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J. Differential Equations 224 (2006) 229–257
Journal of Differential Equations
www.elsevier.com/locate/jde

Smooth solutions to a class of quasilinear wave equations

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Received 19 April 2005; revised 1 June 2005

Available online 27 July 2005

Dedicated to Professor Tai-Ping Liu on his sixtieth birthday

Abstract

This article investigates the existence/nonexistence of smooth solutions of nonlinear vibration equations which arise from the one-dimensional motion of polytropic gas without external forces contained in a finite interval. For any fixed arbitrarily long time, we show that there are smooth small amplitude solutions of the nonlinear equations for which the periodic solutions of the linearized equation are the first-order approximations. On the other hand, when the nonlinearity is strictly convex or concave, there exists no time-periodic solutions which are twice continuously differentiable. An example of possible singularities which occur at the second derivatives is illustrated. We also give another kind of exact solutions with singularity such that shocks occur after a finite time. Furthermore, we get an estimate of the life span of smooth solutions to the initial-boundary value problem.

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[1] The work of C.-H. Hsu was partially supported by the National Science Council of Taiwan and National Center for Theoretical Sciences, Mathematical Division, Taiwan.
[2] The work of S.-S. Lin was partially supported by the National Science Council of Taiwan and National Center for Theoretical Sciences, Mathematical Division, Taiwan.
[3] T. Makino was financially supported by the National Center for Theoretical Sciences, Mathematical Division, Taiwan, and Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan.

doi:10.1016/j.jde.2005.06.007

1. Introduction

The purpose of this work is to investigate the existence/nonexistence of smooth solutions of nonlinear vibration equations. During the study on periodic solutions to the one-dimensional compressible Euler equation under constant gravity (cf. [5]), we were confronted with some difficulties in analyzing the following equation:

$$y_{tt} - \frac{1}{\rho}\big(PG(y_x)\big)_x = 0 \quad \text{for } 0 < x < 1$$

with

$$G(v) = 1 - (1+v)^{-\gamma}, \quad \rho = (1-x)^{\frac{1}{\gamma-1}} \quad \text{and} \quad P = (1-x)^{\frac{\gamma}{\gamma-1}}.$$

Here ρ is density, P is pressure and γ is a constant such that $1 < \gamma \leqslant 2$. In order to clarify the difficulties, we have studied the simplified equation

$$y_{tt} - (G(y_x))_x = 0 \quad \text{for } 0 < x < 1 \tag{1.1}$$

with the following boundary conditions:

$$y(t,0) = y(t,1) = 0. \tag{1.2}$$

Unfortunately, it is very difficult to study the original equation

$$y_{tt} - \frac{1}{\rho}\big(PG(y_x)\big)_x = 0,$$

with which we are confronted during the study of gas dynamics under the gravitational force, because of the singularity at $x = 1$. However, the equations of the form (1.1) are worth studying as equations of vibrating string or other physical models. Section 2 is devoted to derivation of (1.1) and (1.2) from the compressible Euler equations which governs the one-dimensional motion of polytropic gas without external forces contained in a finite interval.

In this article, keeping in mind the case in which $G(v) = 1 - (1+v)^{-\gamma}$, we assume

$$\text{(A)} \qquad \begin{cases} G(v) \ \text{ is real analytic in } |v| < \delta \text{ with} \\ G(0) = 0, \quad G'(0) = \gamma > 0 \quad \text{and} \quad G'(v) > 0 \quad \text{for } |v| < \delta. \end{cases}$$

The most interesting problem is the existence/nonexistence of time periodic solutions of (1.1) and (1.2), since the corresponding linearized problem

$$y_{1,tt} - \gamma y_{1,xx} = 0, \quad y_1(t,0) = y_1(t,1) = 0$$

admits the following smooth time-periodic solution:

$$y_1 = \sum_{n=0}^{N} a_n \sin(n\pi\sqrt{\gamma}(t+\theta_n)) \sin n\pi x, \tag{1.3}$$

where a_n and θ_n are real constants, which is a linear combination of a finite number of simple oscillations. However, as claimed in Keller–Ting [7], there are no hope to have nontrivial small amplitude time-periodic solutions of (1.1) and (1.2) which are twice continuously differentiable when G is strictly convex or concave. For the sake of self-containedness, we will give a proof of this fact in Section 4.

But as a striking fact we should note that Greenberg [2] constructed smooth global solutions of (1.1) under the boundary conditions:

$$y(t,0) = y_x(t,1) = 0,$$

provided that $G(v) = 1 - (1+v)^{-1/3}$. Moreover, Greenberg constructed time-periodic solutions of (1.1) for G of the form

$$G(v) = v^3 \left(1 + \sum_{k \geqslant 1} a_k v^{2k}\right).$$

After this work, there appeared other works: Greenberg and Rascle [3], Greenberg and Peszek [4] and Peszek [9], which constructed time periodic solutions for particular cases of G. However, in this paper we cannot give similar arguments for $G(v) = 1 - (1+v)^{-\gamma}$. Instead of constructing time-periodic or time-global solutions of (1.1) and (1.2), we would like to show that the time-periodic solutions of the linearized equation give a good approximation of solutions to the nonlinear equations (1.1) and (1.2). Roughly speaking, during arbitrarily long time there are small amplitude solutions of the nonlinear equations for which the periodic solutions y_1 of the linearized equation are the first-order approximations. This result is stated and proved in Section 3.

There are no nontrivial uniformly bounded C^2-solutions of (1.1) and (1.2). Hence, we guess that the singularities of the second derivatives of y will develop even if we start from smooth and small initial data. In Section 5, we construct a concrete example of solutions with such singularities. We also give another kind of exact solutions with singularity such that shocks occur after a finite time. The estimate of life span of smooth solutions to the initial-boundary value problem is investigated in Section 6.

2. Derivation of the equation

We consider the one-dimensional motions of a polytropic gas without external forces governed by the compressible Euler equations

$$\rho_t + (\rho u)_x = 0 \quad \text{and} \quad (\rho u)_t + (\rho u^2 + P)_x = 0 \tag{2.1}$$

on a fixed finite interval $0 < x < L$ with the following boundary conditions

$$\rho u|_{x=0} = \rho u|_{x=L} = 0. \tag{2.2}$$

We assume $P = A\rho^\gamma$, where A and γ are positive constants such that $1 < \gamma \leqslant 2$. Equilibria are constant density: $\rho = \bar{\rho} = \text{constant} > 0$ and $u = 0$.

Let us introduce the Lagrangean coordinate

$$m = \int_0^x \rho\, dx.$$

We change the independent variables from (t, x) to (τ, m), where $\tau = t$. Then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \rho u \frac{\partial}{\partial m}, \qquad \frac{\partial}{\partial x} = \rho \frac{\partial}{\partial m}.$$

The equation

$$\rho_t + (\rho u)_x = 0$$

is transformed to

$$\rho_\tau + \rho^2 u_m = 0,$$

and the equation

$$(\rho u)_t + (\rho u^2 + P)_x = 0$$

is transformed to

$$u_\tau + P_m = 0,$$

divided by ρ. On the other hand, we have

$$\begin{pmatrix} t_\tau & t_m \\ x_\tau & x_m \end{pmatrix} = \begin{pmatrix} \tau_t & \tau_x \\ m_t & m_x \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\rho u & \rho \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ u & \frac{1}{\rho} \end{pmatrix},$$

that is,

$$u = \frac{\partial x}{\partial \tau}, \qquad \frac{1}{\rho} = \frac{\partial x}{\partial m}.$$

Thus we get the single equation

$$x_{\tau\tau} + (A(x_m)^{-\gamma})_m = 0, \tag{2.3}$$

where $x = x(\tau, m)$ now is the unknown function. The equilibrium is

$$x = \bar{x}(m) = \frac{L}{M} m \quad \text{with} \quad M = \int_0^L \bar{\rho}\, dx = \bar{\rho} L.$$

We note that

$$M = \int_0^L \rho(t, x)\, dx$$

remains to be constant along solutions by virtue of the boundary conditions.

Now consider the perturbation

$$x(\tau, m) = \bar{x}(m) + y = \frac{L}{M} m + y.$$

Of course $x(\tau, 0) = 0$, $x(\tau, M) = L$ imply the boundary conditions

$$y|_{m=0} = y|_{m=M} = 0.$$

The equation for $y = y(\tau, m)$ is

$$y_{\tau\tau} - A\left(\left(\frac{L}{M}\right)^{-\gamma} - \left(\frac{L}{M} + y_m\right)^{-\gamma}\right)_m = 0.$$

Taking $\bar{x}$ as the independent variable and keeping in mind that $\frac{\partial}{\partial m} = \frac{L}{M}\frac{\partial}{\partial \bar{x}}$, we can write

$$y_{\tau\tau} - A\left(\frac{L}{M}\right)^{1-\gamma} \left(1 - (1 + y_{\bar{x}})^{-\gamma}\right)_{\bar{x}} = 0$$

for $0 < \bar{x} < L$. Rewriting τ by t and $\bar{x}$ by x, Eq. (2.3) turns out to be

$$y_{tt} - A\left(\frac{L}{M}\right)^{1-\gamma} \left(1 - (1 + y_x)^{-\gamma}\right)_x = 0, \quad 0 < x < L, \tag{2.4}$$

and the boundary conditions are

$$y|_{x=0} = y|_{x=L} = 0. \tag{2.5}$$

For the normalization we take the following change of variables:

$$x = L\tilde{x}, \quad y = L\tilde{y} \quad \text{and} \quad t = T\tilde{t}.$$

Then Eq. (2.4) can be written as

$$\frac{L}{T^2}\tilde{y}_{\tilde{t}\tilde{t}} - \frac{A}{L}\left(\frac{L}{M}\right)^{1-\gamma}\left(1-(1+\tilde{y}_{\tilde{x}})^{-\gamma}\right)_{\tilde{x}} = 0, \quad 0 < \tilde{x} < 1, \tag{2.6}$$

since $\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial \tilde{x}}$ and $y_x = \tilde{y}_{\tilde{x}}$. Take

$$T^2 = L^2 A^{-1}\left(\frac{L}{M}\right)^{\gamma-1}$$

and rewrite $\tilde{t}, \tilde{x}, \tilde{y}$ as t, x, y, then (2.5) and (2.6) can be written as the form of (1.1) and (1.2) with $G(v) = 1-(1+v)^{-\gamma}$.

3. Long-time existence of smooth solutions

In this section, we consider the existence of C^2-solutions of Eqs. (1.1) and (1.2). Let $y_1(t,x)$ be a fixed smooth periodic solution of the linearized equation of the form (1.3). Applying the iteration method, we have the following result.

Theorem 1. *Assume G satisfy* (A). *For any positive real number T, there exists $\varepsilon^* > 0$ and a positive constant C depending upon G, the supremum norms of the derivatives of y_1 up to the 4th orders and T such that for any $0 < \varepsilon \leqslant \varepsilon^*$ there is a C^2-solution $y(t,x)$ of* (1.1) *and* (1.2) *for $0 \leqslant t \leqslant T$ such that*

$$|y(t,x) - \varepsilon y_1(t,x)| \leqslant C\varepsilon^2 \quad \text{for } 0 \leqslant t \leqslant T \text{ and } 0 \leqslant x \leqslant 1.$$

Before proving the above theorem, we first consider the problem by extending the solutions as

$$\begin{aligned} y(t,x) &= -y(t,-x) && \text{for } -1 < x < 0, \\ y(t,2n+x) &= y(t,x) && \text{for } n \in \mathbb{Z}. \end{aligned} \tag{3.1}$$

Put

$$u_1 = y_{1,t}, \quad v_1 = y_{1,x}, \quad y_t = \varepsilon u_1 + U \quad \text{and} \quad y_x = \varepsilon v_1 + V. \tag{3.2}$$

then Eq. (1.1) turns out to be the following system:

$$\begin{cases} V_t - U_x = 0, \\ U_t - G'(\varepsilon v_1 + V)V_x = (G'(\varepsilon v_1 + V) - \gamma)\varepsilon v_{1,x}. \end{cases} \tag{3.3}$$

To diagonalize the above system, we introduce

$$W = U + \hat{G}(\varepsilon v_1 + V) - \hat{G}(\varepsilon v_1) \quad \text{and} \quad Z = U - \hat{G}(\varepsilon v_1 + V) + \hat{G}(\varepsilon v_1),$$

where $\hat{G}(v) = \int_0^v \sqrt{G'(\zeta)}\,d\zeta$. Then (3.3) can be represented as

$$\begin{cases} W_t - \Lambda(\varepsilon v_1 + V)W_x = L_-(V), \\ Z_t + \Lambda(\varepsilon v_1 + V)Z_x = L_+(V) \end{cases} \tag{3.4}$$

with $\Lambda(v) = \sqrt{G'(v)}$ and

$$L_\pm(V) = \bigl(-\gamma + \Lambda(\varepsilon v_1 + V)\Lambda(\varepsilon v_1)\bigr)\varepsilon v_{1,x} \pm \bigl(\Lambda(\varepsilon v_1) - \Lambda(\varepsilon v_1 + V)\bigr)\varepsilon u_{1,x}. \tag{3.5}$$

We note that $\Lambda(v)$ is positive and analytic in $|v| < \delta$ with $\Lambda(0) = \sqrt{\gamma}$. Furthermore, it is obvious that

$$W(t,x) = -Z(t,-x) \quad \text{and} \quad Z(t,x) = -W(t,-x) \quad \text{for } -1 < x < 0,$$

and they should be periodic in x. Our purpose is to solve system (3.4) under the following initial conditions:

$$W(0,x) = Z(0,x) = 0. \tag{3.6}$$

Here we take these initial conditions for the sake of simplicity. This means that we want to construct solutions y such that $y(0,x) = \varepsilon y_1(0,x)$. Generally we can take arbitrary initial data $W(0,x) = W^0(x), Z(0,x) = Z^0(x)$ such that W^0, Z^0 are sufficiently smooth and small in order ε^2 with the derivatives and $W^0(x) = -Z^0(-x)$ are $2n$-periodic, and $\int_0^1 V^0(x)\,dx = 0$. We do not perform such a generalization, since there are no essentially new difficulties.

To solve Eqs. (3.4) and (3.6) by using the iteration method, we consider the following systems:

$$\begin{cases} \tilde{W}_t - \Lambda(\varepsilon v_1 + V)\tilde{W}_x = L_-(V), \\ \tilde{Z}_t + \Lambda(\varepsilon v_1 + V)\tilde{Z}_x = L_+(V), \end{cases} \tag{3.7}$$

$$\begin{cases} \tilde{W} = \tilde{U} + \hat{G}(\varepsilon v_1 + \tilde{V}) - \hat{G}(\varepsilon v_1), \\ \tilde{Z} = \tilde{U} - \hat{G}(\varepsilon v_1 + \tilde{V}) + \hat{G}(\varepsilon v_1). \end{cases} \tag{3.8}$$

For each given V, the procedure of iteration method is to solve $\tilde{W}$ and $\tilde{Z}$ of (3.7) first and then solve $\tilde{U}$ and $\tilde{V}$ of (3.8) and continue the same process. Some properties for the solutions $\tilde{U}$ and $\tilde{V}$ of (3.8) are illustrated in the following lemmas.

Lemma 3.1. *There exists $M_0 > 0$ and $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leqslant \varepsilon_0$ and $\|V\|_\infty \leqslant \varepsilon^2 M_0$ then*

$$\|\tilde{U}\|_\infty \leqslant \varepsilon^2 M_0, \quad \|\tilde{V}\|_\infty \leqslant \varepsilon^2 M_0 \quad \textit{and} \quad \varepsilon\|v_1\|_\infty + \varepsilon^2 M_0 \leqslant \delta/2.$$

Here $\|f\|_\infty = \sup\{|f(t,x)| \mid 0 \leqslant t \leqslant T,\ x \in \mathbb{R}\}$.

Proof. Since

$$\tilde{W} + \tilde{Z} = 2\tilde{U}, \tag{3.9}$$

$$\tilde{W} - \tilde{Z} = 2(\hat{G}(\varepsilon v_1 + \tilde{V}) - \hat{G}(\varepsilon v_1)), \tag{3.10}$$

we have

$$\|\tilde{U}\|_\infty = \tfrac{1}{2}\|\tilde{W} + \tilde{Z}\|_\infty \quad \text{and} \quad \|\tilde{V}\|_\infty \leqslant C\|\tilde{W} - \tilde{Z}\|_\infty. \tag{3.11}$$

Here and hereafter C denotes various constants depending upon G, y_1 and T. From (3.11), it is sufficient to estimate $\tilde{W}$ and $\tilde{Z}$.

Fixed (x,t), let $\xi(\tau) = \xi(\tau; t, x)$ be the solution of the following equations:

$$\frac{d\xi}{d\tau} = \Lambda(\varepsilon v_1 + V)(\tau, \xi(\tau)), \quad \xi(t) = x. \tag{3.12}$$

Then

$$\tilde{Z}(t,x) = \int_0^t L_+(V)(\tau, \xi(\tau; t, x))\, d\tau. \tag{3.13}$$

By the assumptions, (3.5) and (3.13), it is easy to see that

$$|L_+(V)| \leqslant C\varepsilon^2(1 + \varepsilon M_0) \quad \text{and} \quad \|\tilde{Z}\|_\infty \leqslant C\varepsilon^2(1 + \varepsilon M_0). \tag{3.14}$$

Applying the same arguments, we have

$$|L_-(V)| \leqslant C\varepsilon^2(1+\varepsilon M_0) \quad \text{and} \quad \|\tilde{W}\|_\infty \leqslant C\varepsilon^2(1+\varepsilon M_0). \tag{3.15}$$

Thus

$$\|\tilde{U}\|_\infty \leqslant C\varepsilon^2(1+\varepsilon M_0) \quad \text{and} \quad \|\tilde{V}\|_\infty \leqslant C\varepsilon^2(1+\varepsilon M_0). \tag{3.16}$$

Hence the results follow by taking

$$M_0 \geqslant 2C, \quad 2C\varepsilon_0 < 1 \quad \text{and} \quad \varepsilon\|v_1\|_\infty + \varepsilon^2 M_0 \leqslant \delta/2. \qquad \square$$

Lemma 3.2. *There exists $M_1 > 0$ and $0 < \varepsilon_1 \leqslant \varepsilon_0$ such that if $0 < \varepsilon \leqslant \varepsilon_1$, $\|V\|_\infty \leqslant \varepsilon^2 M_0$ and $\|V_x\|_\infty \leqslant \varepsilon^2 M_1$ then $\varepsilon^2 M_1 \leqslant 1$ and*

$$\|\tilde{U}_x\|_\infty \leqslant \varepsilon^2 M_1, \quad \|\tilde{U}_t\|_\infty \leqslant \varepsilon^2 M_1, \quad \|\tilde{V}_x\|_\infty \leqslant \varepsilon^2 M_1 \quad \textit{and} \quad \|\tilde{V}_t\|_\infty \leqslant \varepsilon^2 M_1.$$

Proof. By (3.9) and (3.10), we have

$$\tfrac{1}{2}(\tilde{W}+\tilde{Z})_x = \tilde{U}_x, \tag{3.17}$$

$$\tfrac{1}{2}(\tilde{W}-\tilde{Z})_x = \Lambda(\varepsilon v_1+\tilde{V})(\varepsilon v_{1,x}+\tilde{V}_x) - \Lambda(\varepsilon v_1)\varepsilon v_{1,x}. \tag{3.18}$$

Thus

$$|\tilde{U}_x| = \tfrac{1}{2}|(\tilde{W}+\tilde{Z})_x| \quad \text{and} \quad |\tilde{V}_x| \leqslant C|(\tilde{W}-\tilde{Z})_x| + C\varepsilon^3 M_0. \tag{3.19}$$

Hence, it is sufficient to estimate $\tilde{W}_x$ and $\tilde{Z}_x$.

From (3.5) and (3.13), we have

$$\tilde{Z}_x = \int_0^t \frac{\partial L_+}{\partial x}\frac{\partial \xi}{\partial x}(\tau, \xi(\tau))\,d\tau, \tag{3.20}$$

$$\begin{aligned}\frac{\partial L_+}{\partial x} = {} & \Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x)\Lambda(\varepsilon v_1)\varepsilon v_{1,x} + \Lambda(\varepsilon v_1+V)\Lambda'(\varepsilon v_1)\varepsilon^2 v_{1,x}^2 \\ & +(-\gamma+\Lambda(\varepsilon v_1+V)\Lambda(\varepsilon v_1))\varepsilon v_{1,xx} + (\Lambda(\varepsilon v_1)-\Lambda(\varepsilon v_1+V))\varepsilon u_{1,xx} \\ & +(\Lambda'(\varepsilon v_1)\varepsilon v_{1,x} - \Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x))\varepsilon u_{1,x}\end{aligned} \tag{3.21}$$

and $\dfrac{\partial \xi}{\partial x}$ satisfies the following equations:

$$\frac{d}{d\tau}\frac{\partial \xi}{\partial x} = \Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x)\frac{\partial \xi}{\partial x} \quad \text{with} \quad \frac{\partial \xi}{\partial x}(t) = 1. \tag{3.22}$$

Since $\varepsilon^2 M_1 \leqslant 1$, we have $|\Lambda'(\varepsilon v_1 + V)(\varepsilon v_{1,x} + V_x)| \leqslant C\varepsilon$, and

$$\left|\frac{\partial \xi}{\partial x}\right| \leqslant e^{C\varepsilon T} \leqslant C'.$$

On the other hand,

$$\left|\frac{\partial L_+}{\partial x}\right| \leqslant C\varepsilon^2(1 + \varepsilon M_0 + \varepsilon M_1). \tag{3.23}$$

Hence

$$|\tilde{Z}_x(t,x)| \leqslant C\varepsilon^2(1 + \varepsilon M_0 + \varepsilon M_1). \tag{3.24}$$

By the similar arguments, we also obtain

$$|\tilde{W}_x(t,x)| \leqslant C\varepsilon^2(1 + \varepsilon M_0 + \varepsilon M_1). \tag{3.25}$$

On the other hand, by (3.17) we have

$$\begin{aligned}
\Lambda(\varepsilon v_1 + V)\tilde{U}_x &= \tfrac{1}{2}(\Lambda(\varepsilon v_1 + V)\tilde{W}_x + \Lambda(\varepsilon v_1 + V)\tilde{Z}_x) \\
&= \tfrac{1}{2}(\tilde{W}_t - \tilde{Z}_t + L_+(V) - L_-(V)) \\
&= \Lambda(\varepsilon v_1 + \tilde{V})\tilde{V}_t + (\Lambda(\varepsilon v_1 + \tilde{V}) - \Lambda(\varepsilon v_1 + V))\varepsilon u_{1,x}. \qquad (3.26)
\end{aligned}$$

Therefore,

$$|\tilde{V}_t| \leqslant C|\tilde{U}_x| + C\varepsilon^3 M_0 \leqslant C\varepsilon^2(1 + \varepsilon M_0 + \varepsilon M_1). \tag{3.27}$$

Moreover, since

$$2\tilde{U}_t = \tilde{W}_t + \tilde{Z}_t = \Lambda(\varepsilon v_1 + V)\tilde{W}_x + L_-(V) - \Lambda(\varepsilon v_1 + V)\tilde{Z}_x + L_+(V),$$

(3.14), (3.15), (3.24) and (3.25) imply

$$|\tilde{U}_t| \leqslant C(|\tilde{W}_x| + |\tilde{Z}_x| + |L_-| + |L_+|) \leqslant C\varepsilon^2(1 + \varepsilon M_0 + \varepsilon M_1). \tag{3.28}$$

Hence, the results follow by taking

$$M_1 \geqslant 2C(1 + \varepsilon M_0), \quad 2C\varepsilon_1 \leqslant 1 \quad \text{and} \quad \varepsilon_1^2 M_1 \leqslant 1. \qquad \square$$

Lemma 3.3. *There exists $M_2 > 0$ and $0 < \varepsilon_2 \leqslant \varepsilon_1$ such that if $0 < \varepsilon \leqslant \varepsilon_2$, $\|V\|_\infty \leqslant \varepsilon^2 M_0$, $\|V_x\|_\infty \leqslant \varepsilon^2 M_1$ and $\|V_{xx}\|_\infty \leqslant \varepsilon^2 M_2$ then $\varepsilon^2 M_2 \leqslant 1$ and*

$$\|\tilde{U}_{xx}\|_\infty,\ \|\tilde{U}_{xt}\|_\infty,\ \|\tilde{V}_{xx}\|_\infty,\ \|\tilde{V}_{xt}\|_\infty,\ \|\tilde{V}_{tt}\|_\infty \leqslant \varepsilon^2 M_2.$$

Proof. By (3.17) and (3.18), we have

$$\tfrac{1}{2}(\tilde{W}+\tilde{Z})_{xx} = \tilde{U}_{xx}, \tag{3.29}$$

$$\begin{aligned}\tfrac{1}{2}(\tilde{W}-\tilde{Z})_{xx} &= \Lambda'(\varepsilon v_1+\tilde{V})(\varepsilon v_{1,x}+\tilde{V}_x)^2 + \Lambda(\varepsilon v_1+\tilde{V})(\varepsilon v_{1,xx}+\tilde{V}_{xx})\\ &\quad -\Lambda'(\varepsilon v_1)(\varepsilon v_{1,x})^2 - \Lambda(\varepsilon v_1)\varepsilon v_{1,xx}.\end{aligned} \tag{3.30}$$

We first estimate $\tilde{W}_{xx}$ and $\tilde{Z}_{xx}$. By (3.20) and (3.21), we have

$$\tilde{Z}_{xx} = \int_0^t \left(\frac{\partial^2 L_+}{\partial x^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial L_+}{\partial x}\frac{\partial^2 \xi}{\partial x^2}\right)(\tau, \xi(\tau))\, d\tau, \tag{3.31}$$

$$\begin{aligned}\frac{\partial^2 L_+}{\partial x^2} &= \Lambda''(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x)^2\Lambda(\varepsilon v_1)\varepsilon v_{1,x}\\ &\quad +\Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,xx}+V_{xx})\Lambda(\varepsilon v_1)\varepsilon v_{1,x}\\ &\quad +2\Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x)\Lambda'(\varepsilon v_1)(\varepsilon v_{1,x})^2\\ &\quad +2\Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x)\Lambda(\varepsilon v_1)\varepsilon v_{1,xx}\\ &\quad +\Lambda(\varepsilon v_1+V)\Lambda''(\varepsilon v_1)(\varepsilon v_{1,x})^3 + 3\Lambda(\varepsilon v_1+V)\Lambda'(\varepsilon v_1)\varepsilon^2 v_{1,x}v_{1,xx}\\ &\quad +(-\gamma+\Lambda(\varepsilon v_1+V)\Lambda(\varepsilon v_1))\varepsilon v_{1,xxx}\\ &\quad +2(\Lambda'(\varepsilon v_1)\varepsilon v_{1,x} - \Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x))\varepsilon u_{1,xx}\\ &\quad +(\Lambda(\varepsilon v_1)-\Lambda(\varepsilon v_1+V))\varepsilon u_{1,xxx}\\ &\quad +(\Lambda''(\varepsilon v_1)(\varepsilon v_{1,x})^2 - \Lambda''(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x))^2\varepsilon u_{1,x}\\ &\quad +(\Lambda'(\varepsilon v_1)\varepsilon v_{1,xx} - \Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,xx}+V_{xx}))\varepsilon u_{1,x}\end{aligned} \tag{3.32}$$

and $\dfrac{\partial^2 \xi}{\partial x^2}$ satisfies the following equations:

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial^2 \xi}{\partial x^2} &= \big(\Lambda''(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x)^2 + \Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,xx}+V_{xx})\big)\frac{\partial \xi}{\partial x}\\ &\quad +\Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x)\frac{\partial^2 \xi}{\partial x^2} \quad \text{with} \quad \frac{\partial^2 \xi}{\partial x^2}(t) = 0.\end{aligned}$$

Since $\varepsilon M_2 < 1$, it is easy to see

$$\left|\frac{\partial^2 L_+}{\partial x^2}\right| \leqslant C\varepsilon^2(1+\varepsilon M_0+\varepsilon M_1+\varepsilon M_2),$$
$$\left|\frac{\partial^2 \xi}{\partial x^2}\right| \leqslant C\varepsilon(1+\varepsilon M_0+\varepsilon M_1+\varepsilon M_2).$$

Hence (3.23) implies

$$\begin{aligned}\left|\frac{\partial^2 L_+}{\partial x^2}\left(\frac{\partial \xi}{\partial x}\right)^2+\frac{\partial L_+}{\partial x}\frac{\partial^2 \xi}{\partial x^2}\right| &\leqslant (C\varepsilon^2+C\varepsilon^3)(1+\varepsilon M_0+\varepsilon M_1+\varepsilon M_2)\\ &\leqslant C\varepsilon^2(1+\varepsilon M_0+\varepsilon M_1+\varepsilon M_2).\end{aligned}$$

Therefore, we have

$$|\tilde{Z}_{xx}|,\ |\tilde{W}_{xx}| \leqslant C\varepsilon^2(1+\varepsilon M_0+\varepsilon M_1+\varepsilon M_2)$$

and this implies

$$|\tilde{U}_{xx}|,\ |\tilde{V}_{xx}| \leqslant C\varepsilon^2(1+\varepsilon M_0+\varepsilon M_1+\varepsilon M_2). \tag{3.33}$$

Now differentiating (3.26) with respect to x, we have

$$\begin{aligned}&\Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x)\tilde{U}_x+\Lambda(\varepsilon v_1+V)\tilde{U}_{xx}\\ &\quad=\Lambda'(\varepsilon v_1+\tilde{V})(\varepsilon v_{1,x}+\tilde{V}_x)\tilde{V}_t+\Lambda(\varepsilon v_1+\tilde{V})\tilde{V}_{tx}\\ &\qquad+(\Lambda'(\varepsilon v_1+\tilde{V})(\varepsilon v_{1,x}+\tilde{V}_x)-\Lambda'(\varepsilon v_1+V)(\varepsilon v_{1,x}+V_x))\varepsilon u_{1,x}\\ &\qquad+(\Lambda(\varepsilon v_1+\tilde{V})-\Lambda(\varepsilon v_1+V))\varepsilon u_{1,xx}.\end{aligned}$$

Hence,

$$\begin{aligned}|\tilde{V}_{xt}| &\leqslant C|\tilde{U}_{xx}|+C\varepsilon^2(1+\varepsilon M_0+\varepsilon M_1),\\ &\leqslant C\varepsilon^2(1+\varepsilon M_0+\varepsilon M_1+\varepsilon M_2).\end{aligned} \tag{3.34}$$

Since

$$2\tilde{U}_t=\Lambda(\varepsilon v_1+V)\tilde{W}_x+L_-(V)-\Lambda(\varepsilon v_1+V)\tilde{Z}_x+L_+(V),$$

then

$$
\begin{aligned}
2\tilde{U}_{tx} \;=\; & \Lambda'(\varepsilon v_1 + V)(\varepsilon v_{1,x} + V_x)\tilde{W}_x + \Lambda(\varepsilon v_1 + V)\tilde{W}_{xx} + L_{-,x}(V) \\
& -\Lambda'(\varepsilon v_1 + V)(\varepsilon v_{1,x} + V_x)\tilde{Z}_x - \Lambda(\varepsilon v_1 + V)\tilde{Z}_{xx} + L_{+,x}(V).
\end{aligned}
$$

Thus

$$
|\tilde{U}_{tx}| \leqslant C\varepsilon^2(1 + \varepsilon M_0 + \varepsilon M_1 + \varepsilon M_2). \tag{3.35}
$$

Next, differentiating (3.26) with respect to t, we have

$$
\begin{aligned}
& \Lambda'(\varepsilon v_1 + V)(\varepsilon v_{1,t} + V_t)\tilde{U}_x + \Lambda(\varepsilon v_1 + V)\tilde{U}_{xt} \\
& \quad = \Lambda'(\varepsilon v_1 + \tilde{V})(\varepsilon v_{1,t} + \tilde{V}_t)\tilde{V}_t + \Lambda(\varepsilon v_1 + \tilde{V})\tilde{V}_{tt} \\
& \qquad +(\Lambda'(\varepsilon v_1 + \tilde{V})(\varepsilon v_{1,t} + \tilde{V}_t) - \Lambda'(\varepsilon v_1 + V)(\varepsilon v_{1,t} + V_t))\varepsilon u_{1,x} \\
& \qquad +(\Lambda(\varepsilon v_1 + \tilde{V}) - \Lambda(\varepsilon v_1 + V))\varepsilon u_{1,xt}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\tilde{V}_{tt}| \;&\leqslant\; C|\tilde{U}_{tx}| + C\varepsilon^2(1 + \varepsilon M_0 + \varepsilon M_1) \\
&\leqslant\; C\varepsilon^2(1 + \varepsilon M_0 + \varepsilon M_1 + \varepsilon M_2).
\end{aligned} \tag{3.36}
$$

By (3.33)–(3.36), the results follow by taking

$$
M_2 \geqslant 6C, \quad 2C\varepsilon_2 \leqslant 1 \quad \text{and} \quad \varepsilon_2^2 M_2 \leqslant 1. \qquad \square
$$

Basing on the estimates of the above lemmas, we now prove the convergence of the iteration scheme for systems (3.7) and (3.8). Let V^1, V^0 satisfy the estimates of Lemmas 3.1–3.3. We claim that

$$
\|\tilde{V}^1 - \tilde{V}^0\|_\infty \leqslant \tfrac{1}{2}\|V^1 - V^0\|_\infty. \tag{3.37}
$$

Since

$$
|\tilde{V}_1 - \tilde{V}_0| \leqslant C|\tilde{W}^1 - \tilde{Z}^1 - \tilde{W}^0 + \tilde{Z}^0|,
$$

we first estimate $|\tilde{W}^1 - \tilde{W}^0|$ and $|\tilde{Z}^1 - \tilde{Z}^0|$. By (3.13), we have

$$
\tilde{Z}^1(t,x) - \tilde{Z}^0(t,x) = \int_0^t \big(L_+(V^1)(\tau, \xi^1(\tau)) - L_-(V^0)(\tau, \xi^0(\tau))\big)\, d\tau. \tag{3.38}
$$

It is obvious that

$$\begin{aligned}&L_+(V^1)(\tau,\xi^1(\tau))-L_+(V^0)(\tau,\xi^1(\tau))\\&\quad=(\Lambda(\varepsilon v_1+V^1)-\Lambda(\varepsilon v_1+V^0))\Lambda(\varepsilon v_1)\varepsilon v_{1,x}\\&\qquad-(\Lambda(\varepsilon v_1+V^1)-\Lambda(\varepsilon v_1+V^0))\varepsilon u_{1,x}\end{aligned}$$

and this implies

$$|L_+(V^1)(\tau,\xi^1(\tau))-L_+(V^0)(\tau,\xi^1(\tau))|\leqslant C\varepsilon\|V^1-V^0\|_\infty. \tag{3.39}$$

On the other hand, from (3.23) we obtain

$$\begin{aligned}|L_+(V^0)(\tau,\xi^1(\tau))-L_+(V^0)(\tau,\xi^0(\tau))|&\leqslant \left\|\frac{\partial L_+}{\partial x}\right\|_\infty|\xi^1(\tau)-\xi^0(\tau)|\\&\leqslant C\varepsilon^2|\xi^1(\tau)-\xi^0(\tau)|.\end{aligned} \tag{3.40}$$

Denote $\Delta\xi(\tau)=\xi^1(\tau)-\xi^0(\tau)$. Then $\Delta\xi(\tau)$ satisfies the following equations:

$$\begin{aligned}\frac{d}{d\tau}\Delta\xi&=\Lambda(\varepsilon v_1+V^1)(\tau,\xi^0+\Delta\xi)-\Lambda(\varepsilon v_1+V^0)(\tau,\xi^0)\\&=\Lambda(\varepsilon v_1+V^1)(\tau,\xi^0+\Delta\xi)-\Lambda(\varepsilon v_1+V^1)(\tau,\xi^0)\\&\quad+(\Lambda(\varepsilon v_1+V^1)-\Lambda(\varepsilon v_1+V^0))(\tau,\xi^0)\end{aligned}$$

with $\Delta\xi(t)=0$. Since $\|\Lambda'\|_\infty\leqslant C$, then

$$\left|\frac{d}{d\tau}\Delta\xi\right|\leqslant C|\Delta\xi|+C\|V^1-V^0\|_\infty.$$

Applying the Gronwall's inequality, we obtain

$$|\Delta\xi|\leqslant C\|V^1-V^0\|_\infty. \tag{3.41}$$

By (3.38)–(3.41), we derive

$$|\tilde{Z}^1-\tilde{Z}^0|\leqslant\varepsilon C\|V^1-V^0\|_\infty.$$

By the same arguments, we also obtain

$$|\tilde{W}^1-\tilde{W}^0|\leqslant\varepsilon C\|V^1-V^0\|_\infty.$$

Hence,

$$|\tilde{V}^1 - \tilde{V}^0| \leqslant \varepsilon C \|V^1 - V^0\|_\infty.$$

Thus the claim follows by taking $2C\varepsilon \leqslant 1$.

Proof of Theorem 1. Basing on the estimate (3.37), we can construct the solution by using the iteration method. Let

$$V^{(0)} = 0 \quad \text{and} \quad V^{(n+1)} = \widetilde{V^{(n)}}.$$

By (3.37), the sequence $\{V^{(n)}\}$ converges uniformly to a continuous function V. The results of previous lemmas show that $V_t^{(n)}$ and $V_x^{(n)}$ are uniformly bounded and equi-continuous. Therefore, there exist subsequences $\{V_t^{(n_j)}\}$ and $\{V_x^{(n_j)}\}$ which converge uniformly. Thus, V is a C^1 function and this gives a C^2-solution $y(t,x)$ of (1.1) by taking

$$y(t,x) = \varepsilon y_1(t,x) + \int_0^x V(t,x)\,dx.$$

It is easy to show inductively that

$$\begin{aligned} V^{(n)}(t,x) &= V^{(n)}(t,-x), \quad U^{(n)}(t,x) = -U^{(n)}(t,-x), \\ W^{(n)}(t,x) &= -Z^{(n)}(t,-x), \quad Z^{(n)}(t,x) = -W^{(n)}(t,-x), \\ W^{(n)}(t,x+2n) &= W^{(n)}(t,x), \quad Z^{(n)}(t,x+2n) = Z^{(n)}(t,x), \end{aligned}$$

for any integer n. Thus the limit enjoys the same property. Especially, $W(t,1) = -Z(t,1)$, which means that $U(t,1) = 0$ or $y_t(t,1) = 0$. On the other hand, the initial conditions $W(0,x) = Z(0,x) = 0$ implies $V(0,x) = 0$ so that $y(0,1) = 0$. Therefore the boundary condition $y(t,1) = 0$ is satisfied. The proof is complete. □

4. Formulation of singularities

In this section, we investigate the formulation of singularity for solutions of (1.1) and (1.2). We have

Theorem 2. *Assume that G satisfies* (A) *and $G''(v) < 0$ for $|v| < \delta$. If $y(t,x) \in C^2(\mathbb{R} \times [0,1])$ is a solution of* (1.1) *and* (1.2) *such that*

$$|y_x(t,x)| \leqslant \delta_1 \quad \text{for } t \in \mathbb{R},\ x \in [0,1], \tag{4.1}$$

where $0 < \delta_1 < \delta$, then $y = 0$ identically.

Proof. Putting

$$y(t,x) = -y(t,-x) \quad \text{for } -1 < x < 0,$$
$$y(t,2n+x) = y(t,x) \quad \text{for } n \in \mathbb{Z},$$

we can assume that $y \in C^2(\mathbb{R} \times \mathbb{R})$ satisfies (1.1) for all x. Write (1.1) by

$$v_t - u_x = 0 \quad \text{and} \quad u_t - G(v)_x = 0, \tag{4.2}$$

where $u = y_t$ and $v = y_x$. Let

$$w = u + \hat{G}(v) \quad \text{and} \quad z = u - \hat{G}(v). \tag{4.3}$$

Then the system (4.2) is diagonalized to

$$w_t - \sqrt{G'(v)}w_x = 0 \quad \text{and} \quad z_t + \sqrt{G'(v)}z_x = 0. \tag{4.4}$$

From the assumptions, we know that

$$1/C \leqslant \sqrt{G'(v(t,x))} \leqslant C \quad \text{uniformly.}$$

Now let $x = x(t;a)$ satisfy the following equations:

$$\frac{dx}{dt} = \sqrt{G'(v(t,x))}, \quad x|_{t=0} = a. \tag{4.5}$$

Then $X(t;a) = \frac{\partial}{\partial a}x(t;a)$ satisfies

$$\frac{dX}{dt} = \frac{\partial}{\partial x}\sqrt{G'(v(t,x))}X \quad \text{and} \quad X|_{t=0} = 1. \tag{4.6}$$

Here

$$\frac{\partial}{\partial x}\sqrt{G'(v(t,x))} = \frac{1}{2}\frac{G''(v(t,x))}{\sqrt{G'(v(t,x))}}v_x$$

is continuous. Solving (4.6) directly, we obtain

$$X(t;a) = \exp\left(\int_0^t \frac{\partial}{\partial x}\sqrt{G'(v(\tau,x))}\,d\tau\right) > 0. \tag{4.7}$$

On the other hand, let us investigate the right-hand side of (4.6). Since $w-z=2\hat{G}(v)$, denote F as the inverse function of $2\hat{G}(v)$ then

$$v = F(w-z), \qquad \sqrt{G'(v)} = \sqrt{G'\circ F(w-z)},$$
$$\frac{\partial}{\partial a}\sqrt{G'(v)} = \left(\sqrt{G'\circ F}\right)'(w_a - z_a).$$

Since

$$w_a = \frac{\partial}{\partial a} w(t, x(t;a)) = w_x X,$$
$$2\sqrt{G'(v)}w_x = w_t + \sqrt{G'(v)}w_x = \frac{d}{dt} w(t, x(t;a))$$

and $dz/dt = 0$, we have

$$w_x = \frac{1}{2\sqrt{G'(v)}}\frac{d}{dt}(w-z).$$

Thus

$$\begin{aligned}\frac{dX}{dt} &= \frac{\partial}{\partial a}\sqrt{G'(v(t,x))}\\ &= \frac{\left(\sqrt{G'\circ F}\right)'}{2\sqrt{G'\circ F}}\frac{d}{dt}(w-z)X - \left(\sqrt{G'\circ F}\right)'(w-z)z_a\\ &= \frac{d}{dt}(\ln(G'\circ F(w-z))^{1/4})X - \left(\sqrt{G'\circ F}\right)'(w-z)z_a.\end{aligned}$$

In addition, since $dz/dt = 0$, $z(t, x(t;a)) = z(0,a)$, we have

$$z_a = \frac{\partial z}{\partial a}(t, x(t;a)) = \frac{\partial z}{\partial a}(0,a).$$

Thus

$$X = \left(\frac{G'(v(t,x(t;a)))}{G'(v(0,a))}\right)^{1/4}\left(1 + z_a(0,a)\int_0^t Q(\tau;a)\,d\tau\right), \tag{4.8}$$

where

$$Q(\tau;a) = -\left(\frac{G'(v(0,a))}{G'(v(\tau,x(\tau;a)))}\right)^{1/4}\left(\sqrt{G'\circ F}\right)'(w-z)(\tau, x(\tau;a)). \tag{4.9}$$

Now,

$$\frac{G'(v(0,a))}{G'(v(t,x(t;a)))} \geqslant \frac{1}{C} > 0. \tag{4.10}$$

Moreover, elementary computation gives

$$\left(\sqrt{G' \circ F}\right)' = \frac{1}{4}\frac{G'' \circ F}{G' \circ F}. \tag{4.11}$$

Hence, the assumption $G'' < 0$ and (4.9)–(4.11) imply that

$$Q(\tau;a) \geqslant 1/C > 0 \quad \text{and} \quad \int_0^t Q(\tau;a)\,d\tau \longrightarrow \infty \quad \text{as } t \longrightarrow \infty.$$

Therefore, since X should remain positive, we have $z_a(0,a) \geqslant 0$.

Since $z(0,\cdot)$ is periodic, we have $z_x(0,\cdot) = 0$, i.e. $z(0,\cdot)$ is a constant. Furthermore, the equation is invariant with respect to the parallel translation of t, thus $z(t,\cdot)$ is a constant for all t. By the same arguments, $w(t,\cdot)$ is also a constant. (In fact, we consider the solution $x^- = x^-(t;a)$ of

$$\frac{dx^-}{dt} = -\sqrt{G'(v(t,x^-))}, \quad x^-|_{t=0} = a,$$

and $X^-(t;a) = \dfrac{\partial}{\partial a}x^-(t;a)$. Then $X^-(t;a) > 0$ and we have

$$X^- = \left(\frac{G'(v(0,a))}{G'(v(t,x^-(t;a)))}\right)^{1/4}\left(1 - w_a(0,a)\int_0^t Q^-(\tau;a)\,d\tau\right),$$

where

$$Q^-(\tau;a) = -\left(\frac{G'(v(\tau,x^-(\tau;a)))}{G'(v(0,a))}\right)^{1/4}\frac{1}{4}\frac{G''}{G'}(v(\tau,x^-(\tau;a))).$$

Since $G'' < 0$, we have $w_a(0,a) \leqslant 0$. Thus $w_x(0,\cdot) = 0$.)

Therefore, $u(t,\cdot) = (w(t,\cdot) + z(t,\cdot))/2$ is also a constant. Due to $u(t,0) = 0$, we have $u = 0$ and this means $v_x = 0$ since $u_t = G'(v)v_x$. Hence, $v(t,\cdot)$ is a constant. On the other hand, since $y(t,0) = y(t,1) = 0$, we have $v = y_x = 0$ somewhere and therefore everywhere. Since $y(t,0) = 0$, then $y(t,x) = 0$ for any $0 < x < 1$. This completes the proof. □

Remark 4.1. For Theorem 2, if we assume that $G''(v) > 0$ for $|v| < \delta$ instead of $G''(v) < 0$, we can also obtain the same conclusion.

5. Example of singularities

By Theorem 2 there are no nontrivial uniformly bounded time-global C^2-solutions satisfying (1.1) and (1.2). Therefore, we guess that the singularities at the second derivatives of y will develop after a finite time even if we start from smooth and small initial data. In this section we first give an example of possible singularities at the second derivatives y_{tt} and y_{xx}. Then we give another kind of exact solutions to (1.1) with singularity such that shocks occur after a finite time.

5.1.

Let us consider Eq. (1.1) and forget the boundary conditions (1.2). Putting

$$u = y_t, \qquad v = y_x, \tag{5.1}$$

then Eq. (1.1) is equivalent to the system

$$v_t - u_x = 0, \qquad u_t - G'(v)v_x = 0. \tag{5.2}$$

If the mapping $(t, x) \to (u, v)$ is invertible, then we can rewrite (5.2), taking (u, v) as independent variables, formally as the linear system

$$-x_u + t_v = 0, \qquad x_v - G'(v)t_u = 0, \tag{5.3}$$

since

$$\begin{pmatrix} u_t & u_x \\ v_t & v_x \end{pmatrix} = \begin{pmatrix} t_u & t_v \\ x_u & x_v \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} x_v & -t_v \\ -x_u & t_u \end{pmatrix},$$

where

$$\Delta = t_u x_v - t_v x_u.$$

Eliminating x from (5.3) we get the following second-order linear equation:

$$t_{vv} - G'(v)t_{uu} = 0. \tag{5.4}$$

Let us consider the solution of (5.4) of the form

$$t(u, v) = \varepsilon \sin u \Psi(v), \tag{5.5}$$

where ε is a sufficiently small positive constant and $\Psi(v)$ is the solution of the ordinary differential equation

$$\frac{d^2\Psi}{dv^2} = -G'(v)\Psi \quad \text{with} \quad \Psi(0) = 1 \quad \text{and} \quad \Psi'(0) = 0. \tag{5.6}$$

Then we can find a positive number $\delta_0(< \delta)$ such that

$$\Psi(v) > 0 \quad \text{for } |v| \leqslant \delta_0 \quad \text{and} \quad \Psi'(v) \begin{cases} > 0 \text{ for } -\delta_0 \leqslant v < 0, \\ < 0 \text{ for } 0 < v \leqslant \delta_0. \end{cases}$$

We put

$$x(u, v) = \tfrac{1}{2} - \varepsilon \cos u \Psi'(v). \tag{5.7}$$

Then the functions $t = t(u, v)$, $x = x(u, v)$ satisfy the system (5.3).

Now we consider the curvilinear hexagon

$$\tilde{D} = \left\{(u, v) |\ |u| < \frac{\pi}{2}, \quad |v| \leqslant \delta_0, \quad \tan^2 u < H(v)\right\},$$

where

$$H(v) = \frac{G'(v)\Psi(v)^2}{\Psi'(v)^2} \quad \text{and} \quad H(0) = +\infty.$$

By the mapping $(u, v) \to (t(u, v), x(u, v))$ the domain $\tilde{D}$ is transformed onto a curvilinear spindle-shaped hexagon D in (t, x)-plane. The vertices $\{D_k^{\pm}\}_{k=1}^3$ of D are

$$\begin{aligned} D_1^{\pm} &= \left(\pm\varepsilon, \tfrac{1}{2}\right), \\ D_2^{\pm} &= \left(\varepsilon \sin u_{\pm} \Psi(\pm\delta_0), \tfrac{1}{2} - \varepsilon \cos u_{\pm} \Psi'(\pm\delta_0)\right), \\ D_3^{\pm} &= \left(-\varepsilon \sin u_{\pm} \Psi(\pm\delta_0), \tfrac{1}{2} - \varepsilon \cos u_{\pm} \Psi'(\pm\delta_0)\right), \end{aligned}$$

where $u_{\pm} = \arctan\sqrt{H(\pm\delta_0)}$. Hence D contains the interval on the x-axis

$$L_1: \quad t = 0, \qquad \tfrac{1}{2} - \varepsilon\Psi'(-\delta_0) \leqslant x \leqslant \tfrac{1}{2} - \varepsilon\Psi'(\delta_0),$$

and the interval parallel to the t-axis

$$L_2: \quad -\varepsilon < t < \varepsilon, \ \ x = \tfrac{1}{2}.$$

Note that, for any fixed $v \in (-\delta_0, \delta_0)$, $u \to t(u, v)$ is monotone on $\tilde{D}$, since

$$t_u(u, v) = \varepsilon \cos u \Psi(v) > 0$$

and, for any fixed $u \in (-\pi/2, \pi/2)$, $v \to x(u, v)$ is monotone on $\tilde{D}$, since

$$x_v(u, v) = -\varepsilon \cos u \Psi''(v) = \varepsilon G'(v) \cos u \Psi(v) > 0.$$

Therefore it is easy to see that the mapping $(u, v) \to (t(u, v), x(u, v))$ admits a smooth inverse mapping $(t, x) \to (u(t, x), v(t, x))$ from D onto $\tilde{D}$, since $\tan^2 u < H(v)$ and

$$\Delta = \varepsilon^2 (G'(v) \cos^2 u \Psi(v)^2 - \sin^2 u \Psi'(v)^2) > 0 \quad \text{on } \tilde{D}.$$

Then $(u(t, x), v(t, x))$ is a smooth solution on D satisfying the initial conditions on L_1

$$u(0, x) = 0, \qquad v(0, x) = (\Psi')^{-1} \left(\frac{1}{\varepsilon} \left(\frac{1}{2} - x \right) \right).$$

Now let us investigate the behaviors of the derivatives of u, v as $t \to \varepsilon - 0 (u \to \pi/2 - 0)$ along the interval L_2, which corresponds to the interval $\{-\pi/2 < u < \pi/2, v = 0\}$ on the (u, v)-plane.

Putting $v = 0$, we have

$$\begin{aligned} t_u(u, 0) &= \varepsilon \cos u, \qquad t_v(u, 0) = 0, \\ x_u(u, 0) &= 0, \qquad\qquad x_v(u, 0) = \varepsilon \gamma \cos u, \\ \Delta(u, 0) &= \varepsilon^2 \gamma \cos^2 u. \end{aligned}$$

Thus

$$u_t \left(t, \frac{1}{2} \right) = \frac{1}{\varepsilon \cos u}, \quad u_x \left(t, \frac{1}{2} \right) = 0, \quad v_t \left(t, \frac{1}{2} \right) = 0, \quad v_x \left(t, \frac{1}{2} \right) = \frac{1}{\varepsilon \gamma \cos u}.$$

Solving

$$u_t \left(t, \frac{1}{2} \right) = \frac{1}{\varepsilon \cos u \left(t, \frac{1}{2} \right)},$$

we have

$$u \left(t, \frac{1}{2} \right) = \arcsin \left(1 - \frac{1}{\varepsilon} (\varepsilon - t) \right)$$

or

$$\cos u\left(t, \frac{1}{2}\right) = \sqrt{1 - \sin^2 u\left(t, \frac{1}{2}\right)} = \sqrt{\frac{2}{\varepsilon}(\varepsilon - t) - \frac{1}{\varepsilon^2}(\varepsilon - t)^2}$$
$$= \sqrt{\frac{2}{\varepsilon}}\sqrt{\varepsilon - t}\sqrt{1 - \frac{1}{2\varepsilon}(\varepsilon - t)} = \sqrt{\frac{2}{\varepsilon}}\sqrt{\varepsilon - t}\left(1 + O\left(\frac{1}{\varepsilon}(\varepsilon - t)\right)\right).$$

Therefore

$$u_t\left(t, \frac{1}{2}\right) = \frac{1}{\sqrt{2\varepsilon}}\frac{1}{\sqrt{\varepsilon - t}}\left(1 + O\left(\frac{1}{\varepsilon}(\varepsilon - t)\right)\right)$$

and

$$v_x\left(t, \frac{1}{2}\right) = \frac{1}{\gamma\sqrt{2\varepsilon}}\frac{1}{\sqrt{\varepsilon - t}}\left(1 + O\left(\frac{1}{\varepsilon}(\varepsilon - t)\right)\right)$$

as $t \to \varepsilon - 0$. That is, the derivatives $y_{tt} = u_t$, $y_{xx} = v_x$ blow up in this manner. We note that $u(t, \frac{1}{2})$, $v(t, \frac{1}{2})$ themselves have finite limits.

Furthermore, we note that the second derivatives $y_{tt} = u_t$, $y_{xx} = v_x$, $y_{tx} = u_x = v_t$ blow up on the curves $D_2^- D_1^+$, $D_1^+ D_2^+$, too. But the blow up at D_1^+ above observed is typical.

Of course these particular solutions are defined only on the domain D. But if smooth initial data which coincides with one of these particular solutions on L_1, ε being sufficiently small, is given, then a solution of (1.1) and (1.2) which coincides with the particular solution on $D \cap \{t \geqslant 0\}$ may exist by dint of the principle of the finite propagation speed. But this is not the case if G is linear, or $G''(v) = 0$.

5.2.

In this subsection, we give another kind of exact solutions to (1.1) with singularity. In [6], John gave the solution to the initial value problem

$$y_{tt} - (1 + y_x)^2 y_{xx} = 0,$$
$$y(0, x) = f(x), \quad y_t(0, x) = -f'(x) - \tfrac{1}{2} f'(x)^2,$$

which develops a shock at $t = 1/m$, $x = x^*$, where f is an arbitrary function in $C_0^\infty(\mathbb{R})$ with $-m = \min_x f''(x) = f''(x^*) < 0$. y_{xx} blows up at this point, while y, y_t, y_x are still bounded. We try to extend his example to our equation

$$y_{tt} - G'(y_x) y_{xx} = 0. \tag{5.8}$$

Let f be an arbitrary function in $C_0^\infty(\mathbb{R})$. Suppose $v = v(t, x)$ solves the functional equation

$$v = f'(x - \Lambda(v)t). \tag{5.9}$$

Put

$$\Phi(v) = \int_0^v v\Lambda'(v)\,dv = v\Lambda(v) - \hat{G}(v), \tag{5.10}$$

$$y(t, x) = t\Phi(v) + f(x - \Lambda(v)t), \tag{5.11}$$

then $y(t, x)$ solves Eq. (5.8). In fact, we have

$$y_t = \Phi(v) + tv\Lambda'(v)v_t + v(-\Lambda'(v)v_t t - \Lambda(v)) = \Phi(v) - v\Lambda(v) = -\hat{G}(v),$$

$$y_{tt} = -\Lambda v_t.$$

Differentiating (5.9) with respect to t, we have

$$v_t = f''(x - \Lambda(v)t)(-\Lambda'(v)v_t t - \Lambda(v)),$$

that is

$$v_t = -\frac{\Lambda(v) f''(x - \Lambda(v)t)}{1 + f''(x - \Lambda(v)t)\Lambda'(v)t},$$

provided that $1 + f''(x - \Lambda(v)t)\Lambda'(v)t \neq 0$. Hence

$$y_{tt} = \frac{\Lambda^2(v) f''(x - \Lambda(v)t)}{1 + f''(x - \Lambda(v)t)\Lambda'(v)t} = \frac{G'(v) f''(x - \Lambda(v)t)}{1 + f''(x - \Lambda(v)t)\Lambda'(v)t}. \tag{5.12}$$

On the other hand

$$y_x = tv\Lambda'(v)v_x + v(1 - \Lambda'(v)v_x t) = v,$$

$$y_{xx} = v_x.$$

But, differentiating (5.9) with respect to x, we have

$$v_x = f''(x - \Lambda(v)t)(1 - \Lambda'(v)v_x t),$$

that is

$$y_{xx} = v_x = \frac{f''(x - \Lambda(v)t)}{1 + f''(x - \Lambda(v)t)\Lambda'(v)t}. \tag{5.13}$$

By (5.12) and (5.13), $y(t, x)$ solves Eq. (5.8) to the initial value problem

$$y(0, x) = f(x), \quad y_t(0, x) = -\hat{G}(f'(x)).$$

Now we assume that $f \in C_0^\infty(\mathbb{R})$ satisfies that $|f'(\xi)| \leqslant \delta_0 (< \delta)$ for $\xi \in \mathbb{R}$ and

$$-m = \min_\xi f''(\xi)\Lambda'(f'(\xi)) < 0 \leqslant \max_\xi f''(\xi)\Lambda'(f'(\xi)) \leqslant m.$$

For example, the function

$$f(\xi) = \begin{cases} \dfrac{\delta_0}{16} \exp\Big(-\dfrac{1}{1-\xi^2}\Big) & \text{if } |\xi| < 1, \\ 0 & \text{if } |\xi| \geqslant 1, \end{cases}$$

or $-f(\xi)$ is such a function, provided that $\Lambda' = G''/2\sqrt{G'} \neq 0$.

Put $T = 1/m$. Then the functional equation (5.9) can be solved uniquely as long as $x \in \mathbb{R}$, $0 \leqslant t < T$. In fact, Eq. (5.9) is equivalent to

$$\xi = x - \Lambda(f'(\xi))t, \tag{5.14}$$

provided that $\Lambda'(v) = G''(v)/(2\sqrt{G'(v)}) < 0$ or > 0 for $|v| \leqslant \delta_0$. But the right-hand side of (5.14) is a contraction, since

$$\left| \frac{\partial}{\partial \xi}\big(x - \Lambda(f'(\xi))t\big) \right| = \left| \Lambda'(f'(\xi)) f''(\xi)t \right| < 1$$

as long as $|t| < T = 1/m$. Hence $v(t, x)$ is well-defined for $x \in \mathbb{R}$, $0 \leqslant t < T$ and y given by (5.11) is a smooth solution of (5.8) on $x \in \mathbb{R}$, $0 \leqslant t < T$.

As $t \to T - 0$, we see that $y_x = v(t, x)$ and $y_t = u(t, x) = -\hat{G}(v(t, x))$ remain to be bounded. Moreover, if ξ which attain the minimum of $f''(\xi)\Lambda'(f'(\xi))$ are discrete, say ξ_n, then we can claim that $v(t, x)$ tends to a limit $v(T - 0, x)$ for each fixed x. On the other hand

$$y_{xx} = v_x = \frac{f''(x - \Lambda(v)t)}{1 + f''(x - \Lambda(v)t)\Lambda'(v)t} \longrightarrow \infty$$

as $t \to T - 0$ at $x = x_n$ such that $x - \Lambda(v(T - 0, x))T = \xi_n$.

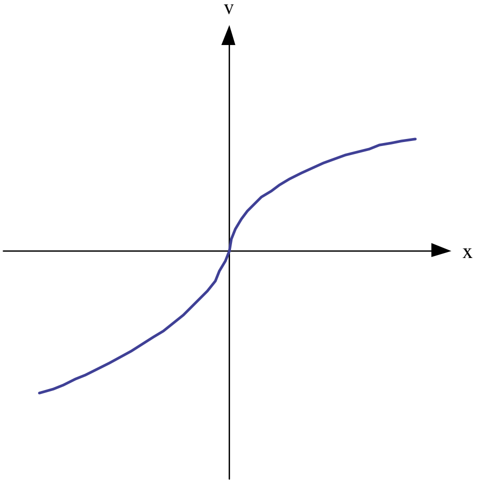

Fig. 1. $t < T$.

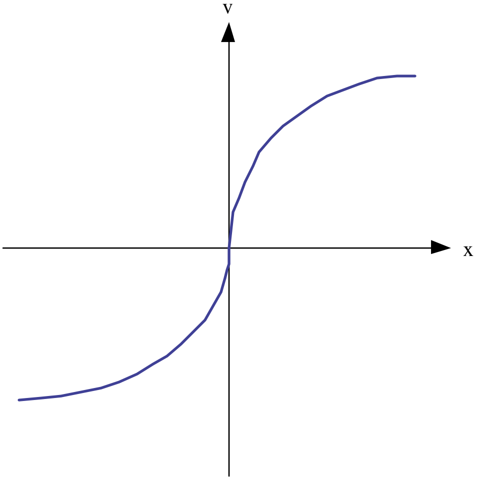

Fig. 2. $t = T$.

Roughly speaking, the situation is as follows. We can approximately write

$$\Lambda(f'(\xi)) \sim \Lambda_n - m(\xi - \xi_n) + \frac{a}{6}(\xi - \xi_n)^3$$

for small $\xi - \xi_n$, where a is a positive constant. Then $\xi = x - \Lambda(f'(\xi))t$ turns out to be

$$x - x_n \sim (1 - mt)(\xi - \xi_n) + \frac{a}{6}(\xi - \xi_n)^3 t.$$

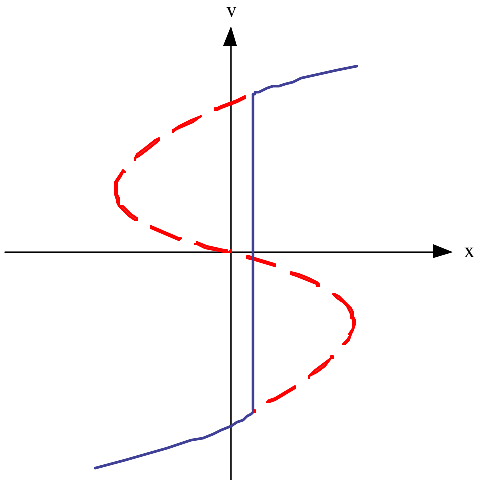

Fig. 3. $t > T$.

At this moment suppose that $f''(\xi) > 0$ near $\xi = \xi_n$. Then, since $v = f'(\xi)$ behaves like ξ, we can see that v increasingly pass through $v(t, x_n)$ at x_n for increasing x if $t < T = 1/m$;

$$v - v_n \sim C(x - x_n)^{1/3}, \quad \text{if } t = T = 1/m;$$

and v has jump discontinuity near x_n, if $t > T = 1/m$, as sketched in Figs. 1–3.

In this way the smooth solution develops shocks after a finite time $T = 1/m$.

6. Estimate of life span of smooth solutions

In this section, we apply the argument of Lax [8] to get an estimate of the life span of smooth solutions to (1.1) and (1.2).

We consider the initial-boundary value problem

$$y_{tt} - (G(y_x))_x = 0 \quad \text{for } 0 < x < 1, \tag{6.1}$$

$$y(t, 0) = y(t, 1) = 0, \tag{6.2}$$

$$y(0, x) = y_0(x) \quad \text{and} \quad y_t(0, x) = y_1(x), \tag{6.3}$$

where y_0 and y_1 are smooth and satisfy the following compatibility conditions:

$$y_0(0) = y_0(1) = y_1(0) = y_1(1) = 0.$$

We have the following results.

Theorem 3. *There are positive constants ε and C such that $|y_{0,x}(x)|,\ |y_1(x)| \leqslant \varepsilon$ and if*

$$|y_{0,xx}(x)|,\ |y_{1,x}(x)| \leqslant M$$

then there is a C^2-solution $y(t,x)$ to (6.1)–(6.3) *as long as $0 \leqslant t < 1/(CM)$.*

Proof. As in Section 3, we consider the problem by extending the solutions periodically as (3.1). Introducing the variables

$$u = y_t, \quad v = y_x,$$

we can write (6.1) as the first-order system

$$v_t - u_x = 0, \quad u_t - G'(v)v_x = 0 \tag{6.4}$$

with the initial conditions

$$u(0,x) = y_1(x), \quad v(0,x) = y_{0,x}(x). \tag{6.5}$$

Let $q = \hat{G}(v)$. The Riemann invariants are

$$w = u + q, \quad z = u - q \tag{6.6}$$

and the diagonalized system is

$$w_t - \Lambda w_x = 0, \quad z_t + \Lambda z_x = 0. \tag{6.7}$$

The initial conditions are

$$w(0,x) = w_0(x), \quad z(0,x) = z_0(x). \tag{6.8}$$

Suppose that there is a C^1-solution (w,z) on $0 \leqslant t < T$. Since w and z are constants along the characteristic curves $\frac{dx}{dt} = \pm\Lambda$, if $|w_0(x)|,\ |z_0(x)| \leqslant \varepsilon_0$ then we have

$$|w(t,x)|,\ |z(t,x)| \leqslant \varepsilon_0 \quad \text{and} \quad |q| = \left|\tfrac{1}{2}(w - z)\right| \leqslant \varepsilon_0.$$

Thus, we can assume that ε_0 is so small that q has an inverse function for $|q| \leqslant \varepsilon_0$ such that $|v| \leqslant \delta_0 < \delta$.

Now let us consider the quantities

$$A := \sqrt{\Lambda} w_x \quad \text{and} \quad B := \sqrt{\Lambda} z_x.$$

By elementary computation we can see that A and B satisfy the equations

$$A_t - \Lambda A_x + \mu A^2 = 0 \quad \text{and} \quad B_t + \Lambda B_x + \mu B^2 = 0, \tag{6.9}$$

respectively. Here

$$\mu = -\frac{1}{2} \frac{1}{\sqrt{\Lambda}} \frac{d\Lambda}{dq} = -\frac{1}{4} G''(v) G'(v)^{-5/4}. \tag{6.10}$$

Since $|v| \leqslant \delta_0 < \delta$, we have $|\mu| \leqslant C$. Let

$$M_1 = \max \left\{ \max_x |A(0,x)|, \; \max_x |B(0,x)| \right\}.$$

Then along the characteristic curve satisfying $\frac{dx}{dt} = -\Lambda$ or $\frac{dx}{dt} = \Lambda$ we have

$$\frac{dA}{dt} + \mu A^2 = 0 \quad \text{or} \quad \frac{dB}{dt} + \mu B^2 = 0, \tag{6.11}$$

respectively. By the comparison theorem of ordinary differential equations, we have the following estimates:

$$|A|, \; |B| \leqslant \frac{M_1}{1 - C M_1 t} \tag{6.12}$$

and since $1/C \leqslant \Lambda \leqslant C$ we have

$$|w_x|, \; |z_x| \leqslant \frac{C M_2}{1 - C M_2 t},$$

where

$$M_2 = \max \left\{ \max_x |w_{0,x}(x)|, \; \max_x |z_{0,x}(x)| \right\}.$$

As Lax said in [8], ‘solution to initial-value problems exists as long as one can place an a priori limitation on the magnitude of their first derivatives’. More precisely we have the following lemma.

Lemma 6.1. *There is a positive number ε such that for any B there exists a positive number h depending upon B such that the initial value problem*

$$w(0,x) = w_0(x), \quad z(0,x) = z_0(x)$$

admits a C^1-solution with Lipschitz conditions first derivatives on $0 \leqslant t \leqslant h$ provided that

$$|w_0(x)|, \ |z_0(x)| \leqslant \varepsilon, \quad |w_{0,x}(x)|, \ |z_{0,x}(x)| \leqslant B$$

and $w_{0,x}$ and $z_{0,x}$ are Lipschitz continuous. Here $w(t,x) = -z(t,-x)$, $z(t,x) = -w(t,-x)$ and are 2-periodic in x.

A proof can be found in the book of Courant–Hilbert (see [1, Chapter V]). We omit the details since the argument is similar to that of Section 3. The essential point is that the iteration converges with the first derivatives on a time interval independent of the magnitude of the second derivatives or the Lipschitz constant of the first derivatives. This completes the proof of the theorem. □

Acknowledgements

The authors express their sincere thanks to the referee for his/her careful reading of the original manuscript and giving of helpful comments.

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