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Discrete Mathematics 306 (2006) 1132–1138
DISCRETE MATHEMATICS
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On the existence of hyper-L triple-loop networks☆

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Received 18 January 2004; received in revised form 16 October 2005; accepted 17 November 2005
Available online 5 April 2006

Abstract

Aguiló-Gost [New dense families of triple loop Networks, Discrete Math. 197/198 (1999) 15–27] has presented a new type of hyper-L tiles and used it to derive a new dense family of triple-loop networks. While Aguiló-Gost's hyper-L tile seems to be a promising tool for studying the triple-loop networks, we need to verify that the hyper-L tile producing good result is indeed the MDD of some triple-loop network. In this paper, we give necessary and sufficient conditions for the existence of Aguiló-Gost's hyper-L triple-loop networks and we correct some flaws in [F. Aguiló-Gost, New dense families of triple loop networks, Discrete Math. 197/198 (1999) 15–27].

Keywords: Triple-loop network; Hyper-L tile; Diameter

1. Introduction

Multi-loop networks were first proposed by Wong and Coppersmith [6] for organizing multimodule memory services. A *multi-loop network* $\mathrm{ML}(N; s_1, s_2, \ldots, s_d)$ has N nodes $0, 1, 2, \ldots, N-1$ and dN links, $i \to (i+s_1) \bmod N$, $i \to (i+s_2) \bmod N, \ldots, i \to (i+s_d) \bmod N$, $i = 0, 1, \ldots, N-1$. Multi-loop networks are now being widely studied because of their relevance to the design of some interconnection networks and communication networks. For details of multi-loop networks, refer to [2,3,5].

A triple-loop network $\mathrm{TL}(N; s_1, s_2, s_3)$ has N nodes $0, 1, 2, \ldots, N-1$ and $3N$ links, $i \to (i+s_1) \bmod N$, $i \to (i+s_2) \bmod N$, $i \to (i+s_3) \bmod N$, $i = 0, 1, \ldots, N-1$. It is an extension of the double-loop network $\mathrm{DL}(N; s_1, s_2)$ with one more fixed step s_3 for each node. It is well known [6] that $\mathrm{DL}(N; s_1, s_2)$ and $\mathrm{TL}(N; s_1, s_2, s_3)$ are strongly connected if and only if $\gcd(N, s_1, s_2) = 1$ and $\gcd(N, s_1, s_2, s_3) = 1$, respectively. In this paper, we are only concerned with strongly connected networks and if the parameters do not satisfy the above conditions, we simply say that the corresponding network does not exist.

A minimum distance diagram $\mathrm{MDD}(v)$ for $\mathrm{DL}(N; s_1, s_2)$ is a 2-dimensional array which gives the shortest paths from node v to every other node. Since $\mathrm{DL}(N; s_1, s_2)$ is node-symmetric, we need only study MDD(0), or simply, MDD. Wong and Coppersmith [6] gave an $O(N)$-time construction of MDD by sequentially adding nodes to the diagram which can be reached from node 0 in i steps for $i = 0, 1, \ldots,$ until every node appears exactly once. Fig. 1 illustrates an MDD where each horizontal step signifies an s_1-step and each vertical, an s_2-step.

☆ This research was partially supported by the National Science Council of the Republic of China under the Grant NSC94-2115-M-009-006.
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doi:10.1016/j.disc.2005.11.023

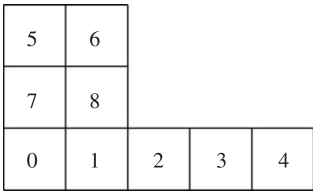

Fig. 1. The MDD of DL(9; 1, 7).

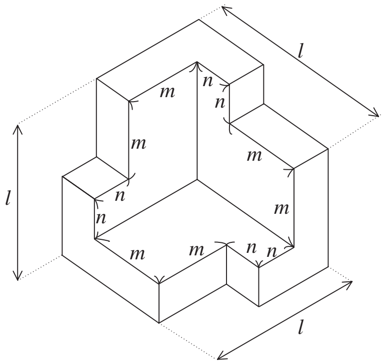

Fig. 2. The hyper-L tile.

It is well known that the MDD for a double-loop network is an L-shape which includes the degenerate form of a rectangle. The L-shape plays a crucial role in proving many desirable properties for a double-loop network.

The MDD for a triple-loop network is a 3-dimensional array with each step in the x_i-axis signifying an s_i-step. Unfortunately, the MDD for a triple-loop network does not have a uniform nice shape like the L-shape, and this fact has really hampered the study of triple-loop networks. Aguiló et al. [2] overcame this difficulty by skipping the triple-loop network and going directly to a nice 3-dimensional shape which they called *hyper-L tile*. This hyper-L tile is characterized by three parameters l, m, n, and is highly structured and symmetrical (see Fig. 2). Note that l, m, n are integers, $m \geqslant n \geqslant 0$, and $l > m + n$. They used the hyper-L tile to derive a dense family of triple-loop networks which has the property

$$N(D) \geqslant \tfrac{2}{27}(D+3)^3 \approx 0.074D^3 + \mathrm{O}(D^2),$$

where $N(D)$ is the maximum number of nodes in a triple-loop network for a fixed diameter D. Note that when a family of triple-loop networks has a good N–D ratio, we say it is *dense*.

Recently, Aguiló-Gost [1] presented a new type of hyper-L tiles which is characterized by three parameters h, m, n, and is also highly structured and symmetrical (see Figs. 3(a) and 4). Aguiló-Gost used it to derive a new dense family of triple-loop networks which has the property

$$N(D) \geqslant \tfrac{1485}{27^3} D^3 \approx 0.075D^3 + \mathrm{O}(D^2). \tag{1.1}$$

For convenience, call this hyper-L tile the *hyper-L_1 tile*, let $\mathrm{HL}_1(h, m, n)$ denote a hyper-L_1 tile with parameters h, m, n, and call a triple-loop network whose MDD is $\mathrm{HL}_1(h, m, n)$ an $\mathrm{HL}_1(h, m, n)$ *triple-loop network*.

While the hyper-L and the hyper-L_1 tiles seem to be promising tools for studying the triple-loop networks, we must be able to verify that those hyper-L tiles producing good results are indeed the MDDs of some triple-loop networks. In [4], Chen et al. have proposed necessary and sufficient conditions for the existence of hyper-L triple-loop networks (see [2] also). In this paper, we will give necessary and sufficient conditions for the existence of hyper-L_1 triple-loop networks.

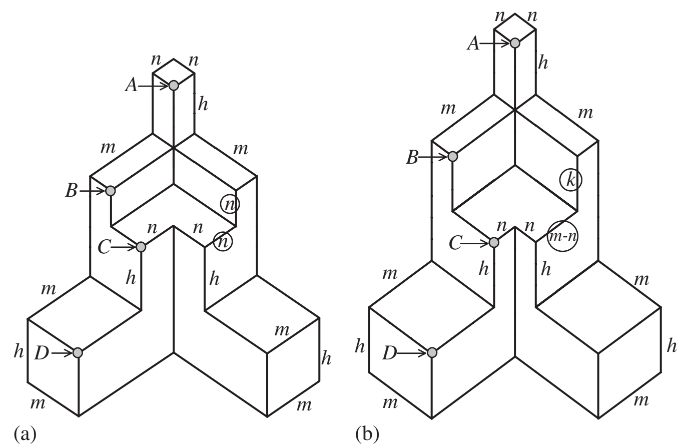

Fig. 3. The hyper-L_1 tile. (a) The hyper-L_1 tile proposed by Aguiló-Gost [1]. (b) Two edges labelled n in (a) are actually not of length n.

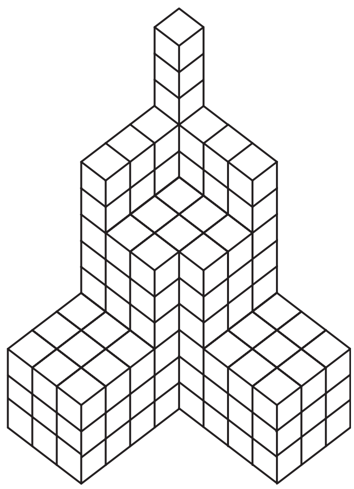

Fig. 4. A hyper-L_1 tile with $h = 3, m = 3, n = 1$; it is the MDD of TL(161; 2, 117, 7).

2. Preliminary

Let $a \mid b$ ($a \nmid b$) denote b is divisible (not divisible) by a. It is well known that

Lemma 1. *If m and n are integers, not both zero, then there exist integers a and b such that* $am - bn = \gcd(m, n)$.

We now prove that

Lemma 2. *If* a, m, b, n *are integers, not all zero, such that* $am - bn = 1$, *then* $\gcd(a, n) = 1$.

Proof. Assume that $am - bn = 1$ and $\gcd(a, n) = k$. Then $k \mid a$ and $k \mid n$. Thus $k \mid am - bn = 1$. So $k = 1$. □

The following lemma will be used in the remaining discussions.

Lemma 3. *If m and n are integers, not both zero, and* $\gcd(m, n) = 1$, *then there exist integers a and b such that* $am - bn = 1$ *and* $\gcd(a, 2m + n) = 1$.

Proof. By Lemma 1, there exist integers a and b such that $am - bn = 1$. By Lemma 2, we have

$$\gcd(a, n) = 1. \tag{2.1}$$

If $\gcd(a, 2m + n) = 1$, then we are done. In the following, assume that $\gcd(a, 2m + n) = d > 1$. Let $a = pd$ and $2m + n = qd$. Then $\gcd(p, q) = 1$. Since $\gcd(m, n) = 1$, we have

$$\gcd(2m + n, n) = \gcd(2m, n) = \begin{cases} 1 & \text{if } n \text{ is odd}, \\ 2 & \text{if } n \text{ is even}. \end{cases}$$

If $\gcd(2m + n, n) = 1$, then clearly $\gcd(qd, n) = 1$ and thus $\gcd(d, n) = 1$. Now suppose that $\gcd(2m + n, n) = 2$. Then $\gcd(qd, n) = 2$; therefore $\gcd(d, n) = 1$ or $\gcd(d, n) = 2$. If $\gcd(d, n) = 2$, then $2 \mid a$ and we have $\gcd(a, n) \geqslant 2$; this contradicts with (2.1). From the above, we have

$$\gcd(d, n) = 1. \tag{2.2}$$

Let $q = st$, where s is the largest factor of q such that

$$\gcd(s, d) = 1. \tag{2.3}$$

That is, s (t) contains those prime factors of q that are relative prime (not relative prime) to d. (For example, if $q = 2^2 \cdot 3^2 \cdot 7$ and $d = 2 \cdot 3^2$, then $s = 7$ and $t = 2^2 \cdot 3^2$.) Then

$$\gcd(s, t) = 1. \tag{2.4}$$

Since $\gcd(p, q) = 1$ and $q = st$, we have

$$\gcd(p, s) = 1. \tag{2.5}$$

Since t contains those prime factors of q that are not relative prime to d, by Eq. (2.2), we have

$$\gcd(t, n) = 1. \tag{2.6}$$

Let

$$a' = a + sn \quad \text{and} \quad b' = b + sm.$$

Then $a'm - b'n = (a + sn)m - (b + sm)n = am - bn = 1$ and

$$\begin{aligned} \gcd(a', 2m + n) &= \gcd(a + sn, 2m + m) = \gcd(pd + sn, qd) \\ &= \gcd(pd + sn, q) \quad \text{(by Eqs. (2.2) and (2.3))} = \gcd(pd + sn, st) \\ &= \gcd(pd + sn, s) \quad \text{(by Eqs. (2.4) and (2.6))} = 1 \ \text{(by Eqs. (2.3) and (2.5))}. \end{aligned}$$

Hence the lemma. □

3. Necessary and sufficient conditions

Aguiló-Gost [1] observed that $\mathrm{HL}_1(h, m, n)$ tessellates the space. By studying the distribution of node 0 in the space, Aguiló-Gost defined a matrix associated with $\mathrm{HL}_1(h, m, n)$ as follows:

$$M_1(h, m, n) = \begin{pmatrix} n & -m & -m \\ n & n + m & -m \\ 2h & h & 2h - n \end{pmatrix}.$$

They also claimed that the diameter of $\mathrm{HL}_1(h, m, n)$ is given by

$$D(h, m, n) = \max\{3m + h + n, 2m + 2h + n, 3h + 3n\} - 3. \tag{3.1}$$

Note that two sides labelled length n in Fig. 3(a) (this figure is Fig. 5 in [1]) are actually *not* of length n; for convenience, we have circled these two n's. One of the two flaws can be verified by checking the lengths of the sides

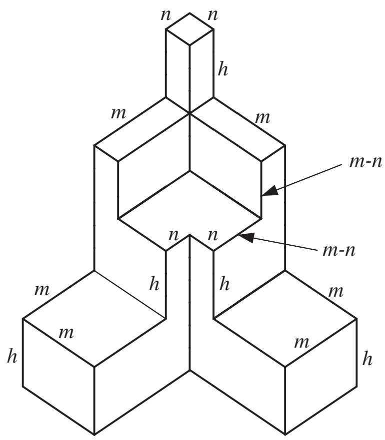

Fig. 5. Setting two of the edges labelled n in Fig. 3(a) to $m-n$.

of the topmost $n \times n$ square and the lengths of the sides of the rightmost $m \times h$ rectangle. The other edge label actually can be anything; suppose it is k (see Fig. 3(b)). We now show that $k = m - n$ is a better choice as far as the diameter is considered. Let A, B, C, D be the four cells shown in Fig. 3(b), and let $d(A), d(B), d(C), d(D)$ be the distance between the origin and cell A, B, C, D, respectively. Then $d(A) = 3h + 2n + k - 3$, $d(B) = m + 2h + 2n + k - 3$, $d(C) = 2m + 2h + n - 3$, and $d(D) = 3m + h + n - 3$. Clearly the diameter of $\mathrm{HL}_1(h, m, n)$ is given by

$$D(h, m, n) = \max\{d(A), d(B), d(C), d(D)\}.$$

One heuristic to derive a better $D(h, m, n)$ is to let $d(A) = d(B) = d(C) = d(D)$ and this occurs when $k = m - n$ and $h = m$. Thus $k = m - n$ is a better choice as far as the diameter is considered. Fig. 5 shows the $\mathrm{HL}_1(h, m, n)$ after setting $k = m - n$.

In the remaining part of this paper, we assume $k = m - n$. Then $d(A) = m + 3h + n - 3$, $d(B) = d(C) = 2m + 2h + n - 3$, and $d(D) = 3m + h + n - 3$. Hence the diameter of $\mathrm{HL}_1(h, m, n)$ is given by

$$D(h, m, n) = \max\{m + 3h + n, 2m + 2h + n, 3m + h + n\} - 3 \tag{3.2}$$

and the matrix associated with $\mathrm{HL}_1(h, m, n)$ is given by

$$M_1(h, m, n) = \begin{pmatrix} n & -m & -m \\ n & n + m & -m \\ 2h & h & h + m - n \end{pmatrix}.$$

Set $M = M_1(h, m, n)$ for easy writing. Recall that Aguiló-Gost [1] observed that $\mathrm{HL}_1(h, m, n)$ tessellates the space. By studying the distribution of node 0 in the space, Aguiló-Gost obtained

$$M^{\mathrm{T}} \times \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} (\mathrm{mod}\, N) \quad \text{or}$$

$$M^{\mathrm{T}} \times \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} N \quad \text{for some integers } \alpha, \beta, \gamma. \tag{3.3}$$

Also, $N = \det M$, i.e.,

$$N = (2m + n)(h(2m + n) + n(m - n)). \tag{3.4}$$

We now give necessary and sufficient conditions for the existence of an $\mathrm{HL}_1(h, m, n)$ triple-loop network.

Theorem 4. *A necessary and sufficient condition for the existence of an* $\mathrm{HL}_1(h, m, n)$ *triple-loop network is* $\gcd(m, n) = 1$ *and* $3 \nmid m - n$.

Proof. Suppose that $\gcd(m,n)=1$ and $3\nmid m-n$. Since $\gcd(m,n)=1$, by Lemma 3, there exist integers a and b such that $am-bn=1$ and $\gcd(a,2m+n)=1$. Since $\gcd(a,2m+n)=1$, $a\neq 0$. Since $\gcd(m,n)=1$, $\gcd(m-n,m)=1$. Since $\gcd(m,n)=1$ and $3\nmid m-n$ and $\gcd(m-n,m)=1$, we have

$$\gcd(m-n,2m+n)=\gcd(m-n,3m)=\gcd(m-n,3)=1. \tag{3.5}$$

By Eqs. (3.3) and (3.4)

$$\begin{aligned}\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix}&=(M^{\mathrm{T}})^{-1}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}N\\ &=\begin{pmatrix}h(2m+n)+(m-n)(m+n) & -(h(2m+n)+n(m-n)) & -h(2m+n)\\ m(m-n) & h(2m+n)+n(m-n) & -h(2m+n)\\ m(2m+n) & 0 & n(2m+n)\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}.\end{aligned}$$

Setting $(\alpha,\beta,\gamma)=(a,0,-b)$, we obtain the solution

$$\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix}=\begin{pmatrix}h(a+b)(2m+n)+a(m-n)(m+n)\bmod N\\ bh(2m+n)+am(m-n)\bmod N\\ 2m+n\end{pmatrix}.$$

Let

$$\phi(a)=\begin{cases}-1 & \text{if } a>0,\\ 1 & \text{if } a<0.\end{cases}$$

From (3.4), $2m+n\mid N$. Therefore there exists an integer k_1 such that

$$h(a+b)(2m+n)+a(m-n)(m+n)\bmod N=k_1(2m+n)+\phi(a)a(m-n)(m+n)$$

and

$$0<k_1(2m+n)+\phi(a)a(m-n)(m+n)<N.$$

Also, there exists an integer k_2 such that

$$bh(2m+n)+am(m-n)\bmod N=k_2(2m+n)+\phi(a)am(m-n)$$

and

$$0<k_2(2m+n)+\phi(a)am(m-n)<N.$$

Therefore

$$\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix}=\begin{pmatrix}k_1(2m+n)+\phi(a)a(m-n)(m+n)\\ k_2(2m+n)+\phi(a)am(m-n)\\ 2m+n\end{pmatrix}.$$

Note that

$$\begin{aligned}&\gcd(s_1,s_2,s_3)\\ &\quad=\gcd(k_1(2m+n)+\phi(a)a(m-n)(m+n),k_2(2m+n)+\phi(a)am(m-n),2m+n)\\ &\quad=\gcd(a(m-n)(m+n),am(m-n),2m+n)\\ &\quad=\gcd(an(m-n),am(m-n),2m+n)\\ &\quad=\gcd(n(m-n),m(m-n),2m+n)\quad(\text{by the fact that }\gcd(a,2m+n)=1)\\ &\quad=\gcd(n,m,2m+n)\quad(\text{by Eq. (3.5)})\\ &\quad=\gcd(m,n)\\ &\quad=1.\end{aligned}$$

So if $\gcd(m,n)=1$ and $3\nmid m-n$, then clearly $\gcd(N,s_1,s_2,s_3)=\gcd(s_1,s_2,s_3)=1$ and $\mathrm{TL}(N;s_1,s_2,s_3)$ exists.

On the other hand, suppose

$$\gcd(m, n) = d > 1 \text{ or } 3 \mid m - n.$$

In the former case, each s_i, $i = 1, 2, 3$, is a linear combination of terms divisible by d. Furthermore, from (3.4), N is also a linear combination of terms divisible by d. Hence

$$\gcd(N, s_1, s_2, s_3) \geqslant d > 1$$

and TL$(N; s_1, s_2, s_3)$ does not exist. In the latter case, since $3 \mid m - n$, we have

$$\gcd(2m + n, m - n) = \gcd(3m, m - n) = r \geqslant 3.$$

Therefore each s_i, $i = 1, 2, 3$, is a linear combination of terms divisible by r. Furthermore, from (3.4), N is also a linear combination of terms divisible by r. Hence

$$\gcd(N, s_1, s_2, s_3) \geqslant r > 1$$

and TL$(N; s_1, s_2, s_3)$ does not exist. □

4. Applications

Aguiló-Gost [1] suggested the ratio $h : m : n = 2 : 2 : 1$ between the dimensions of $\mathrm{HL}_1(h, m, n)$ and used this ratio to derive (1.1). By Theorem 4, a $\mathrm{HL}_1(6t, 6t + 1, 3t + 1)$ triple-loop network does not exist. $\mathrm{HL}_1(6t + 1, 6t + 1, 3t)$ and $\mathrm{HL}_1(6t + 3, 6t + 3, 3t + 1)$ were used in Theorem 1 of [1] to derive (1.1). By Theorem 4, a $\mathrm{HL}_1(6t + 1, 6t + 1, 3t)$ triple-loop network and a $\mathrm{HL}_1(6t + 3, 6t + 3, 3t + 1)$ triple-loop network do exist.

Aguiló-Gost [1] claimed that the number of nodes $N(t)$ of $\mathrm{HL}_1(6t + 1, 6t + 1, 3t)$ equals $1485t^3 + 648t^2 + 90t + 4$ and the diameter $D(t)$ of $\mathrm{HL}_1(6t + 1, 6t + 1, 3t)$ satisfies $D(t) \leq 27t + 1$. Aguiló-Gost then used $N(t)$ and $D(t)$ to derive (1.1). Note that after correcting the two flaws in Fig. 3(a), we still have $N(t) = 1485t^3 + 648t^2 + 90t + 4$ and we will have $D(t) = 27t + 1$. It is not difficult to verify that (1.1) still holds.

Acknowledgement

The authors wish to thank the referees for many constructive comments that greatly improve the presentation of this paper.

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