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Bayesian inference for Rayleigh distribution under progressive censored sample

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SUMMARY

It is often the case that some information is available on the parameter of failure time distributions from previous experiments or analyses of failure time data. The Bayesian approach provides the methodology for incorporation of previous information with the current data. In this paper, given a progressively type II censored sample from a Rayleigh distribution, Bayesian estimators and credible intervals are obtained for the parameter and reliability function. We also derive the Bayes predictive estimator and highest posterior density prediction interval for future observations. Two numerical examples are presented for illustration and some simulation study and comparisons are performed. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The Rayleigh distribution is a special case of the Weibull distribution and has wide applications, such as, in communication engineering [1, 2], in life testing of electrovacum devices [3], etc. The probability density function and reliability function of the Rayleigh distribution, respectively, are given by

$$f(x|\theta) = \frac{x}{\theta^2}\exp\left\{-\frac{x^2}{2\theta^2}\right\}, \quad x > 0 \tag{1}$$

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and

$$R(x|\theta) = \exp\left\{-\frac{x^2}{2\theta^2}\right\}, \quad x > 0$$

where $\theta > 0$ is the parameter. An important characteristic of the Rayleigh distribution is that its failure rate is an increasing linear function of time. This means that when the failure times are distributed according to the Rayleigh law, an intense aging of the equipment takes place. Then as time increases the reliability function decreases at a much higher rate than in the case of exponential distribution (see Reference [3]).

Inferences for the Rayleigh distribution were discussed by several authors. Harter and Moore [4] derived an explicit form for the maximum likelihood estimator of θ based on type II censored data. Dyer and Whisenand [1, 2] provided the best linear unbiased estimator of θ based on complete sample, censored sample and selected order statistics. Doubly censored samples were considered, among other authors, by Lalitha and Mishra [5], and Kong and Fei [6]. Bayesian estimation and prediction problems are also important and have been investigated, among others, by Howlader and Hossain [7], and Fernández [8]. In addition, AL-Hussaini and Ahmad [9, 10] studied Bayesian predictive densities and prediction bounds of generalized order statistics and future records.

In this paper, Bayes estimators and highest posterior density credible intervals for parameter θ and reliability function $R(x|\theta)$ of the Rayleigh distribution, as well as the Bayes predictive estimator and prediction interval for future observations, are obtained based on progressively type II censored samples. The rest of this paper is organized as follows. In Section 2, a brief description of progressive type II censoring is given. In Section 3, the prior density is given and the posterior densities are derived. Section 4 describes how to obtain Bayes estimators and highest posterior density credible intervals. Section 5 is concerned with Bayesian prediction problems from Rayleigh data in the progressive type II censoring case. Section 6 applies the proposed methods to two numerical examples and some simulation studies.

2. PROGRESSIVELY TYPE II CENSORED DATA

A typical experiment in life testing consists of a sample of n units on appropriate devices and subjecting the units to operation under specified conditions until failure of the unit is obtained. In many studies, experiments often must terminate before all units on test have failed. In such cases, exact lifetimes are known for only a portion of the units under study and the remainder of the lifetimes are known only to exceed certain values. Such data are called censored. One of the most common censoring schemes is type II censoring.

In a type II censoring, a total of n units is put on a life test, but instead of continuing until all n units have failed, the life test is stopped at the time of the mth ($1 \leqslant m \leqslant n$) unit failure. If an experimenter desires to remove live units at points other than the final termination point of a life test, the type II censoring scheme will not be of use to the experimenter. The disadvantage of type II censoring is that it does not allow for units to be removed from the life test before the final termination point. However, this allowance will be desirable, as in the case of accidental breakage of test units, in which the loss of units at points other than the termination point may be unavoidable. Intermediate removal may also be desirable when a compromise is sought between time consumption and the observation of some extreme values. These lead us into the

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area of progressive type II censoring. Cohen [11] also mentioned that one of the primary goals of progressive censoring is to save some live units for other tests, which is particularly useful when the units being tested are very expensive. The interested readers may refer to the book by Balakrishnan and Aggarwala [12, Chapter 1] for additional discussions of the need for progressive censoring.

Consider an experiment in which n independent units are placed on a test at time zero, and the failure times of these units are recorded. Suppose that m failures are going to be observed. When the first failure is observed, r_1 of the surviving units are randomly selected and removed. At the second observed failure, r_2 of the surviving units are randomly selected and removed. This experiment stops at the time when the mth failure is observed and the remaining $r_m = n - r_1 - r_2 - \cdots - r_{m-1} - m$ surviving units are all removed. The m ordered observed failure times are called progressively type II censored order statistics of size m from a sample of size n with censoring scheme $(r_1, \ldots, r_m)$.

Suppose that the failure times of the n independent units originally on a test are identically distributed with probability density function $f(x)$ and cumulative distribution function $F(x)$. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be a progressively type II censored sample from $f(x)$ with censoring scheme $(r_1, \ldots, r_m)$. The joint probability density function of all m progressively type II censored order statistics is given by Balakrishnan and Aggarwala [12]

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_{1:m:n}, \ldots, x_{m:m:n}) = c\prod_{i=1}^{m} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{r_i} \tag{2}$$

where

$$c = n(n - r_1 - 1)(n - r_1 - r_2 - 2)\cdots(n - r_1 - \cdots - r_{m-1} - m + 1)$$

When data are obtained by progressive censoring, inference problems for various distributions have been studied by several authors including Wong [13], Yuen and Tse [14], Balasooriya and Saw [15], Balakrishnan and Lin [16], and Wu [17].

3. PRIOR AND POSTERIOR DISTRIBUTIONS

Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be a progressively type II censored sample from a Rayleigh distribution with parameter θ. According to (1) and (2), the likelihood function is given by

$$L(\theta) \propto \frac{1}{\theta^{2m}} \exp\left\{-\frac{1}{2\theta^2}\sum_{i=1}^{m}(r_i + 1)x_{i:m:n}^2\right\} \tag{3}$$

It is easy to obtain the maximum likelihood estimator of θ to be

$$\hat{\theta} = \sqrt{\frac{1}{2m}\sum\nolimits_{i=1}^{m}(r_i + 1)X_{i:m:n}^2} \tag{4}$$

By the invariance property of the maximum likelihood estimator, we can obtain the maximum likelihood estimator of reliability function $R(t|\theta)$ to be

$$\hat{R}_t = \exp\left\{-\frac{t^2}{2\hat{\theta}^2}\right\} \tag{5}$$

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In the Bayesian approach, θ is considered a random variable having some specified distribution. In this paper, we consider conjugate prior distribution of the form

$$\Pi(\theta) = \frac{a^b}{\Gamma(b)2^{b-1}}\theta^{-2b-1}\exp\left\{-\frac{a}{2\theta^2}\right\}, \quad \theta > 0 \tag{6}$$

where $a > 0$ and $b > 0$. This density is known as the square-root inverted-gamma distribution and has expectation and variance, respectively,

$$E(\theta) = \sqrt{\frac{a}{2}}\frac{\Gamma\left(b-\frac{1}{2}\right)}{\Gamma(b)}, \quad b > \frac{1}{2}$$

and

$$\mathrm{Var}(\theta) = \frac{a}{2(b-1)} - \frac{a}{2}\left[\frac{\Gamma\left(b-\frac{1}{2}\right)}{\Gamma(b)}\right]^2, \quad b > 1$$

Most often, the parameters a and b are obtained from the past history. Note that the choice of a square-root inverted-gamma prior for θ is equivalent to selecting a gamma prior for $\lambda = 1/\theta^2$. Waller *et al.* [18] presented a method by which engineering experiences, judgments, and beliefs can be used to assign values to the parameters of a gamma prior distribution. This method requires an engineer to provide two distinct percentiles which are used to determine values for the parameters. We can adopt this method and ask that the engineer provides two percentiles $\theta_1 < \theta_2$ such that $P(\Theta < \theta_1) = p_1$ and $P(\Theta < \theta_2) = p_2$. The simultaneous solution of the above two equations will select the values for a and b which determine the square-root inverted-gamma prior that summarizes the information of engineer.

It follows, from (3) and (6), that the posterior distribution of θ is given by

$$\Pi(\theta|\mathbf{x}) = \frac{\left[a + \sum_{i=1}^m (r_i+1)x_{i:m:n}^2\right]^{b+m}}{2^{b+m-1}\Gamma(b+m)}\theta^{-2(b+m)-1}\exp\left\{-\frac{1}{2\theta^2}\left[a + \sum_{i=1}^m (r_i+1)x_{i:m:n}^2\right]\right\} \tag{7}$$

for $\theta > 0$, zero elsewhere. Substituting $\theta^2 = -t^2/(2\log s)$ into (7), we obtain the posterior probability density function of $s = R(t|\theta)$ as

$$\Pi(s|\mathbf{x}) = \frac{1}{\Gamma(b+m)}\left[\frac{a + \sum_{i=1}^m (r_i+1)x_{i:m:n}^2}{t^2}\right]^{b+m}(-\log s)^{b+m-1}s^{\frac{a+\sum_{i=1}^m (r_i+1)x_{i:m:n}^2}{t^2}-1} \tag{8}$$

for $0 < s < 1$, zero elsewhere.

4. BAYESIAN ESTIMATION

4.1. Bayes estimators

In order to derive Bayes estimators we must first specify a loss function which represents the cost involved in using the estimate $\tilde{\theta}$ when the true value is θ. The loss function is a non-negative function that is taken to be a function of the distance between the estimate and the true value. It generally increases as the distance increases. A commonly used loss function is squared error

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loss, $L(\theta, \tilde{\theta}) = (\tilde{\theta} - \theta)^2$. The squared error loss gives more penalty for large discrepancies. Under squared error loss, the Bayes estimator of θ is the posterior mean

$$\tilde{\theta} = E(\theta|\mathbf{X}) = \sqrt{\frac{1}{2}\left[a + \sum_{i=1}^{m} (r_i + 1)X_{i:m:n}^2\right]}\,\frac{\Gamma(b + m - \frac{1}{2})}{\Gamma(b + m)} \tag{9}$$

Another problem of interest is that of estimating reliability function $R(t|\theta)$ with fixed $t > 0$. For squared error loss, the Bayes estimator of $R(t|\theta)$ is given by

$$\tilde{R}_t = E[R(t|\theta)|\mathbf{X}] = \left[\frac{a + \sum_{i=1}^{m} (r_i + 1)X_{i:m:n}^2}{a + \sum_{i=1}^{m} (r_i + 1)X_{i:m:n}^2 + t^2}\right]^{b+m} \tag{10}$$

The highest posterior density (HPD) estimation is another method in popular use from the Bayesian perspective. This parameter estimation is based on the maximum likelihood principle and, hence the mode of posterior density will be the HPD estimator. Since the posterior density (7) is unimodal, we can obtain the HPD estimator of θ as

$$\theta^* = \sqrt{\frac{a + \sum_{i=1}^{m} (r_i + 1)X_{i:m:n}^2}{2(b + m) + 1}}$$

From (8), the HPD estimator of $R(t|\theta)$ is

$$R_t^* = \exp\left\{-\frac{(b + m - 1)t^2}{a + \sum_{i=1}^{m} (r_i + 1)X_{i:m:n}^2 - t^2}\right\}$$

4.2. HPD credible intervals

A $100(1-\alpha)\%$ Bayesian credible interval for the parameter θ is any interval (ℓ, u) satisfying

$$P(\ell < \theta < u|\mathbf{x}) = 1 - \alpha \tag{11}$$

This two-sided interval (ℓ, u) can be chosen in different ways. The most frequent use is the HPD credible interval. Berger [19, p. 140] defined that a $100(1-\alpha)\%$ HPD credible set for θ, is the subset CR_B of $(0, \infty)$ of the form $\mathrm{CR}_B = \{\theta; \pi(\theta|\mathbf{x}) > c_\alpha\}$, where c_α is the constant such that $P(\theta \in \mathrm{CR}_B \mid \mathbf{x}) = 1 - \alpha$. Hence, a $100(1-\alpha)\%$ HPD credible interval chooses (ℓ, u) to consist of all values of θ with $\Pi(\theta|\mathbf{x}) > c_\alpha$, where c_α is chosen such that (11) holds.

Due to the unimodality of (7), the $100(1-\alpha)\%$ HPD credible interval (ℓ, u) for θ must satisfy the following two equations:

$$\int_{\ell}^{u} \Pi(\theta|\mathbf{x})\,\mathrm{d}\theta = 1 - \alpha \tag{12}$$

and

$$\Pi(\ell|\mathbf{x}) = \Pi(u|\mathbf{x}) \tag{13}$$

From (12) and (13) and after some algebraic computation, the $100(1-\alpha)\%$ HPD credible interval (ℓ, u) for θ is given by the simultaneous solution of the equations

$$\Gamma_{\mathrm{I}}(v_1, b + m) - \Gamma_{\mathrm{I}}(v_2, b + m) = 1 - \alpha$$

and

$$\left(\frac{u}{\ell}\right)^{2(b+m)+1} = \exp\{v_1 - v_2\}$$

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where $v_1 = (1/2\ell^2)[a + \sum_{i=1}^{m} (r_i + 1)x_{i:m:n}^2]$, $v_2 = (1/2u^2)[a + \sum_{i=1}^{m} (r_i + 1)x_{i:m:n}^2]$, and $\Gamma_{\mathrm{I}}(v_i, b + m) = (1/\Gamma(b + m)) \int_0^{v_i} z^{b+m-1} e^{-z}\, \mathrm{d}z$, the incomplete gamma function.

Similarly, the $100(1-\alpha)\%$ HPD credible interval (ℓ_R, u_R) for $R(t|\theta)$ must satisfy

$$\Gamma_{\mathrm{I}}(-\omega \log \ell_R, b + m) - \Gamma_{\mathrm{I}}(-\omega \log u_R, b + m) = 1 - \alpha$$

and

$$\left(\frac{\log u_R}{\log \ell_R}\right)^{b+m-1} = \left(\frac{\ell_R}{u_R}\right)^{\omega-1}$$

where $\omega = [a + \sum_{i=1}^{m} (r_i + 1)x_{i:m:n}^2]/t^2$.

5. PREDICTING FUTURE OBSERVATIONS

It is often of interest to predict the kth failure time in a future sample of size N from the same distribution. A two-sample scheme is used in which the informative sample is a progressively type II censored sample and $Y_{(1)} < \cdots < Y_{(N)}$ are the order statistics of a future sample. Let $Y_{(i)}$, $i = 1, \ldots, N$, be the order statistics in a sample of size N with lifetimes distributed as (1). The probability density function of the kth $(1 \leqslant k \leqslant N)$ order statistic is

$$f(y_{(k)}|\theta) = \frac{N!}{(k-1)!(N-k)!} \frac{y_{(k)}}{\theta^2} \left(1 - \exp\left\{-\frac{y_{(k)}^2}{2\theta^2}\right\}\right)^{k-1} \exp\left\{-(N-k+1)\frac{y_{(k)}^2}{2\theta^2}\right\} \qquad (14)$$

for $y_{(k)} > 0$, zero elsewhere. By forming the product of (7) and (14), and integrating out θ over the set $\{\theta; 0 < \theta < \infty\}$, the predictive distribution of $Y_{(k)}$, given $\mathbf{X}$, is

$$\begin{aligned} f(y_{(k)}|\mathbf{x}) &= \int_0^\infty f(y_{(k)}|\theta)\pi(\theta|\mathbf{x})\, \mathrm{d}\theta \\ &= \frac{2(N!)(b+m)}{(k-1)!(N-k)!} y_{(k)} \left[a + \sum_{i=1}^{m} (r_i + 1)x_{i:m:n}^2\right]^{b+m} \\ &\quad \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left[a + \sum_{i=1}^{m} (r_i + 1)x_{i:m:n}^2 + (N-k+j+1)y_{(k)}^2\right]^{-(b+m+1)} \end{aligned}$$

for $y_{(k)} > 0$, zero elsewhere. Under squared error loss, the Bayes predictive estimator of $Y_{(k)}$ is the expectation of the predictive distribution, that is

$$\tilde{Y}_{(k)} = E(Y_{(k)}|\mathbf{X}) = \frac{N!\sqrt{\frac{\pi}{2}}}{(k-1)!(N-k)!} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (N-k+j+1)^{-3/2}\tilde{\theta}$$

where $\tilde{\theta}$ is the Bayes estimator of θ given by (9).

The $100(1-\alpha)\%$ HPD prediction interval (ℓ_k, u_k) for $Y_{(k)}$ should simultaneously satisfy $\int_{\ell_k}^{u_k} f(y_{(k)}|\mathbf{x})\, \mathrm{d}y_{(k)} = 1 - \alpha$ and $f(\ell_k|\mathbf{x}) = f(u_k|\mathbf{x})$. After some algebraic simplification, the

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$100(1-\alpha)\%$ HPD prediction interval (ℓ_k, u_k) for the future kth order statistic satisfies

$$1-\alpha = \frac{N!}{(k-1)!(N-k)!}\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2\right]^{b+m}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\frac{1}{N-k+j+1}$$
$$\left\{\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2+(N-k+j+1)\ell_k^2\right]^{-(b+m)}\right.$$
$$\left.-\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2+(N-k+j+1)u_k^2\right]^{-(b+m)}\right\}$$

and

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}u_k\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2+(N-k+j+1)u_k^2\right]^{-(b+m+1)}$$
$$=\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\ell_k\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2+(N-k+j+1)\ell_k^2\right]^{-(b+m+1)}$$

6. NUMERICAL EXAMPLES AND SIMULATION STUDY

We apply the Bayesian results to two data sets and also carry out simulations to examine the performance of the proposed method.

6.1. Illustrative examples

Example 1 (Simulated data)

Consider a progressively type II censored sample of size $m = 10$ from a sample of size $n = 20$ with censoring scheme $\mathbf{r} = (2, 0, 0, 2, 0, 0, 0, 2, 0, 4)$ from Rayleigh distribution with parameter θ. It is assumed that the prior distribution of θ is a square-root inverted-gamma distribution given by (6) with $a = 7.0$ and $b = 2.0$. Table I is a progressively type II censored sample with actual generated population values of θ and $R(t = 2|\theta)$ are 1.7238 and 0.5101, respectively. This sample was simulated by using the following algorithm.

Step 1: For the given values of prior parameters (a, b), generate θ from the square-root inverted-gamma distribution.

Table I. Progressively type II censored sample for Example 1.

i	1	2	3	4	5	6	7	8	9	10
x_i	0.1970	0.3029	0.5786	0.9758	1.0066	1.3734	1.4159	1.5209	2.0482	2.2496
r_i	2	0	0	2	0	0	0	2	0	4

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Step 2: Using θ obtained in Step 1, generate a progressively type II censored sample of size m from a sample of size n with censoring scheme $\mathbf{r} = (r_1, \ldots, r_m)$ from Rayleigh distribution according to the algorithm presented in Balakrishnan and Aggarwala [12, pp. 32–33].

From (4) and (5), we obtain the maximum likelihood estimates of θ and $R(t=2|\theta)$ to be $\hat{\theta} = 1.4957$ and $\hat{R}_{t=2} = 0.4090$, respectively. From (9) and (10), we determine the Bayes estimates of θ and $R(t=2|\theta)$ to be $\tilde{\theta} = 1.5163$ and $\tilde{R}_{t=2} = 0.4092$. Similarly, we can calculate the HPD estimates of θ and $R(t=2|\theta)$ to be $\theta^* = 1.4386$ and $R^*_{t=2} = 0.3979$. To obtain the 90% HPD credible intervals for θ and $R(t|\theta)$ we need to use the Newton–Raphson method to solve the equations in Section 4.2. The 90% HPD credible intervals for θ and $R(t=2|\theta)$ are (1.069 9, 1.7393) and (0.1860, 0.5261), respectively.

Furthermore, consider a future sample of size $N = 10$ from the same distribution. Using the formula in Section 5, Bayes predictive estimates and the corresponding 90% HPD prediction intervals for the kth, $1 \leqslant k \leqslant 10$, failure times are shown in Table II. It is easy to see that the length of the HPD prediction interval increases as k increases. This implies that the prediction is less precise as a larger order statistic is considered.

Example 2 (Real life data)

We apply the proposed methods to a real data set presented in Lawless [20, p. 228]. The data arose in tests on the endurance of deep groove ball bearings and were originally discussed by Leiblein and Zelen [21]. They are the number of revolutions to failure for each of $n = 23$ ball bearings in the life test. Raqab and Madi [22] indicated that a one parameter Rayleigh distribution is acceptable for these data. For the purposes of illustrating the methods discussed in this article, a progressively type II censored sample was generated from this data set. The progressively censored sample size is $m = 13$. The observations (in hundreds of millions) and removed numbers are reported in Table III.

Raqab and Madi [22] assumed that the prior distribution of θ is a square-root inverted-gamma distribution and chose the prior parameters to be $a = b = 2$. Using the formulae presented in Section 3, the maximum likelihood estimates of θ and $R(t=1|\theta)$ are $\hat{\theta} = 0.6052$ and $\hat{R}_{t=1} = 0.2554$, respectively. From (9) and (10), the Bayes estimates of θ and $R(t=1|\theta)$ are $\tilde{\theta} = 0.6358$ and $\tilde{R}_{t=1} = 0.2870$. Similarly, we can compute the HPD estimates of θ and $R(t=1|\theta)$ to be $\theta^* = 0.6097$ and $R^*_{t=1} = 0.2644$. The 90% HPD credible intervals for θ and $R(t=1|\theta)$ are

Table II. Bayes predictive estimates and HPD prediction intervals for Example 1.

k	$\tilde{Y}_{(k)}$	(l_k, u_k)
1	0.6010	(0.0719, 1.0068)
2	0.9260	(0.3081, 1.3886)
3	1.1924	(0.5086, 1.6865)
4	1.4379	(0.6897, 1.9625)
5	1.6794	(0.8628, 2.2379)
6	1.9293	(1.0361, 2.5281)
7	2.2014	(1.2171, 2.8513)
8	2.5176	(1.4164, 3.2372)
9	2.9255	(1.6522, 3.7554)
10	3.5933	(1.9719, 4.6693)

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Table III. Progressively type II censored sample for Example 2.

i	1	2	3	4	5	6	7
x_i	0.1788	0.2892	0.3300	0.4212	0.4560	0.4848	0.5184
r_i	0	0	3	0	0	2	0
i	8	9	10	11	12	13	
x_i	0.5196	0.6780	0.6864	0.8412	0.9312	1.2792	
r_i	0	2	0	2	1	0	

Table IV. Bayes predictive estimates and HPD prediction intervals for Example 2.

k	$\tilde{Y}_{(k)}$	(l_k, u_k)
1	0.2520	(0.0304, 0.4118)
2	0.3883	(0.1292, 0.5668)
3	0.5000	(0.2131, 0.6873)
4	0.6030	(0.2892, 0.7987)
5	0.7042	(0.3619, 0.9100)
6	0.8090	(0.4348, 1.0273)
7	0.9232	(0.5109, 1.1579)
8	1.0557	(0.5947, 1.3141)
9	1.2268	(0.6936, 1.5244)
10	1.5068	(0.8270, 1.8967)

(0.4534, 0.6989) and (0.0856, 0.3568), respectively. Consider a future sample of size $N = 10$ from the same distribution. The Bayes predictive estimates and the corresponding 90% HPD prediction intervals for the kth, $1 \leqslant k \leqslant 10$, failure times are reported in Table IV. It can be seen that the HPD prediction interval becomes wider when k increases.

6.2. Simulation results

In the following, the maximum likelihood estimates and Bayes estimates of the parameter θ and the $R(t|\theta)$ are compared via Monte Carlo simulation. Using the method given in Section 6.1, the progressively type II censored samples from Rayleigh distribution with parameter θ having square-root inverted-gamma prior density were generated for $(a, b) = (2, 5), (6, 1.5)$, $t = 0.5$, and different combinations of n, m, and censoring schemes $\mathbf{r}$. For simplicity in notation, we will denote these censoring schemes, for example, by $(4 * 0, 15)$ which represents the censoring scheme $\mathbf{r} = (0, 0, 0, 0, 15)$. Table V provides the estimated risks of the maximum likelihood estimators and Bayes estimators. The estimated risks were calculated as the average of squared deviations. All the results were computed over 10 000 simulations. From Table V, we can see that the Bayes estimates are better than their corresponding maximum likelihood estimates for the considered cases. However, more investigations are needed to see the robustness of the choice of the prior.

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Table V. Estimated risks of the maximum likelihood estimates (MLE) and Bayes estimates.

			Parameter θ		Reliability function $R(t=0.5\|\theta)$	
n	m	Censoring scheme	MLE	Bayes	MLE	Bayes
Prior parameters $(a,b)=(2,5)$						
20	5	$(4*0,15)$	0.0611	0.0268	0.0490	0.0301
		$(15,4*0)$	0.0627	0.0275	0.0500	0.0306
		$(5,5,5,2*0)$	0.0615	0.0275	0.0506	0.0314
		$(3,3,3,3,3)$	0.0620	0.0274	0.0504	0.0309
		$(0,15,3*0)$	0.0604	0.0266	0.0492	0.0301
	10	$(9*0,10)$	0.0538	0.0328	0.0515	0.0382
		$(10,9*0)$	0.0524	0.0319	0.0494	0.0364
		$(4*0,5,5,4*0)$	0.0527	0.0321	0.0498	0.0367
	15	$(14*0,5)$	0.0503	0.0352	0.0505	0.0404
		$(5,14*0)$	0.0492	0.0345	0.0498	0.0399
25	10	$(9*0,15)$	0.0524	0.0319	0.0493	0.0363
		$(15,9*0)$	0.0527	0.0321	0.0500	0.0369
	15	$(14*0,10)$	0.0496	0.0348	0.0505	0.0404
		$(10,14*0)$	0.0512	0.0360	0.0522	0.0419
	20	$(19*0,5)$	0.0485	0.0367	0.0505	0.0423
		$(5,19*0)$	0.0488	0.0370	0.0513	0.0430
50	20	$(19*0,30)$	0.0489	0.0370	0.0514	0.0431
		$(30,19*0)$	0.0484	0.0365	0.0498	0.0417
	25	$(24*0,25)$	0.0474	0.0377	0.0506	0.0437
		$(25,24*0)$	0.0476	0.0379	0.0514	0.0444
	30	$(29*0,20)$	0.0474	0.0391	0.0511	0.0451
		$(20,29*0)$	0.0472	0.0388	0.0507	0.0447
Prior parameters $(a,b)=(6,1.5)$						
20	5	$(4*0,15)$	1.8892	1.8335	0.0037	0.0016
		$(15,4*0)$	1.6789	1.6215	0.0039	0.0016
		$(5,5,5,2*0)$	1.6653	1.6093	0.0040	0.0016
		$(3,3,3,3,3)$	1.4154	1.3472	0.0040	0.0016
		$(0,15,3*0)$	2.2794	2.2232	0.0039	0.0017
	10	$(9*0,10)$	1.2932	1.2659	0.0018	0.0013
		$(10,9*0)$	1.5875	1.5643	0.0019	0.0013
		$(4*0,5,5,4*0)$	1.7606	1.7408	0.0018	0.0013
	15	$(14*0,5)$	1.5042	1.4897	0.0013	0.0011
		$(5,14*0)$	1.6958	1.6834	0.0013	0.0011
25	10	$(9*0,15)$	1.6254	1.6020	0.0018	0.0013
		$(15,9*0)$	1.5901	1.5684	0.0018	0.0013
	15	$(14*0,10)$	1.4278	1.4134	0.0013	0.0011
		$(10,14*0)$	1.4780	1.4635	0.0013	0.0011
	20	$(19*0,5)$	2.8506	2.8469	0.0011	0.0010
		$(5,19*0)$	1.5277	1.5185	0.0011	0.0010
50	20	$(19*0,30)$	1.4811	1.4713	0.0011	0.0010
		$(30,19*0)$	1.5845	1.5744	0.0011	0.0010
	25	$(24*0,25)$	1.3243	1.3160	0.0010	0.0010
		$(25,24*0)$	1.5043	1.4965	0.0010	0.0010
	30	$(29*0,20)$	2.5239	2.5208	0.0010	0.0009
		$(20,29*0)$	1.4260	1.4195	0.0010	0.0009

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