On the Capacity of the Multiantenna Gaussian Cognitive Interference Channel

Stefano Rini and Andrea Goldsmith

***Abstract*—The capacity of the multiantenna Gaussian cognitive interference channel is studied. The cognitive interference channel is a variation of the classical two-users interference channel in which one of the transmitters, the *cognitive* transmitter, is also provided with the message of the second transmitter, the *primary* transmitter. We study the capacity of the multiple-input multiple-output Gaussian model, that is the channel in which the inputs are vectors and the outputs are obtained as linear combinations of the channel inputs plus an additive complex Gaussian noise. This channel models a wireless scenario in which transmitters and receivers have multiple antennas. For this channel, we derive capacity to within an additive gap, that is we show that inner and outer bounds to capacity lie to within a constant distance of each other. The gap between the inner and outer bounds depends on the number of antennas at the cognitive receiver and both bounds can be easily evaluated by considering jointly Gaussian inputs. We also derive capacity to within a constant multiplicative factor of two, that is we show that the ratio between inner and outer bound is at most two. The additive gap well-characterizes the capacity at high SNR, while the multiplicative gap is useful at low SNR. We also derive the exact capacity for a subset of the "strong interference" regime: in this subset, the primary transmitter can decode the cognitive message without loss of optimality. This new capacity result extends and generalizes previously known capacity results, in particular, the capacity in the "very strong interference" and the "primary decodes cognitive" regimes.**

***Index Terms*—Cognitive interference channel; Capacity; Capacity to within a constant gap; Superposition coding; Interference pre-cancellation.**

I. Introduction

The advent of smart wireless devices that can sense and adapt to the surrounding radio environment promises to drastically improve the efficiency in the frequency spectrum utilization. By allowing nodes in the network to overhear the transmissions taking place over the medium, it is possible to develop decentralized and dynamic cooperation strategies which are not possible in centralized networks. The ability of a device to adapt its transmissions to the surrounding RF environment is usually termed *cognition*. The study of cognitive networks is notably difficult: the dependency of the transmission strategy upon the channel conditions introduces a variability in the overall performance which is hard to analyze. For this reason, and despite their relevance in modern communication systems, it has been hard to characterize the optimal performance of general cognitive networks.

We approach the study of cognitive networks from an information theoretic standpoint, that is we attempt to characterize the limiting rate advantages that are provided by cognition in the context of Shannon capacity. More specifically, we focus on an information theoretic model that captures the fundamental features of cooperation in cognitive networks: the *cognitive interference channel* [1]. This channel is obtained from the classic two-user interference channel by providing one of the transmitters, the *cognitive* transmitter, with the message of the other user, the *primary* user. The extra information at the cognitive transmitter models the ability of this node to acquire information about the surrounding nodes by exploiting the broadcast nature of the wireless medium.

The cognitive interference channel idealizes the capabilities of the cognitive transmitter in two ways: (i) it assumes that the extra knowledge of the primary message is available a priori at the cognitive transmitter, instead of causally learned through successive transmissions and (ii) it considers the case in which the cognitive transmitter is able to acquire the primary message in its entirety. Additionally, (iii) full channel knowledge is assumed at every node. Although more realistic channels have been considered in the literature, only this idealized model has been possible to derive capacity regions or bounds on the limiting advantages provided by cognition. Also, the study of this model has provided important insights about the role of cooperation in cognitive networks and the associated optimal transmission strategies that may guide the design of practical communication systems.

Although this model should be considered as an idealization of more practical scenarios, some of the assumptions of this model are also valid in real networks. For instance, (i) the assumption of the a priori knowledge of the primary message is valid in networks in which base stations are connected through a high-speed link or in networks in which one node can overhear the transmissions of the neighbouring nodes; (ii) the assumption of full message knowledge holds in networks which have a long packet size, and thus intercepting the packed of on user gives access to the full message information. Finally, (iii) the full channel knowledge assumption is reasonable in networks which vary slowly over time and in which sufficient feedback rate is available between the receivers and all the transmitters. This is again the case in a downlink system where base stations are connected through a high-capacity link which can be used to quickly share channel state information.

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Indeed Coordinated Multi-Point (CoMP) transmissions and Virtual Base Stations (VBS) protocols are current technologies which point toward a base station architecture where messages and channel knowledge are shared across transmitters.

Literature Overview

The study of the Cognitive InterFerence Channel (CIFC) was initiated in [1] and the capacity of this model, for both the discrete memoryless case and the Gaussian case, remains unknown in general. General outer bounds [2] as well as inner bounds [3] have been derived in the literature and they have been shown to coincide for some classes of channels.

Three channel models are investigate in the literature: the general CIFC, the MIMO CIFC and the Gaussian CIFC. The general CIFC is the most general model in which the outputs are any random function of the inputs. A sub-class of this general channel model is the MIMO CIFC, in which the inputs are vectors of any size and the outputs are linear combination of the inputs plus additive Gaussian noise. The Gaussian CIFC is a sub-class of the MIMO CIFC in which the inputs and outputs are restricted to be scalars.

We next review these results in terms of (a) capacity results, (b) outer bounds, (c) achievable regions and (d) approximate characterizations of capacity; lastly, we focus on the model of interest: (e) the Multiple-Input Multiple-Output CIFC.

The contributions from prior work that are most relevant for this work are also summarized in Fig. 1(a) and in Table I.

In Fig 1(a), three columns represent three classes of channels: the general CIFC, the MIMO CIFC and the Gaussian CIFC. The contributions in the literature are represented using rectangles crossing the different columns: hatched rectangles represent bounds on capacity (such as inner and outer bounds) while solid colors represent exact capacity results. When a rectangle is in a given column, this indicates that a certain result holds for a certain class of channels. The intersection among rectangles indicates that the multiple contributions hold for the same set of channels. The results in Fig. 1(a) are also detailed in Tab. I: here each result is further characterized by the class of channel, type of contribution (inner bound, outer bound etc.) and a bibliographic reference is also provided. Each contribution is identified by roman numerals in both Fig. 1(a) and in Tab. I: this roman numeral is also indicated in parenthesis in the following section, when each contribution is introduced in detail.

a) Capacity Results: Given its full generality, the capacity of the CIFC in which the channel output are any function of the current channel input is currently unknown. For this wide class of channels, capacity is known in the “cognitive more capable” regime (Num. II, Fig. 1(a)), a class of channels which intuitively identifies the models in which there is no loss of optimality in having the cognitive receiver decode both messages. The cognitive more capable regime is a generalization of capacity results that had been previously derived in the literature. The first of such results is the capacity in the “very weak interference” regime [4]. For this set of channels, capacity is achieved by having the primary receiver treat the interference from the cognitive transmitter as noise, while the cognitive receiver decodes both codewords. Capacity was successively derived for the “very strong interference” regime, where it is attained by transmitting the cognitive codeword over the primary codeword and having both receivers decode both messages. This regime is analogous to the “very strong interference” regime for the classical InterFerence Channel (IFC) [5] in which the level of the interference at both receivers is so high that the interfering codeword can be decoded before the intended codeword and stripped from the channel output. The “very weak interference” regime [4] and the “very strong interference” regime [2] were later generalized in the “better cognitive decoding” regime [3]. In this regime capacity is achieved by dividing the cognitive message into two sub-messages: one sub-message, the “cognitive private” message, is decoded only at the cognitive receiver while the other sub-message, the “cognitive public” message, is decoded at both receivers. By setting the rate of the cognitive private message to zero, this result reduces to the “very strong interference” capacity, while setting the rate of the cognitive public message to zero reproduces the “very weak interference” capacity. The rate of the two sub-messages can be varied to achieve a larger attainable region in different channel conditions and target rates, which produces new capacity results. The “cognitive more capable” regime [6] is an refinement of the “better cognitive decoding” regime which considers the same achievable strategy and outer bound but provides a simplification of the attainable region.

Capacity is known for two more specific classes of channels: the semi-deterministic CIFC and the Gaussian CIFC; the semi-determinist CIFC is defined as the CIFC where the channel output at the cognitive decoder is a deterministic function of the inputs, while the output at the primary decoder is any random function. In this channel model, capacity is achieved by having the cognitive transmitter pre-code against the primary interference while simultaneously aiding the primary transmitter [7]. Since the cognitive output is obtained through a deterministic function, the cognitive transmitter can fully pre-cancel the effect of the interference as in the deterministic Gelf’and-Pinsker problem [8].

A larger set of capacity results is available for the Gaussian channel, that is the channel for which the channel inputs are complex numbers and the channel outputs are obtained as linear combinations of the inputs plus Gaussian noise. For this model, capacity is known in two more regimes than for the general CIFC: the “weak interference” regime and the “primary decodes cognitive” regime. In the “weak interference” regime [4], capacity is achieved by having the primary receiver treat the interference as noise while the cognitive transmitter pre-codes its message against the known primary interference. The outer bound for this capacity result utilizes the entropy power inequality and is inspired by the proof of the capacity of the Gaussian broadcast channel. The inner bound also relies on the classical “writing on dirty paper” results [9] which implies that the cognitive transmitter can fully pre-cancel the effect of the interference at the cognitive receiver. The capacity in the “very weak interference” regime for the general CIFC is a subset of the “weak interference” regime for the Gaussian channel in which capacity can also be attained using interference decoding instead of interference pre-cancellation.

The other regime in which capacity is known for the Gaussian case is the "primary decodes cognitive" regime [10] (Num. IV, Fig. 1(a)). Here, capacity is achieved by pre-coding the cognitive codeword against the interference created by the primary transmission and having the primary receiver decode both messages. The primary decoder gains insight over its own message by decoding the cognitive codeword, since the interference against which the cognitive codeword is pre-coded is indeed the primary codeword.

b) Outer Bounds: The tightest available outer bound for the CIFC is derived in [11, Th. 4] using a technique originally developed for the broadcast channel in [12] (Num. I, Fig. 1(a)). This outer bound encompasses and generalizes all the outer bounds which are known to be attainable. In particular, this outer bound corresponds to capacity in the "cognitive more capable regime", the "weak interference regime" and the "better cognitive decoding" regime. Although the outer bound in [11, Th. 4] is the tightest known outer bound, it is difficult to evaluate as it is expressed as the maximization over three auxiliary random variables. This maximization cannot be explicitly solved, not even for some classical channels such as the Gaussian channel. For this reason, an alternative outer bound is derived in [3, Th. 4.1] which is expressed only as a function of the channel inputs and the channel outputs. This outer bound is used to show capacity in the semi-deterministic CIFC [7] but is known not to be tight in other classes of channels. Another simple outer bound is the "strong interference" [2] (Num. III, Fig. 1(a)) outer bound, which is capacity in the "very strong interference" and the "better cognitive decoding" regime. This outer bound is also expressed only as a function of the channel inputs and channel outputs but is again known not to be tight in general.

c) Achievable Schemes: Since the CIFC generalizes both the IFC and the broadcast channel, different achievable schemes can be devised for this model which combine techniques available for these simpler channel models. In particular, we consider the following three main coding strategies for the CIFC: rate-splitting, superposition coding and binning. Rate splitting divides the message of one user into multiple sub-messages which are encoded/decoded by a different set of transmitters/receivers. In the CIFC both cognitive and primary messages can be split into private and common sub-messages: the private messages are decoded only at the intended receivers, while the common messages are decoded at both receivers. Superposition coding is attained by "stacking" the codeword of one user over the codeword of another user: the top codeword can then be decoded only when the bottom codeword is correctly decoded. This reduces the possibility of decoding errors, thus enhancing the performance of the code. Clearly, the two codewords must be known at the same set of encoders for superposition to be feasible. In the CIFC, cognitive codewords can be superposed over primary codewords, since the primary message is known also at the cognitive transmitter but not vice-versa.

Binning consists of pre-coding the codeword of one user against the interference created by another user at the intended decoder. Binning was originally introduced for the Gel'fand-Pinsker problem [8]: a point to point channel in which the outputs is determined by the channel inputs and a sequence of states which is known at the transmitter but not at the receiver. The transmitter can thus design the transmitter codeword so as to "mask" the effect of the channel state to the received output. For binning to be feasible, the interference must be known at the encoder, which implies that cognitive messages can be pre-coded against the interference created by the primary transmission at the cognitive receiver.

A scheme which combines all the possible ways of rate-splitting messages, superposing and binning codewords over and against one another is derived is [3] (Num. I, Fig. 1(a)). This scheme generalizes the achievable rate regions used to prove capacity in those regimes where capacity is known. Unfortunately this scheme also contains an number of auxiliary random variables and can be computed explicitly only under some simple assignments. As a consequence of this, a comparison of the different coding strategies is often not straightforward and no clear conclusion can be drawn over which coding choices are optimal in different sets of channels.

d) Approximate Characterizations of Capacity: Given the difficulties in determining inner and outer bounds in closed form, progress has been slow in determining the exact capacity for models such as the Gaussian CIFC. On the other hand, it has been shown that capacity for the Gaussian case can be attained to within an additive gap of 2 bits/s/Hz and a multiplicative factor of two [13]. That is, the region where the exact capacity lies has been determined up to a finite distance and a finite multiplicative value. The additive gap well characterizes the capacity at high SNR: in this regime the capacity region is large and a small uncertainty on the exact boundary of the capacity is not relevant. On the other hand, the multiplicative gap is useful at low SNR: when the capacity region is small, a multiplicative factor provides a better bound on the exact value of capacity. These two approximate characterizations of the exact capacity have been derived using insights from the high SNR deterministic approximation of the Gaussian cognitive interference channel [14], a deterministic model that captures the behavior of a Gaussian network for large transmit powers [15].

e) Multiple-Input Multiple-Output CIFC: In this work, we focus on the Multiple-Input Multiple-Output CIFC (MIMO CIFC), a generalization of the Gaussian CIFC in which transmitters and receivers possess multiple antennas. The capacity of this channel model is still largely unknown since the capacity results for the "weak interference" and the "primary decodes cognitive" regimes do not extend from the single antenna to the MIMO scenario in a straightforward manner. The authors of [16] were the first to specifically study the capacity of the MIMO CIFC and propose an outer bound and an achievable region based on dirty paper coding [9] (Num. V, Fig. 1(a)). The sum Degrees Of Freedom (DOF) of the MIMO CIFC were studied in [17] where it was shown that the MIMO CIFC has a larger sum DOF than the classical IFC. MIMO CIFCs have also been studied through a game theoretical approach: the cognitive MIMO transceivers compete with each other to maximize their information rate [18] and to allow the coexistence of primary and cognitive users in the presence of perfect channel knowledge [19].

TABLE I
A SUMMARY OF THE CONTRIBUTIONS IN THE LITERATURE, AS DESCRIBED IN SEC. I.

Index	Channel Model	Contribution	References
I	general CIFC	tightest outer bound	[11]
I	general CIFC	largest inner bound	[3]
II	general CIFC	cognitive more capable capacity	[6]
III	general CIFC	strong interference outer bound	[2]
IV	Gaussian CIFC	better cognitive decoding capacity	[3]
V	MIMO CIFC	inner and outer bounds	[16]

Contributions

We now summarize the main contributions of this work on the capacity region of the general CIFC and the MIMO CIFC. The contributions in the paper are also depicted in Fig. 1(b). This figure uses the same formalism as Fig. 1(a): rounded boxes represent channel models, while rectangles represent our contributions in terms of approximate capacity (hatching rectangles) or capacity (solid-colored rectangles). Each contribution is also numbered using a roman numeral: this numeral is used in the enumeration that follows and which presents each contribution in detail.

(I) Derive the approximate capacity for the general CIFC by considering an inner bound with superposition coding and binning and proving a constant gap between the inner and outer bounds. The gap between the inner and outer bounds intuitively relates to the ability of the cognitive transmitter to predict the channel output at the cognitive receiver. This result generalizes the capacity for the semi-deterministic CIFC in which the gap between the inner and outer bounds is zero.

(II) Derive the capacity of the MIMO CIFC to within a constant gap: For the MIMO CIFC the approximate capacity for the general CIFC translates to a gap between the inner and outer bounds which depends on the number of antennas at the cognitive receiver. We also derive a simpler expression of the outer bound by showing that one needs only to consider jointly Gaussian random variables: this greatly simplifies the task of evaluating the approximate capacity for a given channel model.

(III) Derive the capacity of the MIMO CIFC to within a multiplicative factor of two by having the cognitive transmitter pre-cancel the interference created by the primary transmitter while the primary decoder treats the interference as noise. This result is useful when characterizing the capacity of the MIMO CIFC at low SNR.

(VI) Generalize the capacity results for the MIMO CIFC in the "strong interference" regime: Capacity in the "strong interference" regime is known in two cases: the "very strong interference" regime and the "primary decodes cognitive" regime. Both results use the same converse but they differ in the achievability proof. We show that a larger capacity result can be attained by considering an achievable strategy which generalizes the achievable region in the two regimes. In this scheme, the primary message is rate-split into a common and a private part and the private part is then superimposed over the public one. The cognitive message is then superimposed over the common-primary message and binned against the private-primary message.

Paper Organization

The remainder of the paper is organized as follows: Sec. II introduces the channel model under consideration. In Sec. III we review relevant results available in the literature that will be used in subsequent sections and, in Sec. IV, relevant inner bounds are introduced. In Sec. V the approximate capacity for the general CIFC is derived. Capacity for a subset of the MIMO CIFC is derived in Sec. VII. Sec. IX concludes the paper.

II. Channel Model

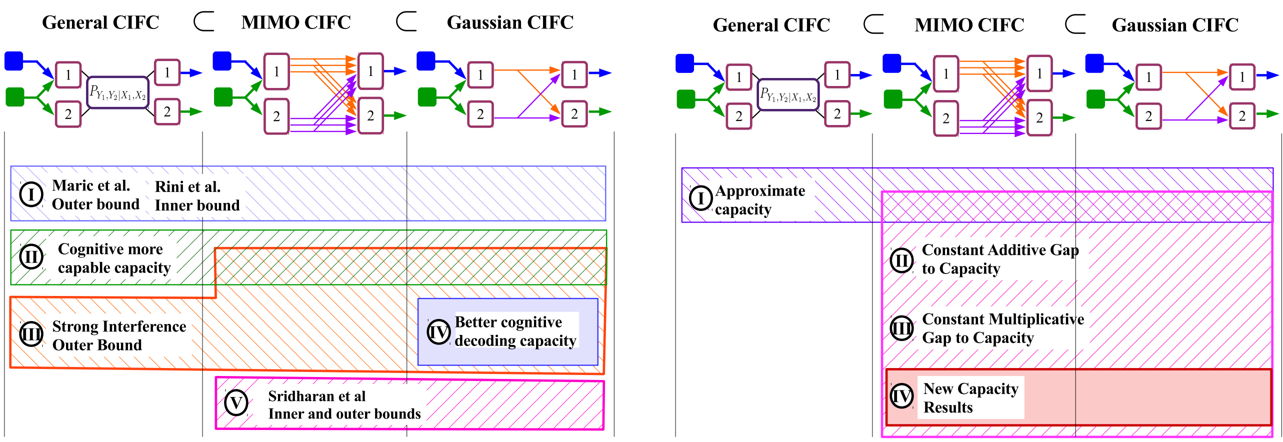
A two user InterFerence Channel (IFC) is a multi-terminal network with two senders and two receivers. Each transmitter i wishes to communicate a message W_i to receiver i, $i \in [1,2]$. In the classical IFC the two transmitters operate independently and have no knowledge of each others' messages. In this paper we consider a variation whereby transmitter 1, the *cognitive* transmitter, in addition to its own message W_1, also knows the message W_2 of transmitter 2, the *primary* transmitter). We refer to transmitter/receiver 1 as the cognitive pair and to transmitter/receiver 2 as the primary pair. This model is termed the Cognitive InterFerence Channel (CIFC) and is an idealized model for unilateral transmitter cooperation.

More precisely, transmitter $i \in [1,2]$ wishes to communicate a message W_i, uniformly distributed on $[1 \cdots 2^{NR_i}]$, to receiver i in N channel uses. The two messages are independent. Transmitter 1 knows both messages while transmitter 2 knows only W_2. The channel is assumed to be memoryless, that is the channel output only depends on the current channel input.

In the following we consider three classes of input/output relationships:

- The **general CIFC**, which is the most general model in which the channel outputs are obtained as any random function of the inputs
- The **Gaussian CIFC**, that is the model in which the channel outputs are obtained as a linear function of the inputs plus an additive white Gaussian noise term. The inputs are additionally subject to a second moment (i.e. power) constraint.
- The **MIMO CIFC** which generalizes the Gaussian CIFC to the case in which inputs and outputs are vectors. The inputs are additionally subject to a covariance constraint.

We now present each model separately to better introduce the notation and the nomenclature used in the following.



(a) The relevant literature, as presented in Sec I.

(b) Our contributions, as presented in Sec. I.

Fig. 1. A pictorial representation of the relevant literature and of our contributions for the general CIFC, the MIMO CIFC and the Gaussian CIFC.

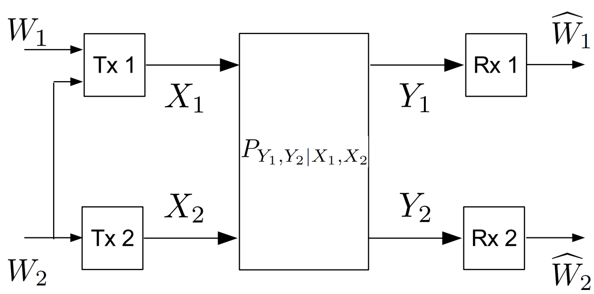


Fig. 2. The general Cognitive Interference Channel (general CIFC).

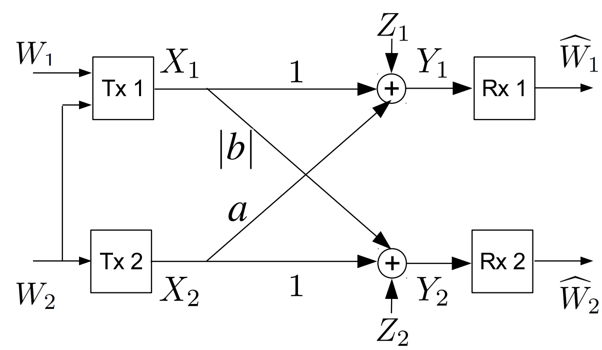


Fig. 3. The Gaussian Cognitive Interference Channel (G IFC).

A. General CIFC

The input/output relationship in each channel use is described by the conditional probability

$$P^N_{Y_1,Y_2|X_1,X_1} = \prod_{i=1}^{N} P_{Y|X}(y_i|x_i). \quad (1)$$

for some conditional probability $P_{Y|X}$ which can be either continuous or discrete. A graphical representation of the general CIFC is provided in Fig. 2.

B. Gaussian CIFC

As we shall see, most of the literature so far has focused on the additive white Gaussian noise model channel; for simplicity we refer to this channel as the Gaussian CIFC (G CIFC).

For the G CIFC $P_{Y|X}$ can be parameterized as

$$Y_1 = X_1 + aX_2 + Z_1 \quad (2a)$$
$$Y_2 = |b|X_1 + X_2 + Z_2, \quad (2b)$$

where $Z \sim \mathcal{CN}(0,1)$, $i \in \{1,2\}$ and for $a, b \in \mathbb{C}$. Note that the phase of b can be disregarded without loss of generality [20, App. A]. This channel model is depicted in Fig. 3.

C. Multiple-Input Multiple-Output CIFC

In the Multiple-Input Multiple-Output CIFC (MIMO CIFC), the input/output relationship $P_{Y|X}$ is described as

$$Y_1 = H_{11}X_1 + H_{12}X_2 + Z_1 \quad (3a)$$
$$Y_2 = H_{21}X_1 + H_{22}X_2 + Z_2, \quad (3b)$$

where Y_i's and X_j's are column vectors of size m_i and n_j respectively, H_{ij}, $i,j \in \{1,2\}$ are complex matrices of size $m_i \times n_j$ and Z_i are iid, zero mean and unitary variance complex Gaussian random column vectors of size m_i. Additionally, the channel inputs X_j are subject to the second moment constraint

$$\mathbb{E}[X_j X_j^H] = \Sigma_j \preceq \mathbf{S}_j \quad j \in \{1,2\}, \quad (4)$$

for some $\mathbf{S} \succeq 0$ where $\succeq$ denotes partial ordering between symmetric matrices, that is $\mathbf{B} \succeq \mathbf{A}$ means that $\mathbf{B} - \mathbf{A}$ is a positive semi-definite matrix. This channel is a special case of the channel model in (1) but generalizes the model in (2). A graphical representation of the MIMO CIFC is provided in Fig. 4.

Any general CIFC in which the channel outputs are obtained as a linear combination of the inputs plus a noise term with any covariance can be reduced to the model in (3) by whitening the noise in the channel output. This can be done through a

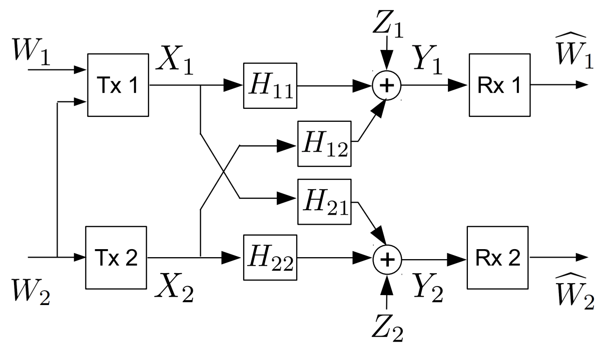


Fig. 4. The Multiple-Input Multiple-Output Cognitive InterFerence Channel (MIMO CIFC)

linear operation which is invertible and thus does not change the capacity of the channel.

D. Relevant Definitions

We next introduce the definition of code, achievable region and capacity for this channel are introduced next. In the following definitions, the calligraphic font indicates the support of a variable.

Definition 1. Code.
A $(2^{NR_1};2^{NR_2};N)$ code for the CIFC consists of two message sets $W_1=[1\ldots 2^{NR_1}]$, $W_2=[1\ldots 2^{NR_2}]$, two encoding functions and two decoding functions, one per each encoder and decoder respectively.

The encoding function at transmitter 1 is defined as

$$X_1^N:\mathscr{W}_1\times\mathscr{W}_2\to\mathscr{X}^N$$
$$X_1^N=X_1^N(W_1,W_2),$$

while the encoding function at transmitter 2 is defined as

$$X_2^N:\mathscr{W}_2\to\mathscr{X}^N$$
$$X_2^N=X_2^N(W_2).$$

Similarly, the decoding functions at the receivers are defined as

$$\widehat{W}_j:\mathscr{Y}\to\mathscr{W}_j$$
$$\widehat{W}_j=\widehat{W}_i(Y_i^N),\quad j\in[1,2].$$

Definition 2. Achievable Rate
The rate pair (R_1,R_2) is achievable if there exists a sequence of codes such that, for W_1 and W_2 uniformly distributed over their support, we have that the probability of a decoding error at both receivers goes to zero as the block length N goes to infinity.

More precisely, the probability of decoding error for a code $(2^{NR_1};2^{NR_2};N)$ is defined as

$$P_e(2^{NR_1};2^{NR_2};N)=\mathbb{P}\left[\widehat{W}_1(Y_1^N)\neq W_1 \text{ or } \widehat{W}_2(Y_2^N)\neq W_2\right] \quad (5)$$

while a rate pair (R_1,R_2) is achievable when there exists a sequence of codes such that

$$\lim_{N\to\infty}P_e(2^{NR_1};2^{NR_2};N)=0 \quad (6)$$

Note that the error probability in (6) *is averaged over all state sequences and all messages.*

Definition 3. Capacity and Approximate Capacity.
The capacity C is the supremum of all the achievable rates. The region R^{IN} is said to be an inner bound to the capacity region if $R^{\mathrm{IN}}\subseteq C$. Similarly, the region R^{OUT} is an outer bound to capacity if $R^{\mathrm{OUT}}\supseteq C$.

An inner and outer bound for which

$$R^{\mathrm{OUT}}-R^{\mathrm{IN}}\leq\Delta, \quad (7)$$

for some constant $\Delta\in\mathbb{R}^+$ are said to characterize the capacity to within an additive gap of Δ bits/channel use (bits/cu).

Similarly, an inner bound and outer bound for which

$$R^{\mathrm{OUT}}/R^{\mathrm{IN}}\leq\Delta, \quad (8)$$

for some constant $\Delta\in\mathbb{R}^+$ are said to characterize the capacity to within a multiplicative gap of Δ.

Capacity to within an additive gap and a multiplicative gap provide an approximate characterization of capacity. This is useful in many channels in which the exact capacity is too hard to characterize exactly. Determining capacity to within an additive is useful when the capacity region is large, since a small difference between the inner and outer bound is negligible in these circumstances. On the other hand, when the capacity region is small, a constant multiplicative gap provides a better bound on the exact capacity.

III. Known Results for the CIFC

In this section we review some of the relevant results derived in the literature that we will use in our derivations. We first present results concerning the general CIFC and then the results for the G CIFC.

A. Results for the general CIFC

A general outer bound to the capacity region of the general CIFC was first derived in [4]. This outer bound is inspired by the Körner - Marton outer bound for the broadcast channel [21] and it has been show to be tight for various classes of channels.

Theorem III.1. Wu et al. Outer Bound [4, Th. 3.2]. *The region*

$$R_1\leq I(Y_1;X_1|X_2) \quad (9a)$$
$$R_2\leq I(Y_2;U,X_2) \quad (9b)$$
$$R_1+R_2\leq I(Y_2;U,X_2)+I(Y_1;X_1|X_2,U), \quad (9c)$$

where the union is over all the distributions P_{U,X_1,X_2}, is an outer bound to the capacity region of a general CIFC.

Note that the outer bound in Th. III.1, just as the outer bound it Körner - Marton, contains an auxiliary Random Variables (RV) U which is not part of the channel description but is introduced to obtain a single letter expression of the bound.

A tighter outer bound than the one in Th. III.1 can be derived in a regime analogous to the “strong interference” regime for the IFC, in which having the primary receiver decode the cognitive receiver can be done without loss of optimality.

Theorem III.2. "Strong Interference" Outer Bound [11, Th. 4]. *Let the "strong interference" regime be defined as the set of CIFCs for which*

$$I(Y_1;X_1|X_2) \le I(Y_2;X_1|X_2), \tag{10}$$

for all the distributions $P_{X_1X_2}$, then the region

$$R_1 \le I(Y_1;X_1|X_2) \tag{11a}$$
$$R_1+R_2 \le I(Y_2;X_1,X_2), \tag{11b}$$

where the union is over all the distributions P_{X_1,X_2}, is an outer bound to the capacity region of a general CIFC in "strong interference".

From a high level perspective, the condition in (10) characterizes those channels for which, given that X_2 has been decoded, X_1 is more easily decoded from Y_2 rather than from Y_1. Since X_2 has to necessarily be decoded at receiver 2, this regime can be thought of as the regime in which receiver 2 can decode both messages without loss of optimality.

Capacity is known in a subset of the "strong interference" regime, the "very strong interference" regime. Here the "strong interference" outer bound is attained by having both receivers decode both messages and by superimposing the cognitive codeword over the primary codeword.

Theorem III.3. Capacity in the "Very Strong Interference" Regime [22, Th. 1]. *Let the "very strong interference" regime be defined as the set of CIFCs in the "strong interference" regime for which*

$$I(Y_2;X_1,X_2) \le I(Y_1;X_1,X_2), \tag{12}$$

for all the distributions P_{X_1,X_2}. In the "very strong interference" regime the region in (11b) *is the capacity region.*

B. Results for the G CIFC

The set of regimes for which the G CIFC capacity is known is larger than for the general CIFC. In the primary G CIFC, binning at the cognitive user can attain perfect interference pre-cancellation and this is the key to achieve capacity in different parameter regimes.

Theorem III.4. Gaussian "Weak Interference" Capacity [4, Lem. 3.6]. *If $|b|<1$, the capacity of the G CIFC is the union over $\alpha \in [0,1]$ of the region*

$$R_1 \le \mathscr{C}(\alpha P_1) \tag{13a}$$
$$R_2 \le \mathscr{C}\left(|b|^2P_1+P_2+2\sqrt{\overline{\alpha}|b|^2P_1P_2}\right)-\mathscr{C}(|b|^2\alpha P_1), \tag{13b}$$

for $\mathscr{C}(x)=\log(1+x)$ and $\overline{\alpha}=1-\alpha$.

Capacity in the Gaussian "weak interference" regime is achieved by pre-coding the cognitive codeword against the interference experienced at the cognitive decoder while treating the interference as noise at the primary decoder. The "strong interference" regime in Th. III.2 takes a particularly simple expression for the G CIFC.

Lemma III.5. Gaussian "Strong Interference" Outer Bound [11, Th. 4]. *If $|b|\ge 1$, then the region*

$$R_1 \le \mathscr{C}(\alpha P_1) \tag{14a}$$
$$R_1+R_2 \le \mathscr{C}\left(|b|^2P_1+P_2+2\sqrt{\overline{\alpha}|b|^2P_1P_2}\right), \tag{14b}$$

where the union is over $\alpha \in [0,1]$, is an outer bound to the capacity region.

In the "strong interference" regime the primary receiver, after having decoded its intended message, can reconstruct an equivalent channel output at the cognitive receiver since

$$\widetilde{Y}_1 \sim \frac{Y_2-(a|b|-1)X_2}{|b|}+\widetilde{Z}_1, \tag{15}$$

with $\widetilde{Z}_1 \sim \mathscr{CN}(0,1-|b|^{-2})$. This observation provides an intuitive interpretation of the sum rate bound in (14b) which suggest that, when $|b|\ge 1$, the primary receiver can decode both messages without loss of optimality. The next lemma translates Th. III.3 to the G CIFC model.

Lemma III.6. Gaussian "Very Strong Interference" Capacity [11, Th. 4]. *If $|b|\ge 1$ and*

$$(1-|b|^2)P_1+(|a|^2-1)P_2 \ge 0 \tag{16a}$$
$$(1-|b|^2)P_1+(|a|^2-1)P_2 \ge 2\left(|b|-\mathrm{Re}\{a^H\}\right)\sqrt{\overline{\alpha}P_1P_2}, \tag{16b}$$

the region in (14) *is the capacity.*

In the "very strong interference" regime, capacity is achieved by superimposing the cognitive message over the primary message and having both decoders decode both messages.

Capacity is also known in another subset of the "strong interference" regime for the G CIFC which is denoted as the "primary decodes cognitive" regime.

Theorem III.7. "Primary Decodes Cognitive" Capacity [10, Th. 3.1]. *If $|b|\ge 1$ and*

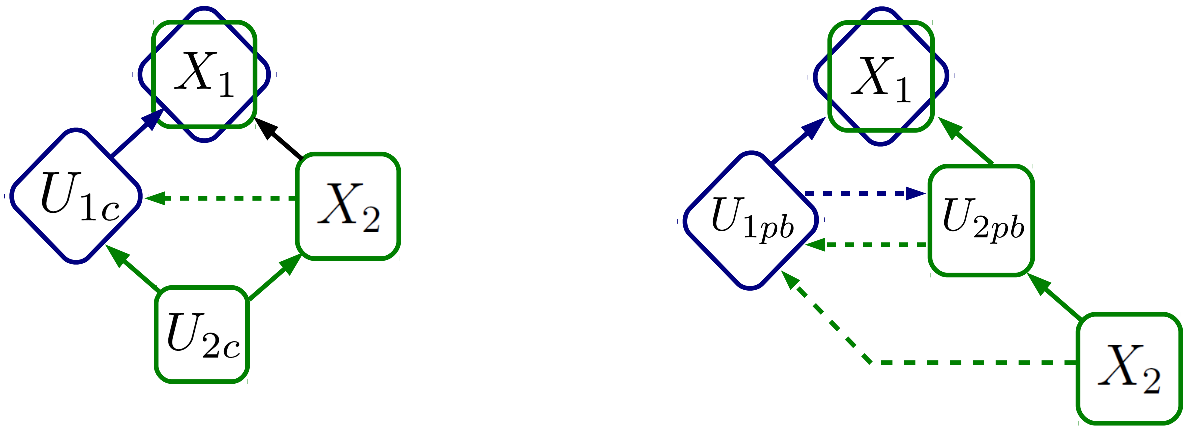
$$P_2|1-a|b||^2(1+P_1) \ge (|b|^2-1)(1+P_1+|a|^2P_2) \tag{17a}$$
$$P_2|1-a|b||^2 \ge (|b|^2-1)(1+P_1+|a|^2P_2-2\mathrm{Re}\{a\}\sqrt{P_1P_2}), \tag{17b}$$

the region in (14) *is the capacity.*

In Th. III.7 capacity is achieved by pre-coding the cognitive message against the interference and having the primary receiver decode codewords. Intuitively, the primary decoder gains insight over its own message by decoding the cognitive codeword, since the interference against which the cognitive codeword is pre-coded is indeed the primary codeword.

IV. Inner Bounds

The largest known inner bound for a general CIFC is obtained in [23] while a compact expression for this region is provided in [20, Sec. IV]. In this section we introduce two sub-schemes of the general transmission scheme in [23] that will be relevant in developing our main results. One scheme generalizes the capacity achieving schemes in the "very strong interference" of Th. III.3 and the "primary decodes cognitive" of Th. III.7. The other scheme is the transmission strategy



(a) The chain graph representation of the inner bound in Th. IV.1 (b) The chain graph representation of the inner bound in Th. IV.2

Fig. 5. The RVs for message 1 are in blue diamond boxes while the RVs for message 2 are in green square boxes. A solid line among RVs indicates that the RVs are superimposed while a dashed line that the RVs are binned against each other.

which achieves capacity in the semi-deterministic CIFC and the capacity of the G CIFC to within one bit [7] .

Theorem IV.1. Achievable Scheme (F) in [20, Sec. IV.F]. *The following region is achievable in a general CIFC*

$$R_1 \leq I(Y_1;U_{1c}|U_{2c}) - I(U_{1c};X_2|U_{2c}) \quad (18a)$$
$$R_1+R_2 \leq I(Y_2;X_1,X_2) \quad (18b)$$
$$R_1+R_2 \leq I(Y_2;X_2|U_{1c},U_{2c}) + I(Y_1;U_{1c},U_{2c}) \quad (18c)$$
$$2R_1+R_2 \leq I(Y_2;U_{1c},X_2|U_{2c}) + I(Y_1;U_{1c},U_{2c}) - I(U_{1c};X_2|U_{2c}), \quad (18d)$$

for any distribution $P_{U_{2c},U_{1c},X_1,X_2}$.

Proof: We provide here a sketch of the proof: the full proof is provided in Appendix A. The message W_1 is associated with the RV U_{1c} and decoded at both receivers ("c" stands for *common*). The message W_2 is rate-split into common and private parts which are associated with U_{2c} and X_2 respectively. ■

The chain graph representation [24] of the achievable scheme in Th. IV.1 is provided in Fig. 5(a). Each box represents a RV in (18), a solid line represents superposition coding, a dashed line binning and a dotted line a deterministic dependence. The blue diamond box contains the message W_1 while the green, square boxes contain part of the message W_2. We next introduce an achievable scheme first considered in [23] and which achieves the capacity for the semi-deterministic CIFC [7], a CIFC in which the channel output at the cognitive receiver is a deterministic function of the inputs, while the output at the primary decoder is any random function. This scheme also approaches the capacity of the Gaussian CIFC to within one bit for channel parameters [7].

Theorem IV.2. Achievable Scheme (C) in [20, Sec. IV.F]. *The following region is achievable in a general CIFC*

$$R_1 \leq I(Y_1;U_{1pb}) - I(U_{1pb};X_2) \quad (19a)$$
$$R_2 \leq I(Y_2;U_{2pb},X_2) \quad (19b)$$
$$R_1+R_2 \leq I(Y_2;U_{2pb},X_2) + I(Y_1;U_{1pb}) - I(U_{1pb};X_2,U_{2pb}), \quad (19c)$$

for any distribution $P_{U_{1pb},U_{2pb},X_1,X_2}$.

A chain graph representation of the inner bound in Th. IV.2 is provided in Fig. 5(b). Encoder 2 transmits W_2 through the RV X_2 while encoder 1 sends W_1 through U_{1pb}. The RV U_{2pb} is superimposed over X_2 and pre-coded at transmitter 1 against the interference created by U_{1pb} at the primary receiver. Similarly, the RV U_{1pb} is pre-coded at transmitter 1 against the interference created by X_2 and U_{2pb} at the cognitive receiver.

V. Approximate Capacity for the General CIFC

In this section we show that the capacity of the general CIFC can be attained to within a constant gap between the inner and outer bounds which depends on how well the cognitive transmitter can forecast the output at the cognitive receiver. This result generalizes the capacity of the semi-deterministic CIFC in which case the gap between the two bounds is zero.

Theorem V.1. Approximate Capacity for the General CIFC *If the rate pair* (R_1,R_2) *belongs to the capacity region of a general CIFC, then the point* $(R_1-\Delta, R_2-\sqrt{2}\Delta)$ *is achievable for*

$$\Delta = \max_{P_{X_1,X_2}} I(Y_1;X_1,X_2|\widetilde{Y}_1), \quad (20)$$

where $\widetilde{Y}_1|X_1,X_2 \sim Y_1|X_1,X_2$.

Proof: We provide here a sketch of the proof: the complete proof is provided in Appendix B. Th. V.3 is graphically represented in Fig. 6: the inner bound in Th. IV.2 is depicted in solid green blue while the outer bound in Th. III.1 is solid green. The distribution of the inner and outer bounds for a fixed distribution in the union operation is shown in dotted

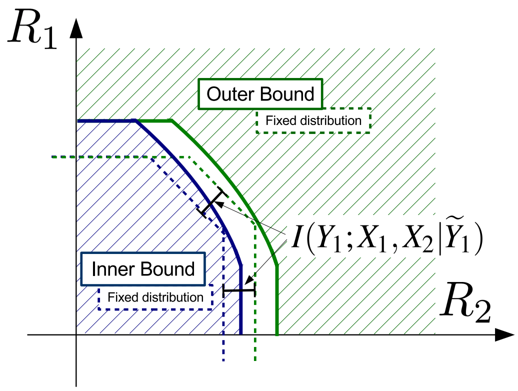


Fig. 6. A visual representation of Th. V.3 in which the distance between the inner and outer bounds for a general CIFC is bounded.

lines, blue and green respectively. The proof is shown by bounding the maximum distance between the inner and outer bounds for all the fixed distributions. The distance is zero for the R_2 bounds while it is $I(Y_1;X_1,X_2|\widetilde{Y}_1)$ for both the R_1 bounds and the sum rate bounds. Since the distance is bounded for each fixed distribution, it is bounded when taking the union over all distributions. ■

The gap between the two bounds for a fixed distribution depends on the term $I(Y_1;X_1,X_2|\widetilde{Y}_1)$ where $\widetilde{Y}_1$ is obtained at the cognitive transmitter by passing X_1 and X_2 through the test channel $P_{Y_1|X_1,X_2}$. The gap between the two bounds therefore intuitively relates to the ability of the cognitive transmitter to reproduce the output at the cognitive receiver. The overall gap between the inner and outer bounds is obtained as the largest among the gaps for each fixed distribution.

Lemma V.2. Semi-Deterministic CIFC *The semi-deterministic channel studied in [10] is a special class of the CIFC in which the channel output at the cognitive interference channel is a deterministic function of X_1 and X_2 while the primary output is any random function of the channel inputs. In this model, it is possible to generate $\widetilde{Y}_1 = \widetilde{Y}$ thus the result in Th. V.3 implies capacity.*

A. Additive Gap in the "Strong Interference" Regime

The "strong interference" outer bound in Th. III.2 can be obtained from the outer bound in Th. III.1 by setting $U = X_1$. This implies that, under condition (10), one can restrict the distribution of U to be equal to X_1. This simplification also results in a reduction of the gap between the inner and outer bounds.

We can refine the gap between the inner and outer bounds in the strong interference regime of Th. III.2 as follows:

Theorem V.3. Approximate Capacity for the General CIFC in "Strong Interference": *If the rate pair (R_1,R_2) belongs to the capacity region of a general CIFC in "strong interference", then the point $(R_1-\Delta,R_2)$ is achievable for*

$$\Delta = \max_{P_{X_1,X_2}} I(Y_1;X_1,X_2|\widetilde{Y}_1), \tag{21}$$

where $\widetilde{Y}_1|X_1,X_2 \sim Y_1|X_1,X_2$.

Proof: Note that the "strong interference" outer bound in Th. III.2 has two Pareto-optimal corner points:

$$A^{\text{out}-\text{SI}} = (I(Y_1;X_1|X_2), I(Y_2;X_1,X_2) - I(Y_1;X_1|X_2)) \tag{22a}$$
$$B^{\text{out}-\text{SI}} = (0, I(Y_2;X_1,X_2)). \tag{22b}$$

The point $B^{\text{out}-\text{SI}}$ is always achievable with a MISO strategy, that is by having both transmitters communicate exclusively with the primary receiver as in a MISO channel. For the point $A^{\text{out}-\text{SI}}$ we will employ the inner bound of (19) with the assignment $U_{1pb} = \widetilde{Y}_1$ and $U_{2pb} = X_1$ as in Th. V.3 to obtain a gap in the R_1 coordinate of

$$R_1^{A-\text{out}} - R_1^{A-\text{in}} = I(Y_1;X_1,X_2|\widetilde{Y}_1). \tag{23}$$

and a gap of zero in the R_2 coordinate since

$$R_2^{A-\text{out}} - R_2^{A-\text{in}} = I(Y_2;X_1,X_2) - I(Y_1;X_1|X_2) + \\ -\Big(I(Y_2;X_1,X_2) - I(\widetilde{Y}_1;X_1|X_2)\Big) = 0. \tag{24}$$

This concludes the proof. ■

VI. Approximate Capacity for the MIMO CIFC

The approximate characterization of the capacity of the MIMO CIFC is readily obtained from Th. V.3.

Lemma VI.1. MIMO CIFC *For the MIMO CIFC we have that Δ in Th. V.3 can be evaluated as*

$$\Delta = \max_{P_{X_1,X_2}} I(Y_1;X_1,X_2|\widetilde{Y}_1) \le n_1, \tag{25}$$

where n_1 is the number of antennas at the cognitive receiver.

Proof: We explicitly write the optimization in (VI.1) as

$$\max_{P_{X_1,X_2}} H(Y_1|\widetilde{Y}_1) - H(Z_1) = \max_{P_{X_1,X_2}} H(Y_1 - \widetilde{Y}_1|\widetilde{Y}_1) - H(Z_1) \tag{26a}$$
$$= \max_{P_{X_1,X_2}} \le H(Z_1 - \widetilde{Z}_1|\widetilde{Y}_1) - H(Z_1) \tag{26b}$$
$$\le H(Z_1 - \widetilde{Z}_1) - H(Z_1) \tag{26c}$$
$$= 2n_1 - n_1 = n_1, \tag{26d}$$

which concludes the proof. ■

Although the result in Lem VI.1 is sufficient to characterize the capacity of the MIMO CIFC to within n_1 bits/cu, the outer bound is expressed as the P_{U,X_1,X_2}. This union is not easily computed and the result does not provide any indication on what distributions produce outer bound points on the convex hull of the union. In the next theorem we show that for the MIMO CIFC it is sufficient to consider only jointly complex Gaussian $[U\ X_1\ X_2]$ in the outer bound in Th.III.1.

Theorem VI.2. MIMO CIFC Outer Bound: *The outer bound in Th. III.1 for the MIMO CIFC can be equivalently obtained by considering the region in* (9) *and taking the union over all the zero mean jointly Gaussian $[U\ X_1\ X_2]$ for which* (4) *is satisfied.*

Proof: The proof involves showing that the union over all possible distributions P_{U,X_1,X_2} is equivalent to the union over jointly complex Gaussian $[U\ X_1\ X_2]$. The complete proof can be found in Appendix C. ■

The characterization of the capacity provided in Lem. VI.1 well bounds the capacity region at high SNR. For low SNR, it is more useful to characterize the region in terms of a multiplicative factor, instead of an additive one. Next we show capacity to within a factor two for any MIMO CIFC.

Theorem VI.3. Capacity of the MIMO CIFC to within a factor two: *If $(R_1, R_2) \in \mathscr{R}^{\text{out}}$, then $(R_1/2, R_2/2)$ is achievable.*

Proof: Consider first the outer bound in (9): by taking the maximum of each bound, we obtain the looser outer bound

$$R_1 \le I(Y_1; X_1 | X_2) \quad (27a)$$
$$R_2 \le I(Y_2; X_1, X_2), \quad (27b)$$

where the union is over all the distributions p_{X_1,X_2}. Since the Gaussian distribution maximizes entropy, it is only necessary to consider jointly Gaussian channel inputs.

Consider then the following two achievable points

$$A^{\text{in,DPC}} = (I(Y_1; X_1|X_2), I(Y_2; X_2)) \quad (28a)$$
$$B^{\text{in,MISO}} = (0, I(Y_2; X_1, X_2)). \quad (28b)$$

The point $A^{\text{in,DPC}}$ is achievable using the inner bound in Thm. IV.2 by setting $U_{2pb} = \emptyset$ and

$$U_{1pb} = X_1 + AX_2 \quad (29a)$$
$$A = \mathbb{V}\text{ar}[X_1] H_{11}^H (H_{11} \mathbb{V}\text{ar}[X_1] H_{11}^H + I)^{-1}, \quad (29b)$$

which completely cancels the effect of the interference at receiver 1. The point $B^{\text{in,MISO}}$ is achieved with the choice $U_{1pb} = U_{2pb} = \emptyset$ which corresponds to having both encoders transmit to Rx 2 as in a MISO channel.

We now show that the simplified outer bound of (27) is to within a factor of two from the convex closure of $A^{\text{in,DPC}}$ and $B^{\text{in,MISO}}$. Since $A^{\text{in,DPC}}$ and $B^{\text{in,MISO}}$ are achievable, with time sharing we can achieve any point (R_1, R_2) such that

$$R_2 = -\frac{I(Y_2; X_1|X_2)}{I(Y_1; X_1|X_2)} R_1 + I(Y_2; X_1, X_2), \quad (30)$$

for $R_1 \in [0 \ldots I(Y_1; X_1|X_2)]$. In particular, the following rate point is achievable,

$$C^{\text{in}} = (R_1^{C-\text{in}}, R_2^{C-\text{in}}) \quad (31a)$$
$$= (1/2 I(Y_1; X_1|X_2), 1/2 (I(Y_2; X_1, X_2) + I(Y_2; X_1, X_2))),$$

For this point we have that $2R_1^{C-\text{in}} = (27a)$ while

$$2R_2^{C-\text{in}} - (27b) = I(Y_2; X_1, X_2) + I(Y_2; X_2) - I(Y_2; X_1, X_2)$$
$$= I(Y_2; X_2) \ge 0. \quad (32a)$$

■

VII. Capacity Results for the MIMO CIFC

After having characterized the approximate capacity of the MIMO CIFC, we now derive a new capacity result for a subset of the parameters regime. This result is obtained by generalizing and expanding the capacity results in the "very strong interference" and the "primary decodes cognitive" regimes for the G CIFC. Both results are obtained by showing the achievability of the same outer bound, the "strong interference" outer bound in Lem. III.5. By considering an achievable scheme which merges the achievable schemes that attain capacity in these two results, it is possible to achieve the "strong interference" outer bound in a regime which is larger than the union of the "very strong interference" and the "primary decodes cognitive" regimes.

Let's begin by deriving the equivalent of the "strong interference" outer bound in Th. III.5 for the MIMO BC.

Lemma VII.1. MIMO "Strong Interference" Outer Bound. *Let the "strong interference" regime be defined as the set of MIMO CIFCs for which*

$$|H_{11}| \le |H_{21}|, \quad (33)$$

then the region

$$R_1 \le \log |\text{Cov}(H_{11}X_1 + Z_1)| \quad (34a)$$
$$R_1 + R_2 \le \log |\text{Cov}(H_{12}X_1 + X_2H_{22} + Z_2)|, \quad (34b)$$

where the union is over all $[X_1 \ X_2] \sim \mathscr{CN}(0, [K_{11}, K_{12}; K_{21}, K_{22}])$ with $K_{ii} \prec S_i$ for $i \in \{1, 2\}$ is an outer bound to the capacity region of the MIMO CIFC in "strong interference".

Proof: The proof consists of showing that the inequality in (10) for the MIMO CIFC can be evaluated only for jointly Gaussian inputs using the extremal inequality. The full proof is provided in Appendix D. ■

In the "very strong interference" regime, the outer bound in Lem. VII.1 is achieved by superimposing the cognitive codeword over the primary one and having both decoders decode both messages. This corresponds to the scheme in Th. IV.1 where $U_{2c} = X_2$, that is, the primary message is set to be common.

In the "primary decodes cognitive" regime, the outer bound in Th. VII.1 is achieved by having the primary message private and pre-coding the cognitive codeword against the primary interference. The primary receiver decodes both codewords and uses the cognitive codeword as a side information to decode its intended message. This corresponds to the scheme in Th. IV.1 where $U_{2c} = \emptyset$, that is, the primary message is set to be private.

The scheme in Th. IV.1 unifies and generalizes the two capacity achieving schemes by considering the case in which the private message is split in two parts, a common and a private part. The cognitive message is public and pre-coded against the private primary message.

Theorem VII.2. Capacity for a Subset of the "Strong Interference" Regime. *If for each $[X_1 \ X_2]$ in Lem. VII.1 there exists an assignment*

$$X_{1c} \sim \mathscr{CN}(\mathbf{0}, K_{11} - K_{12}) \quad (35a)$$
$$X_{2p} \sim \mathscr{CN}(\mathbf{0}, K_{2p}), \quad K_{2p} \preceq K_{22} \quad (35b)$$
$$X_{2c} \sim \mathscr{CN}(\mathbf{0}, K_{22} - K_{2p}) \quad (35c)$$
$$X_2 = X_{2c} + X_{2p} \quad (35d)$$
$$X_1 = X_{1c} + K_{12} K_{22}^{-1/2} X_2 \quad (35e)$$
$$U_{1c} = X_{1c} + \Lambda_{\text{Costa}} H_{12} X_{2p} \quad (35f)$$
$$\Lambda_{\text{Costa}} = K_{1c} H_{11}^H (\mathbf{I} + H_{11} K_{1c} H_{11}^H)^{-1}, \quad (35g)$$

such that

$$I(Y_1;U_{1c},U_{2c}) \geq I(Y_2;U_{1c},U_{2c}) \quad (36a)$$
$$I(Y_1;U_{1c}) \geq I(Y_2;U_{1c}), \quad (36b)$$

then, the region in Lem. VII.1 is the capacity region.

Proof: With the choice of U_{1c} in (35g) the bound in (18a) becomes equal to the bound in (34a) since this corresponds to the assignment in [9]. Since (18b) equals (34b), capacity is shown when the bounds (18d) and (18c) are redundant. The conditions in (36) guarantee that this is indeed the case. ■

We next verify that Th. VII.2 generalizes the "very strong interference" and the "primary decodes cognitive" capacity results.

Lemma VII.3. *The "very strong interference" result corresponds to the case where K_{2p} is a zero matrix in which case the conditions in* (36) *reduce to*

$$I(Y_1;X_1,X_2) \geq I(Y_2;X_1,X_2) \quad (37a)$$
$$I(Y_1;X_1) \geq I(Y_2;X_1). \quad (37b)$$

Since $I(Y_1;X_1|X_2) \geq I(Y_1;X_1|X_2)$ in the "strong interference" regime, condition (37b) *is redundant.*

Similarly, the "primary decodes cognitive" regime, is obtained by setting $K_{2p} = K_{22}$ in which case the conditions in (36) *reduce to*

$$I(Y_1;U_{1c}) \geq I(Y_2;U_{1c}), \quad (38a)$$

which corresponds to the condition in (17) *for the MIMO CIFC.*

A. G CIFC Example

To prove that the result in Th. VII.2 is more general than the union of the "very strong interference" and the "primary decodes cognitive" capacity results, we focus on the G CIFC. For this channel model, the set of input covariances which produce points on the convex hull of the "strong interference" outer bound can be parameterized as in (13). We begin by proving the partial achievability of the "strong interference" outer bound using superposition coding.

Lemma VII.4. Partial achievability of the "strong interference" outer bound with superposition coding [2]. *When $|b| \geq 1$, the "strong interference" outer bound is achievable at the point corresponding to $\alpha = x$ if*

$$(1-|b|^2)P_1 + (|a|^2-1)P_2 \geq 2\left(|b| - \mathrm{Re}\{a^H\}\right)\sqrt{x P_1 P_2}. \quad (39)$$

Proof: When fixing the rate of the private primary message to zero in (18) and for $|b| \geq 1$, the rate bound (18d) can be dropped. The outer bound is achieved when condition (36a) is met, which translates to (39). ■

The capacity result in Th. III.6 is obtained by imposing condition (39) for all $\alpha \in [0,1]$.

We now provide the conditions for the partial achievability of the "strong interference" outer bound with binning.

Lemma VII.5. Partial achievability of the "strong interference" outer bound with binning [10]. *When $|b| \geq 1$, the "strong interference" outer bound is achievable at the point corresponding to $\alpha = x$ if*

$$P_2(1-a|b|)^2(\alpha P_1 + 1) - (|b|^2-1)(P_1 + |a|^2 P_2 + 2a\sqrt{x P_1 P_2} + 1) \geq 0. \quad (40)$$

Proof: When fixing the rate of the common primary message to zero in (18) and for $|b| \geq 1$, the rate bound (18d) can be dropped. The outer bound is achieved when condition (36b) is met, which translates to (40). ■

With the aid of Lem. VII.4 and Lem. VII.5, we now show the achievability of the "strong interference" outer bound for $|b| \geq 1$ using the inner bound in Th. IV.1.

Theorem VII.6. New Achievability of the Strong Interference Outer Bound *Let $(i)|_{\alpha=\gamma}$ indicates that condition (i) holds for the assignment $\alpha = \gamma$ and define*

$$\widetilde{\alpha} = \max\left\{0, \min\left\{1, \frac{(|a|^2-1)P_2 + (1-|b|^2)P_1}{2(\mathrm{Re}\{a^H\} - |b|)\sqrt{P_1 P_2}}\right\}\right\}. \quad (41)$$

If

$$(40)|_{\alpha=0}, \quad (39)|_{\alpha=1}, \quad (40)|_{\alpha=\widetilde{\alpha}}, \quad (42)$$

or

$$(40)|_{\alpha=1}, \quad (39)|_{\alpha=0}, \quad (40)|_{\alpha=\widetilde{\alpha}}, \quad (43)$$

the region in (14) *is the capacity region.*

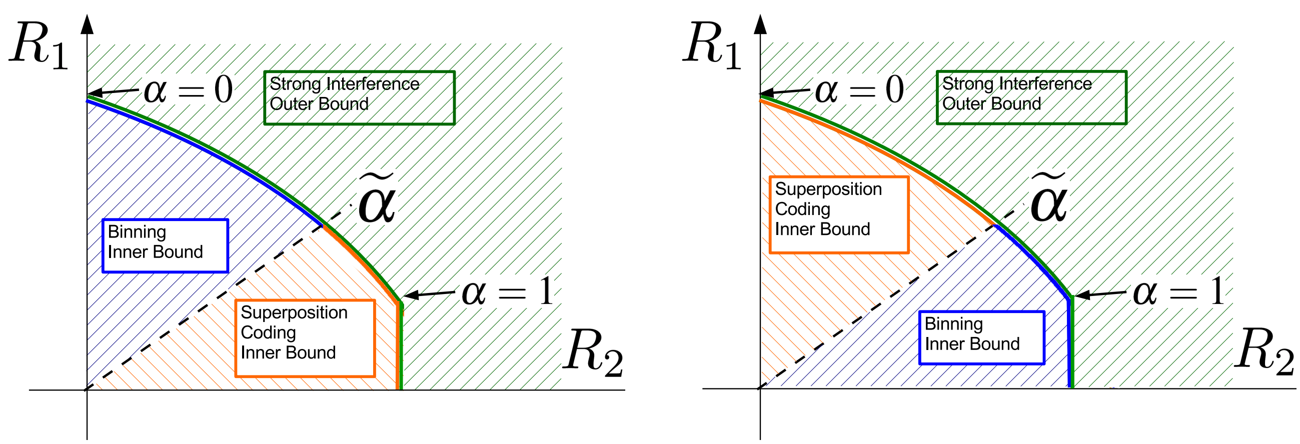
Proof: Capacity is shown by extending the partial achievability results of Lem. VII.4 and Lem. VII.5 to whole range $\alpha \in [0,1]$. To match the inner bound in Th. IV.1 with the assignment in (35) and the outer bound in Th. III.5 for $\beta \geq 1$ we need equations (18c) and (18d) to be redundant, that is

$$I(Y_2;U_{1c},X_2|U_{2c}) \geq I(Y_1;U_{1c}|X_2,U_{2c}) \quad (44a)$$
$$I(Y_1;U_{1c},U_{2c}) \geq I(Y_2;U_{1c},U_{2c}) \quad (44b)$$
$$I(Y_1;U_{2c}) \geq I(Y_2;U_{2c}). \quad (44c)$$

For the G CIFC condition (44c) can be rewritten as

$$\frac{|a|^2P_2 + P_1 + 2\mathrm{Re}\{a^H\}\sqrt{\overline{\alpha}P_1P_2} + 1}{\alpha P_1 + \overline{\beta}\left|\sqrt{\overline{\alpha}P_1} + a\sqrt{P_2}\right|^2 + 1} \geq \frac{|b|^2P_1 + P_2 + 2|b|\sqrt{\overline{\alpha}P_1P_2} + 1}{|b|^2\alpha P_1 + \overline{\beta}\left|\sqrt{|b|^2\overline{\alpha}P_1} + \sqrt{P_2}\right|^2 + 1}. \quad (45)$$

Condition (45) holds only for $\beta = 0$ in the "strong interference" but outside the "very strong interference" regime. This implies that when condition (39) does not hold, one can hope to achieve the outer bound only with the choice $\beta = 0$. With this observation we conclude that capacity can be achieved using $\beta = 0$ for a subset of the $\alpha \in [0,1]$ while using $\beta = 1$ for the remaining subset. That is, either condition (39) or (40) must hold for any $\alpha \in [0,1]$. Note that condition (39) is linear in $\sqrt{x}$, so if it holds for α_1 and α_2, then it holds for the whole interval $[\alpha_1,\alpha_2]$. Similarly, (40) is quadratic and concave in $\sqrt{x}$, so if it holds for α_1 and α_2, then it holds for the whole interval $[\alpha_1,\alpha_2]$. For this reason the outer bound is achievable for any $\alpha \in [0,1]$ when: (i) one condition holds in both zero and one, or (ii) one condition holds in zero and in $\alpha = \widetilde{\alpha}$ and



(a) The capacity result in Th. VII.6 for the condition in (42).

(b) The capacity result in Th. VII.6 for the condition in (43).

Fig. 7. A graphical representation of the capacity result in Th. VII.6.

the other holds in $\alpha = \widetilde{\alpha}$ and one. For simplicity we choose α' to be the α for which condition (39) holds with equality as in (41). ■

The proof of Th. VII.6 is depicted in Fig. 7. Under the condition in Lem. VII.4 the strong interference outer bound can be attained using superposition coding. Similarly, under the condition in Lem. VII.5, it can be attained with binning. When condition (42) holds, the strong interference outer bound is attained by binning in the interval $\alpha \in [0, \widetilde{\alpha}]$ while it in attained by superposition coding in the interval $\alpha \in [\widetilde{\alpha}, 1]$. For condition (43) we have the reverse situation: superposition coding attains the strong interference outer bound in the interval $\alpha \in [0, \widetilde{\alpha}]$, while binning achieves the outer bound in $\alpha \in [\widetilde{\alpha}, 1]$.

In the proof of Th. VII.6, the optimal transmission strategy is obtained by having either a primary public message or a primary private one, depending on the cooperation level between the transmitters. This is somewhat surprising as one would expect rate-splitting to provide some rate advantages. On the other hand, rate-splitting is usually not necessary to achieve the convex hull of the achievable region. The key intuition here is provided by (45): outside the "very strong interference" regime there is a rate penalty in decoding the primary message at the cognitive decoder at some rates. When such a penalty exists, the best thing to do is to set the rate of the private cognitive message to zero. Note that this may not be the case when considering an assignment different from (35). In [10] it is shown that partial interference cancellation, i.e. setting $\lambda \neq \lambda_{\text{Costa 1}}$ in (45), can yield larger achievable regions then full interference cancellation.

VIII. Numerical Results

We now numerically illustrate capacity results of the previous sections for the G CIFC. Since this model is parameterized by two values, a and $|b|$, the classes of channels for which capacity is known, exactly or approximatively, can be conveniently represented on the plane $a \times |b|$. We begin by plotting the result of (42) in Fig. 8(a): in this figure we plot the region where (40) holds for $\alpha = 0$, (39) holds for $\alpha = 1$ and finally where (42) holds. Both conditions (40) for $\alpha = 0$ and (39) for $\alpha = 1$ are necessary but not sufficient conditions for (42) to hold. Unlike for the "weak interference" and "very strong interference" capacity results, the conditions in (42) cannot be intuitively interpreted. More importantly, it is not straight forward to determine where (42) holds and instead numerical simulations are need to determine that region.

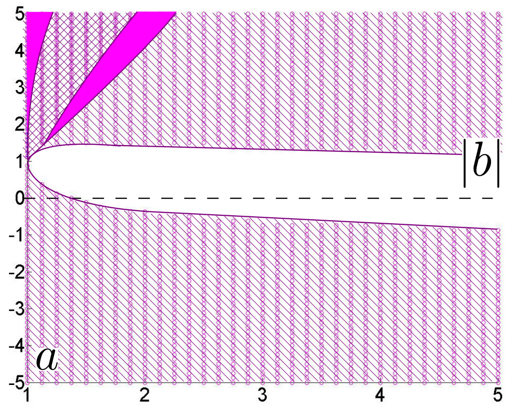
In Fig. 8(b), we present the improvement on the known capacity region that is provided by Th. VII.6. The capacity result in Fig. 8(a) is plotted along side the other regimes in which capacity is known in strong interference, the "very strong interference" and the "primary decodes cognitive" regimes of Th. III.6 and of Th. III.7 respectively. Note that the result in (42) is not necessarily contiguous to any of the above regions, although this can be often observed in the numerical evaluations. In Fig. 8(b), the gap between the "very strong interference" region and the new capacity result is most likely due to numerical precision issues.

Fig. 9(a) is analogous to Fig. 8(a) but for the condition in 8(b): in this figure we plot the region where (40) holds for $\alpha = 1$, (39) holds for $\alpha = 0$ and finally where (43) holds. Note that this region is much smaller than the region in Fig. 8(a) and that the choice of the powers P_1 is different among the two figures. From the numerical simulation, one gathers the impression that indeed the condition in (43) holds for a smaller set of channels than (42). Unfortunately, this intuition cannot yet be shown analytically.

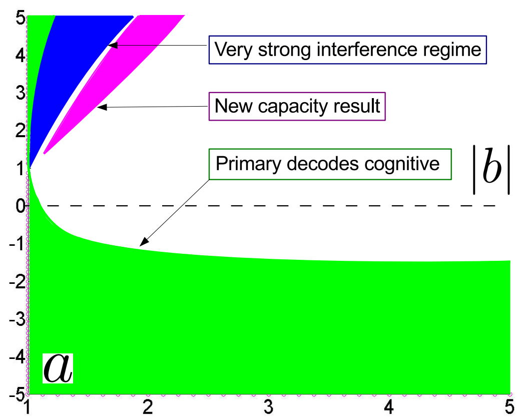
In Fig. 9(b) the new capacity result of Fig. 9(a) is plotted along-side the capacity result in the "very strong" and the "primary decodes cognitive" capacity regions.

IX. Conclusions

We have studied the capacity of the general cognitive interference channel, a variation of the classical interference channel where one of the transmitter, the cognitive transmitter, is provided with the message of the other user, the primary user. We derive the capacity of this channel to within a finite additive gap which relates to the ability of the cognitive transmitter to emulate the channel output at the cognitive

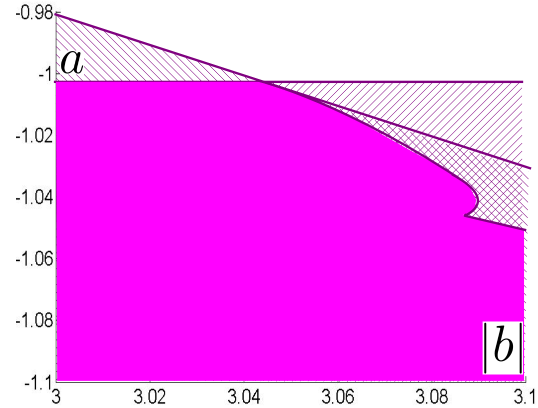


(a) The region where (40) for $\alpha = 0$ (single hatched) holds, where (39) for $\alpha = 1$ (cross hatched)

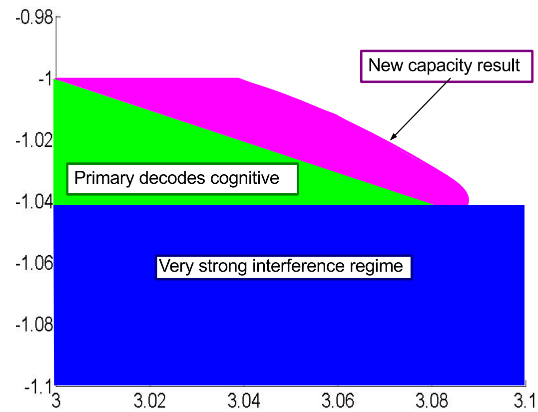


(b) The "very strong interference" capacity region (blue), The "primary decodes cognitive" capacity region (green), holds and where (42) holds (purple)

Fig. 8. The result of Th. VII.6 for the G CIFC with $P_1 = 10$, $P_2 = 1$ and $a \times |b| \in [-5,5] \times [1,5]$



(a) The region where (40) for $\alpha = 0$ (single hatched) holds, where (39) for $\alpha = 1$ (cross hatched)



(b) The "very strong interference" capacity region (blue), The "primary decodes cognitive" capacity region (green), holds and where (42) holds (purple)

Fig. 9. The result of Th. VII.6 for the G CIFC with for $P_1 = 10^{-3}$ with $P_2 = 1$ and $a \times |b| \in [-1.1, -1] \times [3, 3.1]$.

receiver. We specialize this result to the multiple antenna cognitive interference channel and show that, for this model, the gap between the inner and outer bounds is equal to the number of antennas at the cognitive receiver. This result well characterizes the capacity at high SNR; for the low SNR regime we show that the ratio between the inner and the outer bound is at most two. We also derive a new capacity result for the sub-class of the multiantenna cognitive interference channel. This new capacity result is obtained by generalizing the capacity proof for the "very strong interference" regime, in which superposition coding achieves capacity, and for the "primary decodes cognitive" regime, in which binning is optimal. Although this result improves on the class of channels for which capacity is known, the complete characterization of the capacity of this channel is still an open problem. Our results show how cognition benefits both the primary and the cognitive user. The primary user is able to attain much larger rates thanks to the cooperation with the cognitive transmitter. On the other hand, the knowledge of the primary message at the cognitive encoder, allows it to remove the effect of the interference at the cognitive receiver.

Appendix

A. Proof of Th. IV.1

The original expression of scheme (F) in [20, Sec. IV.F] contains a rate bound for R_1 which can be shown redundant with rate-sharing. Let R_{2c} be the rate of the public primary message, R_{2p} that of the private primary message, and R_{1c} the rate of the public cognitive messages, then the original achievable region is expressed as

$$R_{2c} + R_{1c} \le I(Y_1; U_{1c}, U_{2c}) - I(U_{1c}; X_2 | U_{2c}) \quad (46a)$$

$$R_{1c} \le I(Y_1; U_{1c} | U_{2c}) - I(U_{1c}; X_2 | U_{2c}) \quad (46b)$$

$$R_{2c} + R_{1c} + R_{2p} \le I(Y_2; X_1, X_2) \quad (46c)$$

$$R_{1c} + R_{2p} \le I(Y_2; X_2, U_{1c} | U_{2c}) \quad (46d)$$

$$R_{2p} \le I(Y_2; X_2 | U_{2c}, U_{1c}). \quad (46e)$$

Notice now that if the rate vector $[R_{1c}\ R_{2c}\ R_{2p}]$ is achievable, then the rate vector

$$[R_{1c} - \Delta_1\ R_{2c} - \Delta_1\ R_{2p} + \Delta_1 + \Delta_2], \quad (47)$$

for $\Delta_1 \leq R_{1c}$ and $\Delta_2 \leq R_{2c}$ is also achievable. The quantity Δ_1 represents the part of the primary private message transmitter through the public cognitive codeword while Δ_2 is the part of the private primary message embedded in the public primary message. By performing the Fourier-Motzkin elimination of the Δ_1 and Δ_2 and successively setting the rate R_{2c}, one obtains the region in (18).

B. Proof of Th. V.3

The desired result is shown by considering the outer bound in Th. III.1 and the inner bound in Th. IV.2 and bounding the distance between the two bounds for any fixed distribution. We do so by choosing a particular assignment of the RVs in the outer bound for a given distribution of P_{U,X_1,X_2} in the outer bound. In particular, we choose $[X_1\ X_2\ U_{2pb}]$ in Th. IV.2 to equal $[X_1\ X_2\ U]$ respectively in Th. III.1. Additionally, we choose U_{1b} to have the same distribution as $Y_1|X_1,X_2$: this is obtained by passing X_1 and X_2 through the test channel $P_{Y_1|X_1,X_2}$ to generate the RV $\widetilde{Y}_1$. This can be done because the cognitive transmitter has knowledge of both messages and can thus reconstruct both channel inputs. With this assignment, the inner bound becomes

$$R_1 \leq I(Y_1;X_1|X_2) - I(Y_1;X_1,X_2|\widetilde{Y}_1) \tag{48a}$$
$$R_2 \leq I(Y_2;U,X_2) \tag{48b}$$
$$R_1 + R_2 \leq I(Y_2;U,X_2) + I(Y_1;X_1|X_2,U) - I(Y_1;\widetilde{Y}_1), \tag{48c}$$

since the RHS of (48a) can be obtained as

$$\begin{aligned}
&I(Y_1;U_{1pb}) - I(U_{1pb};X_2)\\
&= H(Y_1) - H(Y_1|\widetilde{Y}_1) - H(\widetilde{Y}_1) + H(\widetilde{Y}_1|X_2) && (49a)\\
&= H(\widetilde{Y}_1|X_2) - H(Y_1|\widetilde{Y}_1) && (49b)\\
&= H(Y_1|X_2) - H(Y_1|X_1,X_2) + H(Y_1|X_1,X_2) - H(Y_1|\widetilde{Y}_1) && (49c)\\
&= I(Y_1;X_1|X_2) - I(Y_1;X_1,X_2|\widetilde{Y}_1), && (49d)
\end{aligned}$$

and (48c) can be obtained as

$$\begin{aligned}
&I(Y_1;U_{1pb}) - I(U_{1pb};X_2,U)\\
&= I(Y_1;\widetilde{Y}_1) - I(\widetilde{Y}_1;X_2,U) && (50a)\\
&= H(Y_1|X_2,U) - H(Y_1|\widetilde{Y}_1) && (50b)\\
&= H(Y_1|X_2,U) - H(Y_1|X_1,X_2,U) + H(Y_1|X_1,X_2,\widetilde{Y}_1) - H(Y_1|\widetilde{Y}_1) && (50c)\\
&= I(Y_1;X_1|X_2,U) - I(Y_1;X_1,X_2|\widetilde{Y}_1), && (50d)
\end{aligned}$$

where, in (50c), we have used the Markov chain $Y_1 - X_1,X_2 - U\widetilde{Y}_1$. The largest additive gap between the inner and outer bounds corresponds to the largest gap between the two bound for a fixed distribution P_{U,X_1,X_2}.

C. Proof of Th. VI.2

First of all notice that we can express $\mathscr{R}^{\text{out}}$ as

$$\mathscr{R}^{\text{out}} = \bigcup_{P_{X_1,X_2}} Conv\left\{\mathscr{R}^{\text{out-A}} \cup \left(\bigcup_{P_{U|X_1,X_2}} \mathscr{R}^{\text{out-B}}\right)\right\}, \tag{51}$$

where $Conv(A)$ indicates the convex closure of A, $\mathscr{R}^{\text{out}-\text{A}}$ is defined as

$$R_1 \leq I(Y_1;X_1|X_2) \tag{52a}$$
$$R_2 \leq I(Y_2;X_2,U^{\text{sum}}) \tag{52b}$$
$$R_1 + R_2 \leq I(Y_2;X_2,U^{\text{sum}}) - I(Y_1;X_1|X_2,U^{\text{sum}}), \tag{52c}$$

for

$$U^{\text{sum}} \sim \underset{P_{U|X_1,X_2}}{\operatorname{argmax}} \quad I(Y_2;U,X_2) + I(Y_1;X_1|X_2,U), \tag{53}$$

and $\mathscr{R}^{\text{out}-\text{B}}$ is defined as

$$R_1 \leq I(Y_1;X_1|X_2,U) \tag{54a}$$
$$R_2 \leq I(Y_2;U,X_2). \tag{54b}$$

The argument of the convex closure in the RHS of (51) contains all the points A^{out} and B^{out} which are obtained from (9) when considering the union over $P_{U|X_1,X_2}$ for a fixed P_{X_1,X_2}. Since the bound in (52a) does not depend on U, the largest $R_2^{A-\text{out}}$ is obtained by maximizing the sum rate bound (52c). On the other hand, the coordinates of the point B^{out} depend on $P_{U|X_1,X_2}$ and the region $\mathscr{R}^{\text{out}-\text{B}}$ corresponds to all the points B^{out} generated by varying $P_{U|X_1,X_2}$. Since $\mathscr{R}^{\text{out}-\text{A}}$ contains all the points A^{out} and $\mathscr{R}^{\text{out}-\text{B}}$ contains all the points B^{out}, the equivalence in (51) is shown by considering the convex closure of these two regions.

Consider now the region $\bigcup_{P_{U|X_1,X_2}} \mathscr{R}^{\text{out}-\text{B}}$: the points on the convex hull of this region can be expressed as

$$\begin{aligned}
&\mu R_1 + \overline{\mu} R_2 =\\
&\quad \max_{P_{U|X_1,X_2}} \mu I(Y_1;X_1|X_2,U) + \overline{\mu} I(Y_2;X_2,U) && (55a)\\
&\leq \overline{\mu} H(Y_{2G}) - \mu H(Z_1)\\
&\quad + \mu \left(\max_{P_{U|X_1,X_2}} H(Y_1|X_2,U) - \rho H(Y_2|X_2,U) \right) && (55b)
\end{aligned}$$

for $\mu \in [0,1]$ and $\rho = \overline{\mu}/\mu$ and where Y_{2G} indicates the zero mean Gaussian vector having the same covariance as Y_2. Equality is achieved in (55b) by showing that choosing jointly Gaussian $[X_1\ X_2\ U]$ is optimal. Note that the optimization in (55) for $\mu = .5$ is attained by U^{sum} in (53). Values of $\mu \in (.5\ldots 1]$ need not be considered for (55): these points are already contained in the region $\mathscr{R}^{\text{out}-\text{A}}$ given that $(52\text{c})_{\text{R}} \geq (54\text{a})_{\text{R}} + (54\text{b})_{\text{R}}$ and $(52\text{a})_{\text{R}} \geq (54\text{a})_{\text{R}}$. For this reason we can write the argument of the convex closure in the RHS of (51) as

$$\mu R_1 + \overline{\mu} R_2 = \begin{cases} \max_{P_{U|X_1,X_2}} \mu (54\text{a})_{\text{R}} + \overline{\mu} (54\text{b})_{\text{R}} \\ \qquad \text{for} \quad 0 \leq \mu \leq .5 \\ (52\text{c})_{\text{R}} \\ \qquad \text{for} \quad .5 < \mu \leq 1 \end{cases} \tag{56}$$

For the range $0 \leq \mu \leq .5$, we have $\rho \geq 1$ and thus we can apply the extremal inequality of [25, Th. 8] to conclude that the maximum of (55b) is attained by Gaussian X_1, X_2 and U and that (55b) holds with equality. By the same token, (53) is also maximized by Gaussian inputs and U, since it corresponds to (55b) for $\rho = 1$. This shows that all the points

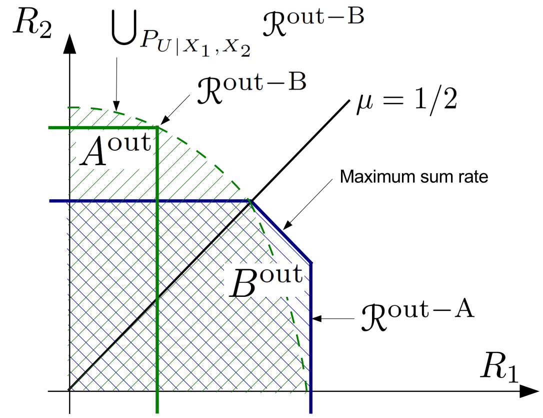


Fig. 10. A graphical representation of the proof of Th. VI.2 which shows the region (52) union over $P_{U|X_1,X_2}$ for a fixed P_{X_1,X_2}.

on the boundary of the region $\mathscr{R}^{\text{out}-\text{A}} \cup \left(\bigcup_{P_{U|X_1,X_2}} \mathscr{R}^{\text{out}-\text{B}} \right)$ are maximized by jointly complex Gaussian $[X_1\ X_2\ U]$.

A graphical representation of the proof is provided in Fig. 10: the region $\mathscr{R}^{\text{out}-\text{A}}$ and the region $\bigcup_{P_{U|X_1,X_2}} \mathscr{R}^{\text{out}-\text{B}}$ intersect for $\mu = .5$ in (55b). For $\mu \leq .5$, points of $\bigcup_{P_{U|X_1,X_2}} \mathscr{R}^{\text{out}-\text{B}}$ are on the boundary of the region while for $\mu > .5$ points of $\mathscr{R}^{\text{out}-\text{A}}$ a re. The maximum sum rate is achieved by the assignment in (53) and for complex Gaussian $[U\ X_1\ X_2]$ according to the extremal inequality of [25].

D. Proof of Th. VII.1

The condition (10) for MIMO CIFC can be rewritten as Eq. 57.

From the conditional version of the extremal inequality of [25, Th. 1], we conclude that the maximization in (57) is attained by jointly complex Gaussian X_1 and X_2.

To further simplify the expression in (57) we require the following lemma:

Lemma A.1. *For A, B positive semi-definite*

$$|I + A| \leq |I + B| \Longleftrightarrow |A| \leq |B|, \tag{58}$$

Proof: Since A and B are positive semi-definite, $|A| = \prod_i \lambda_i^A$, where λ_i is the i^{th} eigenvalue of A. The eigenvalues of $A + I$ are therefore

$$|I + A| = \prod_i (1 + \lambda_i^A), \tag{59}$$

and thus

$$\begin{aligned}
&|I + A| \leq |I + B| && \Longleftrightarrow && (60a)\\
&\prod_i (1 + \lambda_i^A) \leq \prod_i (1 + \lambda_i^B) && \Longleftrightarrow && (60b)\\
&\prod_i \lambda_i^A \leq \prod_i \lambda_i^B && \Longleftrightarrow && (60c)\\
&|A| \leq |B|. && && (60d)
\end{aligned}$$

■

From the extremal inequality we have

$$\max_{P_{X_1|X_2}} H(X_1 + Z_1|X_2) - H(H_{21}X_1 + Z_2|X_2) - (n_1 - n_2)\log(2\pi e) \tag{61a}$$

$$\begin{aligned}
&= \log|I + H_{11}(K_{11} - K_{12}K_{22}^{-1}K_{21})H_{11}^H| + && (61b)\\
&\quad - \log|I + H_{21}(K_{11} - K_{12}K_2^{-1}K_{21})H_{21}^H| && (61c)
\end{aligned}$$

and, using the Lemma A.1, we can conclude that

$$\begin{aligned}
&\log|I + H_{11}(K_1 - K_{12}K_2^{-1}K_{21})H_{11}^T| \leq \\
&\quad \log|I + H_{21}(K_1 - K_{12}K_2^{-1}K_{21})H_{21}^T| && \Longleftrightarrow && (62a)\\
&|I + H_{11}(K_1 - K_{12}K_2^{-1}K_{21})H_{11}^T| \leq \\
&\quad |I + H_{21}(K_1 - K_{12}K_2^{-1}K_{21})H_{21}^T| && \Longleftrightarrow && (62b)\\
&|H_{11}||K_1 - K_{12}K_2^{-1}K_{21}||H_{11}^T| \leq \\
&\quad |H_{21}||K_1 - K_{12}K_2^{-1}K_{21}||H_{21}^T| && \Longleftrightarrow && (62c)\\
&|H_{11}| \leq |H_{21}|. && && (62d)
\end{aligned}$$

The expression in (34) is obtained by noticing that the region in (10) is maximized by jointly complex Gaussian inputs.

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$$
\begin{aligned}
& I(Y_1;X_1|X_2) \le I(Y_2;X_1|X_2) && \Longleftrightarrow \\
& H(Y_1|X_2) - n_1\log(2\pi e) - \le H(Y_2|X_2) - n_2\log(2\pi e) && \Longleftrightarrow \\
& H(Y_1|X_2) - H(Y_2|X_2) - (n_1-n_2)\log(2\pi e) \le 0 && \Longleftrightarrow \\
& H(X_1+Z_1|X_2) - H(H_{21}X_1+Z_2|X_2) - (n_1-n_2)\log(2\pi e) \le 0 && \Longleftrightarrow \\
& \max_{p_{X_1|X_2}} \ (H(X_1+Z_1|X_2) + \\
& \quad -H(H_{21}X_1+Z_2|X_2)) - (n_1-n_2)\log(2\pi e) \le 0 && (57)
\end{aligned}
$$

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