Transition temperature of a weakly interacting Bose gas

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We report a theoretical study of the transition temperature of a trapped interacting dilute Bose gas. The system is treated like a two-fluid model consisting of a thermal component and a condensate component. Through the calculation of the energy spectra, the origins of various effects on the transition temperature are derived. We found that the interactive shift is affected by both the thermal component and the condensate component. The latter effect, which is about 34% of the former, has never been reported so far. With these two effects, our calculated interactive shift agrees very well with the recent measurement.

I. INTRODUCTION

The trap potential, finite number of atoms and the mutual atomic interactions are the three major differences between the current dilute gas BEC system and the ideal Bose gas. Due to the trap potential, the atomic density becomes nonuniform. The finite number of particles, or the so-called finite-size effect, usually drags the system away from the thermodynamic limit; and the Hamiltonian can no longer be described by the single-particle form because of the mutual interactions. Among the effects of the three new features on statistical mechanical properties, the interactive effect is the most hard one to treat. In a recent measurement [1], it was found that the interactive shift of the transition temperature is $\delta T_c/T_c^0 = \alpha N^{1/6}$, with $\alpha = -0.009$, while the theoretical study gives $\alpha = -0.007$ [2]. The goal of this paper is to provide a theoretical method to treat the interactive effect. We will show that the previous theoretical result takes account only the effect of the thermal component. We further take the effect of condensate component into consideration and obtain a very good agreement with the measurement. We will show that the effect of condensate component of interactive shift is about 34% of the thermal component and thus cannot be neglected. With our method, the interactive shift is treated properly.

We briefly summarize the effects on the statistical mechanical properties arise from the three modifications of current BEC systems to the ideal Bose gas [3]. *First*, the thermodynamic limit transition temperature (T_c^0) of ideal Bose gas is in the two-thirds power of the number density, and the number of condensate atoms below T_c^0 is in the 1.5th power of the temperature. With the trap potential, the corresponding T_c^0 changed into the one-third power of the number of atoms, and number of condensate atoms for temperature below T_c^0 is to the third power of the temperature. The transition temperature is usually defined from $n(0)\lambda_T^3 = \zeta(3/2) \simeq 2.612$ as in the

ideal Bose gas. Because the number density n is nonuniform now, it is approximated by the density at the trap center. This gives

$$T_c^0 = \frac{\hbar\omega}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3} \simeq 0.9405\frac{\hbar\omega}{k_B}N^{1/3}, \qquad (1)$$

where $\omega = (\omega_1\omega_2\omega_3)^{1/3}$ is the geometrical mean of the frequencies from trap potential of the form $V_{\text{ext}}(r) = m(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2)/2$, N is the number of particles, and ζ is the Riemann's zeta function. Another major difference is that the curve for the number of condensate atoms versus temperature around the defined T_c^0 has smooth change instead of an abrupt cutoff like the ideal Bose gas [4,5]. Hence, strictly speaking, BEC of a trapped gas is not a phase transition. *Second*, on the finite-size effect, the number of condensate atoms in experiments goes from hundreds to millions. This number is finite and the thermodynamic limit has never been truly reached. For noninteracting atoms in a trap potential, the fractional change of transition temperature T_c relative to the T_c^0 due to the finite-size effect is [6,7]

$$\frac{\delta T_c^N}{T_c^0} \equiv \frac{T_c - T_c^0}{T_c^0} = -\frac{\zeta(2)}{2[\zeta(3)]^{2/3}}\frac{\bar{\omega}}{\omega}N^{-1/3} \simeq -0.73\frac{\bar{\omega}}{\omega}N^{-1/3}, \qquad (2)$$

where $\bar{\omega} = (\omega_1 + \omega_2 + \omega_3)/3$ is the arithmetic mean of the trap frequencies. This shows that the transition temperature is downward shifted with respect to the T_c^0 and vanishes in the large N limit.

Third, the treatment of interaction effects is never straightforward and has been of much interest. Namely, before the realization of BEC in dilute gas, Lee, Yang, and Huang studied the ideal Bose gas with hard sphere interactions [8]. Bagnato *et al.* studied the power-law trap potential with interaction [9]. The dependence of transition temperature T_c on the atomic interactions was studied by mean-field theory. The shift of transition temperature was calculated by Giorgini, Pitaevskii, and Stringari (denoted as GPS hereafter for convenience) [2,10]. Their results were subsequently

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verified by a number of different approaches [11]. We find that GPS counts in *only* the effect of the thermal component and the effect of the condensate component was neglected. Physically, near T_c, there are condensate atoms with order relative small to the number of thermal atoms. But, because the condensate atoms are localized about the trap center, the effect of mutual interactions among condensate atoms is not totally negligible. Throughout this paper, what we discussed are the ultracold atoms. We follow the convention of Ref. [5] and call the atoms in the ground state as the condensate atoms while the atoms in excited states as the thermal atoms. We treat the condensate and thermal gaseous atoms as *two-fluid model* [12]. The energy spectra of condensate and thermal components are calculated through the Hartree-Fock mean-field approximation. We derived that once we obtain the relative shift of the energy gap between thermal and condensate components with respect to the noninteracting Bose gas, the effects of interaction on the shift of transition temperature can then be determined.

The paper is organized as follows: In Sec. II, we briefly describe the mean-field approximation and calculate the energy spectra to the first order of interaction parameter g. We show the different behaviors of the interacting Bose gas with and without the trap potential. In Sec. III, we derive the shift of transition temperature by the shift of energy gap. We verify that the result of GPS is only the effect of thermal component. So we derive the modification induced by the condensate component. Finally, discussions are given in Sec. VI.

II. THE ENERGY SPECTRA

We will show in Sec. III that under the two-fluid model, the energy gap of the thermal and condensate components determines the shift of the transition temperature. So we discuss here the energy spectra first. In the Hartree-Fock (HF) mean-field approximation, the trapped Bose gas in an external potential $V_{ext}(r)$ at finite temperature can be described by the Gross-Pitaevskii equation (GPE) together with the HF approximation [13],

$$\left(-\frac{\nabla^2}{2m} + V_{ext}(r) - \mu\right)\Phi_c(r) + g[n_c(r) + 2n_T(r)]\Phi_c(r) = 0$$

(3)

and

$$i\frac{\partial\Phi_T(r,t)}{\partial t} = \left(-\frac{\nabla^2}{2m} + V_{ext}(r) - \mu\right)\Phi_T(r,t) + 2gn(r)\Phi_T(r,t).$$

(4)

Equation (3) describes the macroscopic condensate wave function $\Phi_c(r) = \langle\psi(r)\rangle$, here $\langle\cdots\rangle$ means the ensemble average and $\psi(r)$ is the particle field operator. Equation (4) is the fluctuation of the condensate, $\Phi_T(r,t) = \psi(r,t) - \Phi_c(r)$, which gives the elementary excitations of the system. In the coupled equations, μ is the chemical potential related to the number of particles and is not to appear in energy spectra directly. $g = 4\pi\hbar^2 a/m$ is the interaction coupling constant

given by the *s*-wave scattering length a. This is the *two-fluid model* where the condensate gas is a superfluid phase and the thermal gas is a normal fluid. $n_c(r) = |\Phi_c(r)|^2$ is the number density of the condensate. $n(r) = \langle\psi(r)^\dagger\psi(r)\rangle$ is the total particle density, and $n_T(r) = n(r) - n_c(r)$ is the thermally excited particle density. The anomalous density $m_T(r) = \langle\psi(r)\psi(r)\rangle - \Phi_c(r)^2$. For higher temperature, $m_T \ll n_T$; while in the low temperature, n_T and m_T are both negligible. The present mean-field approach is expected to provide correct thermodynamic properties of the system.

A. Limit of no trap potential

We consider first the simplified system without external potential $V_{ext} = 0$. In the limit of noninteracting bosons, $g = 0$, and the energy spectrum of the system is given by

$$\epsilon_p^{ide} = \frac{p^2}{2m},$$

(5)

where $p = |\vec{p}|$ and $\vec{p} = 2\pi\hbar\vec{n}/L$ is the momentum eigenvalue of the single particle, in which $\vec{n}$ is a vector whose components are integers and L is the linear size of the system.

Now take the interaction into consideration, under the *two-fluid model*, we denote the condensate state energy as ϵ_c and the thermal component energy spectrum as ϵ_T. Expand to the lowest order of g from Eq. (3) and Eq. (4), one finds

$$\text{limit of no trap potential} \begin{cases} \epsilon_T = \epsilon_T^{ide} + 2gn, \\ \epsilon_c = \epsilon_c^{ide} + 2gn_T + gn_c, \end{cases}$$

(6)

where $n = n_T + n_c$ is the density function, which is independent of r in the large volume limit. Without mutual atomic interaction, the energy gap is $\Delta_g^{ide} = \epsilon_T^{ide} - \epsilon_c^{ide}$. Equation (6) shows that the shift of energy gap of the interacting gas without trap potential is $\delta\Delta_g^{uni} = gn_c$ [14]. We see that the repulsive interaction will increase the energy gap between the condensate and the thermal components.

B. Harmonic trap

Current BEC experiments are mostly carried out in harmonic trap potentials. With the external potential, both the condensate and thermal atomic density profiles are nonuniform. We discuss the effects on the energy spectra as follows.

1. Effect on the thermal gas

The energy eigenvalues of the noninteracting particles in a harmonic trap $V_{ext}(r) = m(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2)/2$ are

$$\epsilon_{\{n_i\}} = \sum_{i=1}^{3}\left(n_i + \frac{1}{2}\right)\hbar\omega_i \quad (n_i = 0, 1, 2, \dots),$$

(7)

Near transition temperature, the condensate component is localized around the trap center ($r = 0$) and its size is much smaller than the thermal component. So in calculating the energy shift of the thermal component, we can neglect the condensate density n_c. We define an effective thermal density $\bar{n}_T$ to be found and write the energy spectrum as

effect on the thermal gas: $\epsilon_T = \epsilon_T^{ide} + 2g\bar{n}_T.$ (8)

To determine $\bar{n}_T$, consider the phase space Bose-Einstein distribution

$$f(p,r) = \frac{1}{\exp[\epsilon(p,r)/k_BT] - 1},$$ (9)

where $\epsilon(p,r)$ is the semiclassical excitation spectrum. By Eq. (4),

$$\epsilon(p,r) \simeq \epsilon^{ide}(p,r) + 2gn(r) - \mu.$$ (10)

Expand Eq. (9) to the first order of g,

$$f(p,r) = f(p,r)|_{g=0} + g\left(\frac{\partial f(p,r)}{\partial g}\right)_{g=0} = f_0(p,r) - 2gn\frac{\partial f_0(p,r)}{\partial \mu},$$ (11)

where $f_0(p,r)$ is distribution function of the noninteracting Bose gas. Integrate over the momentum variables, we get the modification of $n_T(r)$ due to the interaction,

$$n_T^{int}(r) \simeq n_T - 2gn_T\frac{\partial n_T}{\partial \mu},$$ (12)

where $n_T(r)$ is the thermal density distribution of noninteracting Bose gas. Further integrate over the coordinates of Eq. (12), we get the number of thermal particles,

$$N_T^{int} = \int n_T d^3r - 2g\int n_T\frac{\partial n_T}{\partial \mu}d^3r.$$ (13)

On the other hand, by the Taylor's expansion to the first order of g,

$$N_T^{int} \simeq N_T + g\frac{\partial N_T}{\partial g}.$$ (14)

With our assumption

$$\epsilon = \epsilon_T^{ide} + 2g\bar{n}_T - \mu,$$ (15)

we obtain

$$\frac{\partial N_T}{\partial g} = -2\bar{n}_T\frac{\partial N_T}{\partial \mu} = -2\bar{n}_T\int\frac{\partial n_T}{\partial \mu}d^3r.$$ (16)

Set Eq. (16) into Eq. (14) and compare with Eq. (13), the effective density was derived as

$$\bar{n}_T = \frac{\int d^3r\frac{\partial n_T}{\partial \mu}n_T}{\int d^3r\frac{\partial n_T}{\partial \mu}}.$$ (17)

Next, follow the local density approximation (LDA) [15], the spatial distribution of thermal state n_T can be written as

$$n_T(r) = \frac{1}{\lambda_T^3}g_{3/2}(e^{-[V_{ext}(r)-\mu]/k_BT}),$$ (18)

where $\lambda_T = \hbar(2\pi/mk_BT)^{1/2}$ is the thermal wavelength, and $g_\nu(x) = \Sigma_{n=1}^\infty x^n/n^\nu$ is the Bose-Einstein function. With $\mu = 0$, it is easy to obtain

$$\bar{n}_T = Sn_T(0),$$ (19)

where $n_T(0) = \zeta(3/2)/\lambda_T^3$ is the thermal density $n_T(r)$ at trap center $r=0$, and $S = \Sigma_{n,m=1}^\infty(1/n^{1/2}m^{3/2})[1/(n+m)^{3/2}]/\zeta(3/2)\zeta(2) \simeq 0.281$. Thus, we find that the effective density is 0.281 times of the center density in the trap. Finally we obtain the shift of the thermal spectrum relative to the ideal Bose gas as $\Delta\epsilon_T = 2g\bar{n}_T$ with $\bar{n}_T$ derived above.

2. Effect on the condensate gas

Near T_c, n_c is still very small, we can first estimate the effect of interaction on the condensate by using the eigenfunction of the harmonic oscillator,

$$\phi(r) = \left(\frac{1}{\pi a_{ho}^2}\right)^{3/4}e^{-[x^2/(2a_1^2)+y^2/(2a_2^2)+z^2/(2a_3^2)]},$$ (20)

where the oscillator lengths $a_i(i=1,2,3)$ are given by $a_i = \sqrt{\hbar/m\omega_i}$, the condensate density can be written as $n_c(r) = N_c|\phi(r)|^2$. Considering the length scales in harmonic traps, $\xi_i^T = (2\pi m\omega_i^2/k_BT)^{-1/2}$ is the classical oscillation amplitude of a particle in the trap with energy k_BT. It is the size of $n_T(r)$ in the ith direction. $\xi_i^c = a_i$ is the size of the oscillator ground state in the ith direction. We find that

$$\frac{\xi_i^T}{\xi_i^c} = \left(\frac{k_BT}{2\pi\hbar\omega_i}\right)^{1/2} \sim O(N^{1/6}) \gg 1.$$ (21)

This justifies our previous assumption that the size of the condensate component n_c is negligible in calculating the thermal spectrum shift. Now to the condensate component, the thermal component density can be approximated as $n_T(0)$. Because the condensate component is only localized around the trap center, we can effectively treat the condensate component in a trap V_{ext} as an additional constant $2gn_T(0)$.

Now we use a more sophisticated Gaussian variational calculation [16] to obtain the shift of condensate energy. We adopt a trial Gaussion form as Eq. (20),

$$\phi(r) = \frac{e^{-[x^2/(2b_1^2)+y^2/(2b_2^2)+z^2/(2b_3^2)]}}{\pi^{3/4}(b_1b_2b_3)^{1/2}},$$ (22)

but with b_i as variational parameters. Substitution of Eq. (22) into Eq. (3) yields the expression of the condensate energy,

$$E = N_c\sum_i \hbar\omega_i\left(\frac{a_i^2}{4b_i^2} + \frac{b_i^2}{4a_i^2}\right) + 2N_cgn_T(0) + \frac{N_c^2g}{2(2\pi)^{3/2}b_1b_2b_3}.$$ (23)

The nonlinear term N_c^2g can be neglected for small number of N_c. We set the b_i equal to a_i, and find that

$$E = N_c\epsilon_c^{ide} + 2N_cgn_T(0) + \frac{N_c^2}{2}\langle 00|v|00\rangle,$$ (24)

where

$$\langle 00|v|00\rangle = g \int dr |\phi(r)|^4 \qquad (25)$$

is the two-particle interaction energy of the condensate state. The energy level is obtained through

$$\epsilon_c = \frac{\partial E}{\partial N_c} = \epsilon_c^{ide} + 2gn_T(0) + g\bar{n}_c, \qquad (26)$$

where $\bar{n}_c = S_c n_c(0)$ is effective condensate density, $S_c = 2^{-3/2} \simeq 0.354$.

With the typical value of $a/a_{ho} \sim 10^{-3}$, we can see that the linear approximation is justified for $N_c \sim 10^3$ which corresponds to $N \sim 10^6$. This is applicable to many current experiments. Under this condition, our modification to the condensate energy is

$$\text{effect on the condensate:} \quad \epsilon_c = \epsilon_c^{ide} + 2gn_T(0) + g\bar{n}_c, \qquad (27)$$

that is, the shift of condensate energy relative to the ideal Bose gas is $\Delta\epsilon_c = 2gn_T(0) + g\bar{n}_c$. Compared to the ideal Bose gas limit, we find that the shift of energy shows different direction in a trapped potential, and the amount is correlated to the shape of the potential. Also the energy shift is dependent on g.

Note that if the temperature is far below the transition temperature, the number of condensate particles grow up to $O(N)$. For sufficiently large number of condensate atoms, $(N_c \gg 10^3)$, and the interaction energy per particle becomes larger compared to $\hbar\omega_i$; then the kinetic energy terms (proportional to a_i^2/b_i^2) can be neglected in Eq. (23). The leading contribution to the energy level is

$$\epsilon_c = 2gn_T(0) + \frac{7}{4}\left(\frac{2}{\pi}\right)^{1/5}\left(\frac{N_c a}{a_{ho}}\right)^{2/5}\hbar\bar{\omega}. \qquad (28)$$

Or, if we use the Thomas-Fermi approximation [16], the result is

$$\epsilon_c = 2gn_T(0) + \frac{15^{2/5}}{2}\left(\frac{N_c a}{a_{ho}}\right)^{2/5}\hbar\bar{\omega}. \qquad (29)$$

We can see that in this region, the shift of the energy level is not linear to g then.

The shift of energy gap obtained is

$$\delta\Delta_g = \Delta\epsilon_T - \Delta\epsilon_c = 2g\bar{n}_T - 2gn_T(0) - g\bar{n}_c$$
$$= -2\alpha gn_T(0) - S_c gn_c(0), \qquad (30)$$

where $\alpha \simeq 0.719$ and $S_c \simeq 0.354$. The energy shift in the thermal component is denoted as $\delta\Delta_g^T = -2\alpha gn_T(0)$, and the shift in the condensate energy is $\delta\Delta_g^c = -S_c gn_c(0)$. For convenience we define the effective condensate energy level

$$E_c^{\text{eff}} = \epsilon_c^{ide} - \delta\Delta_g. \qquad (31)$$

We summarize our results of energy spectra shifts in Fig. 1. In the ideal Bose gas, the gap is schematically denoted as the energy difference of thermal component and condensate component, that is, $\epsilon_T - \epsilon_c$ under the two-fluid model. Figure 1(b) shows the downward shift of gap for interacting Bose
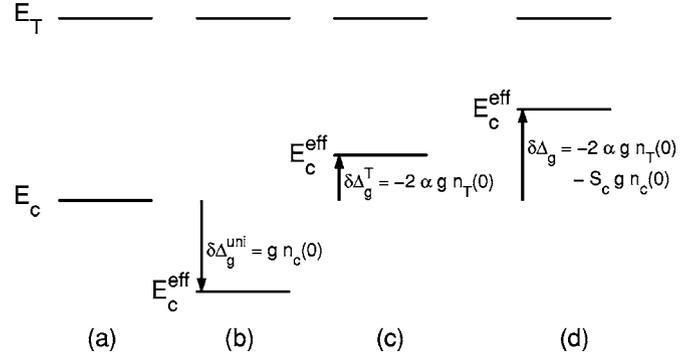

FIG. 1. The shift of the energy gap to the first order of positive g near transition temperature. (a) The ideal Bose gas; (b) the interacting Bose gas without harmonic trap potential; (c) the effect of thermal component for interacting Bose gas under trap potential; (d) the effects including both the thermal and condensate components.

gas with $V_{\text{ext}} = 0$. Figure 1(c) depicts the shift of gap from the thermal gas with interacting Bose gas under harmonic trap potential. In Fig. 1(d), both the effects of thermal component and condensate component to the shift of the gap are included. The plots are for repulsive interactions, for the attractive interactions, the shift directions will be reversed.

III. THE SHIFT OF TRANSITION TEMPERATURE

After the investigation of the shift of energy gap under the two-fluid model, we will discuss its effects on transition temperature. For interacting bosons in a harmonic trap, the total number of thermal particles is given by

$$N_T = \sum_{j=1}^{\infty} (ze^{-2g\bar{n}_T/k_BT})^j \sum_{\{n_i\}} e^{-j\epsilon_{\{n_i\}}/k_BT} = \left(\frac{k_BT}{\hbar\omega}\right)^3 g_3(ze^{-2g\bar{n}_T/k_BT}), \qquad (32)$$

where $z = \exp(\mu/k_BT)$. With $k_BT \gg \hbar\omega_i$, we can apply the Euler-Maclaurin summation formula [6] to obtain the approximate form for the thermal part. The approximation keeps up to $O(k_BT/\hbar\omega) \sim O(N^{1/3})$. During the manipulation, the density of state for the ground state will be zero. So the number of ground state atoms must add in for the total number of particles [17]

$$N = \left(\frac{k_BT}{\hbar\omega}\right)^3 g_3(ze^{-2g\bar{n}_T/k_BT}) + \frac{ze^{-\epsilon_c/k_BT}}{1 - ze^{-\epsilon_c/k_BT}}. \qquad (33)$$

Substitute the effective fugacity $z' = z\exp(-\epsilon_c/k_BT)$ in it, the Bose-Einstein distribution becomes

$$N = \left(\frac{k_BT}{\hbar\omega}\right)^3 g_3(z' e^{(\epsilon_c - 2g\bar{n}_T)/k_BT}) + \frac{z'}{1 - z'}$$
$$\simeq \left(\frac{k_BT}{\hbar\omega}\right)^3 g_3(z') + \frac{\epsilon_c^{ide} - \delta\Delta_g}{\hbar\omega}\left(\frac{k_BT}{\hbar\omega}\right)^2 g_2(z') + \frac{z'}{1 - z'}. \qquad (34)$$

Notice that the phase transition occurs as chemical potential

approaches the ground state energy. That is, $z' \to 1$ for $T \to T_c$. And in our system, we have the conditions ϵ_c, $g\bar{n}_T \ll k_B T$. Equation (34) is then obtained by Taylor expansion of g_3 about z' and use the relationship $\epsilon_c - 2g\bar{n}_T = \epsilon_c^{ide} - \delta\Delta_g$.

By the definition T_c^0 of Eq. (1) and from Eq. (34), we obtain the number of condensate atoms in the interacting Bose gas at the transition temperature T_c^0 as

$$N_c(T = T_c^0) = \frac{\hbar\omega}{E_c^{\text{eff}}}\left(\frac{N}{\zeta(3)}\right)^{1/3}. \tag{35}$$

The result implies that the positive scattering length $g > 0$ will reduce the condensate particle number. In the large N limit, we find $N_c(T = T_c^0) \to O(N^{-1/6})$.

Near transition temperature, $z' \to 1$, using Taylor expansion $g_3(z') \simeq g_3(1) - (1 - z')g_2(1)$ to Eq. (34) gives

$$N = \left(\frac{k_B T}{\hbar\omega}\right)^3 [g_3(1) - (1 - z')g_2(1)]$$
$$+ \frac{E_c^{\text{eff}}}{\hbar\omega}\left(\frac{k_B T}{\hbar\omega}\right)^2 g_2(1) + \frac{z'}{1 - z'}. \tag{36}$$

In the above equation, the last term is the number of condensate atoms N_c and $N_c \ll N$. Near the transition temperature, $(1 - z') \ll 1$. So the leading order terms are used to determine the transition temperature [5,7]:

$$N = \left(\frac{kT_c}{\hbar\omega}\right)^3 g_3(1) + \frac{E_c^{\text{eff}}}{\hbar\omega}\left(\frac{kT_c}{\hbar\omega}\right)^2 g_2(1). \tag{37}$$

This transition temperature is different from T_c^0. Let $T_c = T_c^0 + \delta T_c$, we obtain the fractional correction to the transition temperature. Here both the finite-size effect and the interaction effects are included. This is the extension of Eq. (2),

$$\frac{\delta T_c}{T_c^0} = -\frac{\zeta(2)}{3[\zeta(3)]^{2/3}}\frac{E_c^{\text{eff}}}{\omega}N^{-1/3}. \tag{38}$$

From the meaning of E_c^{eff} defined in Eq. (31), we separate the relative temperature shift into two parts. The one from ϵ_c^{ide} is the so called *finite-size effect* as described in Eq. (2). It vanishes at large N limit. The other is due to the shift of energy gap $\delta\Delta_g$ which is induced by mutual atomic interactions. It enhanced as N increased. And the interactive shift of transition temperature originates from both thermal component and condensate component. We emphasize that *the shift of the transition temperature depends only on E_c^{eff}*.

Also, we have the number of atoms at transition temperature T_c as

$$\left(\frac{k_B T}{\hbar\omega}\right)^3 (1 - z')g_2(1) = \frac{z'}{1 - z'}, \tag{39}$$

as $z' \to 1$, we have $\left(\frac{k_B T}{\hbar\omega}\right)^3 g_2(1) \sim [z'/(1 - z')]^2$, together with Eq. (1) we obtain at T_c,

$$N_c(T = T_c) = \sqrt{\frac{\zeta(2)N}{\zeta(3)}}. \tag{40}$$

It is different from that of $N_c(T = T_c^0)$. Accurancy of all terms

in Eq. (36) are more than $O(N^{1/3})$.

A. Thermal effect

In this paragraph we discuss the shift of the transition temperature due to the energy shift of the thermal component. By our designation of energy shift and Eq. (38) we obtain

$$\frac{\delta T_c^T}{T_c^0} = \frac{\zeta(2)}{3[\zeta(3)]^{2/3}}\frac{\delta\Delta_g^T}{\hbar\omega}N^{-1/3} = -1.33\frac{a}{a_{ho}}N^{1/6}. \tag{41}$$

This result agrees with those of GPS [2], we have shown that it is only the effect of thermal component and the effect of the condensate was neglected.

B. Condensate effect

The shift of transition temperature due to the condensate component from Eq. (38) is

$$\frac{\delta T_c^c}{T_c^0} = -\frac{\zeta(2)}{3[\zeta(3)]^{2/3}}\frac{gS_c n_c(0)}{\hbar\omega}N^{-1/3} = -0.45\frac{a}{a_{ho}}N^{1/6}. \tag{42}$$

Thus, we finally derived that the total shift of the transition temperature, including the effects of both the thermal component and condensate component for the interacting Bose gas is

$$\frac{\delta T_c^{int}}{T_c^0} = \frac{\delta T_c^T + \delta T_c^c}{T_c^0} = -1.78\frac{a}{a_{ho}}N^{1/6}. \tag{43}$$

This result fits the experimental measurement very well [1]. With the form $\delta T_c^{int}/T_c^0 = \alpha N^{1/6}$, the experimental measurement $\alpha = -0.009$, GPS prediction is $\alpha = -0.007$. With the experimental parameters $a = 5.31$ nm, and $a_{ho} = 1.00$ μm, our formula gives $\alpha = -0.009\,45$.

IV. DISCUSSIONS

We show the simple picture of the shift of energy spectra for the thermal and condensate components and the shift of the transition temperature near T_c^0 in first order of g. The change of energy gap is due to the mutual atomic interaction and the trap potential. Near T_c, the number of condensate atoms changes fast; under T_c, the number grows up to $O(N)$; on the other hand, the number of thermal atoms changes slowly. The energy gap changes sharply and is monotonic in this region.

Our results are for atomic number under $N \simeq 10^6$. For $N > 10^6$, the effect of condensate is no longer linear in g anymore, the shift will be smaller than current result. In our calculation, we find the energy spectra through HF approximation Eq. (3) and Eq. (4), and calculate the chemical potential μ through the Bose-Einstein distribution Eq. (40). We find the difference in condensate energy ϵ_c and the chemical potential μ as

$$\frac{\epsilon_c - \mu}{\hbar\omega} = \frac{1}{\sqrt{\zeta(2)}}\left(\frac{\zeta(3)}{N}\right)^{1/6}, \qquad (44)$$

it is not negligible small for many current experimental environments. This difference has not been considered in most theoretical works before.

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