), then the involution has exactly one fixed point at vertex $\frac{n}{2}$. In this case there are two equivalence classes, given by $\frac{n}{2}$ painted or unpainted.

4.5. $E^{(2)}_6$

Label the vertices of $E^{(2)}_6$ as follows:



In the next proposition, we write a typical $v \in V(E^{(2)}_6)$ as

$$v = (v_1, v_2), \quad v_1 \subset \{1, 2, 3\}, \quad v_2 \subset \{4, 5\}.$$

So v_1 is a Vogan diagram of A_3.

Proposition 4.6. *Let $v = (v_1, v_2) \in V(E^{(2)}_6)$. Then there are four equivalence classes of $V(E^{(2)}_6)$, given by*

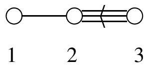
$$(1) \in \{v_2 = \emptyset \text{ and } \phi(v) \text{ is odd}\},$$
$$(2) \in \{v_2 = \emptyset \text{ and } \phi(v) \text{ is even}\},$$
$$(4) \in \{v_2 \neq \emptyset \text{ and } \phi(v) \text{ is even}\},$$
$$(5) \in \{v_2 \neq \emptyset \text{ and } \phi(v) \text{ is odd}\}.$$

Proof. By direct computations, we see that each F_i preserves the parity of $\phi(v)$. Suppose that $v \sim w \in V(E^{(2)}_6)$. Since vertex 4 is longer than vertex 3, $v_2 = \emptyset$ if and only if $w_2 = \emptyset$.

So each of the four subsets in the proposition is a union of equivalence classes. Direct manipulations show that their diagrams are equivalent to (1), (2), (4) and (5), respectively. Hence the proposition follows. □

4.6. $D_4^{(3)}$

By the same arguments as in $G_2^{(1)}$, if we label the vertices of $D_4^{(3)}$ by



then the equivalence classes are given by

$$(1) \sim (1,2) \sim (2,3) \sim (3) \quad \text{and} \quad (2) \sim (1,3) \sim (1,2,3).$$

Acknowledgments

The authors thank the referee for helping to improve the presentation of this article. Also, G.J. Chang showed us the method of switching sequences as used in Appendix A.

Appendix A

In this section, we show that $(0) \nsim (1,7)$ in $E_7^{(1)}$, and that $(1) \nsim (7)$ in $E_8^{(1)}$. This will complete the proofs of Propositions 3.6 and 3.7. We have isolated these remaining steps into two propositions here, because their arguments are very lengthy and purely computational. It would be nice to replace them with more concise and instructive arguments.

Recall that we define the switching sequence $\langle i_1, \ldots, i_k \rangle$ in (3.3). Define the lexicographic ordering on the set of all switching sequences as follows. Given $s = \langle i_1, \ldots, i_k \rangle$ and $u = \langle j_1, \ldots, j_l \rangle$, we declare that $s < u$ by

$$s < u \equiv \begin{cases} k < l, & \text{or} \\ k = l, & i_1 = j_1, \ldots, i_a = j_a \text{ and } i_{a+1} < j_{a+1} \text{ for some } a. \end{cases} \tag{A.1}$$

The following lemma on switching sequences will be useful. We omit the proof, which is obvious. Recall that $N(i)$ is the neighborhood of vertex i, as defined in (1.1).

Lemma A.1.

(a) *If $s = \langle \ldots, i, j_1, \ldots, j_r, i, \ldots \rangle$ and $j_a \neq i$ for all a, then $N(i)$ appears even number of times in $j_1, \ldots, j_r$.*
(b) *If $s = \langle \ldots, i, j, \ldots \rangle$ and the vertices i, j are not adjacent, then they can be interchanged and $s = \langle \ldots, j, i, \ldots \rangle$.*

Proposition A.2. *In* $E_7^{(1)}$, (0) *is not equivalent to* $(1,7)$.

Proof. Since the diagram reflection fixes (0) and $(1,7)$, by Proposition 3.1, it suffices to show that there is no switching sequence for $((0),(1,7))$. Suppose otherwise, let s be a switching sequence for $((0),(1,7))$ which is minimum in the sense of (A.1). We now start our series of arguments to obtain a contradiction. By Lemma 3.3,

$$\sum_{j\in N(i)} t_j = \begin{cases} t_4 \text{ is odd} & \text{for } i=0; \\ t_2 \text{ is odd} & \text{for } i=1; \\ t_6 \text{ is odd} & \text{for } i=7; \\ t_0+t_3+t_5 \text{ is even} & \text{for } i=4; \\ t_{i-1}+t_{i+1} \text{ is even} & \text{for other } i. \end{cases}$$

We will often make use of t_4. So denote it by

$$m = t_4$$

and note that m is odd. Write

$$s = \langle 0, 4_1, X_1, Y_1, Z_1, 4_2, X_2, \ldots, 4_m, X_m, Y_m, Z_m\rangle,$$
$$X_i \subset \{0\}, \qquad Y_i \subset \{1,2,3\}, \qquad Z_i \subset \{5,6,7\}.$$

Here 4_i denotes the ith time entry 4 appears in s. For example if s starts with $\langle 0,4,3,2,5,4,\ldots\rangle$, then the ordered sets satisfy $X_1=\emptyset$, $Y_1=\{3,2\}$, $Z_1=\{5\}$ and so on. We claim that

$$\begin{aligned} Y_i &= \langle 3,2,1\rangle, \langle 3,2\rangle, \langle 3\rangle, \emptyset \quad \text{for all } i=1,\ldots,m, \\ Z_i &= \langle 7,6,5\rangle, \langle 6,5\rangle, \langle 5\rangle, \emptyset \quad \text{for all } i=1,\ldots,m-1. \end{aligned} \tag{A.2}$$

Note that nonempty Y_i has to start with 3. This is because if Y_i starts with $q<3$, then by Lemma A.1(b), $s=\langle \ldots, 4_i, X_i, q, \ldots\rangle = \langle \ldots, q, 4_i, X_i, \ldots\rangle$. This contradicts the assumption that s is a minimum switching sequence (A.1). Similarly, if Z_i ends with $p>5$, then by Lemma A.1(b), $s=\langle \ldots, p, 4_{i+1}, \ldots\rangle = \langle \ldots, 4_{i+1}, p, \ldots\rangle$ again contradicts the assumption that s is a minimum switching sequence. The need for consecutive decreasing integers in Y_i and Z_i comes from the fact that s is minimum. This proves (A.2).

We also claim that

$$\begin{aligned} i \in Y_k, Y_{k+1} \quad &\Rightarrow \quad i-1 \in Y_k \quad \text{for } i=2,3, \\ i \in Z_k, Z_{k+1} \quad &\Rightarrow \quad i+1 \in Z_{k+1} \quad \text{for } i=5,6. \end{aligned} \tag{A.3}$$

Suppose that Y_k and Y_{k+1} contain i, where i is 2 or 3. By Lemma A.1(a), we need $N(i) = \{i-1, i+1\}$ to appear even number of times between $i\in Y_k$ and $i\in Y_{k+1}$. By (A.2) or 4_{k+1}, we know that $i+1\in Y_{k+1}$ definitely appears, so it forces Y_k to contain $i-1$. This proves the first part of (A.3). Similarly, suppose that Z_k and Z_{k+1} contain i, where i is 5 or 6. By Lemma A.1(a), we need $N(i) = \{i-1, i+1\}$ to appear even number of times between $i \in Z_k$ and $i \in Z_{k+1}$. By (A.2) or 4_k, we know that $i-1 \in Z_k$ definitely appears, so it forces Z_{k+1} to contain $i+1$. This completes the proof for (A.3) as claimed. We shall repeatedly apply (A.3) in future arguments.

Consider $s = \langle \ldots, 4_i, X_i, Y_i, Z_i, 4_{i+1}, \ldots \rangle$ for $i = 1, \ldots, m-1$. Since each X_i, Y_i, Z_i contains exactly one element of $N(4) = \{0, 3, 5\}$, it follows from Lemma A.1(a) that

$$\text{for } i = 1, \ldots, m-1, \quad \text{exactly one of } X_i, Y_i, Z_i \text{ is empty.} \tag{A.4}$$

It is clear that $X_1 = \emptyset$. So by (A.4), Y_1 and Z_1 are nonempty. Applying Lemma A.1(a) to $N(0) = \{4\}$, we conclude that

$$\begin{aligned} &\text{for odd } i = 1, \ldots, m, \quad X_i = \emptyset, \\ &\text{for odd } i = 1, \ldots, m-2, \quad Y_i, Z_i \neq \emptyset. \end{aligned} \tag{A.5}$$

We claim that

$$\begin{aligned} &\text{for even } i \leqslant m-3, \quad Y_i \neq \langle 3 \rangle, \\ &\text{for even } i \leqslant m-1, \quad Y_i \neq \langle 3, 2, 1 \rangle. \end{aligned} \tag{A.6}$$

Suppose that $Y_i = \langle 3 \rangle$ for some even $i \leqslant m-3$. By (A.2) and (A.5), $3 \in Y_{i+1}$. By (A.3), $2 \in Y_i$, which is a contradiction. This proves the first part of (A.6). Next suppose that $Y_i = \langle 3, 2, 1 \rangle$ for some even $i \leqslant m-1$. By (A.2) and (A.5), $3 \in Y_{i-1}$. So by (A.3),

$$3 \in Y_{i-1}, Y_i \quad \Rightarrow \quad 2 \in Y_{i-1}, Y_i \quad \Rightarrow \quad 1 \in Y_{i-1}, Y_i. \tag{A.7}$$

The conclusion in (A.7) is impossible, because $N(1) = \{2\}$ appears exactly once between $1 \in Y_{i-1}$ and $1 \in Y_i$. This completes the proof for (A.6).

There are three cases for Z_m, namely $\langle 5, 6, 7 \rangle$, $\langle 6, 7 \rangle$ and $\langle 7 \rangle$. We shall show that each case leads to a contradiction.

Case (I). $Z_m = \langle 5, 6, 7 \rangle$.

By (A.5), $X_m = \emptyset$. So Y_m cannot contain 3 because vertex 4 is unpainted in $(1, 7)$. By (A.2), $Y_m = \emptyset$. Then $Z_{m-1} = \emptyset$, for otherwise $5 \in Z_{m-1}, Z_m$, which contradicts Lemma A.1(a). Therefore, by (A.4), we have

$$X_{m-1}, Y_{m-1} \neq \emptyset, \qquad Z_{m-1} = \emptyset. \tag{A.8}$$

We claim that

$$Y_{m-1} = \langle 3 \rangle, \qquad Y_{m-2} = \langle 3, 2, 1 \rangle, \qquad Y_{m-3} = \emptyset. \tag{A.9}$$

By (A.2), (A.5) and (A.8), $3 \in Y_{m-2}, Y_{m-1}$ and so $2 \in Y_{m-2}$. If $2 \in Y_{m-1}$, then $1 \in Y_{m-1}$ because $Y_m = \emptyset$ and vertex 2 is unpainted in $(1, 7)$. But this is a contradiction because

$N(1) = \{2\}$ appears exactly once between $1 \in Y_{m-1}$ and $1 \in Y_{m-2}$. So Y_{m-1} cannot contain 2, and we have proved the first part of (A.9). Since Y_{m-2} contains 2, and vertex 2 is unpainted in $(1, 7)$, it forces $1 \in Y_{m-2}$, which proves the second part of (A.9). For the last part of (A.9), assume that $Y_{m-3} \neq \emptyset$. Then by (A.6), $Y_{m-3} = \langle 3, 2\rangle$, and so $2 \in Y_{m-2}, Y_{m-3}$. By (A.3), $1 \in Y_{m-3}$, which is a contradiction. This proves the last part of (A.9).

Next we claim that

$$Z_{m-2} = \langle 7, 6, 5\rangle, \qquad Z_{m-3} = \langle 6, 5\rangle, \qquad Z_{m-4} = \langle 5\rangle, \qquad Z_{m-5} = \emptyset. \tag{A.10}$$

We first check that

$$\left.\begin{array}{l} Z_{m-2} \neq \emptyset \Rightarrow 5 \in Z_{m-2}, Z_{m-3} \Rightarrow 6 \in Z_{m-2} \\ Z_{m-4} \neq \emptyset \Rightarrow 5 \in Z_{m-4}, Z_{m-3} \Rightarrow 6 \in Z_{m-3} \end{array}\right\} \quad \Rightarrow \quad 7 \in Z_{m-2},\ 7 \notin Z_{m-3}. \tag{A.11}$$

In (A.11), $Z_{m-2}, Z_{m-4} \neq \emptyset$ follows from (A.5) and $5 \in Z_{m-3}$ because by (A.4) and by (A.9), $Z_{m-3} \neq \emptyset$. Other arguments follow from (A.3). Finally Z_{m-3} cannot contain 7 because otherwise $N(7) = \{6\}$ appears exactly once between $7 \in Z_{m-3}$ and $7 \in Z_{m-2}$. This explains (A.11). By (A.11), we have proved the first two parts of (A.10). If Z_{m-4} contains 6, then (A.3) forces $7 \in Z_{m-3}$, a contradiction. So by (A.2) and (A.5), $Z_{m-4} = \langle 5\rangle$. For the last part of (A.10), if $Z_{m-5} \neq \emptyset$, then $5 \in Z_{m-5}, Z_{m-4}$ and (A.3) implies that $6 \in Z_{m-4}$, a contradiction. This completes the proof for (A.10).

We also claim that

$$Y_{m-4} = \langle 3\rangle, \qquad Y_{m-5} = \langle 3, 2\rangle, \qquad Y_{m-6} = \langle 3, 2, 1\rangle, \qquad Y_{m-7} = \emptyset. \tag{A.12}$$

If $2 \in Y_{m-4}$, then together with (A.9), $2 \in Y_{m-4}, Y_{m-2}$. We need $N(2) = \{1, 3\}$ to appear even number of times between them, so $1 \in Y_{m-4}, Y_{m-2}$. This is a contradiction, because $N(1) = \{2\}$ appears exactly once between $1 \in Y_{m-4}$ and $1 \in Y_{m-2}$. Therefore Y_{m-4} cannot contain 2. Together with (A.5), we get $Y_{m-4} = \langle 3\rangle$. By (A.5) and (A.10), $Z_{m-5} = \emptyset$, so $Y_{m-5} \neq \emptyset$. Together with (A.6), we get $Y_{m-5} = \langle 3, 2\rangle$. To prove that $Y_{m-6} = \langle 3, 2, 1\rangle$, we check that

$$\begin{aligned} X_{m-6} = \emptyset \quad &\Rightarrow \quad Y_{m-6}, Z_{m-6} \neq \emptyset \quad \Rightarrow \quad 3 \in Y_{m-6}, Y_{m-5} \\ &\Rightarrow \quad 2 \in Y_{m-6}, Y_{m-5} \quad \Rightarrow \quad 1 \in Y_{m-6}. \end{aligned} \tag{A.13}$$

The first part of (A.13) follows from (A.4) and (A.5). This implies that $3 \in Y_{m-6}$. The rest of (A.13) follows from (A.3). By (A.13), $Y_{m-6} = \langle 3, 2, 1\rangle$. For the last part of (A.12), assume that $Y_{m-7} \neq \emptyset$. By (A.6), it implies that $Y_{m-7} = \langle 3, 2\rangle$. But by (A.3), $2 \in Y_{m-7}, Y_{m-6}$ implies $1 \in Y_{m-7}$, a contradiction. Hence $Y_{m-7} = \emptyset$. This completes the proof for (A.12).

Repeating the arguments for (A.9), (A.10) and (A.12), we have:

$$\begin{array}{llll} Z_{m-4k-1} = \emptyset, & Z_{m-4k-2} = \langle 7, 6, 5\rangle, & Z_{m-4k-3} = \langle 6, 5\rangle, & Z_{m-4k-4} = \langle 5\rangle, \\ Y_{m-4k-1} = \langle 3, 2\rangle, & Y_{m-4k-2} = \langle 3, 2, 1\rangle, & Y_{m-4k-3} = \emptyset, & Y_{m-4k-4} = \langle 3\rangle, \\ X_{m-4k-1} = \langle 0\rangle, & X_{m-4k-2} = \emptyset, & X_{m-4k-3} = \langle 0\rangle, & X_{m-4k-4} = \emptyset. \end{array}$$

Recall that m is odd. By the above conclusion for X_i, Y_i and Z_i, we have two possibilities to start the switching sequence s:

$$\text{(i)}\quad m \equiv 1 \bmod (4):\quad s = \langle 0, 4_1, 3, 5, 4_2, 0, 6, 5, 4_3, 3, 2, 1, 7, 6, 5, 4_4, \ldots\rangle,$$

$$\text{(ii)}\quad m \equiv 3 \bmod (4):\quad s = \langle 0, 4_1, 3, 2, 1, 7, 6, 5, 4_2, \ldots\rangle.$$

In (i), $N(2) = \{1, 3\}$ appears twice before $2 \in Y_3$. This is impossible, because after $3 \in Y_3$ appears, vertex 2 is unpainted. In (ii), 7 appears in Z_1 while it is still unpainted. Both (i) and (ii) lead to contradictions. Therefore, Case (I) with $Z_m = \langle 5, 6, 7\rangle$ is impossible. We next proceed with Case (II).

Case (II). $Z_m = \langle 6, 7\rangle$.

Vertex 0 is unpainted after 4_m (also by (A.5)), so $X_m = \emptyset$, and therefore

$$Y_m = \langle 3, 2, 1\rangle. \tag{A.14}$$

We claim that

$$Z_{m-1} = \emptyset. \tag{A.15}$$

Suppose otherwise, namely $Z_{m-1} \neq \emptyset$. By (A.2), $5 \in Z_{m-1}$. Then Z_{m-1} cannot contain 6, for otherwise $N(6) = \{5, 7\}$ appears exactly once between $6 \in Z_{m-1}$ and $6 \in Z_m$ via $5 \in Z_{m-1}$. On the other hand, (A.2) and (A.5) imply that $5 \in Z_{m-2}$. Further, by (A.3), $5 \in Z_{m-2}, Z_{m-1}$ implies $6 \in Z_{m-1}$. This is a contradiction. This proves (A.15).

By (A.2) and (A.15), $3 \in Y_{m-1}$. By (A.3) and $Y_m = \langle 3, 2, 1\rangle$,

$$3 \in Y_{m-1}, Y_m \quad\Rightarrow\quad 2 \in Y_{m-1}, Y_m \quad\Rightarrow\quad 1 \in Y_{m-1}, Y_m. \tag{A.16}$$

But the conclusion of (A.16) is impossible, because $N(1) = \{2\}$ appears only once between $1 \in Y_{m-1}$ and $1 \in Y_m$. By this contradiction, we have proved that Case (II) with $Z_m = \langle 6, 7\rangle$ is impossible.

To complete the proof of this proposition, we check that the final case is also impossible.

Case (III). $Z_m = \langle 7\rangle$.

As before, $Y_m = \langle 3, 2, 1\rangle$. If $Y_{m-1} \neq \emptyset$, then we obtain a contradiction by the same argument as (A.16). Together with (A.4), we get

$$X_{m-1} = \langle 0\rangle, \qquad Y_{m-1} = \emptyset, \qquad Z_{m-1} \neq \emptyset. \tag{A.17}$$

We check that

$$\begin{aligned} X_{m-2} = \emptyset \quad&\Rightarrow\quad 5 \in Z_{m-2}, Z_{m-1} \quad\Rightarrow\quad 6 \in Z_{m-1} \\ &\Rightarrow\quad Z_{m-1} = \langle 6, 5\rangle \quad\Rightarrow\quad Z_{m-2} = \langle 5\rangle, Z_{m-3} = \emptyset. \end{aligned} \tag{A.18}$$

Here $X_{m-2} = \emptyset$ follows from (A.5). The next argument is due to (A.2) and (A.4). Also, $5 \in Z_{m-1}$ follows from (A.2) and (A.17). By (A.3), it leads to $6 \in Z_{m-1}$. Observe that $7 \notin Z_{m-1}$, for otherwise $N(7) = \{6\}$ appears once between $7 \in Z_{m-1}$ and $7 \in Z_m$. It follows that $Z_{m-1} = \langle 6, 5\rangle$. Then $6 \notin Z_{m-2}$, for otherwise $7 \in Z_{m-1}$ due to (A.3). So by (A.2) and (A.5), $Z_{m-2} = \langle 5\rangle$. Finally $Z_{m-3} = \emptyset$, for otherwise if $5 \in Z_{m-3}$, then $6 \in Z_{m-2}$ due to (A.3). This explains (A.18).

By $X_{m-2} = \emptyset$ in (A.18), we have $Y_{m-2} \neq \emptyset$. Since $3 \in Y_{m-2}, Y_m$, and since $Y_{m-1} = \emptyset$, it follows that $2 \notin Y_{m-2}$, because $N(3) = \{2, 4\}$ already appears twice between $3 \in Y_{m-2}$ and $3 \in Y_m$ via 4_{m-1} and 4_m. Therefore,

$$Y_{m-2} = \langle 3\rangle. \tag{A.19}$$

Since $Z_{m-3} = \emptyset$, by (A.4) and by (A.6),

$$X_{m-3} = \langle 0\rangle, \qquad Y_{m-3} = \langle 3, 2\rangle. \tag{A.20}$$

By arguments similar to (A.16),

$$\begin{aligned} X_{m-4} = \emptyset \quad &\Rightarrow \quad 3 \in Y_{m-4}, Y_{m-3} \quad \Rightarrow \quad 2 \in Y_{m-4}, Y_{m-3} \\ &\Rightarrow \quad 1 \in Y_{m-4} \quad \Rightarrow \quad Y_{m-4} = \langle 3, 2, 1\rangle. \end{aligned} \tag{A.21}$$

If $Y_{m-5} = \langle 3, 2\rangle$, then $2 \in Y_{m-5}, Y_{m-4}$ and (A.3) imply $1 \in Y_{m-5}$, a contradiction. So $Y_{m-5} \neq \langle 3, 2\rangle$. By (A.6),

$$Y_{m-5} = \emptyset. \tag{A.22}$$

From (A.17) through (A.22), it follows that

$$\begin{array}{llll} Z_{m-4k-1} = \langle 6, 5\rangle, & Z_{m-4k-2} = \langle 5\rangle, & Z_{m-4k-3} = \emptyset, & Z_{m-4k-4} = \langle 7, 6, 5\rangle, \\ Y_{m-4k-1} = \emptyset, & Y_{m-4k-2} = \langle 3\rangle, & Y_{m-4k-3} = \langle 3, 2\rangle, & Y_{m-4k-4} = \langle 3, 2, 1\rangle. \end{array}$$

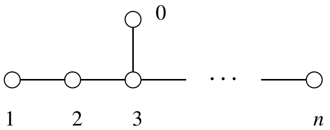
So there are two possibilities, depending on the odd integer m:

$$\begin{array}{lll} \text{(i)} & m \equiv 1 \mod (4): & s = \langle 0, 4_1, 3, 2, 1, 7, 6, 5, \ldots\rangle, \\ \text{(ii)} & m \equiv 3 \mod (4): & s = \langle 0, 4_1, 3, 5, 4_2, 0, 6, 5, 4_3, 3, 2, 1, \ldots\rangle. \end{array}$$

In (i), the entry $7 \in Z_1$ is impossible because vertex 7 is unpainted at that time. In (ii), the entry $2 \in Y_3$ is impossible since vertex 2 is unpainted at that time (because its neighborhood $\{1, 3\}$ appears exactly twice before it). This shows that Case (III) leads to a contradiction too.

We have shown that each of the Cases (I)–(III) on Z_m leads to a contradiction. Therefore, there is no switching sequence between (0) and (1, 7). This proves the proposition. $\square$

Proposition A.2 completes the proof of Proposition 3.6. Finally, we want to show that the $E_8^{(1)}$ diagrams (1) and (7) are not equivalent. To achieve this, we consider the more general diagram



Proposition A.3. *In the above diagram,*

$$(1) \sim (n-1) \quad \Leftrightarrow \quad n \equiv 2 \pmod 4.$$

Proof. Suppose that there is a minimum switching sequence s for $((1), (n-1))$. We want to show that $n \equiv 2 \pmod 4$. We will follow the spirit of Proposition A.2. By Lemma 3.3, it is easy to see that

$$\begin{array}{ll} \text{(a)} & t_i \text{ is even for odd } i \leqslant n-1, \\ \text{(b)} & t_2, t_0 + t_4, t_{n-2} + t_n \text{ are odd}, \\ \text{(c)} & t_{i-1} + t_{i+1} \text{ is even for } i \neq 3, n-1, \\ \text{(d)} & t_{n-1} \text{ is even.} \end{array} \tag{A.23}$$

In particular

$$m = t_3$$

is even, and we write

$$\begin{aligned} &s = \langle 1, 2, 3_1, X_1, Y_1, Z_1, 3_2, X_2, \ldots, 3_m, X_m, Y_m, Z_m \rangle, \\ &X_i \subset \{0\}, \qquad Y_i \subset \{1, 2\}, \qquad Z_i \subset \{4, \ldots, n\}. \end{aligned}$$

We proceed with arguments similar to (A.2). If Y_i starts with 1, then by Lemma A.1(b), $s = \langle \ldots, 3_i, X_i, 1, \ldots \rangle = \langle \ldots, 1, 3_i, X_i, \ldots \rangle$ contradicts the assumption that s is minimum (A.1). So

$$Y_i = \langle 2, 1 \rangle, \langle 2 \rangle, \emptyset \quad \text{for all } i = 1, \ldots, m. \tag{A.24}$$

If Z_i ends with some $p > 4$, then by Lemma A.1(b), $s = \langle \ldots, p, 3_{i+1}, \ldots \rangle = \langle \ldots, 3_{i+1}, p, \ldots \rangle$ again contradicts the assumption that s is minimum. So

$$Z_i = \langle k_i, k_i - 1, k_i - 2, \ldots, 4 \rangle, \emptyset \quad \text{for all } i = 1, \ldots, m-1. \tag{A.25}$$

For nonempty Z_i in (A.25), the need for consecutive decreasing integers comes from the fact that s is minimum.

From $s = \langle \ldots, 3_i, X_i, Y_i, Z_i, 3_{i+1}, \ldots \rangle$ and $N(3) = \{0, 2, 4\}$, we again see that

$$\text{for } i = 1, \ldots, m-1, \quad \text{exactly one of } X_i, Y_i, Z_i \text{ is empty.} \tag{A.26}$$

After $\langle 1, 2, 3_1, X_1, \ldots \rangle$ in the beginning of s, vertex 2 is unpainted. So $Y_1 = \emptyset$ and $X_1, Z_1 \neq \emptyset$. Since $N(0) = \{3\}$ and $X_1 = \langle 0 \rangle$, it follows that $X_2, X_4, \ldots$ are all empty. Together with (A.26), we conclude that

$$\begin{aligned} &X_i = \emptyset \quad \text{for even } i < m, \\ &Y_i, Z_i \neq \emptyset \quad \text{for even } i \leqslant m-2. \end{aligned} \tag{A.27}$$

Consider some odd $i \leqslant m-3$. If $Y_i = \langle 2 \rangle$, then by (A.27), $N(2) = \{3\}$ appears exactly once between $2 \in Y_i$ and $2 \in Y_{i+1}$. This is a contradiction. If $Y_i = \langle 2, 1 \rangle$, then since $2 \in Y_{i-1}, Y_i$, by the argument similar to (A.3), $1 \in Y_{i-1}$. This is a contradiction, because $N(1) = \{2\}$ appears exactly once between $1 \in Y_{i-1}$ and $1 \in Y_i$. We conclude that

$$\text{for odd } i \leqslant m-3, \quad Y_i = \emptyset. \tag{A.28}$$

Since s is a switching sequence for $((1), (n-1))$, we can directly check the end of s that X_m and Y_m are empty, and that $Z_m = \langle 4, 5, \ldots, n-1 \rangle$. If $Z_{m-1} \neq \emptyset$, (A.25) implies $4 \in Z_{m-1}$, then $N(4) = \{3, 5\}$ appears exactly once between $4 \in Z_{m-1}$ and $4 \in Z_m$ via 3_m, which is impossible. So together with (A.26), we conclude that

$$\begin{aligned} &X_{m-1}, Y_{m-1} \neq \emptyset, \qquad Z_{m-1} = \emptyset, \\ &X_m = Y_m = \emptyset, \qquad Z_m = \langle 4, 5, \ldots, n-1 \rangle. \end{aligned} \tag{A.29}$$

By (A.27), (A.28) and (A.29), we see that Y_i is nonempty exactly for even $i \leqslant m-2$, or for $i = m-1$. Recall that m is even. So t_2 is the sum of $\frac{m-2}{2}$ (from the nonempty $Y_2, Y_4, \ldots, Y_{m-2}$) and 2 (entry 2 appears right before 3_1 and in Y_{m-1}). Namely $t_2 = \frac{m}{2} + 1$. By (A.23)(b), t_2 is odd. We conclude that

$$m = 4k \quad \text{for some integer } k. \tag{A.30}$$

By (A.27), (A.28) and (A.29), we see that X_i is empty if and only if i is even. It follows that

$$t_0 = \frac{m}{2} \quad \text{is even.}$$

Therefore, by (A.23)(b) and by (A.23)(c), $t_4 = m - 1$ is odd, and hence t_i is odd for all even $2 \leqslant i \leqslant n-1$. Together with (A.23)(a), we have

$$t_i \text{ is odd} \quad \Leftrightarrow \quad i \text{ is even and } 2 \leqslant i \leqslant n-1. \tag{A.31}$$

By (A.23)(a), (d), $n-1$ is odd, so n is even.

By direct inspection at the beginning of s, we see that Z_1 does not contain 5. Hence $Z_1 = \langle 4\rangle$. Recall that $Z_i = \langle k_i, k_i - 1, \ldots, 4\rangle$ from (A.25). We claim that $k_1, k_2, \ldots$ is strictly increasing, namely

$$k_{i-1} < k_i \tag{A.32}$$

for all $2 \leqslant i \leqslant m-2$. Suppose otherwise, namely $k_i \leqslant k_{i-1}$ for some i. Then $N(k_i) = \{k_i \pm 1\}$ appears only once between $k_i \in Z_{i-1}$ and $k_i \in Z_i$, which is impossible. This proves (A.32) as claimed.

We claim further that $|Z_i| = i$, namely

$$Z_i = \langle i+3, i+2, \ldots, 4\rangle. \tag{A.33}$$

This is proved inductively on i. We have seen from above that $Z_1 = \langle 4\rangle$. By (A.32), we know that $5 \leqslant k_2$. But if $5 < k_2$, then 5, 6 are contained in all of $Z_2, Z_3, \ldots$, this implies that $t_5 = t_6$, which contradicts (A.31). So $5 = k_2$ and $Z_2 = \langle 5, 4\rangle$. We continue this argument and see that if $i+3 < k_i$, then $i+3, i+4$ are contained in all of $Z_i, Z_{i+1}, \ldots$, leading to the impossible $t_{i+3} = t_{i+4}$. Hence $i+3 = k_i$. This proves (A.33) as claimed.

Recall that n is even. By (A.31), t_{n-2} is odd and hence (A.23)(b) implies that t_n is even. We claim that in fact $t_n = 0$. Since $n \in Z_m$ and $Z_{m-1} = \emptyset$ (by (A.29)), n appears even number of times in $Z_1, \ldots, Z_{m-2}$. This is impossible, because (A.32) implies that n can appear exactly once in Z_{m-2} if $t_n \neq 0$. So $t_n = 0$ as claimed.

By (A.23)(d), t_{n-1} is even, since $n-1 \in Z_m$, (A.32) implies that

$$Z_{m-2} = \langle n-1, \ldots, 4\rangle. \tag{A.34}$$

By (A.33) and (A.34), $m+2 = n$, together with (A.30) we have

$$n \equiv 2 \pmod 4.$$

We have proved the first part of Proposition A.3; namely if $(1) \sim (n-1)$, then $n \equiv 2 \pmod 4$.

Conversely, suppose that $n \equiv 2 \pmod 4$. Then the minimum switching sequence for $((1), (n-1))$ is given by

$$\langle 1, 2, 3_1, X_1, Y_1, Z_1, \ldots, 3_{n-3}, X_{n-3}, Y_{n-3}, Z_{n-3}, 3_{n-2}, Z_{n-2}\rangle,$$

where

$$X_i = \begin{cases} \langle 0\rangle, & \text{for odd } i, \\ \emptyset, & \text{for even } i, \end{cases}$$

$$Y_i = \begin{cases} \langle 2\rangle, & \text{for even } i \leqslant n-4, \\ \langle 2, 1\rangle, & \text{for } i = n-3, \\ \emptyset, & \text{for odd } i \leqslant n-5, \end{cases}$$

$$Z_i = \begin{cases} \langle i+3, i+2, \ldots, 4\rangle, & \text{for } 1 \leqslant i \leqslant n-4, \\ \emptyset, & \text{for } i = n-3, \\ \langle 4, 5, \ldots, n-1\rangle, & \text{for } i = n-2. \end{cases}$$

For example if $n = 6$, then the minimum switching sequence for $((1), (5))$ is $\langle 1, 2, 3, 0, 4, 3, 2, 1, 5, 4, 3, 0, 2, 3, 4, 5\rangle$. This completes the proof of Proposition A.3. □

We remark that Proposition A.3 can be extended to

$$(1) \sim (k) \quad \Leftrightarrow \quad k \equiv 1 \pmod 4, \quad 1 \leqslant k \leqslant n.$$

But for the purpose of this paper, we only need the weaker version presented by Proposition A.3. By this proposition, it follows that (1) is not equivalent to (7) in $E_8^{(1)}$. This completes the proof of Proposition 3.7.

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