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On mutually independent hamiltonian paths

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Abstract

Let $P_1 = \langle v_1, v_2, v_3, \ldots, v_n \rangle$ and $P_2 = \langle u_1, u_2, u_3, \ldots, u_n \rangle$ be two hamiltonian paths of G. We say that P_1 and P_2 are *independent* if $u_1 = v_1, u_n = v_n$, and $u_i \neq v_i$ for $1 < i < n$. We say a set of hamiltonian paths $P_1, P_2, \ldots, P_s$ of G between two distinct vertices are *mutually independent* if any two distinct paths in the set are independent. We use n to denote the number of vertices and use e to denote the number of edges in graph G. Moreover, we use $\bar{e}$ to denote the number of edges in the complement of G. Suppose that G is a graph with $\bar{e} \leq n - 4$ and $n \geq 4$. We prove that there are at least $n - 2 - \bar{e}$ mutually independent hamiltonian paths between any pair of distinct vertices of G except $n = 5$ and $\bar{e} = 1$. Assume that G is a graph with the degree sum of any two non-adjacent vertices being at least $n + 2$. Let u and v be any two distinct vertices of G. We prove that there are $\deg_G(u) + \deg_G(v) - n$ mutually independent hamiltonian paths between u and v if $(u, v) \in E(G)$ and there are $\deg_G(u) + \deg_G(v) - n + 2$ mutually independent hamiltonian paths between u and v if otherwise.

Keywords: Hamiltonian; Hamiltonian connected; Hamiltonian path

1. Definitions and notation

For the graph definition and notation we follow [1]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the

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edge set. We use n to denote $|V|$ and use e to denote $|E|$. The complement of G is denoted by $\bar{G}$. We use $\bar{e}$ to denote $|E(\bar{G})|$. Hence, $e + \bar{e} = n(n-1)/2$. For any vertex $x \in V$, $\deg_G(x)$ denotes its degree in G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* P is represented by $\langle v_0, v_1, v_2, \ldots, v_k \rangle$. A path is a *hamiltonian path* if its vertices are distinct and span V. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once.

There are a lot of studies on hamiltonian connected graphs. In this work, we are interested in another aspect of hamiltonian connected graphs. Let $P_1 = \langle v_1, v_2, v_3, \ldots, v_n \rangle$ and $P_2 = \langle u_1, u_2, u_3, \ldots, u_n \rangle$ be any two hamiltonian paths of G. We say that P_1 and P_2 are *independent* if $u_1 = v_1, u_n = v_n$, and $u_i \neq v_i$ for $1 < i < n$. We say a set of hamiltonian paths $P_1, P_2, \ldots, P_s$ of G are *mutually independent* if any two distinct paths in the set are independent. In [4], it is proved that there exist $(k-2)$ mutually independent hamiltonian paths between any two vertices from different bipartite sets of the star graph S_k if $k \geq 4$. The concept of mutually independent hamiltonian arises from the following application. If there are k pieces of data needed to be sent from u to v, and the data needed to be processed at every node (and the process takes times), then we want mutually independent hamiltonian paths so that there will be no waiting time at a processor. The existence of mutually independent hamiltonian paths is useful for communication algorithms. Motivated by this result, we begin the study on graphs with mutually independent hamiltonian paths between every pair of distinct vertices.

In this work, we are interested in two families of graphs. The first family of graphs $\bar{e} \leq n-4$. It was proved [5] that such graphs are hamiltonian connected. In this work, we strengthen this classical result by proving that there are at least $n - 2 - \bar{e}$ mutually independent hamiltonian paths between every pair of distinct vertices of G. The second family of graphs are those graphs with the sum of the degree of any two non-adjacent vertices being at least $n + 1$. It was proved [3] that such graphs are hamiltonian connected. We then further assume that G is a graph with the sum of any two non-adjacent vertices being at least $n + 2$. Let u and v be any two distinct vertices of G. Then there are $\deg_G(u) + \deg_G(v) - n$ mutually independent hamiltonian paths between u and v if $(u, v) \in E(G)$, and there are $\deg_G(u) + \deg_G(v) - n + 2$ mutually independent hamiltonian paths between u and v otherwise.

Throughout this work, we will use $[i]$ to denote i **mod** $(n-2)$.

2. Preliminary

Let G and H be two graphs. We use $G + H$ to denote the disjoint union of G and H. We use $G \vee H$ to denote the graph obtained from $G + H$ by joining each vertex of G to each vertex of H. For $1 \leq m < n/2$, let $C_{m,n}$ denote the graph $(\bar{K}_m + K_{n-2m}) \vee K_m$; see Fig. 1. The following theorem is proved by Chvátal [2].

Theorem 1 ([2]). *Assume that G is a graph with $n \geq 3$ and $\bar{e} \leq n-3$. Then G is hamiltonian. Moreover, the only non-hamiltonian graphs with $\bar{e} \leq n-2$ are $C_{1,n}$ and $C_{2,5}$.*

The following lemma is obvious.

Lemma 1. *Let u and v be two distinct vertices of G. Then there are at most* $\min\{\deg_G(u), \deg_G(v)\}$ *mutually independent hamiltonian paths between u and v if $(u, v) \notin E(G)$, and there are at most* $\min\{\deg_G(u), \deg_G(v)\} - 1$ *mutually independent hamiltonian paths between u and v if $(u, v) \in E(G)$.*

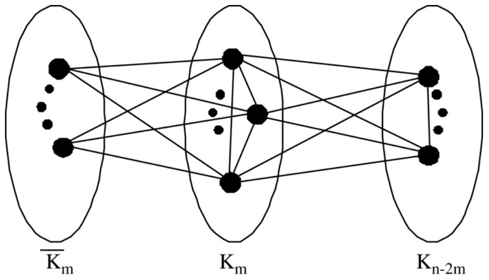

Fig. 1. $C_{m,n}$.

Theorem 2. *Let n be a positive integer with $n \geq 3$. There are $n-2$ mutually independent hamiltonian paths between every two distinct vertices of K_n.*

Proof. Let s and t be two distinct vertices of K_n. We relabel the remaining $(n-2)$ vertices of K_n as $0, 1, 2, \ldots, n-3$. For $0 \leq i \leq n-3$, we set P_i as $\langle s, [i], [i+1], [i+2], \ldots, [i+(n-3)], t\rangle$. It is easy to see that $P_0, P_1, \ldots, P_{n-3}$ form $(n-2)$ mutually independent hamiltonian paths joining s and t. □

Theorem 3 ([5]). *Assume that G is a graph with $\bar{e} \leq n-4$ and $n \geq 4$. Then G is hamiltonian connected.*

Theorem 4 ([5]). *Assume that G is a graph with the sum of any two distinct non-adjacent vertices being at least n with $n \geq 3$. Then G is hamiltonian.*

Theorem 5 ([3]). *Assume that G is a graph with the sum of any two distinct non-adjacent vertices being at least $n+1$ with $n \geq 3$. Then G is hamiltonian connected.*

3. Mutually independent hamiltonian paths

The following result strengthens that of Theorem 3.

Lemma 2. *Assume that G is a graph with $n \geq 4$ and $\bar{e} = n-4$. Then there are two independent hamiltonian paths between any two distinct vertices of G except $n = 5$.*

Proof. For $n = 4$, G is isomorphic to K_4. By Theorem 2, there are two independent hamiltonian paths between any two distinct vertices of G. Assume that $n = 5$. Then G is isomorphic to $K_5 - \{f\}$ for some edge f. Without loss of generality, we assume that $V(G) = \{1, 2, 3, 4, 5\}$ and $f = (1, 2)$. It is easy to check that $P_1 = \langle 3, 2, 5, 1, 4\rangle$ and $P_2 = \langle 3, 1, 5, 2, 4\rangle$ are the only two hamiltonian paths between 3 and 4, but P_1 and P_2 are not independent.

Now, we assume that $n \geq 6$. Let s and t be any two distinct vertices of G. Let H be the subgraph of G induced by the remaining $(n-2)$ vertices of G. We have the following two cases:

Case 1: H is hamiltonian. We can relabel the vertices of H with $\{0, 1, 2, \ldots, n-3\}$ so that $\langle 0, 1, 2, \ldots, n-3, 0\rangle$ forms a hamiltonian cycle of H. Let Q denote the set $\{i \mid (s, [i+1]) \in E(G)$ and $(i, t) \in E(G)\}$. Since $\bar{e} = n-4$, $|Q| \geq n-2-(n-4) = 2$. There are at least two elements in Q. Let q_1 and q_2 be the two elements in Q. For $j = 1, 2$, we set P_j as $\langle s, [q_j+1], [q_j+2], \ldots, [q_j], t\rangle$. Then P_1 and P_2 are two independent hamiltonian paths between s and t.

Case 2: H is non-hamiltonian. There are exactly $(n-2)$ vertices in H. By Theorem 1, there are exactly $(n-4)$ edges in the complement of H and H is isomorphic to $C_{1,n-2}$ or $C_{2,5}$. Since $\bar{e} = n-4$, we

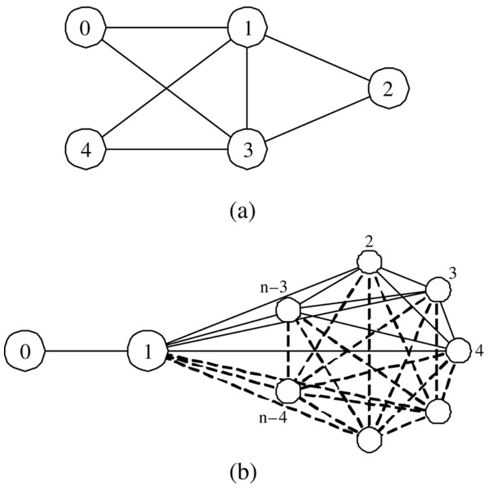

Fig. 2. (a) $C_{2,5}$, (b) $C_{1,n-2}$.

know that $(s, v) \in E(G)$ and $(t, v) \in E(G)$ for every vertex v in H. We can construct two independent hamiltonian paths between s and t as following cases:

Subcase 2.1: H is isomorphic to $C_{2,5}$. We label the vertices of $C_{2,5}$ with $\{0, 1, 2, 3, 4\}$ as shown in Fig. 2(a). Let $P_1 = \langle s, 0, 1, 2, 3, 4, t\rangle$ and $P_2 = \langle s, 2, 3, 4, 1, 0, t\rangle$. Then P_1 and P_2 form the required independent paths.

Subcase 2.2: H is isomorphic to $C_{1,n-2}$. We label the vertices of $C_{1,n-2}$ with $\{0, 1, \ldots, n-3\}$ as shown in Fig. 2(b). Let $P_1 = \langle s, 0, 1, 2, \ldots, n-3, t\rangle$ and $P_2 = \langle s, 2, 3, \ldots, n-3, 1, 0, t\rangle$. Then P_1 and P_2 form the required independent paths. □

We can further strengthen Theorem 3:

Theorem 6. *Assume that G is a graph with $n \geq 4$ and $\bar{e} \leq n-4$. Then there are $n-2-\bar{e}$ mutually independent hamiltonian paths between every two distinct vertices of G except $n = 5$ and $\bar{e} = 1$.*

Proof. With Lemma 2, the theorem for $\bar{e} = n-4$ holds. Now, we need to prove the theorem for $\bar{e} = n-4-r$ with $1 \leq r \leq n-4$. Let s and t be two distinct vertices of G. Let H be the subgraph of G induced by the remaining $(n-2)$ vertices of G.

Then there are exactly $(n-2)$ vertices in H and there are at most $n-4-r$ edges in the complement of H with $1 \leq r \leq n-4$. By Theorem 1, H is hamiltonian. We can label the vertices of H with $\{0, 1, 2, \ldots, n-3\}$ so that $\langle 0, 1, 2, \ldots, n-3, 0\rangle$ forms a hamiltonian cycle of H. Let Q denote the set $\{i \mid (s, [i+1]) \in E(G) \text{ and } (t, i) \in E(G)\}$. Since $\bar{e} = n-4-r$ with $1 \leq r \leq n-4$, we know that $|Q| \geq n-2-(n-4-r) = n-2-\bar{e}$ for $1 \leq r \leq n-4$. Hence, there are at least $n-2-\bar{e}$ elements in Q. Let $q_1, q_2, \ldots, q_{n-2-\bar{e}}$ be the elements in Q. For $j = 1, 2, \ldots, n-2-\bar{e}$, we set $P_j = \langle s, [q_j+1], [q_j+2], \ldots, [q_j], t\rangle$. It is not difficult to see that $P_1, P_2, \ldots, P_{n-2-\bar{e}}$ are mutually independent paths between s and t. □

The following result, in a sense, generalizes that of Theorem 5.

Theorem 7. *Assume that G is a graph such that $\deg_G(x) + \deg_G(y) \geq n+2$ for any two vertices x and y with $(x, y) \notin E(G)$. Let u and v be two distinct vertices of G. Then there are* $\deg_G(u) + \deg_G(v) - n$

mutually independent hamiltonian paths between u and v if $(u, v) \in E(G)$, *and there are* $\deg_G(u) + \deg_G(v) - n + 2$ *mutually independent hamiltonian paths between u and v if* $(u, v) \notin E(G)$.

Proof. Let s and t be two distinct vertices of G, and H be the subgraph of G induced by the remaining $(n-2)$ vertices of G. Let u' and v' be any two distinct vertices in H. We have $\deg_H(u') + \deg_H(v') \geq n + 2 - 4 = n - 2 = |V(H)|$. By Theorem 4, H is hamiltonian. We can label the vertices of H with $\{0, 1, \ldots, n-3\}$, so that $\langle 0, 1, 2, \ldots, n-3, 0\rangle$ forms a hamiltonian cycle of H. Let S denote the set $\{i \mid (s, [i+1]) \in E(G)\}$ and T denote the set $\{i \mid (i, t) \in E(G)\}$. Clearly, $|S \cup T| \leq n - 2$. We have the following two cases:

Case 1: $(s, t) \in E(G)$. Suppose that $|S \cap T| \leq \deg_G(s) + \deg_G(t) - n - 1$. We have $\deg_G(s) + \deg_G(t) - 2 = |S| + |T| = |S \cup T| + |S \cap T| \leq \deg_G(s) + \deg_G(t) - n - 1 + n - 2$. This is a contradiction. Thus, there are at least $w = \deg_G(s) + \deg_G(t) - n$ elements in $S \cap T$. Let $q_1, q_2, \ldots, q_w$ be the elements in $S \cap T$. For $j = 1, 2, \ldots, w$, we set $P_j = \langle s, [q_j + 1], [q_j + 2], \ldots, [q_j], t\rangle$. So $P_1, P_2, \ldots, P_w$ are mutually independent paths between s and t.

Case 2: $(s, t) \notin E(G)$. Assume that $|S \cap T| \leq \deg_G(s) + \deg_G(t) - n + 2 - 1$. We obtain $\deg_G(s) + \deg_G(t) = |S| + |T| = |S \cup T| + |S \cap T| \leq \deg_G(s) + \deg_G(t) - n + 2 - 1 + n - 2$. This is a contradiction. Thus, there are at least $w = \deg_G(s) + \deg_G(t) - n + 2$ elements in $S \cap T$. Let $q_1, q_2, \ldots, q_w$ be the elements in $S \cap T$. For $j = 1, 2, \ldots, w$, we set $P_j = \langle s, [q_j + 1], [q_j + 2], \ldots, [q_j], t\rangle$, and $P_1, P_2, \ldots, P_w$ are mutually independent paths between s and t. $\square$

Example. Let G be the graph $(K_1 \cup K_{n-d-1}) \vee K_d$ where d is an integer with $4 \leq d < n - 1$. So $\bar{e} = n - 1 - d \leq n - 4$. Let x be the vertex corresponding to K_1, y be an arbitrary vertex in K_d, and z be a vertex in K_{n-d-1}. Then $\deg_G(x) = d$, $\deg_G(y) = n - 1$, $\deg_G(z) = n - 2$, $(x, y) \in E(G)$, $(y, z) \in E(G)$, and $(x, z) \notin E(G)$. By Theorem 6, there are $n - 2 - \bar{e} = n - 2 - (n - 1 - d) = d - 1$ mutually independent hamiltonian paths between any two distinct vertices of G. By Lemma 1, there are at most $d - 1$ mutually independent hamiltonian paths between x and y. Hence, the result in Theorem 6 is optimal.

Consider the same example as above; it is easy to check that any two vertices u and v in G, $\deg_G(u) + \deg_G(v) \geq n + 2$. Let x and y be the same vertices as described above; by Theorem 7, there are $\deg_G(x) + \deg_G(y) - n = d + (n - 1) - n = d - 1$ mutually independent hamiltonian paths between x and y. By Lemma 1, there are at most $d - 1$ mutually independent hamiltonian paths between x and y. Hence, the result in Theorem 7 is also optimal.

4. Conjecture

Combining with Theorems 5 and 7, we have the following Corollary.

Corollary 1. *Let r be a positive integer. Assume that G is a graph such that* $\deg_G(x) + \deg_G(y) \geq n + r$ *for any two distinct vertices x and y. Then there are at least r mutually independent hamiltonian paths between any two distinct vertices of G.*

However, we would like to make the following conjecture. Suppose that $r > 1$ and G is a graph such that $\deg_G(u) + \deg_G(v) \geq n + r$ for any two distinct vertices u and v in G. Then there are at least $r + 1$ mutually independent hamiltonian paths between any two distinct vertices of G.

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