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A recursively construction scheme for super fault-tolerant hamiltonian graphs

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Abstract

For the interconnection network topology, it is usually represented by a graph. When a network is used, processors and/or links faults may happen. Thus, it is meaningful to consider faulty networks. We consider k-regular graphs in this paper. We define a k-regular *hamiltonian* and *hamiltonian connected* graph G is *super fault-tolerant hamiltonian* if G remains hamiltonian after removing at most $k-2$ vertices and/or edges and remains hamiltonian connected after removing at most $k-3$ vertices and/or edges. A super fault-tolerant hamiltonian graph has a certain optimal flavor with respect to the *fault-tolerant hamiltonicity* and *fault-tolerant hamiltonian connectivity*. The aim of this paper is to investigate a construction scheme to construct various super fault-tolerant hamiltonian graphs. Along the way, the *recursive circulant graph* is a special case of our construction scheme, and the super fault-tolerant hamiltonian property of recursive circulant graph is obtained.

Keywords: Hamiltonian; Hamiltonian connected; Fault tolerance; Super fault-tolerant hamiltonian; Recursive circulant graphs

1. Introduction

The architecture of an interconnection network is usually represented by a graph $G=(V,E)$, while vertices represent processors and edges represent links between processors. We use terms graphs and networks interchangeable in this paper. There are a lot of mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum for all conditions. One has to design a suitable network depending on the requirements of their properties. The hamiltonian property is one of the major requirements in designing the topology of networks. Fault tolerance is also desirable in massive parallel systems that have relatively high probability of failure. There are many researches on the ring embedding problems in faulty interconnection networks [2–8,10–12].

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In this paper, a network is represented as an undirected graph. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a,b)|(a,b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges incident to v. A graph G is *k-regular* if $\deg(v) = k$ for every vertex in G. Two vertices a and b are *adjacent* if $(a,b) \in E$. A *path* is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct. We also write the path $\langle v_0, v_1, v_2, \ldots, v_m\rangle$ as $\langle v_0, P(v_0, v_i), v_i, v_{i+1}, \ldots, v_j, P(v_j, v_t), v_t, \ldots, v_m\rangle$ where $P(v_0, v_i) = \langle v_0, v_1, \ldots, v_i\rangle$ and $P(v_j, v_t) = \langle v_j, v_{j+1}, \ldots, v_t\rangle$. For our purpose in this paper, a path may contain only one vertex. A path is a *hamiltonian path* if its vertices are distinct and they span V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A cycle is a *hamiltonian cycle* if it traverses every vertex of G exactly once. A graph G is *hamiltonian* if it has a hamiltonian cycle, and G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. Many of the graph definitions and notations we used can be found in [1].

Since vertex faults and edge faults may happen when a network is used, it is practically meaningful to consider faulty networks. A graph G is called *l-fault-tolerant hamiltonian* (*l-fault-tolerant hamiltonian connected* respectively) or simply *l-hamiltonian* (*l-hamiltonian connected* respectively) if it remains hamiltonian (hamiltonian connected respectively), after removing at most l vertices and/or edges. The *fault-tolerant hamiltonicity*, $\mathscr{H}_f(G)$, is defined to be the maximum integer l such that $G - F$ remains hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant l$ if G is hamiltonian, and undefined if otherwise. Obviously, $\mathscr{H}_f(G) \leqslant \delta(G) - 2$, where $\delta(G) = \min\{\deg(v)|v \in V(G)\}$. A regular graph G is *optimal fault-tolerant hamiltonian* if $\mathscr{H}_f(G) = \delta(G) - 2$. *Twisted-cubes*, *crossed-cubes*, *möbius cubes* and *recursive circulant graphs* are proved to be optimal fault-tolerant hamiltonian [2,4–6,11]. All these families of graphs have some good properties in common, including that they can all be recursively constructed. In establishing their fault-tolerant hamiltonicity, another parameter called *fault-tolerant hamiltonian connectivity* is used. The fault-tolerant hamiltonian connectivity, $\mathscr{H}^\kappa_f(G)$, is defined to be the maximum integer l such that $G - F$ remains hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant l$ if G is hamiltonian connected, and undefined if otherwise. Obviously, $\mathscr{H}^\kappa_f(G) \leqslant \delta(G) - 3$. A regular graph G is *optimal fault-tolerant hamiltonian connected* if $\mathscr{H}^\kappa_f(G) = \delta(G) - 3$. Again, twisted-cubes, crossed-cubes, möbius cubes and recursive circulant graphs are proved to be optimal fault-tolerant hamiltonian connected [2,4–6,11]. We call those regular graphs *super fault-tolerant hamiltonian* if $\mathscr{H}_f(G) = \delta(G) - 2$ and $\mathscr{H}^\kappa_f(G) = \delta(G) - 3$.

All the proofs of super fault-tolerant hamiltonicity are done by induction. We observe that there are certain common phenomena behind the recursive structures so that we may construct other super fault-tolerant hamiltonian graphs. In this paper, we try to investigate these phenomena and establish some construction schemes of super fault-tolerant hamiltonian graphs.

The rest of this article is organized as follows. In the next section, a recursively construction scheme and some notations are introduced. The *recursive circulant graph* [9,11] is essentially a special case of this construction scheme. Section 3 describes six lemmas which we shall use in our main results. The main results are proved in Section 4. Finally, the conclusion is given in Section 5.

2. A recursively construction scheme and some notations

Fault tolerance is one of the major requirement on designing a network. A network has higher fault tolerance if it is super fault-tolerant hamiltonian. In this section, we give a construction scheme to recursively construct super fault-tolerant hamiltonian graphs. Let $G_1, G_2, \ldots, G_n$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. We define a new graph $H = G(G_1, G_2, \ldots, G_n, M_{1,2}, M_{2,3}, \ldots, M_{n-1,n}, M_{1,n})$ which has vertex set $V(H) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$, and edge set $E(H) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n) \cup M_{1,2} \cup M_{2,3} \cup \cdots \cup M_{n-1,n} \cup M_{1,n}$, where $M_{i,j}$ is an arbitrary perfect matching between the vertices of G_i and G_j. See Fig. 1. Considering each component G_i as a vertex and each perfect matching $M_{i,j}$ as an edge, then $G(G_1, G_2, \ldots, G_n, M_{1,2}, M_{2,3}, \ldots, M_{n-1,n}, M_{1,n})$ is reduced to a cycle of length n. For the sake of simplicity, we shall abbreviate $G(G_1, G_2, \ldots, G_n, M_{1,2}, M_{2,3}, \ldots, M_{n-1,n}, M_{1,n})$ as $G(G_1, G_2, \ldots, G_n; C_n)$, where C_n stands for a cycle of length n. As an example, the recursive circulant graph, which was proposed by Park and Chwa [9], is essentially constructed as a special case in this way, and it is shown to be super fault-tolerant hamiltonian under a certain condition [11]. In this paper, we show that

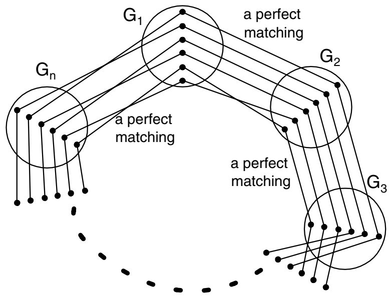

Fig. 1. $H = G(G_1, G_2, \ldots, G_n; C_n)$.

$G(G_1, G_2, \ldots, G_n; C_n)$ is super fault-tolerant hamiltonian for any arbitrary perfect matchings, $M_{1,2}$, $M_{2,3}, \ldots, M_{1,n}$, provided $n \geqslant 3$ and $k \geqslant 5$.

For ease of exposition, we make some convention about our notations we shall use along this paper. Consider the graph $G(G_1, G_2, \ldots, G_n; C_n)$. For each component G_i, we use small letters with subscript i to denote the vertices in G_i, e.g., u_i, v_i, etc. Thus, u_1 is a vertex in G_1, and u_2 is a vertex in G_2. A perfect matching $M_{i,j}$ connecting the vertices of G_i and G_j in pairs, such pairs of vertices are called *matching vertices*, and these edges are called *matching edges*. We shall use the same letter with different subscripts to denote matching vertices of each other; e.g., u_i and u_j are the matching vertices of each other in components G_i and G_j if there is a perfect matching between G_i and G_j.

We need some more terms. We shall consider graphs with some faults. Our objective is to find a fault free hamiltonian cycle (hamiltonian path respectively). In this paper, each fault can be a faulty vertex or a faulty edge. If a vertex v is not faulty, we say v is a *healthy vertex*. We call an edge e (respectively a matching edge e) *healthy* if both edge e and its two endpoints are not faulty. We use F_i to denote the set of faults in G_i, $F_{(i\cdots j)}$ to denote the set of faults in $G(G_i, G_{i+1}, \ldots, G_j, M_{i,i+1}, M_{i+1,i+2}, \ldots, M_{j-1,j})$. Let $f_i = |F_i|$ and $f_{(i\cdots j)} = |F_{(i\cdots j)}|$. Given two distinct healthy vertices x and y, we use *x,y-hamiltonian path* to call a fault free hamiltonian path joining x and y, HP_i to denote a fault free hamiltonian path in $G_i - F_i$, and $HP_{(i\cdots j)}$ to denote a fault free hamiltonian path in $G(G_i, G_{i+1}, \ldots, G_j, M_{i,i+1}, M_{i+1,i+2}, \ldots, M_{j-1,j}) - F_{(i\cdots j)}$ for $i \leqslant j$. A fault free x,y-hamiltonian path in $G_i - F_i$ can be written as $\langle x, HP_i, y\rangle$ and a fault free x,y-hamiltonian path in $G(G_i, G_{i+1}, \ldots, G_j, M_{i,i+1}, M_{i+1,i+2}, \ldots, M_{j-1,j}) - F_{(i\cdots j)}$ can be written as $\langle x, HP_{(i\cdots j)}, y\rangle$. In addition, path $\langle x, HP_i, y\rangle$ and path $\langle x, HP_{(i\cdots j)}, y\rangle$ are cycles if $x = y$.

3. Preliminaries

Consider an interconnection network G, and suppose that there are some faults in it. Let F_G be the set of faults in G, and $f_G = |F_G|$be the number of faults in G. Suppose that G is k-hamiltonian (k-hamiltonian connected respectively) and $f_G \leqslant k$. Let u be a healthy vertex in G. It is clear that some of the edges incident to u is on a hamiltonian cycle (hamiltonian path respectively) in $G - F_G$, but not every edge incident to u is on some hamiltonian cycle (hamiltonian path respectively) in $G - F_G$. In the following two lemmas, [2] proved that at least a fix number of edges incident to vertex u are on some hamiltonian cycles (hamiltonian paths respectively) in $G - F_G$.

Lemma 1 [2]. *Let G be a k-hamiltonian graph, F_G be a set of faults in G with $|F_G| \leqslant k$, and u be a healthy vertex in G. Then there are at least $k - f_G + 2$ edges incident to vertex u, such that each one of them is on some hamiltonian cycle in $G - F_G$.*

Lemma 2 [2]. *Let G be a k-hamiltonian connected graph, F_G be a set of faults in G with $|F_G| \leqslant k$, and $\{x, y, u\}$ be three distinct healthy vertices in G. Then there are at least $k - f_G + 2$ edges incident to vertex u, such that each one of them is on some x,y-hamiltonian path in $G - F_G$.*

Let G_r and G_s be two graphs with the same number of vertices. Let M be an arbitrary *perfect matching* between the vertices of G_r and G_s. [2] has defined graph $G(G_r, G_s; M)$, which has vertex set $V(G(G_r, G_s; M)) = V(G_r) \cup V(G_s)$, and edge set $E(G(G_r, G_s; M)) = E(G_r) \cup E(G_s) \cup M$. The following two lemmas result immediately from the fact that $|V(G_r)| = |V(G_s)| \geqslant k + 1$.

Lemma 3 [2]. *Let G_r and G_s be two k-regular graphs with the same number of vertices. If the total number of faults in $G(G_r, G_s; M)$ is not greater than k, there exists at least one healthy matching edge between G_r and G_s.*

Lemma 4 [2]. *Let G_r and G_s be two k-regular graphs with the same number of vertices, and let x and y be two healthy vertices in $G(G_r, G_s; M)$. If the total number of faults in $G(G_r, G_s; M)$ is not greater than $k - 2$, there exists at least one healthy matching edge between G_r and G_s whose endpoints are neither x nor y.*

The following two lemmas state that the fault-tolerant hamiltonicity $\mathscr{H}_{\mathrm{f}}(G)$ and fault-tolerant hamiltonian connectivity $\mathscr{H}_{\mathrm{f}}^{\kappa}(G)$ of the graph $G(G_r, G_s; M)$, as compared with G_r and G_s, are increased by 1. Hence, $G(G_r, G_s; M)$ is a super fault-tolerant hamiltonian graph.

Lemma 5 [2]. *Assume $k \geqslant 4$. Let G_r and G_s be two k-regular super fault-tolerant hamiltonian graphs and $|V(G_r)| = |V(G_s)|$. Then graph $G(G_r, G_s; M)$ is $(k - 1)$-hamiltonian.*

The fault-tolerant hamiltonian connectivity $\mathscr{H}_{\mathrm{f}}^{\kappa}(G)$ of $G(G_r, G_s; M)$ is also increased by 1, as stated in the following theorem.

Lemma 6 [2]. *Assume $k \geqslant 5$. Let G_r and G_s be two k-regular super fault-tolerant hamiltonian graphs and $|V(G_r)| = |V(G_s)|$. Then graph $G(G_r, G_s;M)$ is $(k - 2)$-hamiltonian connected.*

4. Main results

We make one simple observation first.

Observation 1. To prove that a graph G is l-hamiltonian (respectively l-hamiltonian connected), it suffices to show that $G - F_G$ is hamiltonian (respectively hamiltonian connected) for any faulty set $F_G \subset V(G) \cup E(G)$ with $|F_G| = l$. If the total number of faults $|F_G|$ is strictly less than l, we may arbitrarily designate $l - |F_G|$ healthy edges as faulty to make exactly l faults.

In this section, we shall show that $G(G_1, G_2, \ldots, G_n; C_n)$ is super fault-tolerant hamiltonian. In order to do that, we prove one preliminary result which will simplify our proof later. Consider the graph $G(G_1, G_2, \ldots, G_n, M_{1,2}, M_{2,3}, \ldots, M_{n-1,n}, M_{1,n})$, deleting the perfect matching $M_{1,n}$ from it, the resulting graph is reduced to $G(G_1, G_2, \ldots, G_n, M_{1,2}, M_{2,3}, \ldots, M_{n-1,n})$. For convenience, we shall write it as $G(G_1, G_2, \ldots, G_n; P_n)$, where P_n stands for a path of length $n - 1$. See Fig. 2. In $G(G_1, G_2, \ldots, G_n; P_n)$, $\deg(v) = k + 2$ for all $v \in V(G_2) \cup V(G_3) \cup \cdots \cup V(G_{n-1})$, and $\deg(v) = k + 1$ for $v \in V(G_1) \cup V(G_n)$. The following theorem states that the fault-tolerant hamiltonicity $\mathscr{H}_{\mathrm{f}}(G)$ and fault-tolerant hamiltonian connectivity $\mathscr{H}_{\mathrm{f}}^{\kappa}(G)$ of graph $G(G_1, G_2, \ldots, G_n; P_n)$, as compared with $G_1, G_2, \ldots, G_n$, are increased by 1.

Theorem 1. *Assume $n \geqslant 2$ and $k \geqslant 5$. Let $G_1, G_2, \ldots, G_n$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. Then graph $G(G_1, G_2, \ldots, G_n; P_n)$ is $(k - 1)$-hamiltonian and $(k - 2)$-hamiltonian connected.*

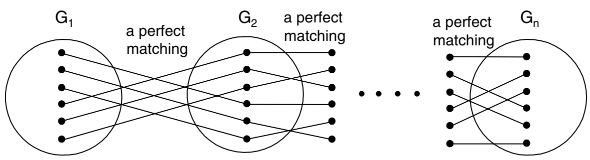

Fig. 2. $G(G_1, G_2, \ldots, G_n; P_n)$.

Proof. We prove it by induction on n. Suppose $n = 2$, then $G(G_1, G_2; P_2) = G(G_1, G_2; M)$. By Lemmas 5 and 6, $G(G_1, G_2; M)$ is $(k-1)$-hamiltonian for $k \geqslant 4$ and $(k-2)$-hamiltonian connected for $k \geqslant 5$.

Assume the theorem is true for n, which means $G(G_1, G_2, \ldots, G_n; P_n)$ is $(k-1)$-hamiltonian and $(k-2)$-hamiltonian connected for $k \geqslant 5$. We shall show that $G(G_1, G_2, \ldots, G_{n+1}; P_{n+1})$ is $(k-1)$-hamiltonian and $(k-2)$-hamiltonian connected for $k \geqslant 5$ and $n+1 \geqslant 3$.

We first prove that the fault-tolerant hamiltonicity $\mathscr{H}_{\mathrm{f}}(G)$ of $G(G_1, G_2, \ldots, G_n, G_{n+1}; P_{n+1})$ is exactly $k-1$. We only consider the situation that the total number of faults is $k-1$. As for the total number of faults is $k' < k-1$, we choose $k-1-k'$ non-faulty edges as faulty edges. Consider G_1 and G_{n+1}, either $f_1 \leqslant k-3$ or $f_{n+1} \leqslant k-3$. If this is not true, then $f_1 \geqslant k-2$ and $f_{n+1} \geqslant k-2$, and $(k-2)+(k-2) \leqslant f_1 + f_{n+1} \leqslant f_{(1\cdots n+1)} = k-1$, so $k \leqslant 3$. It is a contradiction since we assume $k \geqslant 5$. Without loss of generality, we may assume $f_{n+1} \leqslant k-3$. Consequently, $G_{n+1} - F_{n+1}$ is hamiltonian connected. By Lemma 3, there exists at least one healthy matching edge between G_n and G_{n+1}, say (u_n, u_{n+1}). By Lemma 1, there are atleast $(k-1) - f_{(1\cdots n)} + 2 = k + 1 - f_{(1\cdots n)}$ edges incident to vertex u_n, such that each one of them is on some hamiltonian cycle in $G(G_1, \ldots, G_n; P_n) - F_{(1\cdots n)}$. Note that one of the $k+1-f_{(1\cdots n)}$ edges may be a matching edge between G_{n-1} and G_n.Of all these $k+1-f_{(1\cdots n)}-1$ edges, there is at least one, say (u_n, v_n), such that v_n, v_{n+1}, and (v_n, v_{n+1}) are healthy. If it is not true, $f_{(1\cdots n+1)} = f_{(1\cdots n)} + (f_{(1\cdots n+1)} - f_{(1\cdots n)}) \geqslant f_{(1\cdots n)} + ((k+1) - f_{(1\cdots n)} - 1) = k$. This contradicts the fact that $f_{(1\cdots n+1)} = k-1$. We add the matching edge (v_n, v_{n+1}) and delete (u_n, v_n). Then, there exists a u_{n+1}, v_{n+1}-hamiltonian path $\langle u_{n+1}, HP_{n+1}, v_{n+1}\rangle$ in $G_{n+1} - F_{n+1}$ since $f_{n+1} \leqslant k-3$. Therefore, $\langle u_n, HP_{(1\cdots n)}, v_n, v_{n+1}, HP_{n+1}, u_{n+1}, u_n\rangle$ is a fault free hamiltonian cycle in $G(G_1, \ldots, G_{n+1}; P_{n+1}) - F_{(1\cdots n+1)}$. See Fig. 3. This completes the proof that the fault-tolerant hamiltonicity $\mathscr{H}_{\mathrm{f}}(G)$ of $G(G_1, G_2, \ldots, G_n, G_{n+1}; P_{n+1})$ is $k-1$.

Now, we prove that the fault-tolerant hamiltonian connectivity $\mathscr{H}^{\kappa}_{\mathrm{f}}(G)$ of $G(G_1, G_2, \ldots, G_n, G_{n+1}; P_{n+1})$ is $k-2$. Again, we prove this by induction on n. Let x and y be two arbitrary healthy vertices in $G(G_1, \ldots G_{n+1}; P_{n+1})$, we shall find a fault free hamiltonian path joining x and y. We consider the situation that the total number of faults is exactly $k-2$, and the proof is divided with respect to the locations of x and y into two cases.

Case 1

x and y are in different components.

Without loss of generality, we assume x is in G_i and y is in G_j for $i < j$. In this case, we shall separate $G(G_1, \ldots, G_{n+1}; P_{n+1})$ into two parts $G(G_1, \ldots, G_r; P_r)$ and $G(G_{r+1}, \ldots, G_{n+1}; P_{n+1-r})$ for some r, $i \leqslant r < j$, such that x is in $G(G_1, \ldots, G_r; P_r)$ and y is in $G(G_{r+1}, \ldots, G_{n+1}; P_{n+1-r})$. Moreover, we consider the following two subcases.

Subcase 1-1

If there is an r between i and $j-1$, such that both $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$ and $G(G_{r+1}, \ldots, G_{n+1}; P_{n+1-r}) - F_{(r+1\cdots n+1)}$ are hamiltonian connected.

By Lemma 4, there exists at least one healthy matching edge (u_r, u_{r+1}) between G_r and G_{r+1}, such that $u_r \neq x$ and $u_{r+1} \neq y$. There is an x, u_r-hamiltonian path $\langle x, HP_{(1\cdots r)}, u_r\rangle$ in $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$ and a u_{r+1}, y-hamiltonian path $\langle u_{r+1}, HP_{(r+1\cdots n+1)}, y\rangle$ in $G(G_{r+1}, \ldots, G_{n+1}; P_{n+1-r}) - F_{(r+1\cdots n+1)}$ by induction hypothesis. Combining these two fault free paths, we have a fault free x, y-hamiltonian path $\langle x, HP_{(1\cdots r)}, u_r, u_{r+1}, HP_{(r+1\cdots n+1)}, y\rangle$ in this subcase. See Fig. 4.

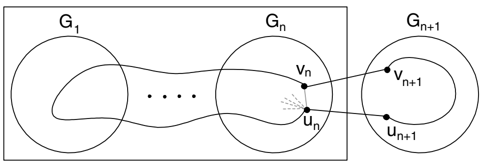

Fig. 3. $G(G_1, G_2, \ldots, G_{n+1}; P_{n+1})$ is $(k-1)$-hamiltonian for $k \geqslant 4$ and $n+1 \geqslant 3$.

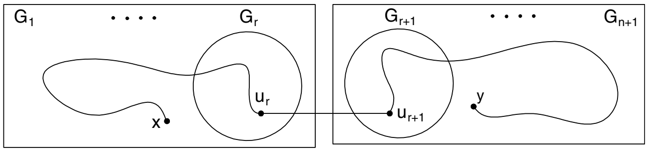

Fig. 4. Subcase 1-1: Both $G(G_1,\ldots,G_r;P_r)-F_{(1\cdots r)}$ and $G(G_{r+1},\ldots,G_{n+1};P_{n+1-r})-F_{(r+1\cdots n+1)}$ are hamiltonian connected.

Subcase 1-2

There does not exist any r between i and $j-1$, to make both $G(G_1,\ldots,G_r;P_r)-F_{(1\cdots r)}$ and $G(G_{r+1},\ldots,G_n;P_{n-r})-F_{(r+1\cdots n)}$ hamiltonian connected.

If both $G(G_1,\ldots,G_r;P_r)$ and $G(G_{r+1},\ldots,G_{n+1};P_{n+1-r})$ contain two or more components, by induction, both of them are fault free hamiltonian connected because the total number of faults is $k-2$. This contradicts our assumption. Hence, we may without loss of generality assume that $r=1$ and $f_1=k-2$. Then $G(G_1,\ldots,G_r;P_r)=G_1$ and G_1 is $(k-2)$-hamiltonian. So there is a hamiltonian cycle in G_1-F_1. Vertex x has two neighboring vertices on this cycle, we choose one that is not matched with y, say u_1. Then, we add matching edge (u_1,u_2) and delete (u_1,x). On the other side, by induction hypothesis, there is a fault free u_2,y-hamiltonian path $\langle u_2,HP_{(2\cdots n+1)},y\rangle$ in $G(G_2,\ldots,G_{n+1};P_n)-F_{(2\cdots n+1)}$. Thus, $\langle x,HP_1,u_1,u_2,HP_{(2\cdots n+1)},y\rangle$ is a fault free x,y-hamiltonian path in this subcase. See Fig. 5.

Case 2

x and y are in the same component.

The proof of this case is further divided into two subcases.

Subcase 2-1

All the $k-2$ faults are in the same component that x and y are in.

Without loss of generality, we may assume x and y are not in G_{n+1}, otherwise we may replace G_{n+1} by G_1. By induction, $G(G_1,\ldots,G_n;P_n)-F_{(1\cdots n)}$ is hamiltonian connected. By Lemma 4, there exists at least one healthy matching edge (u_n,u_{n+1}) between G_n and G_{n+1} such that $u_n\notin\{x,y\}$ and $u_{n+1}\notin\{x,y\}$. By Lemma 2, there are at least $(k-2)-f_{(1\cdots n)}+2=k-f_{(1\cdots n)}$ edges incident to vertex u_n, such that each one of them is on some x,y-hamiltonian path in $G(G_1,\ldots,G_n;P_n)-F_{(1\cdots n)}$. Note that one of the $k-f_{(1\cdots n)}$ edges may be a matching edge between G_{n-1} and G_n. Among these $k-f_{(1\cdots n)}-1$ edges, there is at least one edge (u_n,v_n), such that v_n, v_{n+1}, and (v_n,v_{n+1}) are healthy.If it is not true, then $G(G_1,\ldots,G_{n+1};P_{n+1})$ contains $f_{(1\cdots n+1)}$ faults, and $f_{(1\cdots n+1)}=f_{(1\cdots n)}+(f_{(1\cdots n+1)}-f_{(1\cdots n)})\geqslant f_{(1\cdots n)}+(k-f_{(1\cdots n)}-1)=k-1$. This contradicts the fact that $f_{(1\cdots n+1)}=k-2$. Now, G_{n+1} contains a hamiltonian path $\langle u_{n+1},HP_{n+1},v_{n+1}\rangle$ since $f_{(n+1)}=0$. Therefore,

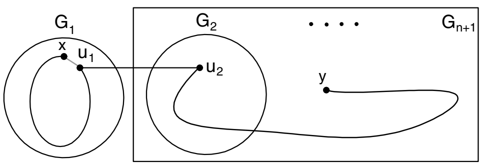

Fig. 5. Subcase 1-2: One of $G(G_1,\ldots,G_r;P_r)-F_{(1\cdots r)}$ and $G(G_{r+1},\ldots,G_n;P_{n+1-r})-F_{(r+1\cdots n)}$ is not hamiltonian connected.

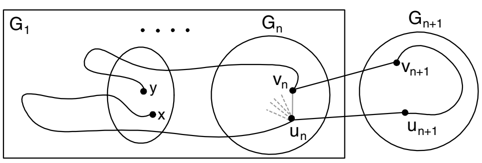

Fig. 6. Subcase 2-1: All the $k-2$ faults are in the same component that x and y are in.

we have a fault free x,y-hamiltonian path $\langle x, P(x,u_n), u_n, u_{n+1}, HP_{n+1}, v_{n+1}, v_n, P(v_n,y), y\rangle$, where $\langle x, P(x,u_n), u_n, v_n, P(v_n,y), y\rangle$ is a hamiltonian path in $G(G_1,\ldots,G_n; P_n) - F_{(1\cdots n)}$. This case is proved. See Fig. 6.

Subcase 2-2

Not all the $k-2$ faults are in the same component that x and y are in.

Without loss of generality, we may also assume x and y are not in G_{n+1}. Let G_r be the component that x and y are in, where $1 \leqslant r \leqslant n$. We separate $G(G_1,\ldots,G_{n+1}; P_{n+1})$ into two parts $G(G_1,\ldots,G_s; P_s)$ and $G(G_{s+1},\ldots,G_{n+1}; P_{n+1-s})$, where:

$$s = r-1, \quad \text{if } r = n \text{ and } f_{n+1} = k-2;$$
$$s = r, \quad \text{otherwise.}$$

In this way of separation, we guarantee that both $G(G_1,\ldots,G_s; P_s) - F_{(1\cdots s)}$ and $G(G_{s+1},\ldots,G_{n+1}; P_{n+1-s}) - F_{(s+1\cdots n+1)}$ are hamiltonian connected. The case $s = r-1$ is similar to the case $s = r$, so we only consider the case $s = r$. By Lemma 4, there exists at least one healthy matching edge (u_s, u_{s+1}) between G_s and G_{s+1}, such that $u_s \notin \{x,y\}$. By Lemma 2, if $s = 1$ ($s \geqslant 2$ respectively), there are at least $(k-3) - f_{(1\cdots s)} + 2 = k - 1 - f_{(1\cdots s)}$ edges ($(k-2) - f_{(1\cdots s)} + 2 = k - f_{(1\cdots s)}$ edges respectively) incident to vertex u_s, such that each one of them is on some x,y-hamiltonian path in $G(G_1,\ldots,G_s; P_s) - F_{(1\cdots s)}$.

Among these $k-1-f_{(1\cdots s)}$ edges ($k - f_{(1\cdots s)}$ edges respectively), there is at least one edge (u_s, v_s), such that v_s, v_{s+1}, and (v_s, v_{s+1}) are healthy. If it is not true, then $G(G_1,\ldots,G_{n+1}; P_{n+1})$ contains at least $f_{(1\cdots s)} + (k-1-f_{(1\cdots s)}) = k-1$ faults when $s = 1$($f_{(1\cdots s)} + (k - f_{(1\cdots s)}) - 1 = k-1$ faults when $s \geqslant 2$ respectively). This contradicts the fact that $f_{(1\cdots n+1)} = k-2$. We then add a matching edge (v_s, v_{s+1}) and delete edge (u_s, v_s). By induction, $G(G_{s+1},\ldots,G_{n+1}; P_{n+1-s}) - F_{(s+1\cdots n+1)}$ is hamiltonian connected. Then, we have a fault free hamiltonian path $\langle x, P(x,u_s), u_s, u_{s+1}, HP_{(s+1\cdots n+1)}, v_{s+1}, v_s, P(v_s,y), y\rangle$ in this subcase, where $\langle x, P(x,u_s), u_s, v_s, P(v_s,y), y\rangle$ is a hamiltonian path in $G(G_1,\ldots,G_s; P_s) - F_{(1\cdots s)}$. See Fig. 7. This completes the proof. □

Now, we consider the graph $G(G_1, G_2,\ldots,G_n; C_n)$, and we shall show that it is a super fault-tolerant hamiltonian graph.

Theorem 2. *Assume $n \geqslant 3$ and $k \geqslant 5$. Let $G_1, G_2,\ldots,G_n$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. Then graph $G(G_1, G_2,\ldots,G_n; C_n)$ is k-hamiltonian.*

Before proving this theorem, we make one remark. In the following proofs of Theorems 2 and 3, we may assume without loss of generality that the faulty set of $G(G_1, G_2,\ldots,G_n; C_n)$ does not contain any matching edge. Otherwise, suppose that there exists one matching edge between G_1 and G_n which is faulty. We simply ignore all the matching edges between G_1 and G_n, then $G(G_1, G_2,\ldots,G_n; C_n)$ is reduced to $G(G_1, G_2,\ldots,G_n; P_n)$. Then the problem of proving $G(G_1, G_2,\ldots,G_n; C_n)$ is k-hamiltonian and $(k-1)$-hamiltonian connected is reduced to show that $G(G_1, G_2,\ldots,G_n; P_n)$ is $(k-1)$-hamiltonian and $(k-2)$-hamiltonian connected. Therefore, the result follows from Theorem 1.

Proof of Theorem 2. We only consider the case that the total number of faults is exactly k and there are no matching edge faults. Without loss of generality, we may assume that $f_1 \geqslant f_i$ for all $2 \leqslant i \leqslant n$. The proof is classified into four cases.

Case 1

$f_1 = k$, all the k faults are in G_1.

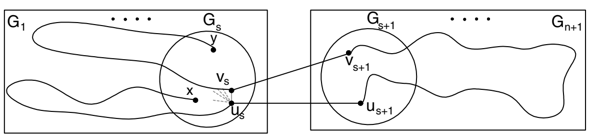

Fig. 7. Subcase 2-2: Not all the $k-2$ faults are in the same component that x and y are in.

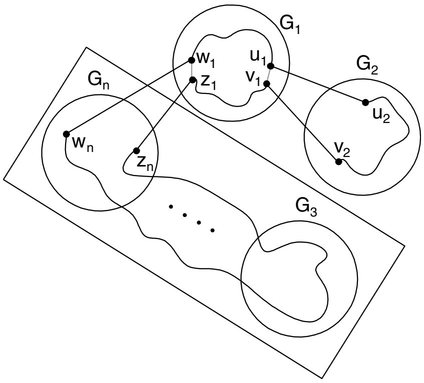

Fig. 8. Case 1: $f_1 = k$ and $|V(G_1 - F_1)| \geqslant 2$.

Let g be a fault in G_1, then, there is a hamiltonian path $\langle u_1, P(u_1, v_1), v_1\rangle$ in $G_1 - (F_1 - \{g\})$. Suppose $|V(G_1 - F_1)| \geqslant 2$. In $G_1 - F_1$, the path $\langle u_1, P(u_1, v_1), v_1\rangle$ is separated into two subpaths, say $\langle u_1, P(u_1, w_1), w_1\rangle$ and $\langle z_1, P(z_1, v_1), v_1\rangle$, which cover all the vertices of $G_1 - F_1$. We then add four matching edges: $(u_1, u_2), (v_1, v_2)$, (w_1, w_n), and (z_1, z_n). In G_2, there is a u_2, v_2-hamiltonian path $\langle u_2, HP_2, v_2\rangle$ since $f_2 = 0$. And in $G(G_3, \ldots, G_n; P_{n-2})$, there is a w_n, z_n-hamiltonian path $\langle w_n, HP_{(3\cdots n)}, z_n\rangle$ since $f_{(3\cdots n)} = 0$. Hence, we have a fault free hamiltonian cycle $\langle u_1, u_2, HP_2, v_2, v_1, P(v_1, z_1), z_1, z_n, HP_{(3\cdots n)}, w_n, w_1, P(w_1, u_1), u_1\rangle$ in this subcase. See Fig. 8. Now, suppose $|V(G_1 - F_1)| = 1$. Let $V(G_1 - F_1) = \{u_1\}$, then the above-mentioned proof does not work. We shall construct a hamiltonian cycle in $G(G_1, \ldots, G_n; C_n) - F_{(1\cdots n)}$ as follows. First, we add two matching edges (u_1, u_2) and (u_1, u_n). In $G(G_2, \ldots, G_n; P_{n-1})$, there is a u_2, u_n-hamiltonian path $\langle u_2, HP_{(2\cdots n)}, u_n\rangle$ since $f_{(2\cdots n)} = 0$. Hence, $\langle u_1, u_2, HP_{(2\cdots n)}, u_n, u_1\rangle$ forms a fault free hamiltonian cycle in $G(G_1, \ldots, G_n; C_n) - F_{(1\cdots n)}$. This case is proved. See Fig. 9.

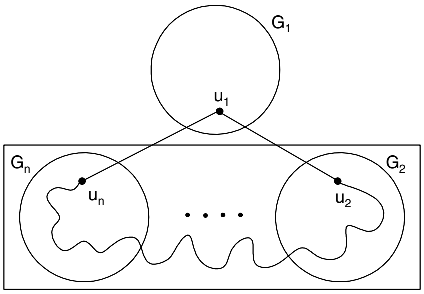

Fig. 9. Case 1: $f_1 = k$ and $|V(G_1 - F_1)| = 1$.

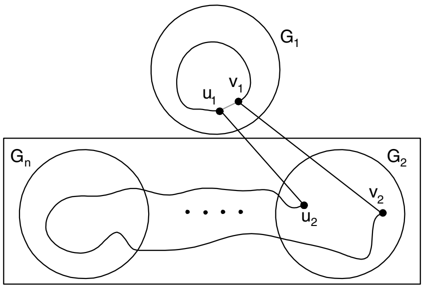

Fig. 10. Case 2: $f_1 = k - 1$.

Case 2
$f_1 = k - 1$.

Since G_1 is $(k-2)$-hamiltonian, there exists a fault free hamiltonian path $\langle u_1, HP_1, v_1\rangle$ in $G_1 - F_1$. Furthermore, $u_1 \neq v_1$ since $|V(G_1)| \geqslant k+1$ and $f_1 = k-1$. Without loss of generality, we may assume the kth fault is not in G_2, otherwise we replace G_2 by G_n. Now, we add matching edges (u_1, u_2) and (v_1, v_2). In $G(G_2, \ldots, G_n; P_{n-1}) - F_{(2\cdots n)}$, there is a u_2, v_2-hamiltonian path $\langle u_2, HP_{(2\cdots n)}, v_2\rangle$ since $f_{(2\cdots n)} = 1$. Therefore, we have a fault free hamiltonian cycle $\langle u_1, u_2, HP_{(2\cdots n)}, v_2, v_1, HP_1, u_1\rangle$ in this case. See Fig. 10.

Case 3
$2 \leqslant f_1 \leqslant k-2$.

We may assume without loss of generality that not all of the faults are in $G(G_1, G_n; M_{1,n})$, or we shall consider $G(G_1, G_2; M_{1,2})$ in place of $G(G_1, G_n; M_{1,n})$. So $f_1 + f_n \leqslant k-1$. By Lemma 3, there is at least one healthy matching edge (u_1, u_n) between G_1 and G_n. In $G_1 - F_1$, by Lemma 1, there are at least $(k-2) - f_1 + 2 = k - f_1$ edges incident to vertex u_1, such that each one of them is on some hamiltonian cycle in $G_1 - F_1$. Of all these $k - f_1$ edges, there is at least one edge (u_1, v_1), such that v_1, v_n, and (v_1, v_n) are healthy. If this is not true, $G(G_1, G_n; M_{1,n})$ contains at least $f_1 + (k - f_1) = k$ faults. But the total number of faults in $G(G_1, G_n; M_{1,n})$ is no greater than $k-1$, causing a contradiction. Since $f_1 \geqslant 2$, we have $f_{(2\cdots n)} \leqslant k-2$, so there is a fault free u_n, v_n-hamiltonian path $\langle u_n, HP_{(2\cdots n)}, v_n\rangle$ in $G(G_2, \ldots, G_n; P_{n-1}) - F_{(2\cdots n)}$. Then, $\langle u_1, HP_1, v_1, v_n, HP_{(2\cdots n)}, u_n, u_1\rangle$ is a fault free hamiltonian cycle in $G(G_1, \ldots, G_n; C_n) - F_{(1\cdots n)}$, and this case is proved. See Fig. 11.

Case 4
$f_1 \leqslant 1$.

Since $f_1 \geqslant f_i$ for all $2 \leqslant i \leqslant n$, we may assume without loss of generality that $f_1 = 1$ and $f_{(1\cdots n)} = k \geqslant 5$, otherwise it is clear that $G(G_1, \ldots, G_n; C_n)$ is hamiltonian. We may further assume that not all of the faults are in $G(G_1, G_n; M_{1,n})$. We choose a minimum number r such that $f_1 + f_2 + \cdots + f_r = 2$. It is clear that $r \leqslant n-1$. By Lemma 3, there is at least one healthy matching edge (u_r, u_{r+1}) between G_r and G_{r+1}. Then, by Lemma 1, there are at least $(k-2) - f_{(1\cdots r)} + 2 = k - f_{(1\cdots r)} = k-2$ edges incident to vertex u_r in $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$, such that each one of them is on some hamiltonian cycle in $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$. We note $k-2 \geqslant 3$ since $k \geqslant 5$. Of these $k-2$ edges incident to u_r, there is at least one edge (u_r, v_r), such that (u_r, v_r) is not a matching edge in $G(G_{r-1}, G_r; M_{r-1,r})$, and the matching vertex v_{r+1} of v_r in G_{r+1} is not faulty. We add the matching edge (v_r, v_{r+1}) and delete (u_r, v_r). Since $f_{(r+1\cdots n)} = k-2 \geqslant 5-2 = 3$ and $f_1 \geqslant f_i$ for all $2 \leqslant i \leqslant n$, $G(G_{r+1}, \ldots, G_n; P_{n-r})$ contains at least three components. Thus, $G(G_{r+1}, \ldots, G_n; P_{n-r}) - F_{(r+1\cdots n)}$ is hamiltonian connected and there is a u_{r+1}, v_{r+1}-hamiltonian path $\langle u_{r+1}, HP_{(r+1\cdots n)}, v_{r+1}\rangle$ in $G(G_{r+1}, \ldots, G_n; P_{n-r}) - F_{(r+1\cdots n)}$. Then we have a fault free hamiltonian cycle $\langle u_r, HP_{(1\cdots r)}, v_r, v_{r+1}, HP_{(r+1\cdots n)}, u_{r+1}, u_r\rangle$ in this case. See Fig. 12. This completes the proof of the theorem. □

The fault-tolerant hamiltonian connectivity $\mathscr{H}_f^\kappa(G)$ in $G(G_1, G_2, \ldots, G_n; C_n)$ is also increased by 2, as stated in the following theorem.

Theorem 3. *Assume $n \geqslant 3$ and $k \geqslant 5$. Let $G_1, G_2, \ldots, G_n$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. Then $G(G_1, G_2, \ldots, G_n; C_n)$ is a $(k-1)$-hamiltonian connected graph.*

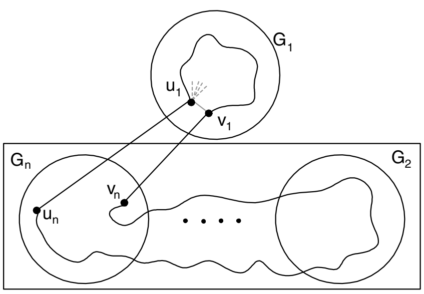

Fig. 11. Case 3: $2 \leqslant f_1 \leqslant k-2$.

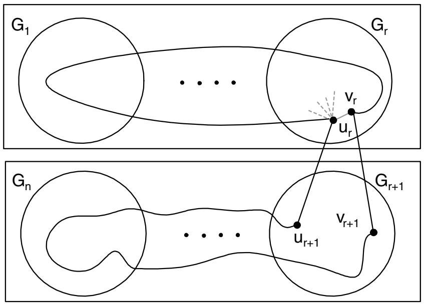

Fig. 12. Case 4: $f_1 \leqslant 1$.

Proof. Let x and y be two healthy vertices in $G(G_1, G_2, \ldots, G_n; C_n)$, we shall find a fault free hamiltonian path joining x and y. Just as before, we only consider the situation that the total number of faults is exactly $k-1$ and there are no matching edge faults. The proof is classified into two cases with respect to the locations of x and y.

Case 1
x and y are in different components.

This case can be further divided into two subcases.

Subcase 1-1
Not all the $k-1$ faults are in the same component.

We may without loss of generality separate $G(G_1, G_2, \ldots, G_n; C_n)$ into two parts $G(G_1, \ldots, G_r; P_r)$ and $G(G_{r+1}, \ldots, G_n; P_{n-r})$ where $1 \leqslant r \leqslant n-1$, such that x in $G(G_1, \ldots, G_r; P_r)$, y in $G(G_{r+1}, \ldots, G_n; P_{n-r})$, $f_{(1\cdots r)} \geqslant 1$, and $f_{(r+1\cdots n)} \geqslant 1$. We shall prove this subcase by considering whether $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$ and $G(G_{r+1}, \ldots, G_n; P_{n-r}) - F_{(r+1\cdots n)}$ are hamiltonian connected. Suppose that both $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$ and $G(G_{r+1}, \ldots, G_n; P_{n-r}) - F_{(r+1\cdots n)}$ are hamiltonian connected. Since the total number of faults is $k-1$, there are at least two healthy matching edges between G_1 and G_n, and there are at least two healthy matching edges between G_r and G_{r+1}. Among these four healthy matching edges, there is at least one (u_r, u_{r+1}) between G_r and G_{r+1}, such that $u_r \notin \{x\}$ and $u_{r+1} \notin \{y\}$. Now, there are an x, u_r-hamiltonian path $\langle x, HP_{(1\cdots r)}, u_r\rangle$ in $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$ and a u_{r+1}, y-hamiltonian path $\langle u_{r+1}, HP_{(r+1\cdots n)}, y\rangle$ in $G(G_{r+1}, \ldots, G_n; P_{n-r}) - F_{(r+1\cdots n)}$. Therefore, we have a fault free x, y-hamiltonian path $\langle x, HP_{(1\cdots r)}, u_r, u_{r+1}, HP_{(r+1\cdots n)}, y\rangle$ in this subcase. See Fig. 13.

Suppose that $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$ or $G(G_{r+1}, \ldots, G_n; P_{n-r}) - F_{(r+1\cdots n)}$ is not hamiltonian connected. We claim that at least one of $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$ and $G(G_{r+1}, \ldots, G_n; P_{n-r}) - F_{(r+1\cdots n)}$ is hamiltonian connected. Suppose not, then $(k-2) + (k-1) \leqslant f_{(1\cdots r)} + f_{(r+1\cdots n)} = f_{(1\cdots n)} = k-1$, so $k \leqslant 2$. But $k \geqslant 5$, it is

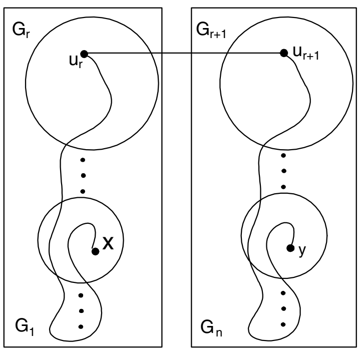

Fig. 13. Subcase 1-1: $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$ and $G(G_{r+1}, \ldots, G_n; P_{n-r}) - F_{(r+1\cdots n)}$ are hamiltonian connected.

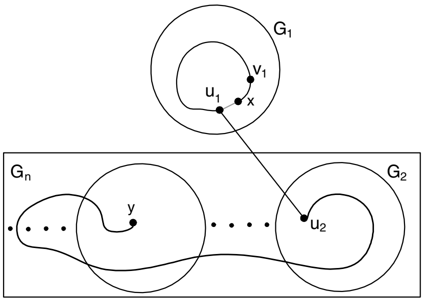

Fig. 14. Subcase 1-1: $G(G_1,\ldots,G_r;P_r)-F_{(1\cdots r)}$ or $G(G_{r+1},\ldots,G_n;P_{n-r})-F_{(r+1\cdots n)}$ is not hamiltonian connected.

a contradiction. Hence, we may assume without loss of generality that $G(G_1,\ldots,G_r;P_r)-F_{(1\cdots r)}$ is not hamiltonian connected, and $G(G_{r+1},\ldots,G_n;P_{n-r})-F_{(r+1\cdots n)}$ is. Now, we know $f_{(1\cdots r)}\leqslant k-2$ and $G(G_1,\ldots,G_r;P_r)-F_{(1\cdots r)}$ is not hamiltonian connected. So by Theorem 1, $f_{(1\cdots r)}=k-2$ and $G(G_1,\ldots,G_r;P_r)=G_1$. Since G_1 is $(k-2)$-hamiltonian, there exists a fault free hamiltonian cycle in G_1-F_1. On this cycle, there are two vertices u_1 and v_1 adjacent to x. Consider the four matching edges incident to u_1 or v_1: (u_1,u_2), (v_1,v_2), (u_1,u_n), and (v_1,v_n). Among these four edges, there is at least one, say (u_1,u_2), which is not matched with y nor with the $(k-1)$th fault, i.e., $u_2\neq y$ and u_2 is healthy. We delete edge (u_1,x) and add edge (u_1,u_2). In $G(G_2,\ldots,G_n;P_{n-1})-F_{(2\cdots n)}$, there is a u_2,y-hamiltonian path $\langle u_2,HP_{(2\cdots n)},y\rangle$ since $f_{(2\cdots n)}\leqslant k-2$. Thus, $\langle x,HP_1,u_1,u_2,HP_{(2\cdots n)},y\rangle$ is a fault free x,y-hamiltonian path in this subcase. See Fig. 14.

Subcase 1-2

All the $k-1$ faults are in the same component.

This subcase can be further divided into two subcases.

Subcase 1-2-1

All the $k-1$ faults are in a single component which contains either x or y.

Without loss of generality, we may assume: (1) x and all the faults are in G_1; and (2) y is in G_r, where $r\neq 1$ and $r\neq 2$. There is a hamiltonian cycle in $G(G_1,G_2;M_{1,2})-F_1$ since $f_1=k-1$. On this cycle, there are two vertices adjacent to x. Of these two vertices, at least one is in G_1, say u_1. Let u_n be the matching vertex of u_1 in G_n. We shall consider the cases: $u_n=y$ or $u_n\neq y$. Suppose $u_n=y$. We add the matching edge (u_1,y) and delete (x,u_1). On any hamiltonian cycle in $G(G_1,G_2;M_{1,2})-F_1$, at least k edges are in G_2 since $|V(G_2)|\geqslant k+1$ and all the faults are in G_1. Of these k edges, there is at least one, say (v_2,w_2), such that the matching vertex v_3 of v_2 (the matching vertex w_3 of w_2 respectively) is not y since $k\geqslant 5$. Then, we delete (v_2,w_2) and add both matching edges (v_2,v_3) and (w_2,w_3). In $G(G_3,\ldots,G_n;P_{n-2})-\{y\}$, there is a v_3,w_3-hamiltonian path $\langle v_3,P(v_3,w_3),w_3\rangle$. Therefore, we have a fault free x,y-hamiltonian path $\langle x,P(x,v_2),v_2,v_3,P(v_3,w_3),w_3,w_2,P(w_2,u_1),u_1,y\rangle$ in this subcase, where $\langle x,P(x,v_2),v_2,w_2,P(w_2,u_1),u_1,x\rangle$ is a hamiltonian cycle in $G(G_1,G_2;M_{1,2})-F_1$. See Fig. 15.

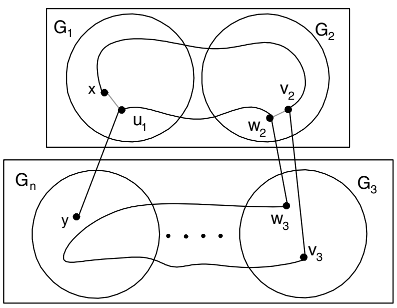

Fig. 15. Subcase 1-2-1: $u_2=y$.

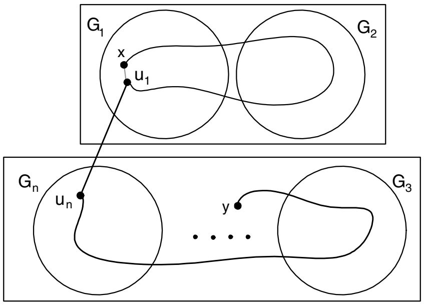

Fig. 16. Subcase 1-2-1: $u_2 \neq y$.

Now, suppose $u_n \neq y$. We then add the matching edge (u_1, u_n) and delete (x, u_1). In $G(G_3, \ldots, G_n; P_{n-2}) - F_{(3\cdots n)}$, there is a u_n, y-hamiltonian path $\langle u_n, HP_{(3\cdots n)}, y\rangle$ since $f_{(3\cdots n)} = 0$. So $\langle x, HP_{(1\cdots 2)}, u_1, u_n, HP_{(3\cdots n)}, y\rangle$ forms a fault free x, y-hamiltonian path in this subcase. See Fig. 16.

Subcase 1-2-2
All the $k-1$ faults are in a single component which does not contain x nor y.

Without loss of generality, we may assume that all the faults are in G_1, x is in G_r, and y is in G_s, where $1 < r < s \leqslant n$. In $G_1 - F_1$, there is a hamiltonian path $\langle u_1, HP_1, v_1\rangle$ since $f_1 = k-1$. We add two matching edges (u_1, u_2) and (v_1, v_2). Note that x may be equal to u_2 or v_2. By Lemma 2, there are at least $(k-3) - f_{(2\cdots r)} + 1$ (we add 1 not 2 since x may be equal to one of u_2 or v_2) edges incident to vertex x, such that each one of them is on some u_2, v_2-hamiltonian path $\langle u_2, HP_{(2\cdots r)}, v_2\rangle$ in $G(G_2, \ldots, G_r; P_{r-1}) - F_{(2\cdots r)}$. Here, $(k-3) - f_{(2\cdots r)} + 1 = (k-3) - 0 + 1 = k-2$. There are at least $k-2$ edges that can be taken into account. Note that $k \geqslant 5$, so $k-2 \geqslant 3$. Of these three edges, there is at least one edge (x, w_r), such that $(x, w_r) \in E(G_r)$ and w_r is not matched with y. Then, we delete (x, w_r) and add a matching edge (w_r, w_{r+1}). In $G(G_{r+1}, \ldots, G_n; P_{n-r})$, there is a w_{r+1}, y-hamiltonian path $\langle w_{r+1}, HP_{(r+1\cdots n)}, y\rangle$ since there is no fault in $G(G_{r+1}, \ldots, G_n; P_{n-r})$. Thus, we have a fault free x, y-hamiltonian path $\langle x, P(x, v_2), v_2, v_1, HP_1, u_1, u_2, P(u_2, w_r), w_r, w_{r+1}, HP_{(r+1\cdots n)}, y\rangle$ in this subcase, where $\langle x, P(x, v_2), v_2, v_1, HP_1, u_1, u_2, P(u_2, w_r), w_r\rangle$ is a fault free hamiltonian path in $G(G_1, \ldots, G_r; P_r) - F_{(1\cdots r)}$. See Fig. 17.

Case 2
x and y are in the same component.

We may assume without loss of generality that x and y are in G_1. This case can be further divided into three subcases.

Subcase 2-1
$k-2 \leqslant f_1 \leqslant k-1$.

Let g and h be two faults in G_1. Since G_1 is $(k-3)$-hamiltonian connected, $G_1 - (F_1 - \{g, h\})$ has a fault free hamiltonian path $\langle x, P(x, y), y\rangle$. Removing g and h, the path $\langle x, P(x, y), y\rangle$ is separated into three subpaths:

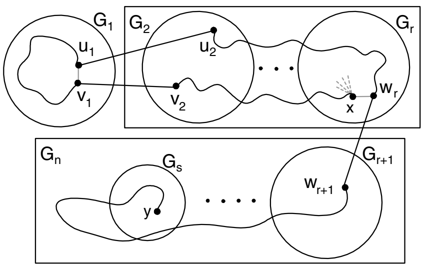

Fig. 17. Subcase 1-2-2: All the faults are in a single component which does not contain x or y.

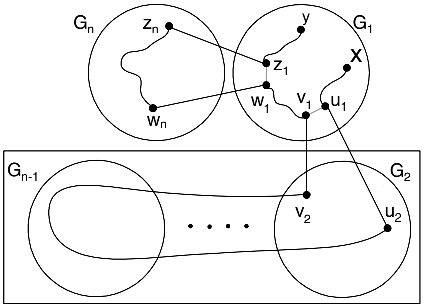

Fig. 18. Subcase 2-1: $\langle x, P(x,y), y\rangle$ is separated into three disjoint subpaths.

$\langle x, P(x,u_1), u_1\rangle$, $\langle v_1, P(v_1,w_1), w_1\rangle$, and $\langle z_1, P(z_1,y), y\rangle$ which cover all the vertices of $G_1 - F_1$. We note that a path could be a single vertex here. There is one exceptional case: $f_1 = k-1$ and $|V(G_1 - F_1)| = 2$, then $V(G_1 - F_1) = \{x,y\}$, and the path $\langle x, P(x,y), y\rangle$ cannot be separated into three disjoint subpaths. Consider the case that the path $\langle x, P(x,y), y\rangle$ is indeed separated into three disjoint subpaths. Since $f_{(2\cdots n)} \leqslant 1$, $F_{(2\cdots n)}$ is either in G_2 or in G_n and this fault is possibly matched with one of u_1, v_1, w_1, or z_1. We may assume without loss of generality that the faulty set $F_{(2\cdots n)}$ is in G_n, and this fault is matched with u_1. We then add four matching edges (u_1,u_2), (v_1,v_2), (w_1,w_n), and (z_1,z_n). In $G(G_2,\ldots,G_{n-1};P_{n-2}) - F_{(2\cdots n-1)}$, there is a u_2,v_2-hamiltonian path $\langle u_2, HP_{(2\cdots n-1)}, v_2\rangle$ since $f_{(2\cdots n-1)} \leqslant 1$. And in $G_n - F_n$, there is an x_n,z_n-hamiltonian path $\langle w_n, HP_n, z_n\rangle$ since $f_n \leqslant 1$. Hence, $\langle x, P(x,u_1), u_1, u_2, HP_{(2\cdots n-1)}, v_2, v_1, P(v_1,w_1), w_1, w_n, HP_n, z_n, z_1, P(z_1,y), y\rangle$ forms a fault free x,y-hamiltonian path in this subcase. See Fig. 18.

Now, suppose $f_1 = k-1$, and $|V(G_1 - F_1)| = 2$. Let $V(G_1 - F_1) = \{x,y\}$. We add matching edges (x,x_2) and (y,y_2). In $G(G_2,\ldots,G_n;P_{n-1})$, there is an x_2,y_2-hamiltonian path $\langle x_2, HP_{(2\cdots n)}, y_2\rangle$ since $f_{(2\cdots n)} = 0$. Therefore, we have a fault free x,y-hamiltonian path $\langle x, x_2, HP_{(2\cdots n)}, y_2, y\rangle$ in this subcase.

Subcase 2-2

$1 \leqslant f_1 \leqslant k-3$.

Without loss of generality, we may assume that not all the faults are in $G(G_1,G_2;M_{1,2})$, or we may replace $G(G_1,G_2;M_{1,2})$ by $G(G_1,G_n;M_{1,n})$. By Lemma 4, there is at least one healthy matching edge (u_1,u_2) between G_1 and G_2 such that $u_1 \notin \{x,y\}$. By Lemma 2, there are at least $(k-3) - f_1 + 2 = k-1-f_1$ edges incident to vertex u_1 in $G_1 - F_1$, such that each one of them is on some x,y-hamiltonian path in $G_1 - F_1$. Among these $k-1-f_1$ edges, there is at least one, say (u_1,v_1), such that all of v_1, v_2, and (v_1,v_2) are healthy. If this is not true, then $G(G_1,G_2;M_{1,2})$ contains at least $f_1 + (k-1-f_1) = k-1$ faults. But we know $f_1 + f_2 \leqslant k-2$, which is a contradiction. Then, we add the matching edge (v_1,v_2). In $G(G_2,\ldots,G_n;P_{n-1}) - F_{(2\cdots n)}$, there is a u_2,v_2-hamiltonian path $\langle u_2, HP_{(2\cdots n)}, v_2\rangle$ since $f_{(2\cdots n)} \leqslant k-2$. Therefore, we have a fault free x,y-hamiltonian path $\langle x, P(x,u_1), u_1, u_2, HP_{(2\cdots n)}, v_2, v_1, P(v_1,y), y\rangle$ in this subcase, where $\langle x, P(x,u_1), u_1, v_1, P(v_1,y), y\rangle$ is a hamiltonian path in $G_1 - F_1$. See Fig. 19.

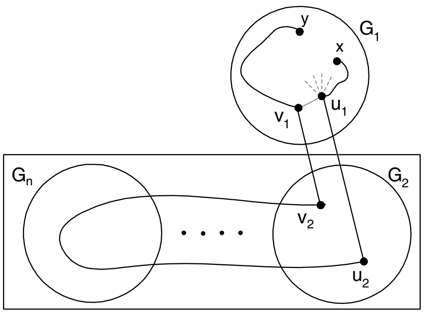

Fig. 19. Subcase 2-2: $1 \leqslant f_1 \leqslant k-3$.

Subcase 2-3
$f_1 = 0$.

Recall that we assume both x add y are in G_1 in this case. There are four matching edges incident to x or y: (x, x_2), (y, y_2), (x, x_n), and (y, y_n). It means that x and y have four matching vertices: x_2, y_2, x_n, and y_n. We divide this subcase into two subcases according to the status of the four matching vertices.

Subcase 2-3-1
Suppose that at least one of the four matching vertices is healthy.

We may assume without loss of generality that x_2 is healthy, then we add edge (x, x_2). In $G(G_2, \ldots, G_n; P_{n-1}) - F_{(2\cdots n)}$, there is a fault free hamiltonian cycle since $f_{(2\cdots n)} \leqslant k - 1$. On this cycle, there are two vertices adjacent to x_2, and at least one of these two is in G_2, say u_2. We consider the cases whether u_2 is matched with y.

Subcase 2-3-1-1
u_2 is not matched with y.

We delete (x_2, u_2), and add (u_1, u_2). In $G_1 - \{x\}$, there is a u_1, y-hamiltonian path $\langle u_1, P(u_1, y), y\rangle$. Therefore, $\langle x, x_2, HP_{(2\cdots n)}, u_2, u_1, P(u_1, y), y\rangle$ forms a fault free x, y-hamiltonian path in this subcase. See Fig. 20.

Subcase 2-3-1-2
u_2 is matched with y.

On the fault free hamiltonian cycle in $G(G_2, \ldots, G_n; P_{n-1}) - F_{(2\cdots n)}$, let v_2 be the vertex adjacent to x_2 and $v_2 \neq u_2$. Note that v_2 is either in G_2 or in G_3. Suppose v_2 is in G_2. We add two matching edges (x, x_2) and (v_1, v_2), and delete (x_2, v_2). In addition, there is a v_1, y-hamiltonian path $\langle v_1, P(v_1, y), y\rangle$ in $G_1 - \{x\}$. Then, $\langle x, x_2, HP_{(2\cdots n)}, v_2, v_1, P(v_1, y), y\rangle$ forms a fault free x, y-hamiltonian path in this subcase. See Fig. 21.

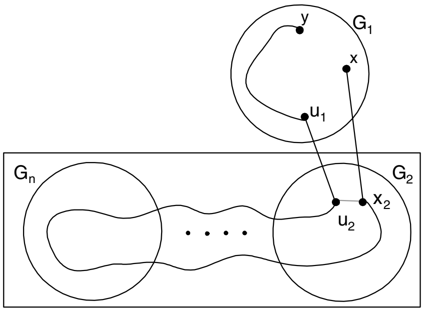

Fig. 20. Subcase 2-3-1-1: u_2 is not matched with y.

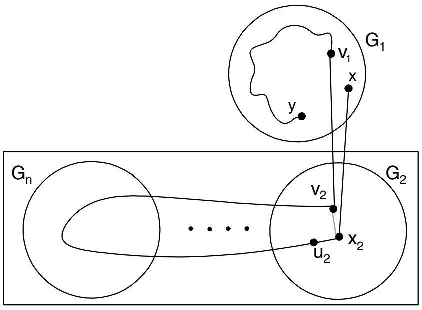

Fig. 21. Subcase 2-3-1-2: v_2 is in G_2.

Suppose v_2 is in G_3. We consider the case that $|V(G_2 - F_2)| > 2$. On the hamiltonian cycle in $G(G_2, \ldots, G_n; P_{n-1}) - F_{(2\cdots n)}$, let w_2 be a vertex in G_2 which is adjacent to u_2 and $w_2 \neq x_2$. We then delete edge (y_2, w_2), and add the matching edge (w_1, w_2). Thus, we have a fault free x, y-hamiltonian path $\langle x, P(x, w_1), w_1, w_2, HP_{(2\cdots n)}, y_2, y\rangle$ in this subcase, where $\langle x, P(x, w_1), w_1\rangle$ is a hamiltonian path in $G_1 - \{y\}$. See Fig. 22.

Suppose $|V(G_2 - F_2)| = 2$, and let $V(G_2 - F_2) = \{x_2, u_2\}$. To construct a fault free x, y-hamiltonian path, we add matching edges (x, x_2) and (u_2, u_3). Then, we have a fault free x, y-hamiltonian path $\langle x, x_2, u_2, u_3, P(u_3, y), y\rangle$ in this subcase, where $\langle u_3, P(u_3, y), y\rangle$ is a hamiltonian path in $G(G_3, G_4, \ldots, G_n, G_1; P_{n-1}) - F_{(3\cdots n)} - \{x\}$. See Fig. 23.

Subcase 2-3-2

Suppose that all of the four matching vertices are faulty.

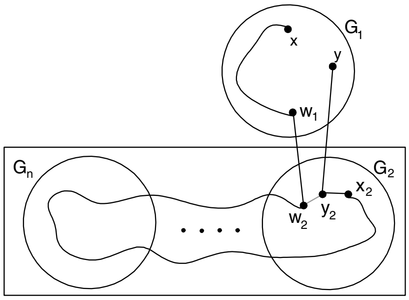

Fig. 22. Subcase 2-3-1-2: v_2 is in G_3.

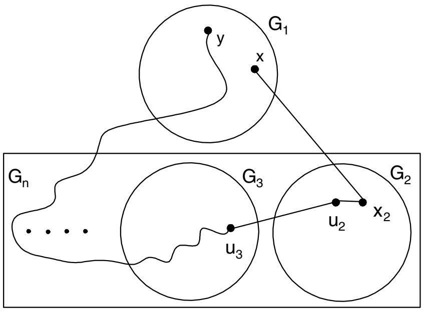

Fig. 23. Subcase 2-3-1-2: $|V(G_2 - F_2)| = 2$.

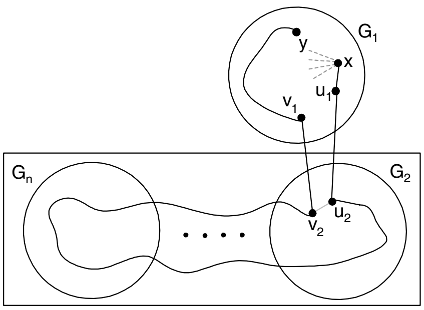

Fig. 24. Subcase 2-3-2: All of the four matching vertices are faulty.

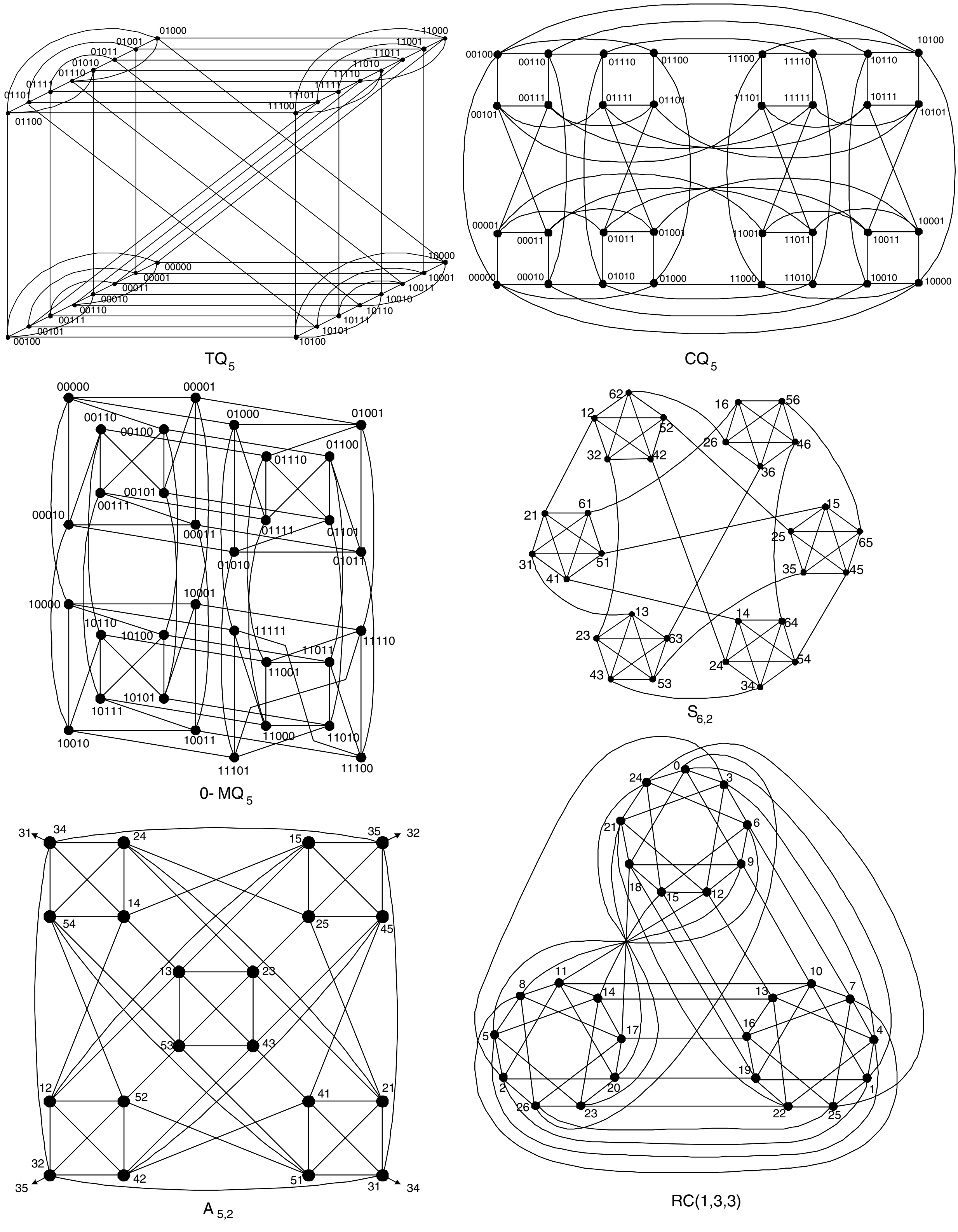

Fig. 25. Super fault-tolerant hamiltonian graphs.

In G_1, excluding possibly y, there are at least $k-1$ vertices adjacent to x. Of these $k-1$ vertices, there is at least one, say u_1, such that u_2 is healthy since $G_2 - \{x_2\}$ has at most $k-2$ faults. Now, we append edge (x, u_1) and the matching edge (u_1, u_2). In $G(G_2, \ldots, G_n; P_{n-1}) - F_{(2\cdots n)}$, there is a fault free hamiltonian cycle since $f_{(2\cdots n)} \leqslant k-1$. On this cycle, neither of the two vertices adjacent to u_2 is matched with y since x_2 and y_2 are faulty. Therefore, we choose one of the two vertices adjacent to u_2, say v_2. We then delete (u_2, v_2) and add the matching edge (v_1, v_2). In $G_1 - \{x, u_1\}$, there is a v_1, y-hamiltonian path $\langle v_1, P(v_1, y), y\rangle$. If it is not true, then $k-3 < 2$, this contradicts the fact that $k \geqslant 5$. So $\langle x, u_1, u_2, HP_{(2\cdots n)}, v_2, v_1, P(v_1, y), y\rangle$ forms a fault free x, y-hamiltonian path in this subcase. See Fig. 24. This completes the proof of this theorem. □

The following corollary results from Theorems 2 and 3.

Corollary 1. *Assume that $G_1, G_2, \ldots, G_n$ are k-regular super fault-tolerant hamiltonian with the same number of vertices where $n \geqslant 3$ and $k \geqslant 5$. Then $G(G_1, G_2, \ldots, G_n; C_n)$ is $(k+2)$-regular super fault-tolerant hamiltonian.*

Among the existing interconnection network topologies, some of them are super fault-tolerant hamiltonian. For example, we use computer program to check that (1) the twisted-cube TQ_5, crossed-cube CQ_5, and möbius cube MQ_5 are 5-regular super fault-tolerant hamiltonian; (2) the (n,k)-star graph $S_{6,2}$ is 5-regular super fault-tolerant hamiltonian; (3) the arrangement graph $A_{5,2}$ is 6-regular super fault-tolerant hamiltonian; and (4) the recursive circulant graph $RC(1,3,3)$ is 6-regular super fault-tolerant hamiltonian. See Fig. 25.

5. Conclusion

The fault-tolerant hamiltonicity and the fault-tolerant hamiltonian connectivity are essential parameters of an interconnection network. In this paper, we propose a family of k-regular, $(k-2)$-hamiltonian, and $(k-3)$-hamiltonian connected graphs. These graphs are maximally fault-tolerant, and we call them super fault-tolerant hamiltonian graphs.

Some of the contributions of this paper are the following. We propose a construction scheme to construct, with flexibility, many k-regular super fault-tolerant hamiltonian graphs for $k \geqslant 6$. The recursive circulant graphs can be recursively constructed using our construction schemes. And therefore, they are in fact a subclass of our proposed family of graphs. Then, we know that they are super fault-tolerant hamiltonian as long as the case is true for initial cases $k \leqslant 5$. With our scheme, some new super fault-tolerant hamiltonian graphs may be constructed.

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