Topological Properties on the Wide and Fault Diameters of Exchanged Hypercubes

Tsung-Han Tsai, Y-Chuang Chen, and Jimmy J.M. Tan

Abstract—The n-dimensional hypercube is one of the most popular topological structure for interconnection networks in parallel computing and communication systems. The exchanged hypercube $\mathrm{EH}(s,t)$, a variant of the hypercube, retains several valuable and desirable properties of the hypercube such as a small diameter, bipancyclicity, and super connectivity. In this paper, we construct $s+1$ (or $t+1$) internally vertex-disjoint paths between any two vertices for parallel routes in the exchanged hypercube $\mathrm{EH}(s,t)$ for $3 \leq s \leq t$. We also show that both the $(s+1)$-wide diameter and s-fault diameter of the exchanged hypercube $\mathrm{EH}(s,t)$ are $s+t+3$ for $3 \leq s \leq t$.

Index Terms—Hypercube, exchanged hypercube, interconnection network, internally vertex-disjoint paths, wide diameter, fault diameter

1 Introduction

A *multiprocessor/multicomputer interconnection network* is usually modeled as a *graph*, in which *vertices* correspond to *processors/computers*, and *edges* correspond to *connections/communication links*. Throughout this paper, the terms networks and graphs are interchangeable. A graph G is a two-tuple (V,E), where V is a nonempty vertex set, and E is a subset of $\{(u,v) \mid (u,v)$ is an unordered pair of $V\}$. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Two vertices, u and v, of a graph G are *adjacent* if $(u,v) \in E(G)$. The *neighborhoods* of a vertex v in graph G, denoted by $N_G(v)$, is $\{x \mid (v,x) \in E(G)\}$. A *path* P of *length* k from vertex u to vertex v in a graph G is a sequence of distinct vertices written as $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k$ where $x_0 = u$, $x_k = v$, and $(x_i, x_{i+1}) \in E(G)$ for every $0 \leq i \leq k-1$ if $k \geq 1$. The path P can be written as $u \rightarrow P \rightarrow v$ to emphasize its first and last vertices. For convenience, P can also be written as $x_0 \rightarrow \cdots \rightarrow x_i \rightarrow Q \rightarrow x_j \rightarrow \cdots \rightarrow x_k$, where $Q = x_i \rightarrow \cdots \rightarrow x_j$. Given a path P from u to v, all vertices in P except u and v are called *internal vertices* of P. Two paths are called *internally vertex-disjoint* (abbreviated as *internally disjoint*) if they share no internal vertex. A *cycle* is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length k is represented by $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k \rightarrow x_1$. The *distance* between two vertices u and v in graph G, denoted by $d_G(u,v)$, is the length of the shortest path between u and v.

To design an interconnection network with desired topologies is an important issue [5]. The *hypercube* is one of the most popular interconnection network structures in parallel computing and communication systems [7], [11], [19], [23]. This is partly because of many attractive properties of the hypercube such as *regularity*, *recursive structure*, *vertex and edge symmetry*, and *maximum connectivity*, as well as *the effective routing* and *broadcasting*. An *n-dimensional hypercube*, denoted by Q_n, is a graph with 2^n vertices and $n \times 2^{n-1}$ edges. Each vertex is labeled by an n-bit binary string $u = u_{n-1}u_{n-2}\cdots u_0$. Two vertices are adjacent if and only if their strings differ exactly in one bit position. Let $u = u_{n-1}u_{n-2}\cdots u_0$ and $v = v_{n-1}v_{n-2}\cdots v_0$ be two n-bit binary strings. The *Hamming distance* between two vertices u and v, denoted by $H(u,v)$, is the number of different bits in the corresponding strings of both vertices. Thus, $H(u,v) = d_{Q_n}(u,v)$. Note that Q_n has diameter n [23].

As a variant of the n-dimensional hypercube, the *exchanged hypercube* $\mathrm{EH}(s,t)$, which was proposed by Loh et al. [13], is defined by removing some edges from the hypercube. To make $\mathrm{EH}(s,t)$ useful in reliable and critical applications, studies have been conducted, which have produced some significant results. $\mathrm{EH}(s,t)$ retains several desirable properties of the hypercube such as *a small diameter* [13], *bipancyclicity* [16], and *super connectivity* [17] and this makes it even better than a hypercube. This is evident in the fact that even though the number of edges of an exchanged hypercube is nearly half of that of a hypercube, their diameters are similar. Thus, exchanged hypercubes have lower link costs than hypercubes. To transfer information safely and quickly between any two vertices in exchanged hypercubes, we need to find as many as possible internally disjoint paths between the two vertices. This idea was proposed in Menger's theorem [18], which states that there are k internally disjoint paths between any two vertices in an interconnection network if k is less than or equal to the connectivity of this network. Moreover, this interconnection network has many benefits such as parallel routing and fault tolerance. In recent years, many literature references discuss the topic of internally disjoint paths in some specific networks, such as hypercubes [20], crossed cubes [8], (n,k)-star graphs [12], folded hypercubes [15], hypercube-like graphs [19], hierarchical hypercubes [21], and augmented k-ary n-cubes [22]. Next, we discuss the *fault* and *wide diameters* of exchanged hypercubes. The *fault diameter*,

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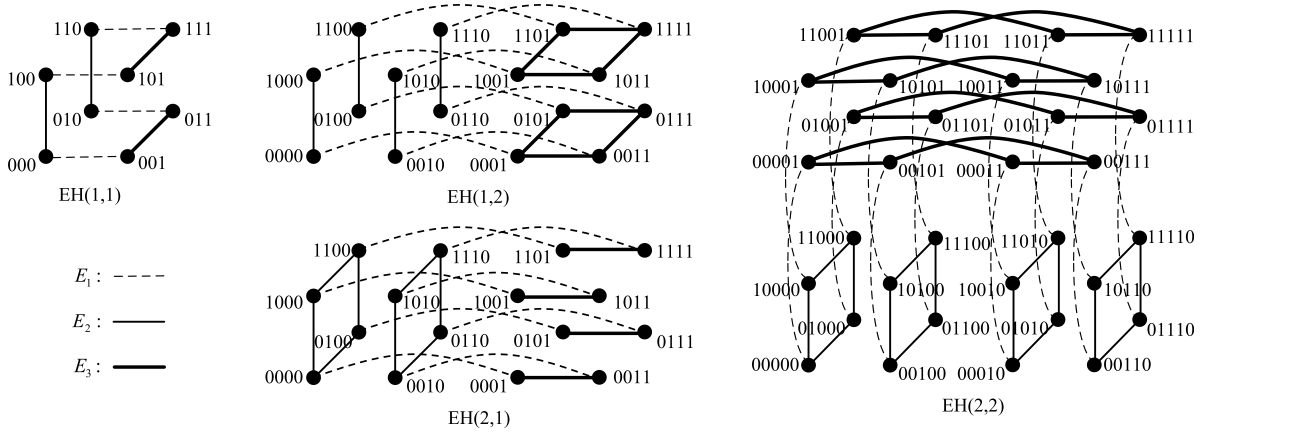

Fig. 1. Four examples of the exchanged hypercubes $\mathrm{EH}(1,1)$, $\mathrm{EH}(1,2)$, $\mathrm{EH}(2,1)$ and $\mathrm{EH}(2,2)$.

which was first proposed in [9], is used to estimate the effects of faults on the diameter, while the *wide diameter* is used to measure the diameter of the connections with prescribed bandwidths, and it is a combination of both the diameter and connectivity. The fault and wide diameters have been discussed in [2], [3], [4], [6], [12], [15], [21], [22]. In this study, we construct $s+1$ (or $t+1$) internally disjoint paths between any two vertices for parallel routes in the exchanged hypercube $\mathrm{EH}(s,t)$ for $3 \le s \le t$. We also prove that both the $(s+1)$-*wide diameter* and s-*fault diameter* are $s+t+3$ for $3 \le s \le t$.

The rest of this paper is organized as follows. In the next section, we provide the definition for exchanged hypercubes and describe their properties. In Section 3, the main results are presented; the internally disjoint paths between any two vertices in $\mathrm{EH}(s,t)$ for $3 \le s \le t$ are discussed and it is demonstrated that both $(s+1)$-wide diameter and s-fault diameter are $s+t+3$ for $3 \le s \le t$. In Section 4, concluding remarks are presented.

2 Preliminaries

The exchanged hypercube is defined as an undirected graph $\mathrm{EH}(s,t) = G(V,E)$, where $s \ge 1$ and $t \ge 1$. The definition of exchanged hypercubes is given as follows.

Definition 1. *The vertex set V of exchanged hypercube $\mathrm{EH}(s,t)$ $(s \ge 1, t \ge 1)$ is the set*

$$\{u_{t+s} \cdots u_{t+1} u_t \cdots u_1 u_0 \mid u_i \in \{0,1\} \text{ for } 0 \le i \le s+t\}.$$

Let $u = u_{t+s} u_{t+s-1} \cdots u_0$ and $v = v_{t+s} v_{t+s-1} \cdots v_0$ be two vertices in $\mathrm{EH}(s,t)$. There is an edge (u,v) in $\mathrm{EH}(s,t)$ if and only if (u,v) is in one of the following sets:

$$E_1 = \{(u,v) \mid u_0 \ne v_0, u_i = v_i \text{ for } 1 \le i \le s+t\},$$
$$E_2 = \{(u,v) \mid u_0 = v_0 = 0, H(u,v) = 1 \text{ with } u_i \ne v_i \text{ for some } t+1 \le i \le s+t\}, \text{ and}$$
$$E_3 = \{(u,v) \mid u_0 = v_0 = 1, H(u,v) = 1 \text{ with } u_i \ne v_i \text{ for some } 1 \le i \le t\},$$

where $H(u,v)$ denotes the Hamming distance between two vertices u and v.

According to the definition of $\mathrm{EH}(s,t)$, the number of vertices is 2^{s+t+1} and the number of edges is $(s+t+2)2^{s+t-1}$. For a vertex x with $x_0 = 0$, the vertex degree is $s+1$, whereas the vertex degree with $x_0 = 1$ is $t+1$. $\mathrm{EH}(s,t)$ is a subgraph of the $(s+t+1)$-dimensional hypercube Q_{s+t+1}, and thus it is also a bipartite graph. Fig. 1 illustrates the exchanged hypercubes $\mathrm{EH}(1,1)$, $\mathrm{EH}(1,2)$, $\mathrm{EH}(2,1)$ and $\mathrm{EH}(2,2)$. Dashed links correspond to the edge set E_1, solid links correspond to the edge set E_2, and bold links correspond to the edge set E_3.

Loh et al. [13] stated the following properties.

Property 1. *The diameter of* $\mathrm{EH}(s,t)$ *is* $s+t+2$.

Property 2. $\mathrm{EH}(s,t)$ *is isomorphic to* $\mathrm{EH}(t,s)$.

Property 3. $\mathrm{EH}(s,t)$ *can be decomposed into two copies of* $\mathrm{EH}(s-1,t)$ *or* $\mathrm{EH}(s,t-1)$.

Property 4. *The subgraphs induced by the vertices of the form* $\overbrace{* \cdots *}^{s} u_t u_{t-1} \cdots u_1 0$ *and* $u_{t+s} u_{t+s-1} \cdots u_{t+1} \overbrace{* \cdots *}^{t} 1$ *in* $\mathrm{EH}(s,t)$ *are isomorphic to* Q_{s} *and* Q_{t}, *respectively, where* $* \in \{0,1\}$.

The subgraphs induced by the vertex sets $V(Q_{\mathrm{s}})$ and $V(Q_{\mathrm{t}})$ are denoted by S and T, respectively. Then, $S \cong Q_{\mathrm{s}}$ and $T \cong Q_{\mathrm{t}}$. Therefore, by Property 4, there are 2^t and 2^s distinct induced subgraphs Q_{s} and Q_{t}, respectively. Denote by Q_{s}^i (respectively, Q_{t}^j) for $0 \le i \le 2^t - 1$ (respectively, $0 \le j \le 2^s - 1$) where i (respectively, j) with radix 10 is the value of $u_t u_{t-1} \cdots u_1$ (respectively, $u_{t+s} u_{t+s-1} \cdots u_{t+1}$). Let $h_{\mathrm{s}}(u,v)$ (respectively, $h_{\mathrm{t}}(u,v)$) denote the number of different bits between u and v in dimensions $t+1$ to $s+t$ (respectively, 1 to t). When the context is clear, $h_{\mathrm{s}}(u,v)$ and $h_{\mathrm{t}}(u,v)$ are simply written as h_{s} and h_{s}, respectively. Moreover, since $\mathrm{EH}(s,t)$ is isomorphic to $\mathrm{EH}(t,s)$ by Property 2, we may, without loss of generality, assume that $s \le t$ in this paper.

A vertex set $S \subseteq V(G)$ is a *separating set* or a *vertex cut* if $G - S$ is disconnected. The *connectivity* of G, written as $\kappa(G)$, is the minimum size of a vertex cut. Let $\delta(G)$ be the minimum degree of G, then it is clear that $\kappa(G) \le \delta(G)$. A graph G is *k-connected* if the connectivity $\kappa(G)$ is at least k. Moreover, a graph G has connectivity k if G is k-connected but not $(k+1)$-connected. This follows from Menger's theorem [18], which states that the connectivity of a graph is at least k if and only if there exist k internally disjoint paths between any two vertices.

Let α and β be two positive integers such that $\alpha \leq \kappa(G)$ and $\beta \leq \kappa(G) - 1$. Given any two vertices u and v of G, let $D_\alpha(u, v)$ denote the set of all α internally disjoint paths between u and v. Each element of $D_\alpha(u, v)$ consists of α internally disjoint paths. $|D_\alpha(u, v)|$ denotes the number of elements in $D_\alpha(u, v)$. Let $l_i(u, v)$ denote the longest length among the α paths of the i-th element of $D_\alpha(u, v)$. Thus, $l_{D_\alpha}(u, v)$ and $d^f_\beta(u, v)$ are defined as follows:

$$l_{D_\alpha}(u, v) = \min_{1 \leq i \leq |D_\alpha(u,v)|} l_i(u, v),$$

$$d^f_\beta(u, v) = \max_{F \subseteq V, |F| = \beta} \{d_{G-F}(u, v) | u, v \notin F\},$$

where $G - F$ denotes the subgraph of G induced by $V - F$. In other words, $d^f_\beta(u, v)$ denotes the longest distance between u and v when any β faulty vertices occur.

Definition 2. [1] *The* α-wide diameter *of* G, *denoted by* $D_\alpha(G)$, *is defined as*

$$D_\alpha(G) = \max_{u,v \in V} \{l_{D_\alpha}(u, v)\}.$$

In particular, $D_{\kappa(G)}(G)$ *is the wide diameter of* G *and* $D_1(G)$ *is simply the diameter* $D(G)$ *of* G.

Definition 3. [1] *The* β-fault diameter *of* G, *denoted by* $D^f_\beta(G)$, *is defined as*

$$D^f_\beta(G) = \max_{u,v \in V} \left\{ d^f_\beta(u, v) \right\}$$

In particular, $D^f_{\kappa(G)-1}(G)$ *is the fault diameter of* G.

Obviously, $D(G) \leq D^f_{\kappa(G)-1}(G) \leq D_{\kappa(G)}(G)$. For the hypercubes Q_n, Latifi [10] proved that $D_n(Q_n) = D^f_{n-1}(Q_n) = n + 1$ for $n \geq 2$. For the crossed cubes CQ_n, Chang et al. [1] proved that $D_n(CQ_n) = D^f_{n-1}(CQ_n) = \lceil \frac{n}{2} \rceil + 2$ for $n \geq 2$. In this paper, we also discuss and prove the wide and the fault diameters of exchanged hypercubes, and proved that $D_{s+1}(\mathrm{EH}(s, t)) = D^f_s(\mathrm{EH}(s, t)) = s + t + 3$ for $3 \leq s \leq t$. The following three theorems are needed in the proofs of our results.

Theorem 1. [14] *The connectivity of the exchanged hypercubes* $\mathrm{EH}(s, t)$ *is* $s + 1$ *for* $1 \leq s \leq t$.

From Menger's theorem, there exist $s + 1$ internal disjoint paths between any two vertices in the exchanged hypercube $\mathrm{EH}(s, t)$.

Theorem 2. [20] *Let* u, v *be any two vertices of the* n*-dimensional hypercube* Q_n *and assume that* $H(u, v) = d$. *Then there are* n *internally disjoint paths between* u *and* v *such that* d *of them are of length* d, *and the remaining* $n - d$ *paths are of length* $d + 2$.

Theorem 3. [16] *In the exchanged hypercube* $\mathrm{EH}(s, t)$ *for* $1 \leq s \leq t$, *the vertices in the set* $V_c = \{u_{t+s} \cdots u_0 \,|\, u_0 = c, u_i \in \{0, 1\}$ *for* $1 \leq i \leq s + t\}$ *are vertex-transitive, where* $c \in \{0, 1\}$.

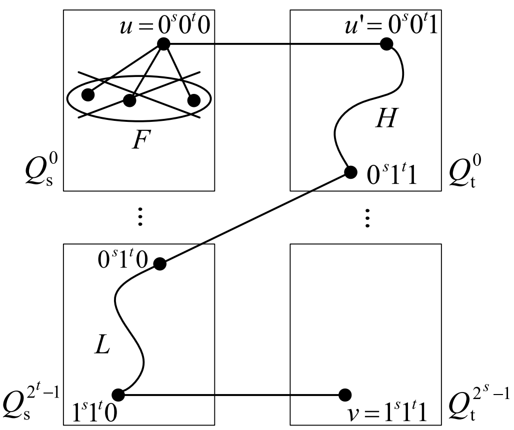

Fig. 2. An illustration for Lemma 1.

For convenience, consecutive i 0's and 1's are denoted by 0^i and 1^i, respectively. That is, $0^i = \overbrace{00\cdots 0}^{i}$ and $1^i = \overbrace{11\cdots 1}^{i}$.

3 Wide and Fault Diameters of Exchanged Hypercubes

In this section, our goal is to prove that $D_{s+1}(\mathrm{EH}(s, t)) = D^f_s(\mathrm{EH}(s, t)) = s + t + 3$ for $3 \leq s \leq t$.

Lemma 1. $D^f_s(\mathrm{EH}(s, t)) \geq s + t + 3$ *for* $1 \leq s \leq t$.

Proof. Let u, u', and v be three vertices of $\mathrm{EH}(s, t)$. We consider that $u = 0^s0^t0$, $u' = 0^s0^t1$ and $v = 1^s1^t1$. See Fig. 2 for illustration. Suppose that F is a faulty vertex set such that $F = N_{\mathrm{EH}(s,t)}(u) - u'$. The shortest path between u and v, denoted by P, in $\mathrm{EH}(s, t) - F$ must pass through u'. Thus, P can be written as $u \to u' \to R \to v$ where R is the shortest path from u' to v in $\mathrm{EH}(s, t) - F$. The subpath R can be written as follows:

$$u' \to H \to 0^s1^t1 \to 0^s1^t0 \to L \to 1^s1^t0 \to v.$$

Note that the length of subpath $u' \to H \to 0^s1^t1$ is t and all vertices of H are in Q^0_{t}; moreover, the length of subpath $0^s1^t0 \to L \to 1^s1^t0$ is s and all vertices of L are in $Q^{2^t-1}_{\mathrm{t}}$. Thus, the length of the subpath $u' \to R \to v$ is $s + t + 2$, and it follows that $d_{\mathrm{EH}(s,t)-F}(u, v) = 1 + (s + t + 2) = s + t + 3$. Therefore, $D^f_s(\mathrm{EH}(s, t)) \geq s + t + 3$ for $1 \leq s \leq t$. □

Next, to show that $D_{s+1}(\mathrm{EH}(s, t)) \leq s + t + 3$ for $3 \leq s \leq t$, internally disjoint paths between any two vertices u and v of $\mathrm{EH}(s, t)$ are constructed in Lemmas 2-11. Table 1 illustrates the conditions of vertices u and v in Lemmas 2-11. For convenience, some symbols are used in the following proofs. Let $u \to P \to v$ be a path from u to v in $\mathrm{EH}(s, t)$. The predecessor vertex of v in P is denoted by $\mathrm{pre}(P, u, i)$ if their i-th bits are different. Similarly, the successor vertex of u in P is denoted by $\mathrm{suc}(P, u, j)$ if their j-th bits are different. We use $l(P)$ to denote the length of P.

Lemma 2. *Let* u *and* v *be two vertices of* $\mathrm{EH}(s, t)$ *for* $3 \leq s \leq t$ *with* $u = u_{t+s}u_{t+s-1} \cdots u_0$ *and* $v = v_{t+s}v_{t+s-1} \cdots v_0$. *If* $u_0 = v_0 = 0$, *and* $h_{\mathrm{t}}(u, v) = 0$, *then there exist* $s + 1$

TABLE 1
The Conditions of Lemmas 2-11 in $\mathrm{EH}(s,t)$

Lemma	Conditions for u_0 and v_0	Conditions for h_s and h_t
2	$u_0 = v_0 = 0$	$h_t = 0$
3	$u_0 = v_0 = 0$	$h_t \neq 0$
4	$u_0 = v_0 = 1$	$h_s = 0$
5	$u_0 = v_0 = 1$	$h_s \neq 0$
6	$u_0 \neq v_0$	$h_s = h_t = 0$
7	$u_0 \neq v_0$	$h_s \neq 0, h_t = 0$
8	$u_0 \neq v_0$	$h_s = 0, h_t \neq 0$
9	$u_0 \neq v_0$	$h_s = s, h_t \neq 0$
10	$u_0 \neq v_0$	$1 \leq h_s \leq s-1, s \leq h_t \leq t$
11	$u_0 \neq v_0$	$1 \leq h_s \leq s-1, 1 \leq h_t \leq s-1$

internally disjoint paths P_i for $1 \leq i \leq s+1$ between u and v such that h_s of them are of length h_s, $s-h_s$ paths are of length h_s+2, and one path is of length h_s+6.

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t0$ and $v = 0^{s-h_s}1^{h_s}0^t0$ are in Q_s^0. See Fig. 3 for illustration. By Theorem 2, in Q_s^0, there exist s internally disjoint paths between u and v such that h_s of them are of length h_s and the remaining $s-h_s$ paths are of length h_s+2. Let $u \rightarrow P_i \rightarrow v$ for $1 \leq i \leq s$ be those s internally disjoint paths. The following sets of $s+1$ internally disjoint paths between u and v in $\mathrm{EH}(s,t)$ can be set:

We construct the paths P_i for $1 \leq i \leq s$ from u to v as follows:

$$u \rightarrow P_i \rightarrow v.$$

Note that all the paths $u \rightarrow P_i \rightarrow v$ are in Q_s^0. Thus, $l(P_i) = h_s$ for $1 \leq i \leq h_s$ and $l(P_i) = h_s+2$ for $h_s+1 \leq i \leq s$.

The path P_{s+1} can be constructed from u to v as follows:

$$\begin{aligned} &u \rightarrow 0^s0^t1 \rightarrow 0^s0^{t-1}11 \\ &\rightarrow 0^s0^{t-1}10 \rightarrow L \rightarrow 0^{s-h_s}1^{h_s}0^{t-1}10 \\ &\rightarrow 0^{s-h_s}1^{h_s}0^{t-1}11 \rightarrow 0^{s-h_s}1^{h_s}0^t1 \rightarrow v. \end{aligned}$$

Note that edge $0^s0^t1 \rightarrow 0^s0^{t-1}11$ is in Q_t^0, subpath $0^s0^{t-1}10 \rightarrow L \rightarrow 0^{s-h_s}1^{h_s}0^{t-1}10$ is in Q_s^1 and edge $0^{s-h_s}1^{h_s}0^{t-1}11 \rightarrow 0^{s-h_s}1^{h_s}0^t1$ is in $Q_t^{2^{h_s}-1}$. It can be seen that P_{s+1} is also internally disjoint to P_i for $1 \leq i \leq s$. Moreover, we have $l(L) = h_s$. Therefore, $l(P_{s+1}) = h_s+6$. This completes the proof. ◻

Lemma 3. *Let u and v be two vertices of $\mathrm{EH}(s,t)$ for $3 \leq s \leq t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 = v_0 = 0$ and $h_t(u,v) \neq 0$, then there exist $s+1$ internally disjoint paths P_i for $1 \leq i \leq s+1$ between u and v such that h_s+1 of them are of length h_s+h_t+2 and $s-h_s$ paths are of length h_s+h_t+4.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t0$ and $v = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}0$ are in Q_s^0 and $Q_s^{2^{h_t}-1}$, respectively. Depending on h_s, two cases are distinguished.

Case 1: $h_s = 0$. Then, $u = 0^s0^t0$ and $v = 0^s0^{t-h_t}1^{h_t}0$. See Fig. 4 for illustration. Let $u^i = u \oplus 0^{s-i}10^{i-1}0^t0 = 0^{s-i}10^{i-1}0^t0$ and $v^i = v \oplus 0^{s-i}10^{i-1}0^t0 = 0^{s-i}10^{i-1}0^{t-h_t}1^{h_t}0$ for $1 \leq i \leq s$ where $\oplus$ is the exclusive-or operation. To construct a path from u^i to v^i, we need the following intermediate vertices: $x^i = u^i \oplus 0^s0^t1$ and $y^i = v^i \oplus 0^s0^t1$. Accordingly, $x^i = 0^{s-i}10^{i-1}0^t1$ and $y^i = 0^{s-i}10^{i-1}0^{t-h_t}1^{h_t}1$. Now we construct path P_i for $1 \leq i \leq s$ from u to v as follows:

$$u \rightarrow u^i \rightarrow x^i \rightarrow R_i \rightarrow y^i \rightarrow v^i \rightarrow v.$$

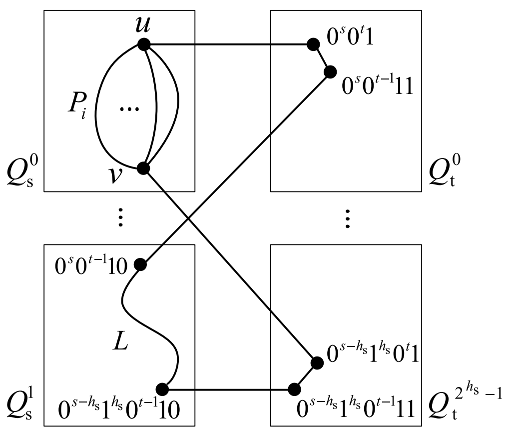

Fig. 3. An illustration for Lemma 2.

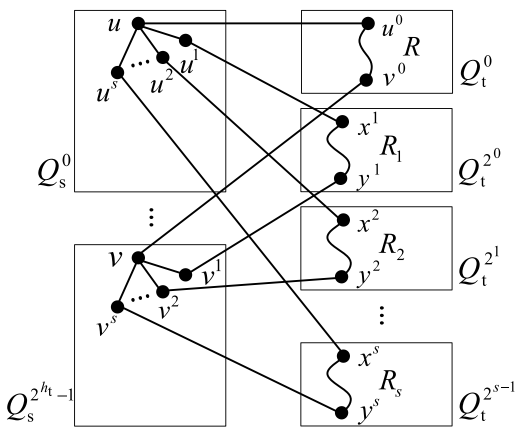

Fig. 4. An illustration for the Case 1 of Lemma 3.

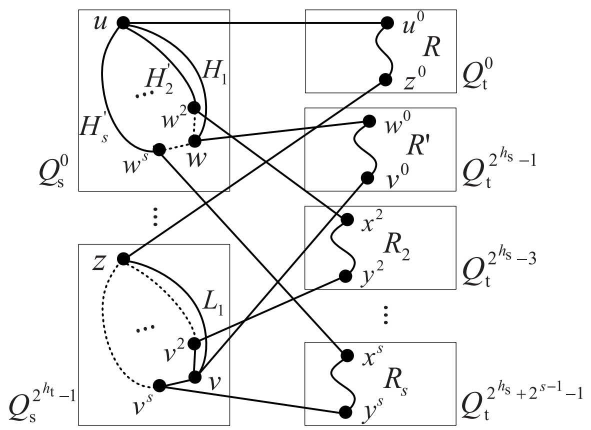

Fig. 5. An illustration for the Case 2 of Lemma 3.

Note that edge $u \to u^i$ is in Q_s^0, subpath $x^i \to R_i \to y^i$ is in $Q_t^{2^{i-1}}$ and edge $v^i \to v$ is in $Q_s^{2^{h_t}-1}$ while $u^i \to x^i$ and $y^i \to v^i$ are two edges in E_1. This can be confirmed that those P_i for $1 \le i \le s$ are internally disjoint.

It remains to construct the $(s+1)$-th internally disjoint path from u to v. Path P_{s+1} can be constructed as follows:

$$u \to u^0 \to R \to v^0 \to v.$$

Note that $u^0 = 0^s0^t1$ and $v^0 = 0^s0^{t-h_t}1^{h_t}1$. We can find that subpath $u^0 \to R \to v^0$ is in Q_t^0 while $u \to u^0$ and $v^0 \to v$ are two edges in E_1.

By inspection, the vertices in Q_t^0 are not in P_i for $1 \le i \le s$. Thus, path P_{s+1} is also internally disjoint to P_i for $1 \le i \le s$. Since both x^i and y^i are in $Q_t^{2^{i-1}}$, $l(R_i) = H(x^i, y^i) = h_t$. Moreover, both u^0 and v^0 are in Q_t^0, and $l(R) = h_t$. Therefore, $l(P_i) = h_t + 4$ for $1 \le i \le s$ and $l(P_{s+1}) = h_t + 2$.

Case 2: $1 \le h_s \le s$. Then, $u = 0^s0^t0$ and $v = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}0$. See Fig. 5 for illustration. Let $w = 0^{s-h_s}1^{h_s}0^t0$ and $z = 0^s0^{t-h_t}1^{h_t}0$ be in Q_s^0 and $Q_s^{2^{h_t}-1}$, respectively. By Theorem 2, in Q_s^0, there exist s internally disjoint paths between u and w such that h_s of them are of length h_s and the remaining $s - h_s$ paths are of length $h_s + 2$. Let $u \to H_i \to w$ for $1 \le i \le s$ be those internally disjoint paths. Similarly, in $Q_s^{2^{h_t}-1}$, there exist s internally disjoint paths between z and v such that h_s of them are of length h_s, and the remaining $s - h_s$ paths are of length $h_s + 2$. We also denote $z \to L_i \to v$ for $1 \le i \le s$ are those internally disjoint paths.

Now the path P_1 can be constructed as follows:

$$u \to H_1 \to w \to w^0 \to R' \to v^0 \to v.$$

Note that $w^0 = w \oplus 0^s0^t1 = 0^{s-h_s}1^{h_s}0^t1$ and $v^0 = v \oplus 0^s0^t1 = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}1$ where $\oplus$ is the exclusive-or operation. We can find that subpath $u \to H_1 \to w$ is in Q_s^0 and subpath $w^0 \to R' \to v^0$ is in $Q_t^{2^{h_s}-1}$ while $w \to w^0$ and $v^0 \to v$ are two edges in E_1. Since both w^0 and v^0 are in $Q_t^{2^{h_s}-1}$, $l(R') = H(w^0, v^0) = h_t$. In addition, we have $l(H_1) = h_s$. Hence $l(P_1) = h_s + h_t + 2$.

Based on H_i and L_i for $2 \le i \le s$, we construct $s - 1$ internally disjoint paths from u to v as follows. Let $w^i = \text{pre}(H_i, w, t+i)$ and $v^i = \text{pre}(L_i, v, t+i)$ where their $(t+i)$-th bits are different. This is, $w^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^t0$ and $v^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^{t-h_t}1^{h_t}0$ when $2 \le i \le h_s$, and $w^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^t0$ and $v^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^{t-h_t}1^{h_t}0$ when $h_s + 1 \le i \le s$. Assume that H_i' is the subpath of H_i without containing w. Clearly, all $u \to H_i' \to w^i$ for $2 \le i \le s$ are internally disjoint. To construct a path from w^i to v^i, we need the following intermediate vertices: $x^i = w^i \oplus 0^s0^t1$ and $y^i = v^i \oplus 0^s0^t1$ for $2 \le i \le s$ where $\oplus$ is the exclusive-or operation. Accordingly, $x^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^t1$ and $y^i = 0^{s-h_s}1^{h_s-i}01^{i-1}\ 0^{t-h_t}1^{h_t}1$ when $2 \le i \le h_s$, and $x^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^t1$ and $y^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^{t-h_t}1^{h_t}1$ when $h_s + 1 \le i \le s$. We construct a path from w^i to v as follows:

$$w^i \to x^i \to R_i \to y^i \to v^i.$$

Combining the subpaths above, we can obtain paths P_i for $2 \le i \le s$ from u to v as follows:

$$u \to H_i' \to w^i \to x^i \to R_i \to y^i \to v^i \to v.$$

Note that, for $2 \le i \le h_s$ (respectively, $h_s + 1 \le i \le s$), subpath $u \to H_i' \to w^i$ is in Q_s^0, subpath $x^i \to R_i \to y^i$ is in $Q_t^{2^{h_s}-2^{i-1}-1}$ (respectively, $Q_t^{2^{h_s}+2^{i-1}-1}$) and edge $v^i \to v$ is in $Q_s^{2^{h_t}-1}$ while $w^i \to x^i$ and $y^i \to v^i$ are two edges in E_1. Moreover, the vertices in $Q_t^{2^{h_s}-1}$ are not in P_i for $2 \le i \le s$. It is easy to verify that all those P_i for $2 \le i \le s$ are internally disjoint.

Since both x^i and y^i are in $Q_t^{2^{h_s}-2^{i-1}-1}$ (or $Q_t^{2^{h_s}+2^{i-1}-1}$), $l(R_i) = H(x^i, y^i) = h_t$. Note that subpath $u \to H_i' \to w^i$ is of length $h_s - 1$ for $2 \le i \le h_s$ and $h_s + 1$ for $h_s + 1 \le i \le s$. As a result, $l(P_i) = h_s + h_t + 2$ for $2 \le i \le h_s$ and $l(P_i) = h_s + h_t + 4$ for $h_s + 1 \le i \le s$.

Next, the path P_{s+1} can be constructed as follows:

$$u \to u^0 \to R \to z^0 \to z \to L_1 \to v.$$

Note that $u^0 = 0^s0^t1$ and $z^0 = 0^s0^{t-h_t}1^{h_t}1$. We can find that subpath $u^0 \to R \to z^0$ is in Q_t^0 and subpath $z \to L_1 \to v$ is in $Q_s^{2^{h_t}-1}$ while $u \to u^0$ and $z^0 \to z$ are two edges in E_1. By inspection, the vertices in Q_t^0 are not in P_i for $1 \le i \le s$. Thus, all paths P_i for $1 \le i \le s+1$ are internally disjoint. Since both u^0 and z^0 are in Q_t^0, $l(R) = H(w^0, v^0) = h_t$. In addition, both z and v are in $Q_s^{2^{h_t}-1}$, and $l(L_1) = h_s$. Therefore, $l(P_{s+1}) = h_s + h_t + 2$. This completes the proof. □

Lemma 4. *Let u and v be two vertices of* $\text{EH}(s,t)$ *for $3 \le s \le t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 = v_0 = 1$, and $h_s(u,v) = 0$, then there exist $t+1$ internally disjoint paths P_i for $1 \le i \le t+1$ between u and v such that h_t of them are of length h_t, $t - h_t$ paths are of length $h_t + 2$, and one path is of length $h_t + 6$.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t1$ and $v = 0^s0^{t-h_t}1^{h_t}1$ are in Q_t^0. By Theorem 2, in Q_t^0, there exist t internally disjoint paths between u and v such that h_t of them are of length h_t, and the remaining $t - h_t$ paths are of length $h_t + 2$. Let $u \to H_i \to v$ for $1 \le i \le t$ be those t internally disjoint paths. The following sets of $t+1$

internally disjoint paths between u and v in $\mathrm{EH}(s,t)$ can be set:

We construct the paths P_i for $1 \le i \le t$ from u to v as follows:

$$u \to P_i \to v.$$

Note that all the paths $u \to P_i \to v$ are in Q_t^0. Thus, $l(P_i) = h_t$ for $1 \le i \le h_t$ and $l(P_i) = h_t + 2$ for $h_t + 1 \le i \le t$.

The path P_{t+1} can be constructed from u to v as follows:

$$\begin{aligned} u &\to 0^s0^t0 \to 0^{s-1}10^t0 \\ &\to 0^{s-1}10^t1 \to L \to 0^{s-1}10^{t-h_t}1^{h_t}1 \\ &\to 0^{s-1}10^{t-h_t}1^{h_t}0 \to 0^s0^{t-h_t}1^{h_t}0 \to v. \end{aligned}$$

Note that edge $0^s0^t0 \to 0^{s-1}10^t0$ is in Q_s^0, subpath $0^{s-1}10^t1 \to L \to 0^{s-1}10^{t-h_t}1^{h_t}1$ is in Q_t^1 and edge $0^{s-1}10^{t-h_t}1^{h_t}0 \to 0^s0^{t-h_t}1^{h_t}0$ is in $Q_s^{2^{h_t}-1}$. It is easy to verify that P_{t+1} is also internally disjoint to P_i for $1 \le i \le t$. Furthermore, we have $l(L) = h_t$. Therefore, $l(P_{t+1}) = h_t + 6$. This completes the proof. □

Lemma 5. *Let u and v be two vertices of $\mathrm{EH}(s,t)$ for $3 \le s \le t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 = v_0 = 1$, and $h_s(u,v) \ne 0$, then there exist $t+1$ internally disjoint paths P_i for $1 \le i \le t+1$ between u and v such that $h_t + 1$ of them are of length $h_s + h_t + 2$ and $t - h_t$ paths are of length $h_s + h_t + 4$.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t1$ and $v = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}1$ are in Q_t^0 and $Q_t^{2^{h_s}-1}$, respectively. Depending on h_t, two cases are distinguished.

Case 1: $h_t = 0$. Then, $u = 0^s0^t1$ and $v = 0^{s-h_s}1^{h_s}0^t1$. Let $u^i = u \oplus 0^s0^{t-i}10^{i-1}0 = 0^s0^{t-i}10^{i-1}1$ and $v^i = v \oplus 0^s0^{t-i}10^{i-1}0 = 0^{s-h_s}1^{h_s}0^{t-i}10^{i-1}1$ for $1 \le i \le t$ where $\oplus$ is the exclusive-or operation. To construct a path from u^i to v^i, we need the following intermediate vertices: $x^i = u^i \oplus 0^s0^t1 = 0^s0^{t-i}10^{i-1}0$ and $y^i = v^i \oplus 0^s0^t1 = 0^{s-h_s}1^{h_s}0^{t-i}10^{i-1}0$. Now we construct path P_i for $1 \le i \le t$ from u to v as follows:

$$u \to u^i \to x^i \to R_i \to y^i \to v^i \to v.$$

Note that edge $u \to u^i$ is in Q_t^0, subpath $x^i \to R_i \to y^i$ is in $Q_s^{2^{i-1}}$ and edge $v^i \to v$ is in $Q_t^{2^{h_s}-1}$ while $u^i \to x^i$ and $y^i \to v^i$ are two edges in E_1.

It remains to construct the $(t+1)$-th internally disjoint path from u to v. Path P_{t+1} can be constructed as follows:

$$u \to u^0 \to R \to v^0 \to v.$$

Note that $u^0 = 0^s0^t0$ and $v^0 = 0^{s-h_s}1^{h_s}0^t0$. We can find that subpath $u^0 \to R \to v^0$ is in Q_s^0 while $u \to u^0$ and $v^0 \to v$ are two edges in E_1. This can be confirmed that those P_i for $1 \le i \le t+1$ are internally disjoint. Moreover, we have $l(R_i) = l(R) = h_s$. Therefore, $l(P_i) = h_s + 4$ for $1 \le i \le t$ and $l(P_{t+1}) = h_s + 2$.

Case 2: $1 \le h_t \le t$. Then, $u = 0^s0^t1$ and $v = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}1$. Let $w = 0^s0^{t-h_t}1^{h_t}1$ and $z = 0^{s-h_s}1^{h_s}0^t1$ be in Q_t^0 and $Q_t^{2^{h_s}-1}$, respectively. By Theorem 2, in Q_t^0, there exist t internally disjoint paths between u and w such that h_t of them are of length h_t, and the remaining $t - h_t$ paths are of length $h_t + 2$. Let $u \to H_i \to w$ for $1 \le i \le t$ be those internally disjoint paths. Similarly, there exist t internally disjoint paths between z and v such that h_t of them are of length h_t, and the remaining $t - h_t$ paths are of length $h_t + 2$. We also denote $z \to L_i \to v$ for $1 \le i \le t$ are those internally disjoint paths.

Now the path P_1 can be constructed as follows:

$$u \to H_1 \to w \to w^0 \to R' \to v^0 \to v.$$

Note that $w^0 = 0^s0^{t-h_t}1^{h_t}0$ and $v^0 = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}0$. We can find that subpath $u \to H_1 \to w$ is in Q_t^0 and subpath $w^0 \to R' \to v^0$ is in $Q_s^{2^{h_t}-1}$ while $w \to w^0$ and $v^0 \to v$ are two edges in E_1. Additionally, we have $l(R') = h_s$ and $l(H_1) = h_t$. Hence $l(P_1) = h_s + h_t + 2$.

Based on H_i and L_i for $2 \le i \le t$, we construct $t-1$ internally disjoint paths from u to v as follows. Let $w^i = \mathrm{pre}(H_i, w, i)$ and $v^i = \mathrm{pre}(L_i, v, i)$ where their i-th bits are different. This is, $w^i = 0^s0^{t-h_t}1^{h_t-i}01^{i-1}1$ and $v^i = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t-i}01^{i-1}1$ when $2 \le i \le h_t$, and $w^i = 0^s0^{t-h_t}1^{h_t-i}01^{i-1}1$ and $v^i = 0^{s-h_s}1^{h_s}0^{t-i}10^{i-h_t-1}1^{h_t}1$ when $h_t + 1 \le i \le t$. Assume that H_i' is the subpath of H_i without containing w. Clearly, all $u \to H_i' \to w^i$ for $2 \le i \le t$ are internally disjoint. To construct a path from w^i to v^i, we need the following intermediate vertices: $x^i = 0^s0^{t-h_t}1^{h_t-i}01^{i-1}0$ and $y^i = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t-i}01^{i-1}0$ when $2 \le i \le h_s$, and $x^i = 0^s0^{t-i}10^{i-h_t-1}1^{h_t}0$ and $y^i = 0^{s-h_s}1^{h_s}0^{t-i}10^{i-h_t-1}1^{h_t}0$ when $h_s + 1 \le i \le t$. We construct paths P_i for $2 \le i \le t$ from u to v as follows:

$$u \to H_i' \to w^i \to x^i \to R_i \to y^i \to v^i \to v.$$

Note that, for $2 \le i \le h_t$ (respectively, $h_s + 1 \le i \le t$), subpath $u \to H_i' \to w^i$ is in Q_t^0, subpath $x^i \to R_i \to y^i$ is in $Q_s^{2^{h_t}-2^{i-1}-1}$ (respectively, $Q_s^{2^{h_t}+2^{i-1}-1}$) and edge $v^i \to v$ is in $Q_t^{2^{h_s}-1}$ while $w^i \to x^i$ and $y^i \to v^i$ are two edges in E_1. Additionally, we have $l(H_i') = h_t - 1$ for $2 \le i \le h_t$ and $l(H_i') = h_t + 1$ for $h_t + 1 \le i \le t$, and $l(R_i) = h_s$ for $2 \le i \le t$, As a result, $l(P_i) = h_s + h_t + 2$ for $2 \le i \le h_t$ and $l(P_i) = h_s + h_t + 4$ for $h_t + 1 \le i \le t$.

Next, the path P_{t+1} can be constructed as follows:

$$u \to u^0 \to R \to z^0 \to z \to L_1 \to v.$$

Note that $u^0 = 0^s0^t0$ and $z^0 = 0^{s-h_s}1^{h_s}0^t0$. We can find that subpath $u^0 \to R \to z^0$ is in Q_s^0 and subpath $z \to L_1 \to v$ is in $Q_t^{2^{h_s}-1}$ while $u \to u^0$ and $z^0 \to z$ are two edges in E_1. Moreover, we have $l(R) = h_s$ and $l(L_1) = h_t$. Therefore, $l(P_{t+1}) = h_s + h_t + 2$.

It is easy to verify that all those P_i for $1 \le i \le t+1$ are internally disjoint. This completes the proof. □ □

Lemma 6. *Let u and v be two vertices of $\mathrm{EH}(s,t)$ for $3 \le s \le t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 \ne v_0$ and $h_s(u,v) = h_t(u,v) = 0$, then there exist $s+1$ internally disjoint paths P_i for $1 \le i \le s+1$ between u and v such that s of them are of length 7 and one path is of length 1.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t0$ and $v = 0^s0^t1$ are in Q_s^0 and Q_t^0, respectively. Let $u^i = u \oplus 0^{s-i}10^{i-1}0^t0 = 0^{s-i}10^{i-1}0^t1$ and

$v^i = v \oplus 0^s0^{t-i}10^{i-1}0 = 0^s0^{t-i}10^{i-1}1$ for $1 \le i \le t$ where $\oplus$ is exclusive-or operation. To construct a path from u^i to v^i, we need the following intermediate vertices: $x^i = u^i \oplus 0^s0^t1 = 0^{s-i}10^{i-1}0^t1$, $y^i = x^i \oplus 0^s0^{t-i}10^{i-1}0 = 0^{s-i}10^{i-1}0^{t-i}10^{i-1}1$, $z^i = w^i \oplus 0^{s-i}\,10^{i-1}0^t0 = 0^{s-i}10^{i-1}0^{t-i}10^{i-1}0$ and $w^i = v^i \oplus 0^s0^t1 = 0^s\,0^{t-i}10^{i-1}0$. Now we construct path P_i for $1 \le i \le t$ from u to v as follows:

$$u \rightarrow u^i \rightarrow x^i \rightarrow y^i \rightarrow z^i \rightarrow w^i \rightarrow v^i \rightarrow v.$$

Note that, for $1 \le i \le s$, edge $u \rightarrow u^i$ is in Q_{s}^0, edge $x^i \rightarrow y^i$ is in $Q_{\mathrm{t}}^{2^{i-1}}$, edge $z^i \rightarrow w^i$ is in $Q_{\mathrm{s}}^{2^{i-1}}$ and edge $v^i \rightarrow v$ is in Q_{t}^0 while $u^i \rightarrow x^i$, $y^i \rightarrow z^i$ and $w^i \rightarrow v^i$ are three edges in E_1. Thus, $l(P_i) = 7$ for $1 \le i \le s$. This can be confirmed that all those P_i for $1 \le i \le s$ are internally disjoint.

Finally, the path P_{s+1} is $u \rightarrow v$, and $l(P_{s+1}) = 1$. This completes the proof. □

Lemma 7. *Let u and v be two vertices of $\mathrm{EH}(s,t)$ for $3 \le s \le t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 \ne v_0$, $h_{\mathrm{s}}(u,v) \ne 0$ and $h_{\mathrm{t}}(u,v) = 0$, then there exist $s+1$ internally disjoint paths P_i for $1 \le i \le s+1$ between u and v such that h_{s} of them are of length $h_{\mathrm{s}}+5$, $s-h_{\mathrm{s}}$ paths are of length $h_{\mathrm{s}}+7$, and one path is of length $h_{\mathrm{s}}+1$.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t0$ and $v = 0^{s-h_s}1^{h_s}0^t1$ are in Q_{s}^0 and $Q_{\mathrm{t}}^{2^{h_s}-1}$, respectively. Let $w = 0^{s-h_s}1^{h_s}0^t0$ be in Q_{s}^0. By Theorem 2, in Q_{s}^0, there exist s internally disjoint paths between u and w such that h_{s} of them are of length h_{s}, and the remaining $s-h_{\mathrm{s}}$ paths are of length $h_{\mathrm{s}}+2$. Let $u \rightarrow H_i \rightarrow w$ for $1 \le i \le s$ be those internally disjoint paths.

The path P_1 can be constructed as follows:

$$u \rightarrow H_1 \rightarrow w \rightarrow v.$$

Note that subpath $u \rightarrow H_1 \rightarrow w$ is in Q_{s}^0 while $w \rightarrow v$ is an edge in E_1. Thus, $l(P_1) = h_{\mathrm{s}}+1$.

Based on H_i for $2 \le i \le s$, we construct $s-1$ internally disjoint paths from u to v as follows. Let $w^i = \mathrm{pre}(H_i, w, i)$ where their $(t+i)$-th bits are different. This is, $w^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^t0$ when $2 \le i \le h_{\mathrm{s}}$ and $w^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^t0$ when $h_{\mathrm{s}}+1 \le i \le s$. Assume that H_i' is the subpath of H_i without containing w. Clearly, all $u \rightarrow H_i' \rightarrow w^i$ for $2 \rightarrow i \rightarrow s$ are internally disjoint. Let $v^i = v \oplus 0^{s-h_s}1^{h_s}0^t0 = 0^{s-h_s}1^{h_s}0^{t-i}10^{i-1}1$ where $\oplus$ is the exclusion-or operation. To construct a path from w_i to v_i, we need the following intermediate vertices: $a^i = 0^{s-h_s}\,1^{h_s-i}01^{i-1}0^t1$, $b^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^{t-i}10^{i-1}1$ and $c^i = 0^{s-h_s}\,1^{h_s-i}01^{i-1}0^{t-i}10^{i-1}0$ when $2 \le i \le h_{\mathrm{s}}$, and $a^i = 0^{s-i}\,10^{i-h_s-1}1^{h_s}0^t1$, $b^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^{t-i}10^{i-1}1$ and $c^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^{t-i}10^{i-1}0$ when $h_{\mathrm{s}}+1 \le i \le s$. $d^i = 0^{s-h_s}\,1^{h_s}0^{t-i}10^{i-1}1$ when $2 \le i \le s$. We construct paths P_i for $2 \le i \le s$ from u to v as follows:

$$u \rightarrow H_i' \rightarrow w^i \rightarrow a^i \rightarrow b^i \rightarrow c^i \rightarrow d^i \rightarrow v^i \rightarrow v.$$

Note that, for $2 \le i \le h_{\mathrm{s}}$ (respectively, $h_{\mathrm{s}}+1 \le i \le s$), subpath $u \rightarrow H_i' \rightarrow w^i$ is in Q_{s}^0, edge $a^i \rightarrow b^i$ is in $Q_{\mathrm{t}}^{2^{h_s}-2^{i-1}-1}$ (respectively, $Q_{\mathrm{t}}^{2^{h_s}+2^{i-1}-1}$), edge $c^i \rightarrow d^i$ is in $Q_{\mathrm{s}}^{2^{i-1}}$ and edge $v^i \rightarrow v$ is in $Q_{\mathrm{s}}^{2^{h_s}-1}$ while $w^i \rightarrow a^i$, $b^i \rightarrow c^i$ and $d^i \rightarrow v^i$ are three edges in E_1. Moreover, we have $l(H_i') = h_{\mathrm{s}}-1$ for $2 \le i \le h_{\mathrm{s}}$ and $l(H_i') = h_{\mathrm{s}}+1$ for $h_{\mathrm{s}}+1 \le i \le s$. As a result, $l(P_i) = h_{\mathrm{s}}+5$ for $2 \le i \le h_{\mathrm{s}}$ and $l(P_i) = h_{\mathrm{s}}+7$ for $h_{\mathrm{s}}+1 \le i \le s$.

The path P_{s+1} can be constructed from u to v as follows:

$$u \rightarrow u^0 \rightarrow x \rightarrow y \rightarrow L \rightarrow z \rightarrow v^1 \rightarrow v.$$

Note that $u^0 = 0^s0^t1$, $x = 0^s0^{t-1}11$, $y = 0^s0^{t-1}10$ and $z = 0^{s-h_s}1^{h_s}0^{t-1}10$. We can find that edge $u^0 \rightarrow x$ is in Q_{t}^0, subpath $y \rightarrow L \rightarrow z$ is in Q_{s}^1 and edge $v^1 \rightarrow v$ is in $Q_{\mathrm{s}}^{2^{h_s}-1}$ while $u \rightarrow u^0$, $x \rightarrow y$ and $z \rightarrow v^1$ are three edges in E_1. Furthermore, we have $l(L) = h_{\mathrm{s}}$. Therefore, $l(P_{s+1}) = h_{\mathrm{s}}+5$.

It is easy to verify that all those P_i for $1 \le i \le s+1$ are internally disjoint. This completes the proof. □

Lemma 8. *Let u and v be two vertices of $\mathrm{EH}(s,t)$ for $3 \le s \le t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 \ne v_0$, $h_{\mathrm{s}}(u,v) = 0$ and $h_{\mathrm{t}}(u,v) \ne 0$, then there exist $s+1$ internally disjoint paths P_i for $1 \le i \le s+1$ between u and v such that the following two cases are distinguished.*

1) *If $h_t(u,v) \ge s$, then s of them are of length h_t+5 and one path is of length h_t+1.*
2) *If $h_t(u,v) \le s-1$, then h_{t} of them are of length $h_{\mathrm{t}}+5$, $s-h_t$ paths are of length h_t+7, and one path is of length h_t+1.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t0$ and $v = 0^s0^{t-h_t}1^{h_t}1$ are in Q_{s}^0 and Q_{t}^0, respectively. Let $u^i = u \oplus 0^{s-i}10^{i-1}0^t0 = 0^{s-i}10^{i-1}0^t0$ for $1 \le i \le s$ where $\oplus$ is the exclusive-or operation.

Now the path P_1 can be constructed as follows:

$$u \rightarrow u^1 \rightarrow x \rightarrow L \rightarrow y \rightarrow z \rightarrow v^0 \rightarrow v.$$

Note that $x = 0^{s-1}10^t1$, $y = 0^{s-1}10^{t-h_t}1^{h_t}1$, $z = 0^{s-1}\,10^{t-h_t}1^{h_t}0$ and $v^0 = 0^s0^{t-h_t}1^{h_t}0$. We can find that edge $u \rightarrow u^1$ is in Q_{s}^0, subpath $x \rightarrow L \rightarrow y$ is in Q_{t}^1, edge $z \rightarrow v^0$ is in $Q^{2^{h_t}-1}$ while $u^1 \rightarrow x$, $y \rightarrow z$ and $v^0 \rightarrow v$ are three edges in E_1. Moreover, we have $l(L) = h_{\mathrm{t}}$. Hence $l(P_1) = h_{\mathrm{t}}+5$.

Let $w = 0^s0^t1$ be in Q_{t}^0. By Theorem 2, in Q_{t}^0, there exist t internally disjoint paths between w and v such that h_{t} of them are of length h_{t}, and the remaining $t-h_{\mathrm{t}}$ paths are of length $h_{\mathrm{t}}+2$. Let $w \rightarrow H_i \rightarrow v$ for $1 \le i \le t$ be those internally disjoint paths. Based on H_i for $2 \le i \le s$, we construct $s-1$ internally disjoint paths from u to v as follows. Let $w^i = \mathrm{suc}(H_i, w, i)$ where their i-th bits are different. This is, $w^i = 0^s0^{t-i}10^{i-1}1$ when $2 \le i \le s$. Assume that H_i' is the subpath of H_i without containing w. Clearly, all $w^i \rightarrow H_i' \rightarrow v$ for $2 \le i \le s$ are internally disjoint. To construct a path from u^i to w^i, we need the following intermediate vertices: $a^i = 0^{s-i}10^{i-1}0^t1$, $b^i = 0^{s-i}\,10^{i-1}0^{t-i}10^{i-1}1$, $c^i = 0^{s-i}10^{i-1}0^{t-i}10^{i-1}0$ and $d^i = 0^s0^{t-i}\,10^{i-1}0$ when $2 \le i \le s$. We construct paths P_i for $2 \le i \le s$ from u to v as follows:

$$u \rightarrow u^i \rightarrow a^i \rightarrow b^i \rightarrow c^i \rightarrow d^i \rightarrow w^i \rightarrow H_i' \rightarrow v.$$

Note that edge $u \rightarrow u^i$ is in Q_{s}^0, edge $a^i \rightarrow b^i$ is in $Q_{\mathrm{t}}^{2^{i-1}}$, edge $c^i \rightarrow d^i$ is in $Q_{\mathrm{s}}^{2^{i-1}}$ and subpath $w^i \rightarrow H_i' \rightarrow v$ is in Q_{t}^0

while $u^i \to a^i$, $b^i \to c^i$ and $d^i \to w^i$ are three edges in E_1. In addition, if $h_t(u,v) \geq s$, then $l(H_i') = h_t - 1$ for $2 \leq i \leq s$; otherwise, $l(H_i') = h_t - 1$ for $2 \leq i \leq h_t$ and $l(H_i') = h_t + 1$ for $h_t + 1 \leq i \leq s$. Thus, if $h_t(u,v) \geq s$, we have $l(P_i) = h_t + 5$ for $2 \leq i \leq s$; otherwise, $l(P_i) = h_t + 5$ for $2 \leq i \leq h_t$ and $l(P_i) = h_t + 7$ for $h_t + 1 \leq i \leq s$.

The path P_{s+1} can be constructed as follows:

$$u \to w \to H_1 \to v.$$

Note that subpath $w \to H_1 \to v$ is in Q_t^0 while $u \to w$ is an edge in E_1. Moreover, we have $l(H_1) = h_t$. Therefore, $l(P_1) = h_t + 1$.

It is easy to verify that all those P_i for $1 \leq i \leq s+1$ are internally disjoint. This completes the proof. □

Lemma 9. *Let u and v be two vertices of* $\mathrm{EH}(s,t)$ *for $3 \leq s \leq t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 \neq v_0$, $h_s(u,v) = s$ and $h_t(u,v) \neq 0$, then there exist $s+1$ internally disjoint paths P_i for $1 \leq i \leq s+1$ between u and v such that the following two cases are distinguished.*

1) *If $h_t(u,v) \geq s$, then s of them are of length $s + h_t + 3$ and one path is of length $s + h_t + 1$.*
2) *If $h_t(u,v) \leq s-1$, then $h_t + 2$ of them are of length $s + h_t + 3$ and $s - h_t - 1$ paths are of length $s + h_t + 5$.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t0$ and $v = 1^s0^{t-h_t}1^{h_t}1$ are in Q_s^0 and $Q_t^{2^s-1}$, respectively. Let $w = 1^s0^t0$ and $x = 1^s0^t0$ be in Q_s^0 and $Q_t^{2^s-1}$, respectively. By Theorem 2, in Q_s^0, there exist s internally disjoint paths between u and w such that s of them are of length s. In addition, there exist t internally disjoint paths between x and v such that h_t of them are of length h_t, and the remaining $t - h_t$ paths are of length $h_t + 2$. Let $u \to H_i \to w$ for $1 \leq i \leq s$ and $x \to L_i \to v$ for $1 \leq i \leq t$ be those internally disjoint paths. Based on H_i for $1 \leq i \leq s-1$, we construct $s-1$ internally disjoint paths from u to v as follows. Let $w^i = \mathrm{pre}(H_i, w, i)$ where their $(t+i)$-th bits are different. This is, $w^i = 1^{s-i}01^{i-1}0^t0$ when $1 \leq i \leq s-1$. Assume that H_i' is the subpath of H_i without containing w. Clearly, all $u \to H_i' \to w^i$ for $1 \leq i \leq s-1$ are internally disjoint. Similarly, based on L_i for $1 \leq i \leq s-1$, we construct $s-1$ internally disjoint paths from u to v as follows. Let $x^i = \mathrm{suc}(L_i, x, i)$ where their i-th bits are different. This is, $x^i = 1^s0^{t-i}10^{i-1}1$ when $1 \leq i \leq s-1$. Assume that L_i' is the subpath of L_i without containing x. Clearly, all $w^i \to L_i' \to v$ for $1 \leq i \leq s-1$ are internally disjoint. To construct a path from w^i to x^i, we need the following intermediate vertices: $a^i = 1^{s-i}01^{i-1}0^t1$, $b^i = 1^{s-i}01^{i-1}0^{t-i}10^{i-1}1$, $c^i = 1^{s-i}01^{i-1}0^{t-i}10^{i-1}0$ and $d^i = 1^s0^{t-i}10^{i-1}0$ when $1 \leq i \leq s-1$. We construct paths P_i for $1 \leq i \leq s-1$ from u to v as follows:

$$u \to H_i' \to w^i \to a^i \to b^i \to c^i \to d^i \to x^i \to L_i' \to v.$$

Note that subpath $u \to H_i' \to w^i$ is in Q_s^0, edge $a^i \to b^i$ is in $Q_t^{2^s-2^{i-1}-1}$, edge $c^i \to d^i$ is in $Q_s^{2^{i-1}}$, and subpath $x^i \to L_i' \to v$ is in $Q_t^{2^s-1}$ while $w^i \to a^i$, $b^i \to c^i$ and $d^i \to x^i$ are three edges in E_1. Furthermore, we have $l(H_i') = s-1$ for $1 \leq i \leq s-1$. In addition, if $h_t(u,v) \geq s$, then $l(L_i') = h_t - 1$ for $1 \leq i \leq s-1$; otherwise, $l(L_i') = h_t - 1$ for $1 \leq i \leq h_t$ and $l(L_i') = h_t + 1$ for $h_t + 1 \leq i \leq s-1$. Thus, if $h_t(u,v) \geq s$, we have $l(P_i) = s + h_t + 3$ for $1 \leq i \leq s-1$; otherwise, $l(P_i) = s + h_t + 3$ for $1 \leq i \leq h_t$ and $l(P_i) = s + h_t + 5$ for $h_t + 1 \leq i \leq s-1$.

The path P_s can be constructed as follows:

$$u \to H_s \to w \to x \to L_s \to v.$$

Note that subpath $u \to H_s \to w$ is in Q_s^0, subpath $x \to L_s \to v$ is in $Q_t^{2^s-1}$ while $w \to x$ is an edge in E_1. Moreover, if $h_t(u,v) \geq s$, then $l(L_s) = h_t - 1$; otherwise, $l(L_s) = h_t + 1$. Hence, if $h_t(u,v) \geq s$, we have $l(P_s) = s + h_t + 1$; otherwise, $l(P_s) = s + h_t + 3$.

It remains to construct the $(s+1)$-th internally disjoint path from u to v. Path P_{s+1} can be constructed as follows:

$$u \to u^0 \to R \to y \to z \to K \to v^0 \to v.$$

Note that $u^0 = 0^s0^t1$, $y = 0^s0^{t-h_t}1^{h_t}1$, $z = 0^s0^{t-h_t}1^{h_t}0$ and $v^0 = 1^s0^{t-h_t}1^{h_t}0$. We can find that subpath $u^0 \to R \to y$ is in Q_t^0 and $z \to K \to v^0$ is in $Q_s^{2^{h_t}-1}$ while $u \to u^0$, $y \to z$ and $v^0 \to v$ are three edges in E_1. Furthermore, we have $l(R) = h_t$ and $l(K) = s$. Therefore, $l(P_{s+1}) = s + h_t = 3$.

It is easy to verify that all those P_i for $1 \leq i \leq s+1$ are internally disjoint. This completes the proof. □

Lemma 10. *Let u and v be two vertices of* $\mathrm{EH}(s,t)$ *for $3 \leq s \leq t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 \neq v_0$, $1 \leq h_s(u,v) \leq s-1$ and $s \leq h_t(u,v) \leq t$, then there exist $s+1$ internally disjoint paths P_i for $1 \leq i \leq s+1$ between u and v such that $h_s + 2$ of them are of length $h_s + h_t + 3$ and $s - h_s - 1$ paths are of length $h_s + h_t + 5$.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t0$ and $v = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}1$ are in Q_s^0 and $Q_t^{2^{h_s}-1}$, respectively. Let $w = 0^{s-h_s}1^{h_s}0^t0$ and $x = 0^{s-h_s}1^{h_s}0^t1$ be in Q_s^0 and $Q_t^{2^{h_s}-1}$, respectively. By Theorem 2, in Q_s^0, there exist s internally disjoint paths between u and w such that h_s of them are of length h_s, and the remaining $s - h_s$ paths are of length $h_s + 2$. In addition, there exist t internally disjoint paths between x and v such that h_t of them are of length h_t, and the remaining $t - h_t$ paths are of length $h_t + 2$. Let $u \to H_i \to w$ for $1 \leq i \leq s$ and $x \to L_i \to v$ for $1 \leq i \leq t$ be those internally disjoint paths. Based on H_i for $1 \leq i \leq s-1$, we construct $s-1$ internally disjoint paths from u to w as follows. Let $w^i = \mathrm{pre}(H_i, w, i)$ where their $(t+i)$-th bits are different. This is, $w^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^t0$ when $1 \leq i \leq h_s$ and $w^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^t0$ when $h_s + 1 \leq i \leq s-1$. Assume that H_i' is the subpath of H_i without containing w. Clearly, all $u \to H_i' \to w^i$ for $1 \leq i \leq s-1$ are internally disjoint. Similarly, based on L_i for $1 \leq i \leq s-1$, we construct $s-1$ internally disjoint paths from x to v as follows. Let $x^i = \mathrm{suc}(L_i, x, i)$ where their i-th bits are different. This is, $x^i = 0^{s-h_s}1^{h_s}0^{t-i}10^{i-1}1$ when $1 \leq i \leq s-1$. Assume that L_i' is the subpath of L_i without containing x. Clearly, all $x^i \to L_i' \to v$ for $1 \leq i \leq s-1$ are internally disjoint. To construct a path from w^i to x^i, we need the following intermediate vertices: $a^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^t1$, $b^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^{t-i}10^{i-1}1$,

$c^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^{t-i}10^{i-1}0$ when $1 \le i \le h_s$, and $a^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^t1$, $b^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^{t-i}10^{i-1}1$, $c^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^{t-i}10^{i-1}0$ when $h_s+1 \le i \le s-1$. $d^i = 0^{s-h_s}1^{h_s}0^{t-i}10^{i-1}0$ when $1 \le i \le s-1$. We construct paths P_i for $1 \le i \le s-1$ from u to v as follows:

$$u \to H_i' \to w^i \to a^i \to b^i \to c^i \to d^i \to x^i \to L_i' \to v.$$

Note that, for $1 \le i \le h_s$ (respectively, $h_s+1 \le i \le s-1$), subpath $u \to H_i' \to w^i$ is in Q_s^0, edge $a^i \to b^i$ is in $Q_t^{2^{h_s}-2^{i-1}-1}$ (respectively, $Q_t^{2^{h_s}+2^{i-1}-1}$), edge $c^i \to d^i$ is in $Q_s^{2^{i-1}}$, and subpath $x^i \to L_i' \to v$ is in $Q_t^{2^{h_s}-1}$ while $w^i \to a^i$, $b^i \to c^i$ and $d^i \to x^i$ are three edges in E_1. Moreover, we have $l(H_i') = h_s - 1$ for $1 \le i \le h_s$ and $l(H_i') = h_s + 1$ for $h_s+1 \le i \le s-1$, and $l(L_i') = h_s - 1$ for $1 \le i \le s-1$. Thus, we have $l(P_i) = h_s + h_t + 3$ for $1 \le i \le h_s$ and $l(P_i) = h_s + h_t + 5$ for $h_s+1 \le i \le s-1$.

The path P_s can be constructed as follows:

$$u \to H_s \to w \to x \to L_s \to v.$$

Note that subpath $u \to H_s \to w$ is in Q_s^0 and subpath $x \to L_s \to v$ is in $Q_t^{2^{h_s}-1}$ while $w \to x$ is an edge in E_1. Moreover, we have $l(H_s) = h_s + 2$ and $l(L_s) = h_t$. Hence $l(P_s) = h_s + h_t + 3$.

It remains to construct the $(s+1)$-th internally disjoint path from u to v. Path P_{s+1} can be constructed as follows:

$$u \to u^0 \to R \to y \to z \to K \to v^0 \to v.$$

Note that $u^0 = 0^s0^t1$, $y = 0^s0^{t-h_t}1^{h_t}1$, $z = 0^s0^{t-h_t}1^{h_t}0$ and $v^0 = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}0$. We can find that subpath $u^0 \to R \to y$ is in Q_t^0 and $z \to K \to v^0$ is in $Q_s^{2^{h_t}-1}$ while $u \to u^0$, $y \to z$ and $v^0 \to v$ are three edges in E_1. Furthermore, we have $l(R) = h_t$ and $l(K) = h_s$. Therefore, $l(P_{s+1}) = h_s + h_t = 3$.

It is easy to verify that all those P_i for $1 \le i \le s+1$ are internally disjoint. This completes the proof. □

Lemma 11. *Let u and v be two vertices of $\mathrm{EH}(s,t)$ for $3 \le s \le t$ with $u = u_{t+s}u_{t+s-1}\cdots u_0$ and $v = v_{t+s}v_{t+s-1}\cdots v_0$. If $u_0 \ne v_0$, $1 \le h_s(u,v) \le s-1$ and $1 \le h_t(u,v) \le s-1$, then there exist $s+1$ internally disjoint paths P_i for $1 \le i \le s+1$ between u and v such that the following three cases are distinguished.*

1) *If $h_s + h_t \ge t+1$, then $s+t-h_s-h_t-1$ of them are of length h_s+h_t+5 and h_s+h_t-t+2 paths are of length h_s+h_t+3.*
2) *If $s \le h_s+h_t \le t$, then $s-1$ of them are of length h_s+h_t+5 and two paths are of length h_s+h_t+3.*
3) *If $h_s+h_t \le s-1$, then h_s+h_t+1 of them are of length h_s+h_t+5, $s-h_s-h_t-1$ of them are of length h_s+h_t+7, and one path is of length h_s+h_t+3.*

Proof. By Theorem 3, we may assume without loss of generality that $u = 0^s0^t0$ and $v = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}1$ are in Q_s^0 and $Q_t^{2^{h_s}-1}$, respectively. Let $w = 0^{s-h_s}1^{h_s}0^t0$ and $x = 0^{s-h_s}1^{h_s}0^t1$ be in Q_s^0 and $Q_t^{2^{h_s}-1}$, respectively. By Theorem 2, in Q_s^0, there exist s internally disjoint paths between u and w such that h_s of them are of length h_s, and the remaining $s-h_s$ paths are of length h_s+2. In addition, there exist t internally disjoint paths between x and v such that h_t of them are of length h_t, and the remaining $t-h_t$ paths are of length h_t+2. Let $u \to H_i \to w$ for $1 \le i \le s$ and $x \to L_i \to v$ for $1 \le i \le t$ be those internally disjoint paths. Based on H_i for $1 \le i \le s-1$, we construct s internally disjoint paths from u to w as follows. Let $w^i = \mathrm{pre}(H_i, w, i)$ where their $(t+i)$-th bits are different. This is, $w^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^t0$ when $1 \le i \le h_s$ and $w^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^t0$ when $h_s+1 \le i \le s$. Assume that H_i' is the subpath of H_i without containing w. Clearly, all $u \to H_i' \to w^i$ for $1 \le i \le s$ are internally disjoint. Similarly, based on L_k for $1 \le k \le t$, we construct t internally disjoint paths from x to v as follows. Let $x^k = \mathrm{suc}(L_k, x, k)$ where their k-th bits are different. This is, $x^k = 0^{s-h_s}1^{h_s}0^{t-k}10^{k-1}1$ when $1 \le k \le t$. Assume that L_k' is the subpath of L_k without containing x. Clearly, all $x^k \to L_k' \to v$ for $1 \le k \le t$ are internally disjoint. To construct a path from w^i to x^i, we need the intermediate vertices: $a^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^t1$, $b^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^{t-k}10^{k-1}1$ and $c^i = 0^{s-h_s}1^{h_s-i}01^{i-1}0^{t-k}10^{k-1}0$ when $1 \le i \le h_s$, and $a^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^t1$, $b^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^{t-k}10^{k-1}1$ and $c^i = 0^{s-i}10^{i-h_s-1}1^{h_s}0^{t-k}10^{k-1}0$ when $h_s+1 \le i \le s-1$. $d^k = 0^{s-h_s}1^{h_s}0^{t-k}10^{k-1}0$ when $1 \le i \le s-1$. Now we construct paths P_i for $1 \le i \le s-1$ from u to v as follows:

$$u \to H_i' \to w^i \to a^i \to b^i \to c^i \to d^k \to x^k \to L_k' \to v.$$

Note that, for $1 \le i \le h_s$ (respectively, $h_s+1 \le i \le s-1$), subpath $u \to H_i' \to w^i$ is in Q_s^0, edge $a^i \to b^i$ is in $Q_t^{2^{h_s}-2^{i-1}-1}$ (respectively, $Q_t^{2^{h_s}+2^{i-1}-1}$), edge $c^i \to d^k$ is in $Q_s^{2^{k-1}}$, and subpath $x^i \to L_k' \to v$ is in $Q_t^{2^{h_s}-1}$ while $w^i \to a^i$, $b^i \to c^i$ and $d^k \to x^k$ are three edges in E_1. Moreover, we have $l(H_i') = h_s - 1$ for $1 \le i \le h_s$ and $l(H_i') = h_s + 1$ for $h_s+1 \le i \le s$, and $l(L_i') = h_t - 1$ for $1 \le i \le h_t$ and $l(L_i') = h_t + 1$ for $h_t+1 \le i \le t$.

Then the path P_s can be constructed as follows:

$$u \to H_s \to w \to x \to L_k \to v.$$

Note that subpath $u \to H_s \to w$ is in Q_s^0, subpath $x \to L_k \to v$ is in $Q_t^{2^{h_s}-1}$ while $w \to x$ is an edge in E_1. Moreover, we have $l(H_s) = h_s + 2$ and $l(L_k)$ is equal to h_t or $h_t + 2$ depending on k.

Next, we calculate the length of path P_i for $1 \le i \le s$.

Case 1: $h_s + h_t \ge t+1$.

For $1 \le i \le t-h_t$, let $k = h_t + i$ and then $l(P_i) = h_s + h_t + 5$. For $t-h_t+1 \le i \le h_s$, let $k = h_t + i - t$ and then $l(P_i) = h_s + h_t + 3$. For $h_s+1 \le i \le s-1$, let $k = h_t + i - t$ and then $l(P_i) = h_s + h_t + 5$. For $i = s$, let $k = h_t + s - t$ and then $l(P_s) = h_s + h_t + 3$.

Case 2: $s \le h_s + h_t \le t$.

For $1 \le i \le h_s$, let $k = h_t + i$ and then $l(P_i) = h_s + h_t + 5$. For $h_s+1 \le i \le s-1$, let $k = i - h_s$ and then $l(P_i) = h_s + h_t + 5$. For $i = s$, let $k = s - h_s$ and then $l(P_s) = h_s + h_t + 3$.

Case 3: $h_s + h_t \le s-1$.

For $1 \le i \le h_s$, let $k = h_t + i$ and then $l(P_i) = h_s + h_t + 5$. For $h_s+1 \le i \le h_s+h_t$, let $k = i - h_s$ and then $l(P_i) = h_s + h_t + 5$. For $h_s+h_t+1 \le i \le s-1$, let $k = i$

TABLE 2
Comparison of Some Properties on Q_n, CQ_n and $\mathrm{EH}(s,t)$

Network	Vertices	Edges	Minimum degree	Diameter	IDP	Wide diameter	Fault diameter
Q_n	2^n	$n2^{n-1}$	n	n	n	$n+1$	$n+1$
CQ_n	2^n	$n2^{n-1}$	n	$\lceil \frac{n+1}{2} \rceil$	n	$\lceil \frac{n}{2} \rceil + 2$	$\lceil \frac{n}{2} \rceil + 2$
$\mathrm{EH}(s,t)$	2^{s+t+1}	$(s+t+2)2^{s+t-1}$	$s+1$	$s+t+2$	$s+1$ or $t+1$	$s+t+3$	$s+t+3$

IDP: the number of internally disjoint paths between any two vertices.

and then $l(P_i) = h_s + h_t + 7$. For $i = s$, let $k = s$ and then $l(P_s) = h_s + h_t + 5$.

Finally, the path P_{s+1} can be constructed as follows:

$$u \to u^0 \to R \to y \to z \to K \to v^0 \to v.$$

Note that $u^0 = 0^s0^t1$, $y = 0^s0^{t-h_t}1^{h_t}1$, $z = 0^s0^{t-h_t}1^{h_t}0$ and $v^0 = 0^{s-h_s}1^{h_s}0^{t-h_t}1^{h_t}0$. We can find that subpath $u^0 \to R \to y$ is in Q_t^0 and $z \to K \to v^0$ is in $Q_s^{2^{h_t}-1}$ while $u \to u^0$, $y \to z$ and $v^0 \to v$ are three edges in E_1. Furthermore, we have $l(R) = h_t$ and $l(K) = h_s$. Therefore, $l(P_{s+1}) = h_s + h_t = 3$.

It is easy to verify that all those P_i for $1 \le i \le s+1$ are internally disjoint. This completes the proof. □

By Lemmas 2-11 above, $s+1$ or $t+1$ internally disjoint paths between any two vertices of the exchanged hypercubes $\mathrm{EH}(s,t)$ can be constructed, and it can be verified that the length of each the internally disjoint paths is at most $s+t+3$. Take Lemma 2 for instance, one of the $s+1$ internally disjoint paths is of length $h_s + 6$. Then, $h_s + 6 \le s+t+3$ since $3 \le s \le t$. Additionally, take Lemma 3 for instance, $s - h_s$ of the $s+1$ internally disjoint paths is of length $h_s + h_t + 4$. Suppose that $h_s \le s-1$, then $h_s + h_t + 4 \le (s-1) + t + 4 = s+t+3$. Suppose that $h_s = s$, then no path is of length $h_s + h_t + 4$ since $s - h_s = 0$. Therefore, the following corollary can be obtained.

Corollary 1. *$D_{s+1}(\mathrm{EH}(s,t)) \le s+t+3$ for $3 \le s \le t$.*

The wide diameter and fault diameter of the exchanged hypercubes $\mathrm{EH}(s,t)$ for $3 \le s \le t$ are stated in Theorem 4.

Theorem 4. *$D_{s+1}(\mathrm{EH}(s,t)) = D_s^f(\mathrm{EH}(s,t)) = s+t+3$ for $3 \le s \le t$.*

Proof. Clearly, $D_s^f(\mathrm{EH}(s,t)) \le D_{s+1}(\mathrm{EH}(s,t))$. Additionally, by Lemma 1 and Corollary 1, we have that $s+t+3 \le D_s^f(\mathrm{EH}(s,t)) \le D_{s+1}(\mathrm{EH}(s,t)) \le s+t+3$ for $3 \le s \le t$. Therefore, $D_{s+1}(\mathrm{EH}(s,t)) = D_s^f(\mathrm{EH}(s,t)) = s+t+3$ for $3 \le s \le t$, and this completes the proof. □

For the cases of $s = 1, 2$ on the wide diameter and fault diameter of the exchanged hypercubes $\mathrm{EH}(s,t)$, the statement of Theorem 4 is not true. The following is a counterexample. Consider that $u = 0^s0^t1$, $u' = 0^s0^t0$, and $v = 0^s1^t1$. Assume that F is a faulty vertex set such that $F = N_{\mathrm{EH}(s,t)}(u) - u'$. The shortest path P between u and v in $\mathrm{EH}(s,t) - F$ can be written as the following:

$$\begin{aligned} P : u = 0^s0^t1 &\to u' \to 0^{s-1}10^t0 \to 0^{s-1}10^t1 \to L \\ &\to 0^{s-1}11^t1 \to 0^{s-1}11^t0 \to 0^s1^t0 \\ &\to v = 0^s1^t1. \end{aligned}$$

Note that the path L is of length t. It follows that $d_{\mathrm{EH}(s,t)-F}(u,v) = t+6$. Therefore, we have $D_s^f(\mathrm{EH}(s,t)) \ge t+6 > s+t+3$ for $s = 1, 2$.

4 Concluding Remarks

The topology of a network is an important consideration in the design of interconnection networks since it affects many key properties such as efficiency and fault tolerance. The exchanged hypercube $\mathrm{EH}(s,t)$, which is beneficial in parallel computing and communication systems, constitutes nearly half the number of edges in comparison with the hypercube Q_{s+t+1} and yet retains the advantages of many topologies; furthermore, it provides good application to support. In this paper, we focus on constructing $s+1$ internally disjoint paths between any two vertices u and v in the exchanged hypercube $\mathrm{EH}(s,t)$. However, if $u_0 = v_0 = 1$, $t+1$ $(\ge s+1)$ internally disjoint paths between u and v in $\mathrm{EH}(s,t)$ can be constructed. We also discuss the wide and fault diameters of the exchanged hypercube $\mathrm{EH}(s,t)$. We proved that $D_{s+1}(\mathrm{EH}(s,t)) = D_s^f(\mathrm{EH}(s,t)) = s+t+3$ for $3 \le s \le t$. These properties demonstrate that interconnection networks modeled by the exchanged hypercube $\mathrm{EH}(s,t)$ are extremely robust. They have high fault tolerance and reliability on a topological structure for interconnection networks. Finally, Table 2 illustrates the comparison of some properties on the n-dimension hypercube Q_n, crossed hypercube CQ_n and exchanged hypercube $\mathrm{EH}(s,t)$.

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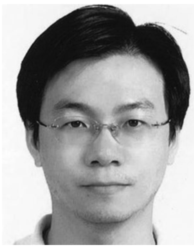

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