A closed form solution for constant flux pumping in a well under partial penetration condition

Shaw-Yang Yang,[1] Hund-Der Yeh,[2] and Pin-Yuan Chiu[2]

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[1] An analytical model for the constant flux pumping test is developed in a radial confined aquifer system with a partially penetrating well. The Laplace domain solution is derived by the application of the Laplace transforms with respect to time and the finite Fourier cosine transforms with respect to the vertical coordinates. A time domain solution is obtained using the inverse Laplace transforms, convolution theorem, and Bromwich integral method. The effect of partial penetration is apparent if the test well is completed with a short screen. An aquifer thickness 100 times larger than the screen length of the well can be considered as infinite. This solution can be used to investigate the effects of screen length and location on the drawdown distribution in a radial confined aquifer system and to produce type curves for the estimation of aquifer parameters with field pumping drawdown data.

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1. Introduction

[2] A partially penetrating well is commonly situated in an aquifer that is relatively thick. *Hantush* [1962] presented a point source solution for the drawdown distribution around a partially penetrating well under constant flux pumping in an infinite confined aquifer. *Sternberg* [1973] provided a graphical solution for evaluating the total drawdown. *Strelsova-Adams* [1979] reported an analysis of the transient pressure response for a well with limited flow entry produced from an oil reservoir with a gas cap in a zone of low permeability and that with impermeable top and bottom boundaries. *Ruud and Kabala* [1997] developed a two-dimensional integrated well face flux (IWFF) model for computing the drawdown at the well face and around a fully or partially penetrating well with the wellbore storage situated in multilayer confined aquifers. For a partially penetrating well situated in a homogeneous isotropic aquifer, they found that the differences between the IWFF model and Hantush's model [*Hantush*, 1964] were insignificant for wellbore drawdowns but pronounced for the well face flux. Such differences may arise for a partially penetrating well situated in multilayer aquifers, especially if the screen is not located in the most conductive layer. *Cassiani and Kabala* [1998] developed a semianalytical solution to the mixed-type boundary value problem via the dual integral equations. The pumping and slug tests are performed on a partially penetrating well with wellbore storage, infinitesimal skin, and aquifer anisotropy. They stated that their solution is computationally more efficient than the corresponding finite difference solution. In addition, their solutions described accurately the point flux distribution along the well screen of a partially penetrating well.

[3] The objective of this study is to develop a closed form solution for a mathematical model describing the constant flux pumping test performed in a partially penetrating well with finite radius in a homogeneous (single zone) confined aquifer system. This closed form solution can be used to investigate the effects of screen length and location on the drawdown distribution and to produce type curves for the estimation of aquifer parameters with field drawdown data. On the basis of the mathematical model, the Laplace domain solution is derived using the Laplace transforms with respect to time and the finite Fourier cosine transforms with respect to the vertical coordinates. The modified Crump algorithm [*de Hoog et al.*, 1982; *Visual Numerics*, 1997] is adopted to invert the Laplace domain solution. The closed form solution is then obtained using the Bromwich integral method. A numerical approach, including a root search scheme, a numerical integration method, and the Shanks method [*Shanks*, 1955], is proposed to evaluate this solution.

2. Mathematical Derivations

2.1. Analytical Model

[4] In order to account for the effect of partial penetration, a term representing the vertical flow is included in the governing equation of a radial confined aquifer system. The well and aquifer configurations for a radial confined aquifer system are depicted in Figure 1. The assumptions are made for the solution in terms of drawdown; they are: (1) The aquifer is homogeneous, anisotropic, infinite extent, and with a constant thickness; (2) The well is partially penetrated with a finite thickness radius; (3) The pumping rate is maintained constant throughout the whole test period.

[1]Department of Civil Engineering, Vanung University, Chungli, Taiwan.
[2]Institute of Environmental Engineering, National Chiao Tung University, Hsinchu, Taiwan.

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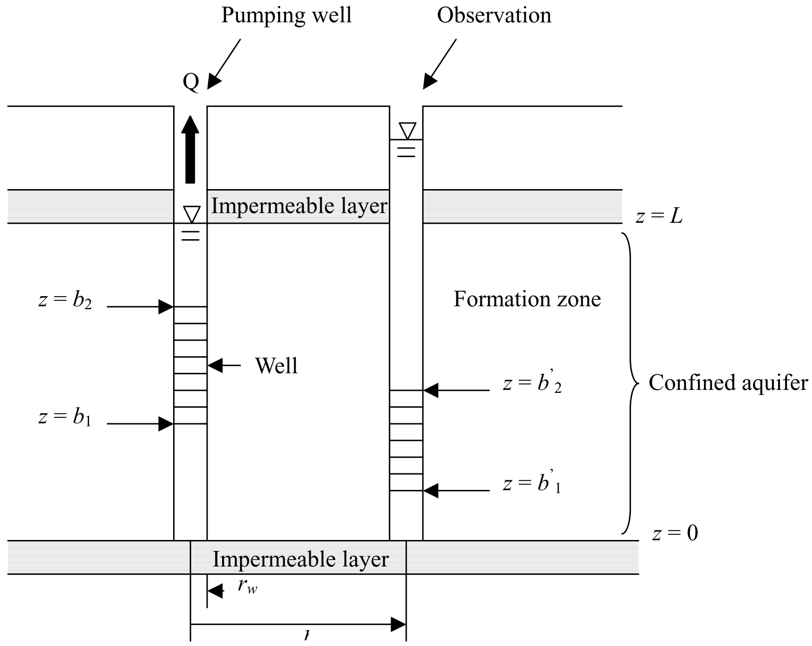

Figure 1. Schematic diagram of the well and aquifer configurations.

According to these assumptions, the governing equation of drawdown, $s(r, z, t)$, can be written as

$$K_r \frac{\partial^2 s(r,z,t)}{\partial r^2} + \frac{K_r}{r}\frac{\partial s(r,z,t)}{\partial r} + K_z \frac{\partial^2 s(r,z,t)}{\partial z^2} = S_s \frac{\partial s(r,z,t)}{\partial t} \quad (1)$$

where K_r and K_z are the hydraulic conductivities in the radial and vertical directions, respectively; S_s is the specific storage; r is the radial distance from the centerline of well; z is the vertical distance from the lower impermeable layer; r_w is the well radius; and t is the time from the start of test.

[5] The drawdown is initially assumed to be zero within the formation zone, this is

$$s(r,z,0) = 0 \quad (2)$$

[6] As r approaches infinity, the drawdown tends to be zero. Therefore the outer boundary condition is

$$s(\infty,z,t) = 0 \quad (3)$$

The lower and upper boundary conditions in the vertical direction are, respectively,

$$\frac{\partial s(r,0,t)}{\partial z} = 0 \quad (4)$$

and

$$\frac{\partial s(r,L,t)}{\partial z} = 0 \quad (5)$$

where L is the thickness of the confined aquifer.

[7] On the basis of Darcy's law, the boundary condition for the flux across the screen is assumed uniform and expressed as

$$\frac{\partial s(r_w,z,t)}{\partial r} = -\frac{Q}{2\pi r_w K_r (b_2 - b_1)}[U(z-b_1) - U(z-b_2)], \quad 0 \le z \le L \quad (6)$$

where Q is the pumping rate, b_1 and b_2 are respectively the lower and upper vertical coordinates of the well screen, and $U(\cdot)$ is a unit step function defining that $U(z - b_i)$ equals one when $b_i \leqq z$ but equals 0 otherwise for $i = 1$ or 2. Notice that *Neuman* [1974, p. 304, equations (15) and (16)] also made the same assumption as (6) but in different form for a constant pumping at a partially penetrating well in an unconfined aquifer.

2.2. Laplace Domain Solution

[8] To solve the boundary value problem, the Laplace and finite Fourier cosine transforms are applied to the governing equation and the boundary conditions. The detailed derivation of the solution is given in Appendix A and the final result is

$$\bar{s}(r,z,p) = \frac{Q}{4\pi T}\frac{2}{r_w}\frac{K_0(q_2 r)}{p q_2 K_1(q_2 r_w)} + \frac{Q}{4\pi T}\frac{4}{(b_2-b_1) r_w} \cdot \sum_{n=1}^{\infty}\left[\frac{K_0(q_1 r)}{p q_1 K_1(q_1 r_w)}\right] W(b_1,b_2)\cos(w_n z) \quad (7)$$

where p is the Laplace variable, $T = LK_r$ is the transmissivity of the aquifer, $q_1 = \sqrt{\alpha w_n^2 + \beta p}$, $q_2 = \sqrt{\beta p}$, $\alpha = K_z/K_r$, $\beta = S_S/K_r$, $w_n = n\pi/L$, $n = 1, 2, \ldots$, $W(b_1, b_2) = [\sin(w_n b_2) - \sin(w_n b_1)]/w_n$, and $K_0(\cdot)$ and $K_1(\cdot)$ are the modified Bessel functions of the second kind of order zero and one. Notice that the right-hand side (RHS) of (7) has two terms; the first term represents the solution for a radial confined flow, and the second term contains a summation term accounting for the effect of partial penetration.

[9] The water level in an observation well represents the average drawdown in the aquifer profile that is in contact with the well screen (or perforated section). The average drawdown in an observation well that is screened between the depths of b_1' and b_2' can be obtained by integrating the drawdown equation with respect to z between the limits of b_1' and b_2', and then dividing the result by the screen length ($b_2' - b_1'$). Thus the average drawdown can be expressed as

$$\bar{s}(r,p) = \frac{Q}{4\pi T}\frac{2}{r_w}\frac{K_0(q_2 r)}{pq_2K_1(q_2r_w)} + \frac{Q}{4\pi T}\frac{4}{(b_2-b_1)\left(b_2'-b_1'\right)r_w} \cdot \sum_{n=1}^{\infty}\left[\frac{K_0(q_1 r)}{pq_1K_1(q_1r_w)}\right]W(b_1,b_2)W\left(b_1',b_2'\right) \tag{8}$$

2.3. Closed Form Solution

[10] The solution of (7) in the time domain is obtained via the inverse Laplace transforms, convolution theorem, and the Bromwich integral [*Hildebrand*, 1976]. Detailed derivation is given in Appendix B and the time domain solution is

$$s(r,z,t) = \frac{Q}{4\pi T}\left[f_1(r,u) + \sum_{n=1}^{\infty} f_2(r,u)W(b_1,b_2)\cos(w_n z)\right] \tag{9}$$

with

$$f_1(r,u) = \frac{4}{r_w\pi}\int_0^{\infty}\left[1-e^{-\left(\frac{u^2}{\beta}\right)t}\right]\frac{Y_0(ru)J_1(r_wu)-J_0(ru)Y_1(r_wu)}{Y_1^2(r_wu)+J_1^2(r_wu)} \cdot \frac{du}{u^2} \tag{10}$$

and

$$f_2(r,u) = \frac{8}{r_w\pi(b_2-b_1)}\int_0^{\infty}\frac{1}{u^2+\alpha w_n^2} \cdot \left[1-e^{-\left(\frac{u^2}{\beta}+\frac{\alpha}{\beta}w_n^2\right)t}\right]\frac{Y_0(ru)J_1(r_wu)-J_0(ru)Y_1(r_wu)}{Y_1^2(r_wu)+J_1^2(r_wu)}du \tag{11}$$

where u is a dummy variable. $J_0(\cdot)$ and $Y_0(\cdot)$ are respectively the Bessel functions of the first and second kind of order zero, and $J_1(\cdot)$ and $Y_1(\cdot)$ are respectively the Bessel functions of the first and second kind of order one.

[11] The average drawdown in an observation well is

$$s(r,t) = \frac{Q}{4\pi T}\left[f_1(r,u) + \sum_{n=1}^{\infty} f_2(r,u)W(b_1,b_2)W\left(b_1',b_2'\right)\right] \tag{12}$$

2.4. Dimensionless Solutions

[12] The dimensionless variables are defined as $\tau = K_r t/S_s r_w^2$, $\rho = r/r_w$, $L_D = L/r_w$, $B_1 = b_1/r_w$, $B_2 = b_2/r_w$, $B_1' = b_1'/r_w$, $B_2' = b_2'/r_w$, $q_{1D} = q_1 r_w$, $q_{2D} = q_2 r_w$, $w_{nD} = n\pi/L_D$, $\bar{\sigma} = \bar{s}(4\pi T)/Q$, and $\sigma = s(4\pi T)/Q$. The Laplace domain solution for dimensionless drawdown of (8) is

$$\bar{\sigma}(\rho,p) = \frac{2K_0(q_{2D}\rho)}{pq_{2D}K_1(q_{2D})} + \frac{4}{(B_2-B_1)\left(B_2'-B_1'\right)} \cdot \sum_{n=1}^{\infty}\left[\frac{K_0(q_{1D}\rho)}{pq_{1D}K_1(q_{1D})}\right]W(B_1,B_2)W\left(B_1',B_2'\right) \tag{13}$$

Similarly, the time domain solution for dimensionless drawdown of (12) can be expressed as

$$\sigma(\rho,\tau) = \frac{4}{\pi}\left[f_{1D}(\rho,w) + \sum_{n=1}^{\infty} f_{2D}(\rho,w)W(B_1,B_2)W\left(B_1',B_2'\right)\right] \tag{14}$$

where

$$f_{1D}(\rho,w) = \int_0^{\infty}\left(1-e^{-\tau w^2}\right)\frac{Y_0(\rho w)J_1(w)-J_0(\rho w)Y_1(w)}{Y_1^2(w)+J_1^2(w)}\frac{dw}{w^2} \tag{15}$$

and

$$f_{2D}(\rho,w) = \frac{2}{(B_2-B_1)}\int_0^{\infty}\frac{1}{w^2+\alpha w_{nD}^2}\left[1-e^{-\left(w^2+\alpha w_{nD}^2\right)\tau}\right] \cdot \frac{Y_0(\rho w)J_1(w)-J_0(\rho w)Y_1(w)}{Y_1^2(w)+J_1^2(w)}dw \tag{16}$$

3. Numerical Evaluations

3.1. Numerical Inversion for the Laplace Domain Solution

[13] The Laplace transforms are commonly used to solve the differential and integral equations. In many engineering problems, the Laplace domain solutions for mathematical models are tractable, yet the corresponding solutions in the time domain may not be easily solved. Under such circumstances, the methods of numerical Laplace inversion such as Stehfest method [*Stehfest*, 1970], the Crump method [*Crump*, 1976], or the Talbot method [*Talbot*, 1979] may be used. The Laplace inversion transform of (13) is performed with five significant digits using the routine INLAP of IMSL [*Visual Numerics*, 1997] developed according to the work of *de Hoog et al.* [1982].

3.2. Evaluation of the Closed Form Solution

[14] The derived dimensionless solution of (14) is an integral that covers a range from zero to infinity and has an integrand comprising many product terms of the Bessel functions. The proposed numerical approach, including a root search scheme, a numerical integration method, and the Shanks method, is used to evaluate this solution. The root search scheme employs the Newton method to find the root of the integrand, which is oscillatory along the horizontal axis. Both the six-point and 10-point formulas of the Gaussian quadrature are employed at the same time to perform the numerical integration within the chosen interval between two consecutive roots. Notice that the integrands in (14) exhibit oscillatory behavior because of the nature of the Bessel functions. The resultant infinite series may have the

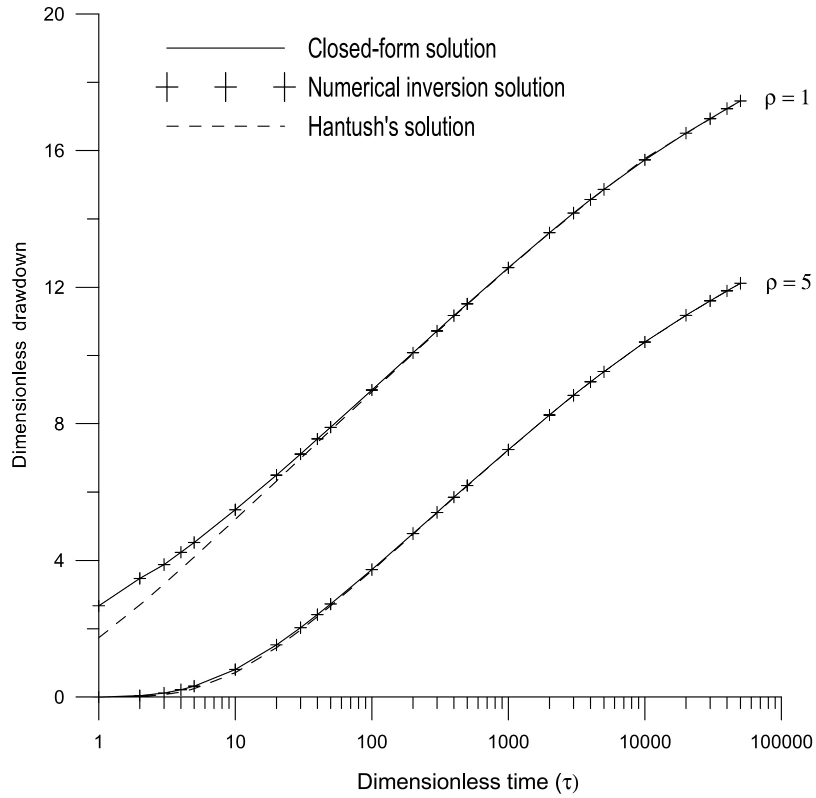

Figure 2. Dimensionless drawdown versus dimensionless time (τ) estimated by the closed form solution, the numerical inversion from the Laplace domain solution, and the Hantush solution [*Hantush*, 1964] for $\alpha = 0.1$, $B_1 = 40$, $B_2 = 160$, $B_1' = 40$, $B_2' = 160$, and $L_D = 200$ when $\rho = 1$ or 5.

problem of slow convergence. The Shanks method [*Shanks*, 1955] is employed to accelerate the convergence in summing the alternating series. Detailed numerical evaluation processes can be found in *Yeh et al.* [2003].

4. Results and Discussion

4.1. Comparison Between the Closed Form Solution and Other Solutions

[15] The comparison of results obtained from the closed form solution of (14) with the inversion results from the Laplace domain solution of (13) may provide a cross check for the validity and accuracy of both solutions and numerical evaluations. The curves of dimensionless drawdown versus dimensionless time evaluated by the proposed numerical approach for (14), the modified Crump algorithm for (13), and the Hantush solution [*Hantush*, 1964] are shown in Figure 2. Figure 2 gives the values of dimensionless drawdowns for, $\alpha = 0.1$, $B_1 = 40$, $B_2 = 160$, $B_1' = 40$, $B_2' = 160$, and $L_D = 200$ when $\rho = 1$ or 5. The dimensionless drawdowns of the closed form solution agree well with the results estimated by the numerical inversion from the Laplace domain solution. It indicates that the closed form solution was correctly evaluated by the proposed numerical approach. However, the drawdowns of the Hantush solution differ from these of the closed form solution because the well radius is neglected when the dimensionless time (τ) is very small. In other words, neglecting the effect of well radius causes errors in dimensionless drawdown estimation for small ρ, especially when τ is small.

4.2. Effects of Partial Penetration

[16] The drawdown in a partially penetrating well reflects the sum of the formation losses associated with a horizontal flow to a well and the loss associated with the converging and vertical flow near a well screen due to well partial penetration [*Charbeneau*, 2000]. Define the penetration ratio $\Phi = (B_2 - B_1)/L_D$. Figure 3 shows the dimensionless drawdown versus dimensionless distance for $\alpha = 0.1$, $\tau = 10^5$, $L_D = 200$, and Φ ranging from 0.1 to 1 when the half length of well screen is situated at the center or 1/4 thickness of confined aquifer. Note that $\Phi = 1$ represents the case of a well under full penetration. As can be seen, the differences between these three curves are very small when $\rho \geq 300$. This implies that the effects of partial penetration are uniform along the z direction when $r/L \geq 1.5$. This result coincides with Hantush's assertion that the second term on the RHS of (14) is negligible when $r/L \geq 1.5$ [*Hantush*, 1964].

[17] Figure 4 shows the effect of screen length, i.e., penetration portion, on dimensionless drawdown. When ϕ approaches one, the effect of partial penetration is inconspicuous and the dimensionless drawdown curve approaches the Theis line [*Theis*, 1935] where the dimensionless drawdown is linearly proportional to the logarithm of dimensionless time. When $\phi \leq 0.01$, the curve approaches an asymptotic value at large dimensionless time (i.e., $\tau \geq 10^3$). In other words, the pumping well can maintain the discharge rate at a constant drawdown. Therefore the thickness of the aquifer can be considered as

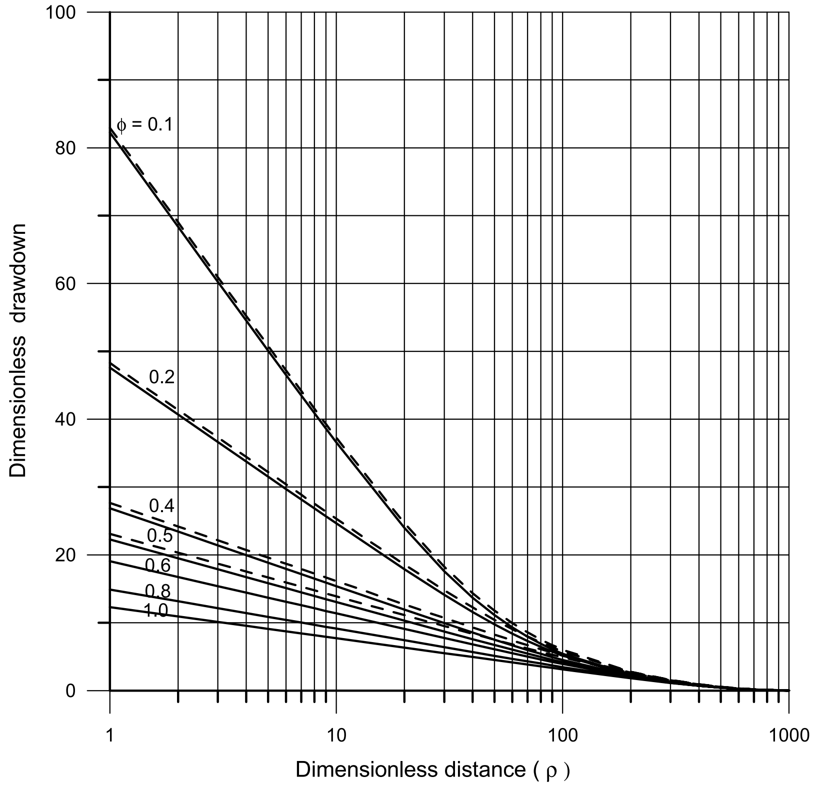

Figure 3. Dimensionless drawdown versus dimensionless distance (ρ) for $\alpha = 0.1$, $\tau = 10^5$, and L_D = 200 and the penetration ratio (ϕ) ranging from 0.1 to 1 when the middle of well screen is situated at the center (solid line) or 1/4 thickness (dashed line) of confined aquifer.

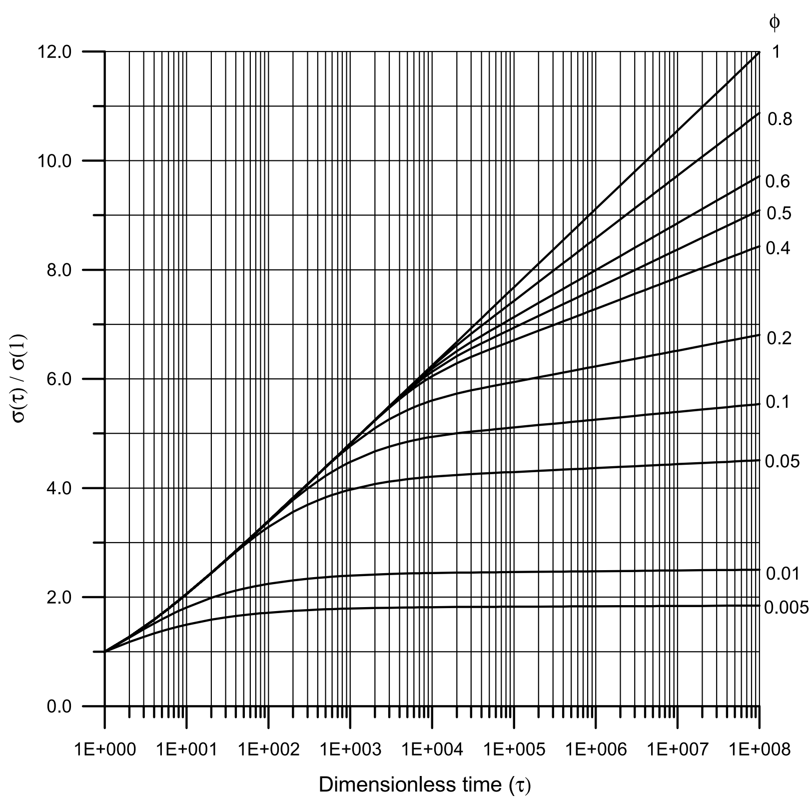

Figure 4. Dimensionless drawdown $\sigma(\tau)/\sigma(1)$ versus dimensionless time (τ) for $\alpha = 0.1$, $\rho = 1$, and $L_D = 200$ when ϕ ranges from 0.005 to 1.

infinite at large dimensionless time when $\Phi \leqq 0.01$, and the dimensionless drawdown remains a constant.

5. Conclusions

[18] A closed form solution has been developed for constant flux pumping tests in a radial confined aquifer system with a partially penetrating well. This solution considers the effects of well radius and partial penetration and provides appropriate mathematical models for the pumping test data analyses. The Laplace domain solution can reduce to that presented by *Hantush* [1964] if the well radius is neglected. This article demonstrates that the Hantush solution has minor errors in drawdown estimation when the observation well is close to the pumping well and/or when the pumping time is small. The effect of partial penetration is apparent if the test well is completed with a short screen. An aquifer thickness 100 times greater than the screen length of the well can be considered as infinite. The effect of partial penetration on the drawdown decreases with increasing distance. For $r/L \geq 1.5$, such an effect is negligible as suggested by *Hantush* [1964].

Appendix A: Derivation of Laplace Domain Solution (7)

[19] The solution of drawdown is derived via the Laplace transform with respect to time variable t and the finite Fourier transform with respect to spatial variable z. The appropriate finite Fourier transform is given by [*Kreyszig*, 1993]

$$F\{s(z)\} = \tilde{s}(w_n) = \int_0^L s(z)\cos(w_n z)dz,\ 0 \leq z \leq L \qquad \text{(A1)}$$

The Laplace transform has following operational property

$$F\left\{\frac{d^2 s(z)}{dz^2}\right\} = (-1)^n \frac{ds(z)}{dz}\bigg|_{z=L} - \frac{ds(z)}{dz}\bigg|_{z=0} - w_n^2 \tilde{s}(w_n) \qquad \text{(A2)}$$

[20] With the boundary conditions of (4) and (5) and applying the Laplace and finite Fourier cosine transforms, (1) gives the following subsidiary formulas of the transformed drawdown $\hat{\tilde{s}}$

$$\frac{d^2\hat{\tilde{s}}(r,w_n,p)}{dr^2} + \frac{1}{r}\frac{d\hat{\tilde{s}}(r,w_n,p)}{dr} = q_1^2\hat{\tilde{s}}(r,w_n,p) \qquad \text{(A3)}$$

The transformed boundary conditions are

$$\hat{\tilde{s}}(\infty, w_n, p) = 0 \qquad \text{(A4)}$$

and

$$\frac{d\hat{\tilde{s}}(r_w,w_n,p)}{dr} = \frac{-Q}{2\pi r_w K_r(b_2-b_1)p}\int_{b_1}^{b_2}\cos(w_n z)dz \qquad \text{(A5)}$$

[21] The general solution of (A3) is

$$\hat{\tilde{s}}(r,w_n,p) = C_1 I_0(q_1 r) + C_2 K_0(q_1 r) \qquad \text{(A6)}$$

where C_1 and C_2 are the undetermined constants. Substituting (A6) into (A4) and (A5), one obtains

$$C_1 = 0 \qquad \text{(A7)}$$

and

$$C_2 = \left(\frac{Q}{2\pi r_w K_r(b_2-b_1)}\right)\frac{1}{pq_1K_1(q_1r_w)}\int_{b_1}^{b_2}\cos(w_n z)dz \qquad \text{(A8)}$$

Consequently, the solution of drawdown can be obtained by substituting the constants of (A7) and (A8) into (A6) as

$$\hat{\tilde{s}}(r,w_n,p) = \left(\frac{Q}{2\pi r_w K_r(b_2-b_1)}\right)\frac{K_0(q_1 r)}{pq_1K_1(q_1r_w)}\int_{b_1}^{b_2}\cos(w_n z)dz \qquad \text{(A9)}$$

[22] The inverse finite Fourier transform is given by [*Kreyszig*, 1993]

$$F^{-1}\left\{\hat{\tilde{s}}(r,w_n,p)\right\} = \frac{1}{L}\hat{\tilde{s}}(r,w_0,p) + \frac{2}{L}\sum_{n=1}^{\infty}\hat{\tilde{s}}(r,w_n,p)\cos(w_n z) \qquad \text{(A10)}$$

Applying the above formula, the Laplace domain solution (7) for drawdown can then be obtained.

Appendix B: Derivation of Time Domain Solution (9)

[23] The convolution theorem [*Hildebrand*, 1976, p.63] states that

$$L^{-1}\{f(p)g(p)\} = \int_0^t F(t-\eta)G(\eta)d\eta \qquad \text{(B1)}$$

Let the Laplace domain solution of (7) be

$$\bar{s}(r,z,p) = \frac{Q}{4\pi T}\frac{2}{r_w}\overline{S}_A + \frac{Q}{4\pi T}\frac{4}{(b_2-b_1)r_w} \cdot \sum_{n=1}^{\infty}\overline{S}_B W(b_1,b_2)\cos(w_n z) \qquad \text{(B2)}$$

where $\overline{S}_A = K_0(q_2 r)/[pq_2K_1(q_2r_w)]$ and $\overline{S}_B = K_0(q_1 r)/[pq_1K_1(q_1r_w)]$.

[24] The Laplace inversion of $\overline{S}_A$ can be expressed as

$$S_A = L^{-1}\left\{\overline{S_A}\right\} = L^{-1}\{f_1(p)\cdot g_1(p)\} \qquad \text{(B3)}$$

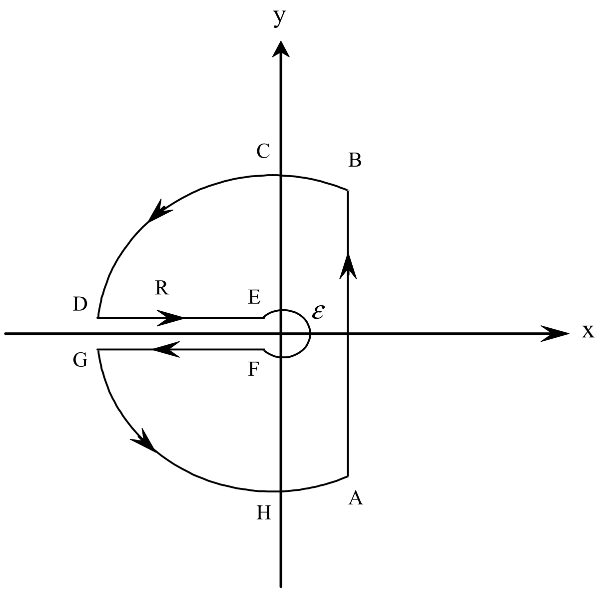

Figure B1. Closed contour integration of $\bar{s}$ for the Bromwich integral [*Hildebrand*, 1976].

Let $f_1(p) = 1/p$ and $g_1(p)$ represents the term on $\overline{S}_A$ except $1/p$. Applying the Bromwich integral with $L^{-1}\{f_1(p)\} = F(t) = 1$ yields [*Hildebrand*, 1976, p. 624]

$$L^{-1}\{g_1(p)\} = \frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta-i\infty} e^{pt} g_1(p)dp = 0 \tag{B4}$$

where p is a complex variable, i is an imaginary unit, and ζ is a large, real, and positive constant so that all the poles lie to the left of line $(\zeta - i\infty, \zeta + i\infty)$.

[25] A single branch point with no singularity (pole) at $p = 0$ exists in the integrand of $\overline{S}_A$. Thus this integration may require using a contour integral for the Laplace inversion. The contour of integrand is shown in Figure B1 with a cut of p plane along a negative real axis, where ε is taken sufficiently small to exclude all poles from the circle about the origin. Along the small circle EF, the integration around the origin when ε approaches zero is carried out by using the Cauchy integral and the value of integration is equal to zero. The integrals taken along BCD and GHA tend to zero when R approaches infinity. Therefore $\overline{S}_A$ can be superseded by the sum of integrals along DE and FG. In other words, (B4) can be written as

$$G_1(t) = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \frac{-1}{2\pi i}\left[\int_{DE} e^{pt} g_1(p)dp + \int_{FG} e^{pt} g_1(p)dp\right] \tag{B5}$$

The result of contour integral can then be obtained by following the method of *Yeh et al.* [2003] as

$$G_1(t) = \frac{2}{\pi\beta}\int_0^\infty e^{-\frac{u^2}{\beta}t}\,\frac{Y_0(ru)J_1(r_w u) - J_0(ru)Y_1(r_w u)}{Y_1^2(r_w u) + J_1^2(r_w u)}du \tag{B6}$$

Therefore the complete solution for a constant flux pumping obtained by the convolution is

$$S_A(r,t) = \int_0^t 1 \cdot G_1(\eta)d\eta \tag{B7}$$

The result of (B7) after the integration is

$$S_A = \frac{2}{\pi}\int_0^\infty \left(1 - e^{-\left(\frac{u^2}{\beta}\right)t}\right)\frac{Y_0(ru)J_1(r_w u) - J_0(ru)Y_1(r_w u)}{Y_1^2(r_w u) + J_1^2(r_w u)}\frac{du}{u^2} \tag{B8}$$

[26] The first shifting theorem of the Laplace transforms states that

$$L^{-1}\{p - a\} = e^{at}L^{-1}\{p\} \tag{B9}$$

On the basis of $\overline{S}_B(p) = f_1(p)g_1\ (p + \frac{\alpha}{\beta}w_n^2)$, the Laplace inversion of $\overline{S}_B\ (p)$ is

$$S_B(r,t) = \int_0^t 1 \cdot e^{-\frac{\alpha}{\beta}w_n^2\eta}G_1(\eta)d\eta \tag{B10}$$

Thus the result of (B10) after the integration is

$$S_B = \frac{2}{\pi}\int_0^\infty \left(1 - e^{-\left(\frac{u^2}{\beta}+\frac{\alpha}{\beta}w_n^2\right)t}\right)\frac{1}{\left(u^2 + \alpha w_n^2\right)} \cdot \frac{Y_0(ru)J_1(r_w u) - J_0(ru)Y_1(r_w u)}{Y_1^2(r_w u) + J_1{}^2(r_w u)}du \tag{B11}$$

Combining (B8) and (B11), the result of time domain solution (9) can then be obtained.

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P. Y. Chiu and H.-D. Yeh, Institute of Environmental Engineering, National Chiao Tung University, 75 Po-Ai Street, Hsinchu, 300, Taiwan. (chiu.pinyuan@msa.hinet.net; hdyeh@mail.nctu.edu.tw)

S.-Y. Yang, Department of Civil Engineering, Vanung University, 1 Van-Nung Road, Chungli City, Taoyuan, 320, Taiwan. (shaoyang@msa.vnu.edu.tw)