



LQG optimal control of discrete stochastic systems under parametric and noise uncertainties

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Received 20 July 2005; received in revised form 15 February 2006; accepted 21 February 2006

Abstract

In this paper, the linear-quadratic-Gaussian (LQG) optimal control problem is considered and a robust minimax controller composed of the Kalman filter and the optimal regulator is synthesized to guarantee the asymptotic stability of the discrete time-delay systems under both parametric uncertainties and uncertain noise covariances. Designed procedures are finally elaborated with an illustrative example.

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Keywords: LQG optimal control; Minimax controller; Kalman filter

1. Introduction

A well-known property of the discrete linear-quadratic-Gaussian (LQG) optimal control problem is that the optimal regulator, synthesized by the LQ optimal technique, is

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generated from the estimated state which is the output of the Kalman filter. The LQG optimal control design for a linear stochastic dynamic system requires not only an accurate description of the statistical characteristic of noise signal but also an exact system model. Nevertheless, neither noise nor plant parameters may be precisely known in a real control system. Therefore, it is of interest to consider the robust LQG optimal problem for those systems whose noise covariances and plant parameters are known only to be within some classes.

Time delay is commonly encountered in various engineering systems; for example, systems with computer control have delays, as it takes time for the computer to execute numerical operations. Besides, remote working, radar, electric networks, transport process, metal rolling systems, etc. all have delays. The output in these systems responds only to an input after some time interval. The introduction of time-delay factor is often a source of instability and generally complicates the analysis. Hence, the problem of stability analysis of time-delay systems has been one of the main concerns of researchers wishing to inspect the properties of such systems and there have been several research efforts [1–11] on this issue. For instance, Basin et al. proposed the optimal control and filtering algorithms for time-delay systems, and discussed the delay-dependent stability for linear discrete stochastic systems [10–14].

In this paper, the LQG optimal control problem is considered and Minimax theory and Bellman–Gronwall lemma are employed to derive a robust criterion which guarantees the asymptotic stability of the discrete time-delay systems under both parametric uncertainties and uncertain noise covariances. On the basis of this criterion, a robust minimax controller composed of the Kalman filter and the optimal regulator is synthesized to stabilize the uncertain stochastic systems.

The organization of this paper is as follows. The system description is presented in Section 2. The design procedure of a robust minimax controller is proposed in Section 3. An example is provided in Section 4 to illustrate our main results. A conclusion is finally drawn in Section 5.

2. Problem formulation

A discrete time-delay system is depicted by the following difference equations:

$$x_p(k+1) = A_0 x_p(k) + \Delta A_0(k) x_p(k) + \sum_{i=1}^m A_i x_p(k-i) + \sum_{i=1}^m \Delta A_i(k) x_p(k-i) + B_p u(k) + \Delta B_p(k) u(k) + v(k), \quad (2.1a)$$

$$y(k) = C_p x_p(k) + \Delta C_p(k) x_p(k) + e(k) \quad (2.1b)$$

where $x_p(k)$ is an $n \times 1$ state vector; $u(k)$ is an $r \times 1$ input vector; $y(k)$ is a $p \times 1$ output vector; $v(k)$ is an $n \times 1$ random process vector; $e(k)$ is a $p \times 1$ random process vector; and A_0 , A_i , B_p ($\text{rank}(B_p) = r$) and C_p ($\text{rank}(C_p) = p$) are constant matrices with appropriate dimensions. Moreover, $\Delta A_0(k)$, $\Delta A_i(k)$, $\Delta B_p(k)$ and $\Delta C_p(k)$ denote linear time-varying parametric uncertainties with the following upper norm-bounds:

$$\|\Delta A_0(k)\| \leq \sigma, \quad \|\Delta A_i(k)\| \leq \eta_i, \quad i = 1, 2, \dots, m, \quad \|\Delta B_p(k)\| \leq \delta, \quad \|\Delta C_p(k)\| \leq \rho \quad (2.2)$$

where σ , η_i , δ , and ρ are given constants. The process noise $v(k)$ and measurement noise $e(k)$ are uncorrelated random sequences with zero mean. Meanwhile, they have no time correlation or are “white” noises, that is,

$$E\{v(i)v^T(j)\} = 0 \quad \text{if } i \neq j, \tag{2.3a}$$

$$E\{e(i)e^T(j)\} = 0 \quad \text{if } i \neq j, \tag{2.3b}$$

and their covariances or “noise levels” are defined by

$$E\{v(k)v^T(k)\} \equiv R_1, \tag{2.3c}$$

$$E\{e(k)e^T(k)\} \equiv R_2. \tag{2.3d}$$

In (2.3c)–(2.3d), R_1 and R_2 are symmetric, positive definite matrices and have the following norm-bounds:

$$\|R_1 - R_{10}\| \leq \varepsilon_1, \tag{2.3e}$$

$$\|R_2 - R_{20}\| \leq \varepsilon_2 \tag{2.3f}$$

where ε_1 , ε_2 are given positive constants and R_{10} , R_{20} denote the nominal parts of the actual covariances of the process noise $v(k)$ and measurement noise $e(k)$, respectively.

By defining an $n(m + 1) \times 1$ new state vector

$$\bar{x}(k) = [x_p^T(k - m) \quad x_p^T(k - m + 1) \quad \dots \quad x_p^T(k - 1) \quad x_p^T(k)]^T, \tag{2.4}$$

system (2.1) can then be transformed into the following system:

$$\bar{x}(k + 1) = \bar{A}\bar{x}(k) + \Delta\bar{A}(k)\bar{x}(k) + \bar{B}u(k) + \Delta\bar{B}(k)u(k) + \bar{v}(k), \tag{2.5a}$$

$$y(k) = \bar{C}\bar{x}(k) + \Delta\bar{C}(k)\bar{x}(k) + e(k) \tag{2.5b}$$

where

$$\bar{A} = \begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I \\ A_m & A_{m-1} & A_{m-2} & \dots & A_1 & A_0 \end{bmatrix}_{\bar{n} \times \bar{n}},$$

$$\Delta\bar{A}(k) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \Delta A_m(k) & \Delta A_{m-1}(k) & \dots & \Delta A_1(k) & \Delta A_0(k) \end{bmatrix}_{\bar{n} \times \bar{n}}, \tag{2.6a}$$

$$\bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ B_p \end{bmatrix}_{\bar{n} \times r}, \quad \Delta \bar{B}(k) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Delta B_p(k) \end{bmatrix}_{\bar{n} \times r}, \tag{2.6b}$$

$$\bar{C} = [0 \ 0 \ \cdots \ 0 \ C_p]_{p \times \bar{n}}, \quad \Delta \bar{C}(k) = [0 \ 0 \ \cdots \ 0 \ \Delta C_p(k)]_{p \times \bar{n}}, \quad \bar{v}(k) = Gv(k), \tag{2.6c}$$

where $\bar{n} \equiv n(m + 1)$ and

$$G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}_{\bar{n} \times n}.$$

According to Eqs. (2.3c) and (2.6c), the covariance of $\bar{v}(k)$ is given by

$$E\{\bar{v}(k)\bar{v}^T(k)\} = GR_1G^T \equiv \bar{R}_1 \tag{2.7}$$

and \bar{R}_1 has the following norm-bound:

$$\|\bar{R}_1 - \bar{R}_{10}\| \leq \bar{\varepsilon}_1 \tag{2.8}$$

where $\bar{R}_{10} \equiv GR_{10}G^T$ and $\bar{\varepsilon}_1 \equiv \varepsilon_1 \|G\| \|G^T\|$.

Lemma 2.1. *If $\text{rank}(B_p) = r$, the pair $\{\bar{A}, \bar{B}\}$ is controllable.*

Proof. From Eq. (2.6b), obtained here is $\text{rank}(\bar{B}) = \text{rank}(B_p)$. If $\text{rank}(B_p) = r$, the pair $\{\bar{A}, \bar{B}\}$ is then controllable if and only if $\text{rank}[\bar{B}, \bar{A}\bar{B}, \dots, \bar{A}^{\bar{n}-r}\bar{B}] \equiv \text{rank}(U_{\bar{n}-r}) = \bar{n}$ [12]. The matrix $U_{\bar{n}-r}$ is easily found to have the following form:

$$U_{\bar{n}-r} = \begin{bmatrix} 0 & 0 & 0 & \cdots & B_p & \cdots & \times \\ 0 & 0 & 0 & \vdots & A_0 B_p & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots & \times & \vdots & \vdots \\ \vdots & \vdots & 0 & \cdots & \vdots & \cdots & \times \\ \vdots & 0 & B_p & \cdots & \vdots & \cdots & \times \\ 0 & B_p & A_0 B_p & \cdots & \vdots & \cdots & \times \\ B_p & A_0 B_p & (A_0^2 + A_1)B_p & \cdots & \times & \cdots & \times \end{bmatrix}_{\bar{n} \times (\bar{n}-r+1)r} \tag{2.9}$$

and $U_{\bar{n}-r}$ has \bar{n} linearly independent columns. Thus, $\text{rank}(U_{\bar{n}-r}) = \bar{n}$. The proof is then complete. \square

Lemma 2.2. *If $\text{rank}(C_p) = p$, the pair $\{\bar{A}, \bar{C}\}$ is observable.*

Proof. From Eq. (2.6c), obtained here is $\text{rank}(\bar{C}) = \text{rank}(C_p)$. If $\text{rank}(C_p) = p$, the pair $\{\bar{A}, \bar{C}\}$ is then observable if and only if [15]

$$\text{rank} \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{\bar{n}-p} \end{bmatrix}_{(\bar{n}-p+1)p \times \bar{n}} \equiv \text{rank}(O_{\bar{n}-p}) = \bar{n}. \tag{2.10}$$

The matrix $O_{\bar{n}-p}$ is easily observed to have the following form:

$$O_{\bar{n}-p} = \begin{bmatrix} 0 & 0 & \cdots & 0 & C_p \\ C_p A_m & C_p A_{m-1} & \cdots & C_p A_1 & C_p A_0 \\ C_p A_0 A_m & C_p A_0 A_{m-1} & \cdots & C_p A_0 A_1 & C_p A_0^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \times & \times & \cdots & \times & \times \end{bmatrix}_{(\bar{n}-p+1)p \times \bar{n}} \tag{2.11}$$

and $O_{\bar{n}-p}$ has \bar{n} linearly independent rows. Thus, $\text{rank}(O_{\bar{n}-p}) = \bar{n}$. The proof is then complete. \square

3. Robust minimax controller

Prior to discussing the design of robust minimax controller for the uncertain stochastic system (2.5), its nominal system is first considered (i.e., $\Delta\bar{A}(k) = 0, \Delta\bar{B}(k) = 0, \Delta\bar{C}(k) = 0$ and $\bar{\varepsilon}_1 = \varepsilon_2 = 0$):

$$\bar{x}(k + 1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{v}(k), \tag{3.1a}$$

$$y(k) = \bar{C}\bar{x}(k) + e(k). \tag{3.1b}$$

The performance index to be minimized is chosen as

$$J = \sum_{k=0}^{\infty} E\{\bar{x}^T(k)Q\bar{x}(k) + u^T(k)Wu(k)\}^1 \tag{3.2}$$

where $Q = Q^T \geq 0, W = W^T > 0$ and the triple $\{\bar{A}, \bar{R}_1^{1/2}, Q^{1/2}\}$ is assumed here to be controllable and observable. The optimal admissible control $u_{\text{opt}}(t)$, which minimizes the performance index J in Eq. (3.2) subject to dynamic system (3.1), is given by

$$u_{\text{opt}}(k) = -G_f \hat{x}(k) \tag{3.3a}$$

where

$$G_f = (W + \bar{B}^T S \bar{B})^{-1} \bar{B}^T S \bar{A} \tag{3.3b}$$

¹ $E(\cdot)$ denotes the expected value of (\cdot) .

and S is the symmetric positive definite solution of the following discrete Riccati equation:

$$S = \bar{A}^T S \bar{A} + Q - (\bar{B}^T S \bar{A})^T (W + \bar{B}^T S \bar{B})^{-1} (\bar{B}^T S \bar{A}). \quad (3.3c)$$

Meanwhile, the estimated state \hat{x} is the output of the Kalman filter:

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + \bar{B}u(k) + F[y(k) - \bar{C}\hat{x}(k)], \quad (3.4a)$$

$$F = \bar{A}P\bar{C}^T [R_2 + \bar{C}P\bar{C}^T]^{-1} \quad (3.4b)$$

where the Kalman gain F is chosen to minimize the state reconstruction error $\tilde{x}(k) \equiv \bar{x}(k) - \hat{x}(k)$ and P is the steady-state solution of the following equation:

$$P(k+1) = \bar{A}P(k)\bar{A}^T + \bar{R}_1 - \bar{A}P(k)\bar{C}^T [R_2 + \bar{C}P(k)\bar{C}^T]^{-1} \bar{C}P(k)\bar{A}^T. \quad (3.4c)$$

The objective here lies in formulating a robust controller for a given controllable and observable system (3.1) such that the optimal regulator (3.3) still minimizes the performance index J in Eq. (3.2) and the Kalman filter (3.4) also asymptotically tracks the actual states in the presence of parametric uncertainties and uncertain noise covariances.

The approach for the design of a robust controller is divided into two steps. In the first step, we only consider system (3.1) under uncertain noise covariances, i.e.

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{v}(k), \quad (3.5a)$$

$$y(k) = \bar{C}\bar{x}(k) + e(k), \quad (3.5b)$$

with

$$\bar{R}_1 \in S_1 = \{\bar{R}_1 : \|\bar{R}_1 - \bar{R}_{10}\| \leq \bar{\varepsilon}_1, \bar{R}_1 > 0\}, \quad (3.5c)$$

$$R_2 \in S_2 = \{R_2 : \|R_2 - R_{20}\| \leq \varepsilon_2, R_2 > 0\}. \quad (3.5d)$$

The design of a robust controller for system (3.5) can therefore be considered as a saddlepoint problem which treats the uncertain (but bounded) noise covariances problem. By means of minimax theory, the following lemma is obtained here as:

Lemma 3.1 (Looze et al. [16], Chen and Dong [17]). *The robust controller for system (3.5) is a minimax controller that solves the saddlepoint problem with the worst noise covariances, $\bar{R}_{10} + \bar{\varepsilon}_1 I$ and $R_{20} + \varepsilon_2 I$, i.e.*

$$u(k) = -G_f \hat{x}(k), \quad (3.6a)$$

$$G_f = (W + \bar{B}^T S \bar{B})^{-1} \bar{B}^T S \bar{A}, \quad (3.6b)$$

$$S = \bar{A}^T S \bar{A} + Q - (\bar{B}^T S \bar{A})^T (W + \bar{B}^T S \bar{B})^{-1} (\bar{B}^T S \bar{A}), \quad (3.6c)$$

and

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + \bar{B}u(k) + \bar{F}[y(k) - \bar{C}\hat{x}(k)], \quad (3.7a)$$

$$\bar{F} = \bar{A}P\bar{C}^T [(R_{20} + \varepsilon_2 I) + \bar{C}P\bar{C}^T]^{-1}, \quad (3.7b)$$

$$\begin{aligned}
 P(k+1) &= \bar{A}P(k)\bar{A}^T + (\bar{R}_{10} + \bar{\varepsilon}_1 I) - \bar{A}P(k)\bar{C}^T [(R_{20} + \varepsilon_2 I) \\
 &\quad + \bar{C}P(k)\bar{C}^T]^{-1} \bar{C}P(k)\bar{A}^T.
 \end{aligned}
 \tag{3.7c}$$

The minimax controller in Eqs. (3.6) and (3.7) may still not be robust if system (3.5) is perturbed not only by noise uncertainties but also by parametric uncertainties, i.e. the uncertain system (2.5) is considered. More constraints must consequently be imposed to let the minimax controller in Eqs. (3.6) and (3.7) become robust under parametric uncertainties.

Introducing the control law (3.6a) and subtracting (3.7a) from (2.5a) yield

$$\begin{aligned}
 \bar{x}(k+1) - \hat{x}(k+1) \equiv \tilde{x}(k+1) &= (\bar{A} - \bar{F}\bar{C})\tilde{x}(k) + (\Delta\bar{A}(k) - \Delta\bar{B}(k)G_f - \bar{F}\Delta\bar{C}(k))\bar{x}(k) \\
 &\quad + \Delta\bar{B}(k)G_f\tilde{x}(k) + \bar{v}(k) - \bar{F}e(k).
 \end{aligned}
 \tag{3.8}$$

Combining Eq. (2.5a) with (3.8), we have

$$x(k+1) = Ax(k) + \Delta A(k)x(k) + Hn(k)
 \tag{3.9}$$

where

$$x(k) = \begin{bmatrix} \bar{x}(k) \\ \tilde{x}(k) \end{bmatrix},
 \tag{3.10a}$$

$$A = \begin{bmatrix} \bar{A} - \bar{B}G_f & \bar{B}G_f \\ 0 & \bar{A} - \bar{F}\bar{C} \end{bmatrix},
 \tag{3.10b}$$

$$\Delta A(k) = \begin{bmatrix} \Delta\bar{A}(k) - \Delta\bar{B}(k)G_f & \Delta\bar{B}(k)G_f \\ \Delta\bar{A}(k) - \Delta\bar{B}(k)G_f - \bar{F}\Delta\bar{C}(k) & \Delta\bar{B}(k)G_f \end{bmatrix},
 \tag{3.10c}$$

$$H = \begin{bmatrix} I & 0 \\ I & -\bar{F} \end{bmatrix},
 \tag{3.10d}$$

$$n(k) = \begin{bmatrix} \bar{v}(k) \\ e(k) \end{bmatrix}.
 \tag{3.10e}$$

A robust criterion is derived in the following to guarantee the asymptotic stability of system (3.9). Before proceeding to examine robust stability, Bellman–Gronwall lemma in discrete form which will be used in the proof of the next theorem is given below.

Lemma 3.2 (Desoer and Vidyasagar [18]). *Let Z_+ denote the set of nonnegative integers: $\{0, 1, 2, \dots\}$ and $u(k), f(k), h(k)$ be real-valued sequence on Z_+ . Let*

$$h(k) \geq 0, \quad \forall k \in Z_+.$$

Under these conditions, if

$$u(k) \leq f(k) + \sum_{i=0}^{k-1} h(i)u(i), \quad k = 0, 1, 2, \dots,
 \tag{3.11}$$

then

$$u(k) \leq f(k) + \sum_{i=0}^{k-1} \left\{ \prod_{j=i+1}^{k-1} [1 + h(j)] h(i) f(i) \right\}, \quad k = 0, 1, 2, \dots, \tag{3.12}$$

in which $\prod_{j=i+1}^{k-1} [1 + h(j)]$ is set equal to 1 when $i = k - 1$.

Note that:

(a) If for some constant h_M , $h(i) \leq h_M, \forall i$, then (3.12) becomes

$$u(k) \leq f(k) + h_M \sum_{i=0}^{k-1} (1 + h_M)^{k-i-1} f(i). \tag{3.13}$$

(b) If for some constant f_M , $f(i) \leq f_M, \forall i$, then (3.12) becomes

$$u(k) \leq f_M \prod_{i=0}^{k-1} [1 + h(i)]. \tag{3.14}$$

Theorem 1. Assume that the matrix A in Eq. (3.10b) is diagonalizable and Hurwitz (i.e., all the eigenvalues of A are inside the unit circle) so that the state transition matrix A^k satisfies the inequality:

$$\|A^k\| \leq Mr^{k2}, \quad k = 0, 1, 2, \dots, \tag{3.15}$$

in which $M \geq 1$ and $0 \leq r < 1$. If the following inequality holds:

$$r(1 + h) < 1 \tag{3.16}$$

with

$$h \equiv \frac{M}{r} \left[2 \left(\sigma + \sum_{i=1}^m \eta_i \right) + 4\delta \|G_f\| + \rho \|\bar{F}\| \right], \tag{3.17}$$

then system (3.9) is asymptotically stable. Namely, the minimax controller in Eqs. (3.6) and (3.7) is a robust LQG optimal controller under both parametric uncertainties and uncertain noise covariances.

Proof. See Appendix. \square

4. Example

A fourth-order model of a fluid catalytic cracking unit (this model was obtained by linearization, followed by normalization, of the Lee and Kugelman [19] model around a nominal stable operating point, as described in Oliveira [20]) is given to illustrate the designed procedures. Using a sampling time $T = 0.05$ s leads to the following discrete

²If A is a $q \times q$ diagonalizable and Hurwitz matrix, the following result is obtained: $\|A^k\| = \|N \hat{A}^k N^{-1}\| \leq \|N\| \|\hat{A}^k\| \|N^{-1}\| \leq Mr^k$ where $\hat{A} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_q)$ and $M = \|N\| \|N^{-1}\|$ (the condition number of A). The i th column of N is an eigenvector corresponding to λ_i . Moreover, r is chosen as the distance from the origin to the eigenvalue (of the matrix A) nearest to the unit circle.

model:

$$x_p(k + 1) = A_0x_p(k) + A_1x_p(k - 1) + B_pu(k) + \Delta A_0(k)x_p(k) + \Delta A_1(k)x_p(k - 1) + \Delta B_p(k)u(k) + v(k), \tag{4.1a}$$

$$y(k) = C_px_p(k) + \Delta C_p(k)x_p(k) + e(k), \tag{4.1b}$$

with

$$A_0 = \begin{bmatrix} 0.07362 & 0.1148 & 0.01044 & 0.4390 \\ 0.03940 & 0.1492 & 0.00894 & 0.8111 \\ 0.27940 & -0.7511 & -0.003253 & -6.127 \\ 0.04545 & 0.1559 & 0.009798 & 0.8384 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.0014 & 0.0498 & 0.0085 & -0.0022 \\ 0.0166 & 0.0058 & 0.0043 & -0.0011 \\ 0.0027 & 0.7024 & 0.0017 & -5.0004 \\ 0.0005 & 0.0034 & 0.0202 & -0.8006 \end{bmatrix},$$

$$B_p = \begin{bmatrix} -0.2495 & 0.6030 \\ 0.0168 & 0.2505 \\ -5.785 & 7.2139 \\ 0.01157 & 0.1217 \end{bmatrix}, \quad C_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and

$$\Delta A_0(k) = \begin{bmatrix} 0.019 & 0.01 & -0.004 & 0.002 \\ 0.004 & 0.008 \sin(k) & 0 & -0.0026 \\ -0.002 & 0.003 & 0.009 & 0.006 \exp(-k) \\ 0.001 \cos(k) & -0.003 & 0 & 0.021 \end{bmatrix},$$

$$\Delta A_1(k) = \begin{bmatrix} 0.0064 & -0.01 & 0.0027 \cos(k) & 0.0041 \\ 0.0038 \sin(k) & 0.0035 & 0 & -0.0012 \\ -0.0015 & 0.002 & 0.001 & 0 \\ -0.003 & 0.004 \sin(k) & 0.005 & 0.0089 \end{bmatrix},$$

$$\Delta B_p(k) = \begin{bmatrix} 0.015 \cos(k) & 0.013 & -0.007 & 0 \\ -0.02 & 0.011 & 0 & -0.013 \end{bmatrix}^T,$$

$$\Delta C_p(k) = \begin{bmatrix} 0.01 & 0.023 \sin(k) & 0 & -0.014 \\ -0.012 & 0 & -0.023 \cos(k) & 0.011 \end{bmatrix}. \tag{4.2}$$

In Eq. (4.1), $x_p = [C_{sc} \ T_{rx} \ C_{rg} \ T_{rg}]^T$ and $u = [F_a \ F_c]^T$. Here C_{sc} denotes the coke content in the spent catalyst, T_{rx} the reactor bed temperature, C_{rg} the coke content in the regenerated catalyst and T_{rg} is regenerator bed temperature. The manipulated variables

are F_a (air flow rate) and F_c (catalyst circulation rate). Moreover,

$$E\{v(k)\} = E\{e(k)\} = 0, \quad E\{v(k)v^T(k)\} \equiv R_1, \quad E\{e(k)e^T(k)\} \equiv R_2, \tag{4.3a}$$

where

$$R_1 \in S_1 = \left\{ R_1 : R_{10} - \begin{bmatrix} 0.15 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0 & 0.15 \end{bmatrix} \leq R_1 \leq R_{10} + \begin{bmatrix} 0.15 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0 & 0.15 \end{bmatrix} \right\}, \tag{4.3b}$$

$$R_2 \in S_2 = \left\{ R_2 : R_{20} - \begin{bmatrix} 0.23 & 0 \\ 0 & 0.23 \end{bmatrix} \leq R_2 \leq R_{20} + \begin{bmatrix} 0.23 & 0 \\ 0 & 0.23 \end{bmatrix} \right\}, \tag{4.3c}$$

and

$$R_{10} = \begin{bmatrix} 0.22 & 0 & 0 & 0 \\ 0 & 0.27 & 0 & 0 \\ 0 & 0 & 0.21 & 0 \\ 0 & 0 & 0 & 0.02 \end{bmatrix}, \quad R_{20} = \begin{bmatrix} 0.35 & 0 \\ 0 & 0.43 \end{bmatrix}. \tag{4.3d}$$

Meanwhile, the weighting matrices are assumed here to be

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}. \tag{4.4}$$

It is desired to design a robust LQG optimal controller to stabilize the uncertain stochastic time-delay system (4.1).

Solution: By defining a new state vector

$$\bar{x}(k) = [x_p^T(k-1) \ x_p^T(k)]^T, \tag{4.5}$$

the system (4.1) is transformed into the following system:

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \Delta\bar{A}(k)\bar{x}(k) + \bar{B}u(k) + \Delta\bar{B}(k)u(k) + \bar{v}(k), \tag{4.6a}$$

$$y(k) = \bar{C}\bar{x}(k) + \Delta\bar{C}(k)\bar{x}(k) + e(k) \tag{4.6b}$$

where

$$\bar{A} = \begin{bmatrix} 0 & I \\ A_1 & A_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.0014 & 0.0498 & 0.0085 & -0.0022 & 0.07362 & 0.1148 & 0.01044 & 0.4390 \\ 0.0166 & 0.0058 & 0.0043 & -0.0011 & 0.03940 & 0.1492 & 0.00894 & 0.8111 \\ 0.0027 & 0.7024 & 0.0017 & -5.0004 & 0.27940 & -0.7511 & -0.003253 & -6.127 \\ 0.0005 & 0.0034 & 0.0202 & -0.8006 & 0.04545 & 0.1559 & 0.009798 & 0.8348 \end{bmatrix}, \tag{4.7a}$$

$$\Delta \bar{A}(k) = \begin{bmatrix} 0 & 0 \\ \Delta A_1(k) & \Delta A_0(k) \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0 \\ B_p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.6030 & 0.2505 & 7.2139 & 0.1217 \\ 0 & 0 & 0 & 0 & -0.2495 & 0.0168 & -5.785 & 0.01157 \end{bmatrix}^T, \tag{4.7b}$$

$$\Delta \bar{B}(k) = \begin{bmatrix} 0 \\ \Delta B_p(k) \end{bmatrix}, \quad \bar{v}(k) = Gv(k), \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, \tag{4.7c}$$

$$\bar{C} = [0 \ C_p] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Delta \bar{C}(k) = [0 \ \Delta C_p(k)]. \tag{4.7d}$$

The pairs (\bar{A}, \bar{B}) , (\bar{A}, \bar{C}) are definitely observed from the fact of Lemmas 2.1 and 2.2 to be controllable and observable, respectively.

Based on Lemma 3.1, the minimax controller is described as follows:

$$u(k) = -G_f \hat{x}(k), \tag{4.8a}$$

$$\hat{x}(k + 1) = \bar{A}\hat{x}(k) + \bar{B}u(k) + \bar{F}[y(k) - \bar{C}\hat{x}(k)] \tag{4.8b}$$

where

$$G_f = \begin{bmatrix} 0.0017 & 0.0545 & 0.0065 & -0.5871 & 0.0402 & 0.0024 & 0.0092 & -0.2879 \\ 0.0011 & -0.0453 & 0.0061 & 0.1219 & -0.0008 & 0.1053 & 0.0081 & 0.6084 \end{bmatrix} \tag{4.8c}$$

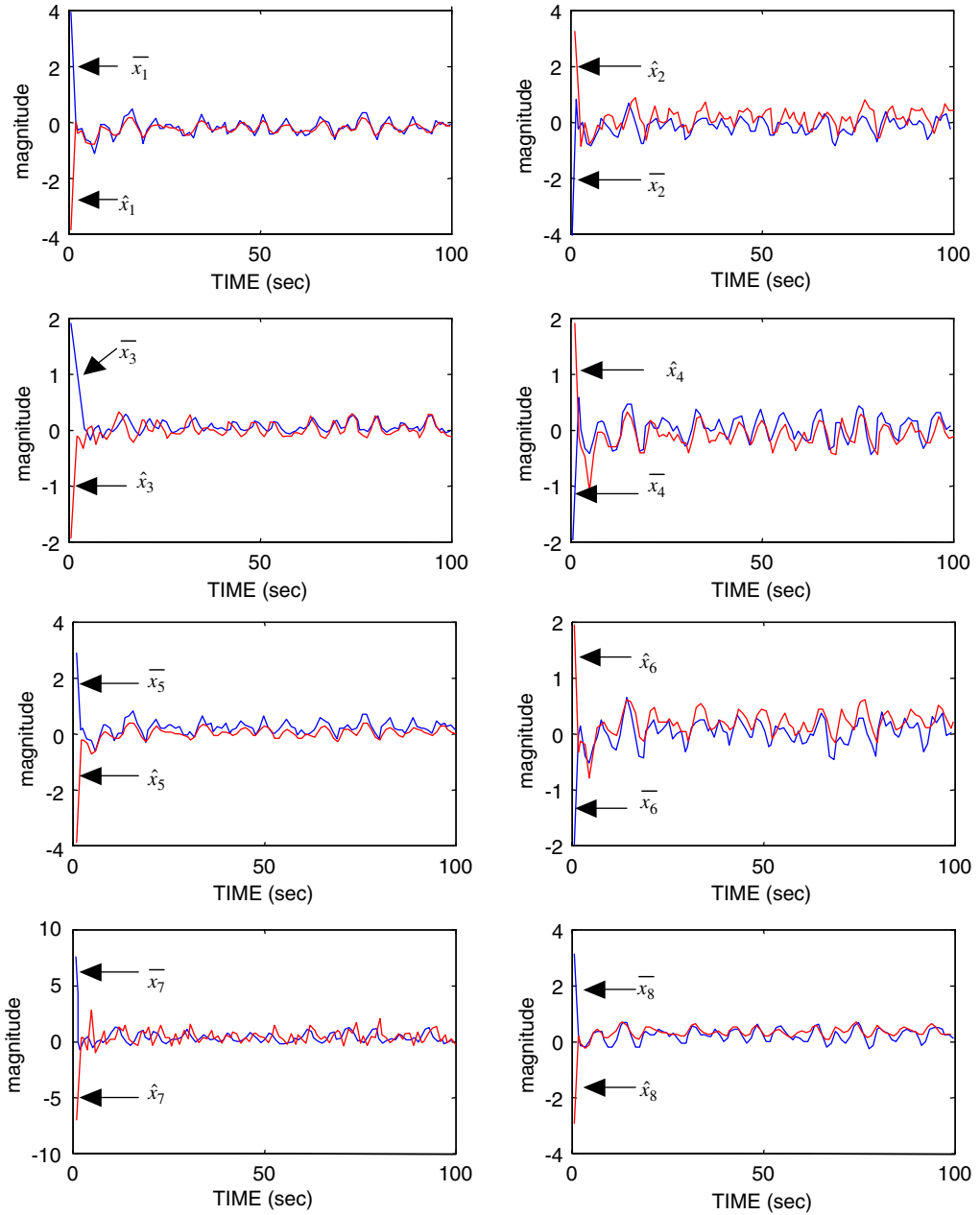


Fig. 1. Simulation of time response for true state $\bar{x} = [\bar{x}_1 \ \bar{x}_2 \ \bar{x}_3 \ \bar{x}_4 \ \bar{x}_5 \ \bar{x}_6 \ \bar{x}_7 \ \bar{x}_8]^T$ with initial condition $[4 \ -4 \ 2 \ -2 \ 3 \ -2 \ 8 \ 3]^T$ and state estimate $\hat{x} = [\hat{x}_1 \ \hat{x}_2 \ \hat{x}_3 \ \hat{x}_4 \ \hat{x}_5 \ \hat{x}_6 \ \hat{x}_7 \ \hat{x}_8]^T$ with initial condition $[-4 \ 3 \ -2 \ 2 \ -4 \ 2 \ -7 \ -3]^T$.

and

$$\bar{F} = \begin{bmatrix} -0.0451 & -0.0474 & 0.9732 & -0.0288 & -0.0121 & -0.0237 & 0.6607 & -0.0531 \\ 0.0356 & 0.4535 & -0.0540 & 0.1063 & 0.1020 & 0.1554 & -1.2869 & 0.1098 \end{bmatrix}^T \tag{4.8d}$$

The condition number $M = 2.7752^3$ and $r = 0.2926$ obtained here through substituting G_f and \bar{F} into Eq. (3.10b) to obtain the matrix A and then applying the inequality (3.15). In accordance with Eqs. (2.2) and (4.2), we have $\sigma = 0.023$, $\eta_1 = 0.015$, $\delta = 0.032$ and $\rho = 0.03$. Substituting $\|G_f\|$ and $\|\bar{F}\|$ into (3.17) yields $h = (M/r)[2(\sigma + \eta_1) + 4\delta\|G_f\| + \rho\|\bar{F}\|] = 2.1715$ and then $r(1 + h) = 0.928 < 1$.

The robust stability condition (3.16) is hence satisfied. Namely, the minimax controller in Eq. (4.8) is a robust LQG optimal controller in the presence of parametric uncertainties and uncertain noise covariances. The result of simulation is shown in Fig. 1.

5. Conclusion

In this paper, the LQG optimal control problem is considered and Minimax theory and Bellman–Gronwall lemma are employed to derive a robust criterion which guarantees the asymptotic stability of the discrete time-delay systems under both parametric uncertainties and uncertain noise covariances. On the basis of this criterion, a robust minimax controller composed of the Kalman filter and the optimal regulator is synthesized to stabilize the uncertain stochastic systems. Designed procedures are finally elaborated with an illustrative example.

Acknowledgments

The authors wish to express sincere gratitude to Prof. M.F. Aburdene and Prof. J. English for their help and the anonymous reviewers for their constructive comments and helpful suggestions, which lead to substantial improvements of this paper.

Appendix

The solution $x(k)$ to Eq. (3.9) is expressed as

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} \Delta A(j)x(j) + \sum_{j=0}^{k-1} A^{k-j-1} Hn(j). \tag{A.1}$$

Taking norms on both sides of Eq. (A.1), we obtain

$$\begin{aligned} \|x(k)\| &= \left\| A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} \Delta A(j)x(j) + \sum_{j=0}^{k-1} A^{k-j-1} Hn(j) \right\| \\ &\leq \|A^k\| \|x(0)\| + \sum_{j=0}^{k-1} \|A^{k-j-1}\| \|\Delta A(j)\| \|x(j)\| + \sum_{j=0}^{k-1} \|A^{k-j-1}\| \|H\| \|n(j)\|. \end{aligned} \tag{A.2}$$

³Euclidean norm case is considered in this example.

Similarly, taking norms on both sides of Eqs. (3.10c)–(3.10e), we have the following inequalities:

$$\begin{aligned}
 \|\Delta A(k)\| &\leq \|\Delta \bar{A}(k) - \Delta \bar{B}(k)G_f\| + \|\Delta \bar{A}(k) - \Delta \bar{B}(k)G_f - \bar{F}\Delta \bar{C}(k)\| + 2\|\Delta \bar{B}(k)\| \|G_f\| \\
 &\leq \|\Delta \bar{A}(k)\| + \|\Delta \bar{B}(k)\| \|G_f\| + \|\Delta \bar{A}(k)\| + \|\Delta \bar{B}(k)\| \|G_f\| + \|\bar{F}\| \|\Delta \bar{C}(k)\| \\
 &\quad + 2\|\Delta \bar{B}(k)\| \|G_f\| \\
 &\leq 2\|\Delta \bar{A}(k)\| + 4\|\Delta \bar{B}(k)\| \|G_f\| + \|\bar{F}\| \|\Delta \bar{C}(k)\| \\
 &\leq 2 \sum_{i=0}^m \|\Delta A_i(k)\| + 4\|\Delta B_p(k)\| \|G_f\| + \|\bar{F}\| \|\Delta C_p(k)\| \\
 &\leq 2 \left(\sigma + \sum_{i=1}^m \eta_i \right) + 4\delta \|G_f\| + \rho \|\bar{F}\|, \tag{A.3}
 \end{aligned}$$

$$\|H\| \leq 2 + \|\bar{F}\|, \tag{A.4}$$

$$\|n(k)\| \leq \|\bar{v}(k)\| + \|e(k)\| \leq \|v(k)\| + \|e(k)\| \leq [\text{tr}(R_{10} + \varepsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2}. \tag{A.5}$$

Substituting Eq. (3.15) and inequalities (A.3)–(A.5) into Eq. (A.2) yields

$$\begin{aligned}
 \|x(k)\| &\leq Mr^k \|x(0)\| + \sum_{j=0}^{k-1} Mr^{k-j-1} \left[2 \left(\sigma + \sum_{i=1}^m \eta_i \right) + 4\delta \|G_f\| + \rho \|\bar{F}\| \right] \|x(j)\| \\
 &\quad + \sum_{j=0}^{k-1} Mr^{k-j-1} (2 + \|\bar{F}\|) \{ [\text{tr}(R_{10} + \varepsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2} \}. \tag{A.6}
 \end{aligned}$$

Multiplying both sides of Eq. (A.6) by r^{-k} leads to

$$\begin{aligned}
 \|x(k)\| r^{-k} &\leq M \|x(0)\| + \sum_{j=0}^{k-1} Mr^{-j-1} \left[2 \left(\sigma + \sum_{i=1}^m \eta_i \right) + 4\delta \|G_f\| + \rho \|\bar{F}\| \right] \|x(j)\| \\
 &\quad + \sum_{j=0}^{k-1} Mr^{-j-1} (2 + \|\bar{F}\|) \{ [\text{tr}(R_{10} + \varepsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2} \}. \tag{A.7}
 \end{aligned}$$

Inequality (A.7) can be changed to

$$\begin{aligned}
 \|x(k)\| r^{-k} &\leq M \|x(0)\| + M \frac{1 - r^{-k}}{r - 1} (2 + \|\bar{F}\|) \{ [\text{tr}(R_{10} + \varepsilon_1 I)]^{1/2} + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2} \} \\
 &\quad + \sum_{j=0}^{k-1} Mr^{-1} \left[2 \left(\sigma + \sum_{i=1}^m \eta_i \right) + 4\delta \|G_f\| + \rho \|\bar{F}\| \right] \|x(j)\| r^{-j}. \tag{A.8}
 \end{aligned}$$

Applying Lemma 3.2 to Eq. (A.8), we obtain the following inequality:

$$\begin{aligned} \|x(k)\|r^{-k} &\leq M\|x(0)\| + M\frac{1-r^{-k}}{r-1}(2 + \|\bar{F}\|)\{\text{tr}(R_{10} + \varepsilon_1 I)\}^{1/2} + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2} \\ &\quad + M\|x(0)\|[(1+h)^k - 1] + hM(2 + \|\bar{F}\|)\{\text{tr}(R_{10} + \varepsilon_1 I)\}^{1/2} \\ &\quad + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2}\left\{\frac{(1+h)^k - 1}{h(r-1)} - \frac{r(1+h)^k - r^{1-k}}{(r-1)[r(1+h) - 1]}\right\}, \end{aligned} \tag{A.9}$$

where

$$h = \frac{M}{r} \left[2 \left(\sigma + \sum_{i=1}^m \eta_i \right) + 4\delta\|G_f\| + \rho\|\bar{F}\| \right].$$

Multiplying r^k to both sides of Eq. (A.9), we get the following result:

$$\begin{aligned} \|x(k)\| &\leq Mr^k\|x(0)\| + M\frac{r^k - 1}{r-1}(2 + \|\bar{F}\|)\{\text{tr}(R_{10} + \varepsilon_1 I)\}^{1/2} + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2} \\ &\quad + M\|x(0)\|[r(1+h)]^k - Mr^k\|x(0)\| + M(2 + \|\bar{F}\|)\{\text{tr}(R_{10} + \varepsilon_1 I)\}^{1/2} \\ &\quad + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2}\left\{\frac{[r(1+h)]^k - r^k}{r-1} - M(2 + \|\bar{F}\|)\{\text{tr}(R_{10} + \varepsilon_1 I)\}^{1/2}\right. \\ &\quad \left. + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2}\frac{h\{[r(1+h)]^k r - r\}}{(r-1)[r(1+h) - 1]}\right\}. \end{aligned} \tag{A.10}$$

Since $0 \leq r < 1$ and $r(1+h) < 1$, $\|x(k)\|$ will approach to a certain value

$$\begin{aligned} &\frac{M(2 + \|\bar{F}\|)}{1-r} \left\{ 1 + \frac{M[2(\sigma + \sum_{i=1}^m \eta_i) + 4\delta\|G_f\| + \rho\|\bar{F}\|]}{r + M[2(\sigma + \sum_{i=1}^m \eta_i) + 4\delta\|G_f\| + \rho\|\bar{F}\|] - 1} \right\} \{\text{tr}(R_{10} + \varepsilon_1 I)\}^{1/2} \\ &\quad + [\text{tr}(R_{20} + \varepsilon_2 I)]^{1/2}, \end{aligned} \tag{A.11}$$

as $k \rightarrow \infty$. Thus, system (3.9) is asymptotically stable.

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