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The Appell function F_1 and Regge string scattering amplitudes

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ABSTRACT

We show that each 26D open bosonic Regge string scattering amplitude (RSSA) can be expressed in terms of one single Appell function F_1 in the Regge limit. This result enables us to derive infinite number of recurrence relations among RSSA at arbitrary mass levels, which are conjectured to be related to the known $SL(5, C)$ dynamical symmetry of F_1. In addition, we show that these recurrence relations in the Regge limit can be systematically solved so that all RSSA can be expressed in terms of one amplitude. All these results are dual to high energy symmetries of fixed angle string scattering amplitudes discovered previously [4–8].

1. Introduction

In contrast to the low energy string, the importance of high energy behavior of string theory was pointed out by Gross [1–3] more than two decades ago. Recently a saddle point method was invented to explicitly calculate string scattering amplitudes for string states at arbitrary mass levels in the fixed angle regime [4–8]. It was found that the ratios of string scattering amplitudes at each fixed mass level were independent of the scattering energy and the scattering angle, and the ratios can be extracted at each mass level. Alternatively, this infinite number of ratios can be recalculated algebraically by solving linear relations or stringy Ward identities derived from decoupling of two types of zero-norm states [9–11] in the string spectrum. These infinite linear relations are so powerful that all fixed angle high energy string scattering amplitudes can be expressed in terms of one amplitude, say, four tachyon amplitude.

There is another high energy regime of string scattering amplitudes, namely, the fixed momentum transfer or Regge regime [12–18]. It was shown that there existed intimate link between high energy string scattering amplitudes in the fixed angle regime and in the Regge regime. Indeed, the ratios among scattering amplitudes of different string states in the fixed angle regime can be extracted from the Kummer functions which are closely related to the Regge string scattering amplitudes (RSSA) [17,18]. Note that the number of RSSA is much more numerous than that of high energy fixed angle string scattering amplitudes. For example, there are only 4 high energy fixed angle string scattering amplitudes while there are 22 RSSA at mass level $M^2 = 4$ [17]. More recently [19], it was discovered that each RSSA can be expressed in terms of a finite sum of Kummer functions. One can then solve these Kummer functions at each mass level and express them in terms of RSSA. Recurrence relations of Kummer functions can then be used to derive some recurrence relations among RSSA [19]. Recurrence relations of higher spin generalization of the BPST vertex operators [15] can also be constructed [20].

Since in general each RSSA was expressed in terms of more than one Kummer function, it was awkward to derive the complete recurrence relations at arbitrary higher mass levels. In this letter, we show that each 26D open bosonic RSSA can be expressed in terms of one single Appell function F_1. In contrast to the case of a sum of Kummer functions, this result enables us to derive the complete infinite number of recurrence relations among RSSA at arbitrary mass levels, which are conjectured to be related to the known $SL(5, C)$ dynamical symmetry of F_1 [21]. In addition, we show that these recurrence relations in the Regge limit can be systematically solved so that all RSSA can be expressed in terms of one amplitude. All these results are dual to high energy symmetries of fixed angle string scattering amplitudes discovered previously [4–8]. In sum, the duality of the scattering amplitudes between the two high energy regimes implies amplitudes of the two regimes share many important properties, and there is an intimate link (ratios stated above) connecting the two regimes.

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2. Regge string scattering amplitudes

We first review recent results for high energy string scatterings in the fixed angle regime,

$$s,-t\to\infty, t/s\approx-\sin^2\frac{\phi}{2}=\text{fixed}\quad(\text{but }\phi\neq 0) \tag{1}$$

where $s=-(k_1+k_2)^2$, $t=-(k_2+k_3)^2$ and $u=-(k_1+k_3)^2$ are the Mandelstam variables and ϕ is the center of mass scattering angle. It was shown [6,7] that, at mass level $M_2^2=2(N-1)$, states to the leading order in energy are of the form (the second state of the four point function is chosen to be the higher spin string state)

$$|N,2m,q\rangle\equiv\left(\alpha_{-1}^T\right)^{N-2m-2q}\left(\alpha_{-1}^L\right)^{2m}\left(\alpha_{-2}^L\right)^q|0,k\rangle \tag{2}$$

where the polarizations of the 2nd particle with momentum k_2 on the scattering plane were defined to be $e^P=\frac{1}{M_2}(E_2,\mathrm{k}_2,0)=\frac{k_2}{M_2}$, $e^L=\frac{1}{M_2}(\mathrm{k}_2,E_2,0)$ and $e^T=(0,0,1)$. $\eta_{\mu\nu}=\text{diag}(-1,1,1)$. N,m and q in Eq. (2) are non-negative integers and $N\geq 2m+2q$. Since e^P approaches to e^L in the fixed angle regime [5], we did not put e^P components in Eq. (2). For simplicity, we choose the particles associated with momenta k_1, k_3 and k_4 to be tachyons. The $s-t$ channel high energy fixed angle string scattering amplitudes can be calculated [6] to be

$$\mathcal{T}^{(N,2m,q)}=\sqrt{\frac{2\pi}{Kf_0''}}e^{-Kf_0}\left[(-1)^{N-q}\frac{2^{N-q-2m}(2m)!}{m!M_2^{q+2m}}\right.$$
$$\left.\times\,\tau^{-\frac{N}{2}}(1-\tau)^{\frac{3N}{2}}E^N+O\left(E^{N-2}\right)\right] \tag{3}$$

where $K\equiv-k_1.k_2\to 2E^2$, $f(x)\equiv\ln x-\tau\ln(1-x)$, $\tau\equiv-\frac{k_2.k_3}{k_1.k_2}\to\sin^2\frac{\phi}{2}$, and the saddle point for the integration of moduli is defined by $f'(x_0)=0$ with $x_0=\frac{1}{1-\tau}$. The ratios among high energy fixed angle string scattering amplitudes of different string states at each fixed mass level can be extracted from Eq. (3) to be [6]

$$\frac{T^{(N,2m,q)}}{T^{(N,0,0)}}=\left(-\frac{1}{M_2}\right)^{2m+q}\left(\frac{1}{2}\right)^{m+q}(2m-1)!!. \tag{4}$$

Alternatively, it was discovered that [6,7] the ratios above can be recalculated by using the decoupling of two types of high energy fixed angle zero norm states

$$L_{-1}|n-1,2m-1,q\rangle$$
$$\simeq M|n,2m,q\rangle+(2m-1)|n,2m-2,q+1\rangle, \tag{5}$$
$$L_{-2}|n-2,0,q\rangle\simeq\frac{1}{2}|n,0,q\rangle+M|n,0,q+1\rangle. \tag{6}$$

Eqs. (5) and (6) give infinite number of linear relations among high energy fixed angle string scattering amplitudes of different string states at each fixed mass level. It turned out that these linear relations can be systematically solved so that all high energy fixed angle string scattering amplitudes can be expressed in terms of four tachyon amplitude [4–8].

We now turn to another high energy regime of string scatterings, namely the Regge regime, which contains complementary information [17] of the theory. That is in the kinematic regime $s\to\infty$ with $-t$ is finite and fixed. It was found [17] that the number of high energy scattering amplitudes for each fixed mass level in the Regge regime is much more numerous than that of fixed angle regime in Eq. (2). The leading order high energy open string states in the Regge regime at each fixed mass level $N=\sum_{n,m,l>0}np_n+mq_m+lr_l$ are [19]

$$|p_n,q_m,r_l\rangle=\prod_{n>0}\left(\alpha_{-n}^T\right)^{p_n}\prod_{m>0}\left(\alpha_{-m}^P\right)^{q_m}\prod_{l>0}\left(\alpha_{-l}^L\right)^{r_l}|0,k\rangle. \tag{7}$$

We first set up and calculate the kinematics of the Regge scattering. The momenta of the four particles on the scattering plane are

$$k_1=\left(+\sqrt{p^2+M_1^2},-p,0\right), \tag{8}$$
$$k_2=\left(+\sqrt{p^2+M_2^2},+p,0\right), \tag{9}$$
$$k_3=\left(-\sqrt{q^2+M_3^2},-q\cos\phi,-q\sin\phi\right), \tag{10}$$
$$k_4=\left(-\sqrt{q^2+M_4^2},+q\cos\phi,+q\sin\phi\right) \tag{11}$$

where $p\equiv|\vec{\mathrm{p}}|$, $q\equiv|\vec{\mathrm{q}}|$ and $k_i^2=-M_i^2$. The relevant kinematics in the Regge regime are

$$e^P\cdot k_1\simeq-\frac{s}{2M_2},\qquad e^P\cdot k_3\simeq-\frac{\tilde{t}}{2M_2}=-\frac{t-M_2^2-M_3^2}{2M_2}; \tag{12}$$
$$e^L\cdot k_1\simeq-\frac{s}{2M_2},\qquad e^L\cdot k_3\simeq-\frac{\tilde{t}'}{2M_2}=-\frac{t+M_2^2-M_3^2}{2M_2}; \tag{13}$$
$$e^T\cdot k_1=0,\qquad e^T\cdot k_3\simeq-\sqrt{-t} \tag{14}$$

where $\tilde{t}=t-M_2^2-M_3^2$ and $\tilde{t}'=t+M_2^2-M_3^2$. We are now ready to calculate the Regge scattering amplitudes. In the high energy limit, we only need to consider higher spin vertex with polarizations on the scattering plane. The $s-t$ channel one higher spin and three tachyons string scattering amplitudes in the Regge limit can then be calculated as

$$A^{(p_n;q_m;r_l)}$$
$$=\int_0^1 dx\,x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\cdot\left[\frac{e^P\cdot k_1}{x}-\frac{e^P\cdot k_3}{1-x}\right]^{q_1}$$
$$\times\left[\frac{e^L\cdot k_1}{x}+\frac{e^L\cdot k_3}{1-x}\right]^{r_1}\cdot\prod_{n=1}\left[\frac{(n-1)!e^T\cdot k_3}{(1-x)^n}\right]^{p_n}$$
$$\times\prod_{m=2}\left[\frac{(m-1)!e^P\cdot k_3}{(1-x)^m}\right]^{q_m}\prod_{l=2}\left[\frac{(l-1)!e^L\cdot k_3}{(1-x)^l}\right]^{r_l}$$
$$=\prod_{n=1}\left[(n-1)!\sqrt{-t}\right]^{p_n}\prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{q_m}$$
$$\times\prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l}\cdot\sum_{j=0}^{r_1}\sum_{i=0}^{q_1}\binom{r_1}{j}\binom{q_1}{i}\left(-\frac{s}{\tilde{t}}\right)^i\left(-\frac{s}{\tilde{t}'}\right)^j$$
$$\times B\left(-\frac{s}{2}+N-1-i-j,-\frac{t}{2}-1+i+j\right) \tag{15}$$

where in the Regge limit the beta function B can be further reduced to

$$B\left(-\frac{s}{2}-1+N-i-j,-\frac{t}{2}-1+i+j\right)$$
$$\simeq B\left(-\frac{s}{2}-1,-\frac{t}{2}-1\right)\frac{(-1)^{i+j}(-\frac{t}{2}-1)_{i+j}}{(\frac{s}{2})_{i+j}}, \tag{16}$$

where $(a)_n=a\cdot(a+1)\cdots(a+n-1)$ is the rising Pochhammer symbol. Thus

$$A^{(p_n;q_m;r_l)}=B\left(-\frac{s}{2}-1,-\frac{t}{2}-1\right)$$

$$\cdot \sum_{j=0}^{r_1}\sum_{i=0}^{q_1}\binom{r_1}{j}\binom{q_1}{i}\frac{(-\frac{t}{2}-1)_{i+j}}{(\frac{s}{2})_{i+j}}\left(\frac{s}{\tilde{t}}\right)^i\left(\frac{s}{\tilde{t}'}\right)^j$$
$$\cdot \prod_{n=1}\left[(n-1)!\sqrt{-t}\right]^{p_n}\prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{q_m}$$
$$\times \prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l} \tag{17}$$

in which the double summation can be expressed in terms of the Appell function F_1 as

$$\sum_{j=0}^{r_1}\sum_{i=0}^{q_1}\binom{r_1}{j}\binom{q_1}{i}\frac{(-\frac{t}{2}-1)_{i+j}}{(\frac{s}{2})_{i+j}}\left(\frac{s}{\tilde{t}}\right)^i\left(\frac{s}{\tilde{t}'}\right)^j$$
$$=\sum_{j=0}^{r_1}\sum_{i=0}^{q_1}\frac{(-q_1)_i(-r_1)_j}{i!j!}\frac{(-\frac{t}{2}-1)_{i+j}}{(\frac{s}{2})_{i+j}}\left(-\frac{s}{\tilde{t}}\right)^i\left(-\frac{s}{\tilde{t}'}\right)^j$$
$$=F_1\left(-\frac{t}{2}-1;-q_1,-r_1;\frac{s}{2};-\frac{s}{\tilde{t}},-\frac{s}{\tilde{t}'}\right). \tag{18}$$

The Appell function F_1 is one of the four extensions of the hypergeometric function ${}_2F_1$ to two variables and is defined to be

$$F_1(a;b,b';c;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_{m+n}}x^m y^n \tag{19}$$

Note that when a or $b(b')$ is a nonpositive integer, the Appell function truncates to a polynomial. This is the case for the Appell function in the RSSA calculated in Eq. (20) in the following

$$A^{(p_n;q_m;r_l)}$$
$$=\prod_{n=1}\left[(n-1)!\sqrt{-t}\right]^{p_n}\prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{q_m}$$
$$\times\prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l}\cdot F_1\left(-\frac{t}{2}-1;-q_1,-r_1;\frac{s}{2};-\frac{s}{\tilde{t}},-\frac{s}{\tilde{t}'}\right)$$
$$\cdot B\left(-\frac{s}{2}-1,-\frac{t}{2}-1\right). \tag{20}$$

Note that the B function above is power-law behaved in energy s in the Regime limit as in the usual case, and so is the Appell polynomial function F_1. Alternatively, it is interesting to note that the result calculated in Eq. (20) can be directly obtained from an integral representation of F_1 due to Emile Picard (1881) [22]

$$F_1(a;b_1,b_2;c;x,y)$$
$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 dt\, t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}, \tag{21}$$

which was later generalized by Appell and Kampe de Feriet (1926) [23] to n variables

$$F_D^{(n)}(a;b_1,b_2,\ldots,b_n;c;x_1,x_2,\ldots,x_n)$$
$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 dt\, t^{a-1}(1-t)^{c-a-1}$$
$$\cdot(1-x_1t)^{-b_1}\cdots(1-x_nt)^{-b_n} \tag{22}$$

where $F_D^{(n)}$ is one of the Lauricella functions introduced in 1893 [24]. Note that $F_D^{(2)}=F_1$. Eq. (22) may have application for higher point RSSA [25]. To apply the Picard formula in Eq. (21), we do the transformation $x\to(1-x)$, and RSSA can be calculated to be

$$A^{(p_n;q_m;r_l)}$$
$$=\int_0^1 dx\,(1-x)^{-\frac{s}{2}+N-2}x^{-\frac{t}{2}-2}\cdot\left[1-\frac{s}{\tilde{t}}\frac{x}{1-x}\right]^{q_1}$$
$$\times\left[1-\frac{s}{\tilde{t}'}\frac{x}{1-x}\right]^{r_1}\cdot\prod_{n=1}\left[(n-1)!\sqrt{-t}\right]^{p_n}$$
$$\times\prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{q_m}\prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l}$$
$$\simeq B\left(-\frac{t}{2}-1,-\frac{s}{2}-1\right)\cdot F_1\left(-\frac{t}{2}-1,-q_1,-r_1,-\frac{s}{2};\frac{s}{\tilde{t}},\frac{s}{\tilde{t}'}\right)$$
$$\cdot\prod_{n=1}\left[(n-1)!\sqrt{-t}\right]^{p_n}\prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{q_m}$$
$$\times\prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l}, \tag{23}$$

which is consistent with the result calculated in Eq. (20). It is important to note that although F_1 in Eq. (20) is a polynomial in s, the result in Eq. (20) is valid only for the *leading order* in s in the Regge limit. Note that in contrast to the previous calculation [19] in Eq. (31) where a finite sum of Kummer functions was obtained, here we get only one single Appell function in Eq. (20). This simplification will greatly simplify the calculation of recurrence relations among RSSA to be discussed in the next section.

3. Recurrence relations

The Appell function F_1 entails four recurrence relations among contiguous functions

$$(a-b_1-b_2)F_1(a;b_1,b_2;c;x,y)-aF_1(a+1;b_1,b_2;c;x,y)$$
$$+b_1F_1(a;b_1+1,b_2;c;x,y)+b_2F_1(a;b_1,b_2+1;c;x,y)=0, \tag{24}$$

$$cF_1(a;b_1,b_2;c;x,y)-(c-a)F_1(a;b_1,b_2;c+1;x,y)$$
$$-aF_1(a+1;b_1,b_2;c+1;x,y)=0, \tag{25}$$

$$cF_1(a;b_1,b_2;c;x,y)+c(x-1)F_1(a;b_1+1,b_2;c;x,y)$$
$$-(c-a)xF_1(a;b_1+1,b_2;c+1;x,y)=0, \tag{26}$$

$$cF_1(a;b_1,b_2;c;x,y)+c(y-1)F_1(a;b_1,b_2+1;c;x,y)$$
$$-(c-a)yF_1(a;b_1,b_2+1;c+1;x,y)=0. \tag{27}$$

All other recurrence relations can be deduced from these four relations. We can easily solve the Appell function in Eq. (20) and express it in terms of the RSSA

$$F_1\left(-\frac{t}{2}-1;-q_1,-r_1;\frac{s}{2};-\frac{s}{\tilde{t}},-\frac{s}{\tilde{t}'}\right)$$
$$=\frac{A^{(p_n;q_m;r_l)}}{B(-\frac{s}{2}-1,-\frac{t}{2}-1)}\prod_{n=1}\left[(n-1)!\sqrt{-t}\right]^{-p_n}$$
$$\times\prod_{m=1}\left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{-q_m}\prod_{l=1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{-r_l}. \tag{28}$$

Note that among the set of integers (p_n,q_m,r_l) on the right hand side of Eq. (28), only $(-q_1,-r_1)$ dependence shows up on the

Appell function F_1 on the left hand side of Eq. (28). Indeed, for those highest spin string states at the mass level $M_2^2 = 2(N-1)$, i.e. $|N; q_1, r_1\rangle \equiv (\alpha_{-1}^T)^{N-q_1-r_1}(\alpha_{-1}^P)^{q_1}(\alpha_{-1}^L)^{r_1}|0,k\rangle$, the string amplitudes reduce to

$$A^{(N;q_1,r_1)} = (\sqrt{-t})^{N-q_1-r_1}\left(-\frac{\tilde{t}}{2M_2}\right)^{q_1}\left(\frac{\tilde{t}'}{2M_2}\right)^{r_1} \cdot F_1\left(-\frac{t}{2}-1; -q_1, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) B\left(-\frac{s}{2}-1, -\frac{t}{2}-1\right), \tag{29}$$

which can be used to solve easily the Appell function F_1 in terms of the RSSA $A^{(N;q_1,r_1)}$.

We now proceed to show that the recurrence relations of the Appell function F_1 *in the Regge limit* in Eq. (20) can be systematically solved so that all RSSA can be expressed in terms of one amplitude. As the first step, we note that in [19] the RSSA was expressed in terms of finite sum of Kummer functions. There are two equivalent expressions [19]

$$\begin{aligned} A^{(p_n;q_m;r_l)} &= \prod_{n>0}\left[(n-1)!\sqrt{-t}\right]^{p_n} \cdot \prod_{m>0}\left[-(m-1)!\frac{\tilde{t}}{2M_2}\right]^{q_m} \\ &\quad \cdot \prod_{l>1}\left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l} \cdot B\left(-\frac{s}{2}-1, -\frac{t}{2}+1\right)\left(\frac{1}{M_2}\right)^{r_1} \\ &\quad \cdot \sum_{i=0}^{q_1}\binom{q_1}{i}\left(\frac{2}{\tilde{t}}\right)^i\left(-\frac{t}{2}-1\right)_i U\left(-r_1, \frac{t}{2}+2-i-r_1, \frac{\tilde{t}'}{2}\right) \end{aligned} \tag{30}$$

$$\begin{aligned} &= \prod_{n>0}\left[(n-1)!\sqrt{-t}\right]^{p_n} \cdot \prod_{m>1}\left[-(m-1)!\frac{\tilde{t}}{2M}\right]^{q_m} \\ &\quad \cdot \prod_{l>0}\left[(l-1)!\frac{\tilde{t}'}{2M}\right]^{r_l} \cdot B\left(-\frac{s}{2}-1, -\frac{t}{2}+1\right)\left(-\frac{1}{M_2}\right)^{q_1} \\ &\quad \cdot \sum_{j=0}^{r_1}\binom{r_1}{j}\left(\frac{2}{\tilde{t}'}\right)^j\left(-\frac{t}{2}-1\right)_j U\left(-q_1, \frac{t}{2}+2-j-q_1, \frac{\tilde{t}}{2}\right) \end{aligned} \tag{31}$$

It is easy to see that, for $q_1 = 0$ or $r_1 = 0$, the RSSA can be expressed in terms of only one single Kummer function $U(-r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2})$ or $U(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2})$, which are thus related to the Appell function $F_1(-\frac{t}{2} - 1; 0, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'})$ or $F_1(-\frac{t}{2} - 1; -q_1, 0; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'})$ respectively in the Regge limit in Eq. (20). Indeed, one can easily calculate

$$\lim_{s\to\infty} F_1\left(-\frac{t}{2}-1; 0, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) = \left(\frac{2}{\tilde{t}'}\right)^{r_1} U\left(-r_1, \frac{t}{2}+2-r_1, \frac{\tilde{t}'}{2}\right), \tag{32}$$

$$\lim_{s\to\infty} F_1\left(-\frac{t}{2}-1; -q_1, 0; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) = \left(\frac{2}{\tilde{t}}\right)^{q_1} U\left(-q_1, \frac{t}{2}+2-q_1, \frac{\tilde{t}}{2}\right). \tag{33}$$

On the other hand, it was shown in [19] that the ratio

$$\frac{U(\alpha,\gamma,z)}{U(0,z,z)} = f(\alpha,\gamma,z), \quad \alpha = 0, -1, -2, -3, \ldots \tag{34}$$

is determined and $f(\alpha,\gamma,z)$ can be calculated by using recurrence relations of $U(\alpha,\gamma,z)$. Note that $U(0,z,z) = 1$ by explicit calculation. We thus conclude that in the Regge limit $s \to \infty$, i.e. $c, x, y \to \infty$ and a, b_1, b_2 fixed, the Appell functions $F_1(a; 0, b_2; c; x, y)$ and $F_1(a; b_1, 0; c; x, y)$ are determined up to an overall factor by recurrence relations. The next step is to derive the recurrence relation

$$\begin{aligned} &yF_1(a; b_1, b_2; c; x, y) - xF_1(a; b_1+1, b_2-1; c; x, y) \\ &\quad + (x-y)F_1(a; b_1+1, b_2; c; x, y) = 0, \end{aligned} \tag{35}$$

which can be obtained from Eqs. (26) and (27). We are now ready to show that the recurrence relations of the Appell function F_1 *in the Regge limit* in Eq. (20) can be systematically solved so that all RSSA can be expressed in terms of one amplitude. We will use the short notation $F_1(a; b_1, b_2; c; x, y) = F_1(b_1, b_2)$ in the following. For $b_2 = -1$, by using Eq. (35) and the known $F_1(b_1, 0)$ and $F_1(0, b_2)$, one can easily show that $F_1(b_1, -1)$ are determined for all $b_1 = -1, -2, -3, \ldots$. Similarly, $F_1(b_1, -2)$ are determined for all $b_1 = -1, -2, -3, \ldots$ if one uses the result of $F_1(b_1, -1)$ in addition to Eq. (35) and the known $F_1(b_1, 0)$ and $F_1(0, b_2)$. This process can be continued and one ends up with the result that $F_1(b_1, b_2)$ are determined for all $b_1, b_2 = -1, -2, -3, \ldots$. This completes the proof that the recurrence relations of the Appell function F_1 in the Regge limit in Eq. (20) can be systematically solved so that all RSSA can be expressed in terms of one amplitude.

With the result calculated in Eq. (20), one can easily derive many recurrence relations among RSSA at arbitrary mass levels. For example, the identity in Eq. (35) leads to

$$\sqrt{-t}\left[A^{(N;q_1,r_1)} + A^{(N;q_1-1,r_1+1)}\right] - M_2 A^{(N;q_1-1,r_1)} = 0, \tag{36}$$

which is the generalization of Eq. (3.90) in [19] for mass level $M_2^2 = 4$ to arbitrary mass levels $M_2^2 = 2(N-1)$. Incidentally, one should keep in mind that the recurrence relations among RSSA are valid only in the Regge limit. We give one example to illustrate the calculation. By using Eqs. (24)–(27), and taking the leading term of s in the Regge limit, we end up with the recurrence relation for b_2

$$\begin{aligned} &cx^2 F_1(a; b_1, b_2; c; x, y) \\ &\quad + \left[(a-b_1-b_2-1)xy^2 + cx^2 - 2cxy\right]F_1(a; b_1, b_2+1; c; x, y) \\ &\quad - \left[(a+1)x^2y - (a-b_2-1)xy^2 - cx^2 + cxy\right] \\ &\quad \times F_1(a; b_1, b_2+2; c; x, y) \\ &\quad - (b_2+2)x(x-y)yF_1(a; b_1, b_2+3; c; x, y) = 0, \end{aligned} \tag{37}$$

which leads to a recurrence relation for RSSA at arbitrary mass levels

$$\begin{aligned} &\tilde{t}'^2 A^{(N;q_1,r_1)} \\ &\quad + \left[\tilde{t}'^2 + \tilde{t}\left(t - 2\tilde{t}' - 2q_1 - 2r_1 + 4\right)\right]\left(\frac{\frac{\tilde{t}'}{2M_2}}{\sqrt{-t}}\right) A^{(N;q_1,r_1+1)} \\ &\quad + \left[\tilde{t}'^2 - \tilde{t}'(\tilde{t}+t) + \tilde{t}(t - 2r_1 + 4)\right]\left(\frac{\frac{\tilde{t}'}{2M_2}}{\sqrt{-t}}\right)^2 A^{(N;q_1,r_1+2)} \\ &\quad - 2(r_1-2)\left(\tilde{t}' - \tilde{t}\right)\left(\frac{\frac{\tilde{t}'}{2M_2}}{\sqrt{-t}}\right)^3 A^{(N;q_1,r_1+3)} = 0. \end{aligned} \tag{38}$$

More higher recurrence relations which contain general number of $l \geq 3$ Appell functions can be found in [26].

4. Conclusion

In this paper, we show that open bosonic RSSA can be expressed in terms of one single Appell function F_1 in the Regge limit. This result enables us to derive recurrence relations among RSSA at arbitrary mass levels. In addition, we show that these recurrence relations of RSSA are so powerful that one can solve them and all RSSA can be expressed in terms of one single amplitude. All these results are dual to high energy symmetries of fixed angle string scattering amplitudes conjectured by Gross in 1988 [2] which were explicitly proved in [4–8] previously.

Since it was shown that [21] the Appell function F_1 are basis vectors for models of irreducible representations of $sl(5, C)$ algebra, it seems reasonable to believe that the spacetime symmetry of Regge string theory is closely related to $SL(5, C)$ non-compact group. In particular, the recurrence relations of RSSA studied in this paper are related to the $SL(5, C)$ group as well. Further investigation remains to be done and more evidences need to be uncovered.

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