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Tessellating polyominos in the plane

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Abstract

Let the $\mathbb{R}^2$ space be divided into unit squares where a *polyomino* is a finite, connected set of unit squares. In this paper, we give a necessary and sufficient condition on tessellating polyominos by observing an unexpected relation between such tessellations and systems of arithmetic progressions.
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1. Introduction

Let the $\mathbb{R}^2$ space be divided into unit squares, i.e., the four corners of a square have coordinates (x, y), $(x+1, y)$, $(x, y+1)$, $(x+1, y+1)$ for some integers x and y. A *polyomino* is defined as a finite, connected set of unit squares where connection is established by sharing a side. A polyomino P is said to tessellate the plane if the plane can be decomposed into polyominos all translations of P. For example, Fig. 1(a) shows a tessellation, but not Fig. 1(b), which uses a 180° rotation.

But does the polyomino in Fig. 2 tessellate the plane?

One motivation of studying the above tessellation problem comes from the study of *double-loop network* which is a popular topology for computer networks (see [5] for a recent survey). The double-loop network $\mathrm{DL}(N; a, b)$ is a digraph with N nodes labeled by $0, 1, \ldots, N-1$ and $2N$ links of two types $i \to i+a \pmod N$, $i \to i+b \pmod N$ for all $i \in \mathbb{Z}_N$. When $\mathrm{DL}(N; a, b)$ is strongly connected, we can define a *minimum distance diagram* (MDD) which shows the shortest routes from a given node to all other nodes. It was proved [4,6] that the MDD corresponds to an L-shape or a rectangle, where both are polyominos and tessellate the plane.

In fact, one approach to find a good double-loop network is to first select a good L-shape (including the rectangle as a degenerating case) and then check whether there exists a corresponding double-loop network. Tessellation of the L-shape plays a crucial role in this checking [1–5,7].

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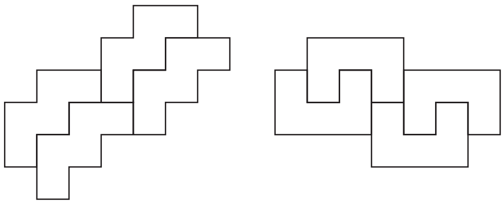

(a) (b)

Fig. 1. Two polyominos in $\mathbb{R}^2$.

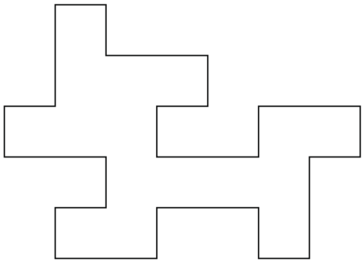

Fig. 2. A puzzle polyomino.

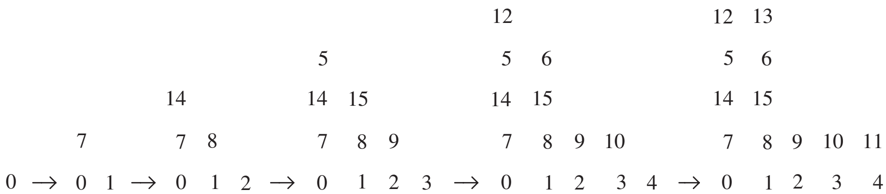

Fig. 3. The drawing of the MDD of DL(16; 1, 7).

In this paper, we give a necessary and sufficient condition for a polyomino to be tessellating in $\mathbb{R}^2$, and also an extension to unions of polyominos.

2. Some preliminary results

Throughout this paper, it is assumed that the digraph DL$(N; a, b)$ is strongly connected, i.e., $\gcd(N, a, b) = 1$. Due to symmetry, we may consider only routes from node 0 to all other nodes in a DL$(N; a, b)$. An MDD for these routes can be drawn by first inserting node 0 at cell (0, 0), and then sequentially inserting nodes to the diagram in the order of their closeness to node 0. Suppose a node v is of distance d from node 0, and can be reached by taking i a-links and j b-links, i.e., $v \equiv ia + jb \pmod N$ and $d = i + j$. Then v is placed at cell (i, j). Fig. 3 illustrates the drawing of the MDD of DL(16; 1, 7).

Wong and Coppersmith [6] proved that the MDD of DL$(N; 1, b)$ is an L-shape (degeneration into a rectangle allowed) as shown in Fig. 4. Fiol et al. [4] then extended the result to DL$(N; a, b)$ for any $a \geqslant 1$. They also showed such an L-shape always tessellates the plane.

Conversely, if an L-shape with N unit squares is given, then we would like to know whether there exists a DL$(N; a, b)$ realizing the L-shape. Let ℓ, h, p, n denote the side-lengths of the L-shape as indicated in Fig. 4 and v denote a specific position (an unit square) in the L-shape. Then $N = \ell h - pn$ and by inspecting the relative positions of v in neighboring L-shapes in the plane tessellation, Fiol et al. [4] obtained the important equations

$$\ell a - nb \equiv 0 \pmod N \quad \text{and} \quad -pa + hb \equiv 0 \pmod N.$$

They also gave the following result.

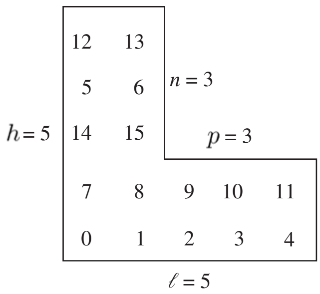

Fig. 4. An L-shape with parameters ℓ, h, p, n.

Lemma 2.1. *Given non-negative integers ℓ, h, p, n. If $\gcd(\ell, h, p, n) = 1$ and $N = \ell h - pn > 0$, then there exist $a > 0$ and $b > 0$ in $\mathbb{Z}_N$ satisfying $\ell a - nb \equiv 0 \pmod N$, $-pa + hb \equiv 0 \pmod N$ and $\gcd(N, a, b) = 1$.*

Chen and Hwang [2] showed that $\ell > n$ and $h \geqslant p$ are also necessary for the existence of $\mathrm{DL}(N; a, b)$. They gave a constructive proof of Lemma 2.1 by using a "sieve" method to solve for (a, b). Esqué et al. [3] gave a "Smith normal form" method to solve for (a, b). For a comparison of efficiency of these two methods, see [1].

Moreover, since the solution (a, b) in $\mathbb{Z}_N$ satisfies $\gcd(N, a, b) = 1$, which guarantees the existence of (s, t) in $\mathbb{Z}_N$ such that $sa + tb \equiv 1 \pmod N$. Namely, node v can reach node $v + 1$ by taking s a-links and t b-links, hence v can reach all nodes.

Now, we extend the result of Lemma 2.1 to the case $\gcd(\ell, h, p, n) = f > 1$ by following the constructive proof of Chen and Hwang.

Lemma 2.2. *Given non-negative integers ℓ, h, p, n. If $\gcd(\ell, h, p, n) = f > 1$ and $N = \ell h - pn > 0$, then there exist A and B in $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ satisfying*

$$\ell A - nB \equiv (0, 0) \quad (\mathrm{mod}\, \mathbb{Z}_f \times \mathbb{Z}_{N/f}),$$

$$-pA + hB \equiv (0, 0) \quad (\mathrm{mod}\, \mathbb{Z}_f \times \mathbb{Z}_{N/f}),$$

$$\alpha A + \beta B \equiv (0, 1) \quad (\mathrm{mod}\, \mathbb{Z}_f \times \mathbb{Z}_{N/f}),$$

and

$$\gamma A + \delta B \equiv (1, 0) \quad (\mathrm{mod}\, \mathbb{Z}_f \times \mathbb{Z}_{N/f})$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_{N/f}$.

Proof. First, suppose $f = \prod_i p_i^{\alpha_i} > 1$, where the p_i's are distinct primes. For each p_i, one of ℓ, h, p, n must be divisible by $p_i^{\alpha_i}$, but not by $p_i^{\alpha_i+1}$. Partition p_i into two groups G_1, G_2 depending on whether it is ℓ or n containing the smallest power of p_i, versus h or p (ties are broken arbitrarily). Define $p(G_1) = \prod_{p_i \in G_1} p_i^{\alpha_i}$ and $p(G_2) = \prod_{p_i \in G_2} p_i^{\alpha_i}$. If $G_1 = \phi$ or $G_2 = \phi$, then set $p(G_1) = 1$ or $p(G_2) = 1$.

Let $\ell' = \ell / p(G_1)$, $h' = h / p(G_2)$, $p' = p / p(G_2)$ and $n' = n / p(G_1)$. Then $\gcd(\ell', h', p', n') = 1$ and $N/f = \ell' h' - p' n' > 0$. Hence, by Lemma 2.1, we can obtain $a > 0$ and $b > 0$ in $\mathbb{Z}_{N/f}$ satisfying $\ell' a - n' b \equiv 0 \pmod{N/f}$, $-p' a + h' b \equiv 0 \pmod{N/f}$ and $\gcd(N/f, a, b) = 1$. Note that $N = \ell h - pn$ is divisible by f^2, hence N/f is divisible by f. And since $\gcd(N/f, a, b) = 1$, there exist s and t in $\mathbb{Z}_{N/f}$ satisfying $sa + tb \equiv 1 \pmod{N/f}$, which also implies $s' a + t' b \equiv 1 \pmod f$ where $s' = s \pmod f$ and $t' = t \pmod f$.

Next, let $A = (-t', a)$, $B = (s', b)$, $\alpha = s$, $\beta = t$, $\gamma = -b$ and $\delta = a$. Then

$$\ell A - nB = \begin{pmatrix} f\left(\dfrac{-\ell t'}{f} - \dfrac{ns'}{f}\right) \\ p(G_1)(\ell' a - n' b) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (\mathrm{mod}\, \mathbb{Z}_f \times \mathbb{Z}_{N/f}),$$

$$-pA + hB = \begin{pmatrix} f\left(\dfrac{pt'}{f} + \dfrac{hs'}{f}\right) \\ p(G_2)(-p'a + h'b) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod{\mathbb{Z}_f \times \mathbb{Z}_{N/f}},$$

$$\alpha A + \beta B = \begin{pmatrix} -st' + ts' \\ sa + tb \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pmod{\mathbb{Z}_f \times \mathbb{Z}_{N/f}}$$

and

$$\gamma A + \delta B = \begin{pmatrix} bt' + as' \\ -ba + ab \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod{\mathbb{Z}_f \times \mathbb{Z}_{N/f}}. \qquad \square$$

The conditions $\alpha A + \beta B \equiv (0, 1) \pmod{\mathbb{Z}_f \times \mathbb{Z}_{N/f}}$ and $\gamma A + \delta B \equiv (1, 0) \pmod{\mathbb{Z}_f \times \mathbb{Z}_{N/f}}$ are to guarantee that, by taking certain numbers of A-links and B-links, a node (i, j) can reach the nodes $(i, j) + (0, 1)$ and $(i, j) + (1, 0)$, hence it can reach all nodes.

3. The main results

We apply the results of the last section to the polyomino tessellation problem.

Theorem 3.1. *A polyomino P with N unit squares tessellates $\mathbb{R}^2$ if and only if we can label the unit squares of P by the set of residues modulo $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ for some $f \geqslant 1$ and $f \mid N$, in such a way that each row is an arithmetic progression with skip parameter A and each column an arithmetic progression with skip parameter B for some (A, B) in $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ ($A = B$ allowed).*

Proof. *Sufficiency*: Using the arithmetic progressions with skip parameters A and B, the unit squares of $\mathbb{R}^2$ can be labeled by the residues modulo $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$. Let P^* denote a labeled polyomino P as described such that each residue of $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ appears exactly once. Form as many P^* (through translations only) as possible in $\mathbb{R}^2$. Suppose to the contrary that there exists an unit square z with residue i in $\mathbb{R}^2$ which is not in any P^* formed. Let P' be a labeled translation of P containing z. Then any unit square in P' cannot lie in any formed P^* since the relative position between residue i and any other residue j is fixed in an P^*. Therefore, we can add P' to the set of formed P^*, contradicting the assumption that the existing set of formed P^* is already as large as possible.

Necessity: Let x denote a specific position (an unit square) in P. Since P is tessellating in $\mathbb{R}^2$, we can find two vectors $(u, -v)$ and $(-w, y)$ satisfying $N = uy - wv$, which are the independent vectors characterizing the distribution of x in neighboring P's in the plane tessellation. Let $f = \gcd(u, y, w, v)$. If $f = 1$, then from Lemma 2.1, set $A = (0, a)$ and $B = (0, b)$ which belong to $\mathbb{Z}_1 \times \mathbb{Z}_N$. Otherwise, if $f > 1$, then by Lemma 2.2, we obtain (A, B) in $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$. Hence we can label the unit squares of $\mathbb{R}^2$ using the arithmetic progressions with skip parameters A and B. Since (A, B) is the solution of $uA - vB \equiv (0, 0) \pmod{\mathbb{Z}_f \times \mathbb{Z}_{N/f}}$, $-wA + yB \equiv (0, 0) \pmod{\mathbb{Z}_f \times \mathbb{Z}_{N/f}}$ and P tessellates $\mathbb{R}^2$, the polyomino P must contain each residue modulo $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ exactly once. $\square$

Example. Fig. 5 shows a tessellating polyomino. Two independent vectors are $(4, 0)$, $(-2, 2)$ and $N = 8 = 4 \cdot 2 - 2 \cdot 0$. Since $\gcd(4, 2, 2, 0) = 2 > 1$, we consider $G_1 = \phi$ and $G_2 = \{2\}$. Hence $p(G_1) = 1$ and $p(G_2) = 2$. Then we obtain $(a, b) = (1, 1)$ satisfying $4a \equiv 0 \pmod 4$, $-a + b \equiv 0 \pmod 4$ and $\gcd(4, a, b) = 1$. Since $s = s' = 1$ and $t = t' = 0$ satisfy $sa + tb \equiv 1 \pmod 4$ and $s'a + t'b \equiv 1 \pmod 2$, we have $A = (-t', a) = (0, 1)$ and $B = (s', b) = (1, 1)$ in $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Note that the unit squares of the polyomino in Fig. 5 cannot be labeled by the residues in $\mathbb{Z}_8$ (with every residue appearing exactly once). To see this, suppose we fix an unit square of the polyomino and assign the label 0 to it (see Fig. 6(a)). Since the number of unit squares is even from one 0 to the next 0, there are only two possible arithmetic progressions for row (see Fig. 6(b)). It is easily verified that every arithmetic progression for column has one unit square whose label is repeated.

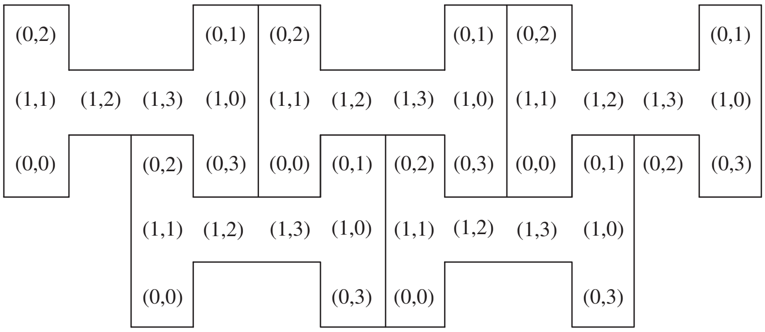

Fig. 5. A tessellating polyomino.

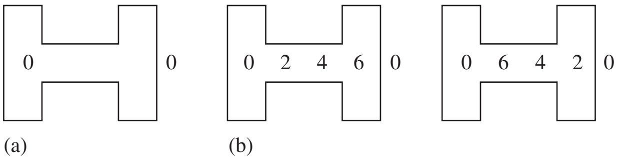

Fig. 6. A counterexample against $\mathbb{Z}_N$ labeling.

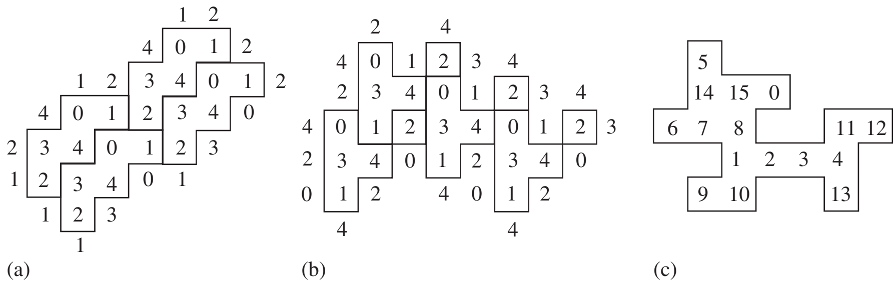

Fig. 7. Two tessellating polyominos and a tessellating configuration.

A *configuration* generalizes the notion of polyomino by dropping the requirement "being connected", i.e., a configuration can be viewed as a disjoint union of polyominos.

Corollary 3.2. *Theorem* 3.1 *can be extended to a configuration.*

Proof. The proof of Theorem 3.1 does not depend on the connection of the given polyomino. □

Fig. 7 illustrates Theorem 3.1 and Corollary 3.2. The configuration in Fig. 7(b) consists of a polyomino of four unit squares and a polyomino of one unit square. Note that if $f = 1$ then $\mathbb{Z}_1 \times \mathbb{Z}_N$ can be seen as $\mathbb{Z}_N$.

4. Some concluding remarks

One would expect that tessellation in $\mathbb{R}^3$ (or even in higher dimension) can be similarly characterized as in Theorem 3.1 except $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ would have to be replaced by $\mathbb{Z}_{f_1} \times \mathbb{Z}_{f_2} \times \mathbb{Z}_{f_3}$ with $f_1 \cdot f_2 \cdot f_3 = N$. The sufficiency part of the proof should remain intact, but there are still some technical difficulties in the necessity part.

How does Theorem 3.1 help the question "does the polyomino in Fig. 2 tessellate the plane"? First, it is not difficult to show that the MDD of the digraph, which has N nodes labeled by the elements in $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ and $2N$ links $u \to u + A$

(mod $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$), $u \to u + B$ (mod $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$) for each node $u \in \mathbb{Z}_f \times \mathbb{Z}_{N/f}$, is also an L-shape (see [3,6]). Hence we can solve for (A, B), respectively, for each L-shape with size N and check whether these arithmetic progressions (A, B) satisfy the condition that every residue of $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ appears exactly once in the given polyomino. Then the given polyomino tessellates $\mathbb{R}^2$ if and only if there exists at least one (A, B) satisfying this condition. Fig. 7(c) gives the answer.

Granted, the above method can be time consuming, but we establish the fact that "whether a polyomino in $\mathbb{R}^2$ is tessellating" can be determined, which might open the door for more efficient methods in the future, and perhaps, for generalization to polyominos in $\mathbb{R}^n$.

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