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# Optimal ordering policies for periodic-review systems with replenishment cycles

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## Abstract

In this paper, we consider inventory models for periodic-review systems with replenishment cycles, which consist of a number of periods. By replenishment cycles, we mean that an order is always placed at the beginning of a cycle. We use dynamic programming to formulate both the backorder and lost-sales models, and propose to charge the holding and shortage costs based on the ending inventory of periods (rather than only on the ending inventory of cycles). Since periods can be made any time units to suit the needs of an application, this approach in fact computes the holding cost based on the average inventory of a cycle and the shortage cost in proportion to the duration of shortage (for the backorder model), and remedies the shortcomings of the heuristic or approximate treatment of such systems (Hadley and Whitin, *Analysis of Inventory Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1963). We show that a base-stock policy is optimal for the backorder model, while the optimal order quantity is a function of the on-hand inventory for the lost-sales model. Moreover, for the backorder model, we develop a simple expression for computing the optimal base-stock level; for the lost-sales model, we derive convergence conditions for obtaining the optimal operational parameters.

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## 1. Introduction

Despite the attractiveness of continuous-review inventory systems, periodic-review models are still applied in many situations, especially for inventory systems where many different items are purchased from the same supplier and the coordination of ordering and transportation is important. See, e.g., Chiang

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and Gutierrez [5] and Silver et al. [16] for other reasons of adopting periodic-review systems. Often, periodic-review systems have the review periods that are one or few weeks (or months) long and the supply lead-time is shorter than a review period.

Studies on the periodic-review inventory models (see, e.g., Porteus [15] and references therein) often assume that the supply lead-time is a (integer) multiple of a review period, and develop optimal policies (for backorder models) that are of either the base-stock type or the  $(s, S)$  type (i.e., whenever the inventory is reviewed or drops to  $s$  at the beginning of a period, an order is placed to raise the inventory to a predetermined level  $S$ ), depending on whether or not a fixed cost of ordering is present. Efficient procedures are also developed to find the optimal  $s$  and  $S$  (see, e.g., Zheng and Federgruen [18]) or near-optimal solutions for the lost-sales model with positive lead times (see, e.g., Morton [11], Nahmias [13], van Donselaar et al. [6], and Johansen and Hill [9]). As Chiang [1] and Chiang and Gutierrez [4,5] point out in the two-supply-mode setting, such periodic models could be regarded as an approximation of continuous-review inventory systems, for the review periods are typically modeled as small as one day and the holding and shortage costs are computed based on the ending inventory of periods.

For periodic inventory systems where the review periods are one or few weeks long, it is appropriate to compute the holding and shortage costs based on respectively, the average period inventory and the duration of shortage (for the backorder model). Exact analysis of such systems assumes the  $(R, T)$  policy and derives the average annual cost expression (for specific cases only) that is difficult to compute, especially for the lost-sales model (see, e.g., Hadley and Whitin [7, Sections 5–6 and 5–13]). Consequently, the heuristic or approximate treatment of such systems is often used by standard textbooks (see, e.g., [7, Section 5–2] and [16, Section 7.9.4]) and research papers (see, e.g., Chiang [3] and Moses and Seshadri [12]) to obtain easy-to-implement solutions.

There are many shortcomings for the approximate treatment of periodic inventory systems with one-or-few-weeks-long review periods. First, the average on-hand inventory is derived by assuming that backorders or lost sales are incurred in very small quantities. This approximation is poor if backorders or lost sales are not an insignificant portion of demand, or if demand is highly volatile as discussed by Nahmias and Smith [14] for the lost-sales model. To overcome this shortcoming, van der Heijden and de Kok [17] propose an improved approximate method to estimate the mean physical stock given a target fill rate. Second, for the backorder model, the shortage cost is charged per unit of shortage irrespective of the duration of shortage [7, p. 238], while for the lost-sales model, the effect of lost sales occurring between the time an order is placed and the time it arrives is ignored [7, p. 241]. Third, the lost-sales model assumes the base-stock policy, which is in general not optimal, as demonstrated by Karlin and Scarf [10]. In this paper, we study periodic inventory systems with one-or-few-weeks-long review periods, and use dynamic programming to formulate both the backorder and lost-sales models.

To be somewhat consistent with the periodic-review literature, we will rename the one-or-few-weeks-long review periods as *replenishment cycles* (or simply *cycles*), and let a cycle consist of a number of periods. By replenishment cycles, we mean that an order is always placed at the beginning of a cycle (i.e., at a review epoch), as in the  $(R, T)$  policy. This assumption is reasonable if the fixed cost of ordering is small or negligible, which is especially true if an order for a specific item is part of a joint order and the ordering cost for a joint order (which is incurred every time a joint order is placed) is irrelevant to individual items. We assume that the length of cycles is handled outside our models (e.g., determined by the need of coordinating replenishments of many different items), as in Chiang [1] and Chiang and Gutierrez [4,5]. The holding and shortage costs will be computed based on the ending inventory of periods (rather than only on the ending inventory of cycles). As periods can be defined to be any time units for the purpose of an application, this approach actually computes the holding cost based on the average inventory of a cycle and the shortage cost in proportion to the duration of shortage (for the backorder model), and thus remedies the shortcomings mentioned above for the approximate treatment of periodic systems with replenishment cycles.

It should be noted that Chiang and Gutierrez [5] and Chiang [2] also consider periodic inventory systems with replenishment cycles that consist a number of periods. However, periods in their models are defined to be such that an emergency order can be placed at the beginning of them. Also, they develop only the back-order models.

We notice that the proposed dynamic programming approach to computing the holding and shortage costs has several economic implications. First, the proposed backorder and lost-sales models that minimize the expected discounted cost over a planning horizon consider the time value of money, while the approximate models [7, Section 5–2] that minimize the expected annual cost do not take it into account. Second, as we discuss above, periods can be defined to suit the needs of an application. For example, if customers consider the level of disservice in proportion to its time expressed in days (or even hours), periods are defined as days (or hours). This is often true of inventory systems for the service parts of equipment or cars and computer products. Customers usually escalate their unhappiness as their waiting for service parts to arrive continues. In such situations, the shortage cost should be charged in proportion to the duration of shortage (in addition to the amount of shortage). By contrast, the approximate backorder model computes the shortage cost irrespective of the duration of shortage and may not be applicable to these situations.

We will show that a base-stock policy is optimal for the backorder model. This agrees with the periodic-review inventory literature. We also develop a simple expression for computing the optimal base-stock level. Moreover, we show that the optimal order quantity is a function of the on-hand inventory for the lost-sales model. This generalizes Theorem 4 of Karlin and Scarf [10] for the one-period-lag inventory problem. We also derive the convergence conditions of stopping dynamic programming computation and obtaining the optimal operational parameters for the infinite-horizon lost-sales model. As computational results indicate that it takes only a few cycles for operational parameters to converge, we advocate that firms use the proposed method (of computing the holding and shortage costs) and implement the optimal policy.

## 2. The backorder model

Suppose that a replenishment cycle, whose length is exogenously determined, consists of  $m$  periods. Assume first that all demand not immediately satisfied is backordered. Let  $c$  denote the unit item cost. The inventory holding and shortage costs will be charged at the end of each period. Let  $h$  be the inventory cost per unit held per period, and  $p$  the shortage cost per unit per period. Also let  $\varphi(\xi)$  denote the probability density function of demand  $\xi$  during a period with mean  $\mu$ . Demand is assumed to be non-negative and independently distributed in disjoint time intervals.

Suppose the net inventory (i.e., inventory on hand minus backorder) at the beginning of a period is  $X$ ; then the expected holding and shortage costs incurred in that period are given by

$$L_0(X) = \int_0^{X^+} h(X - \xi)\varphi(\xi) d\xi + \int_{X^+}^{\infty} p(\xi - X)\varphi(\xi) d(\xi), \quad (1)$$

where  $X^+$  denotes  $\max\{X, 0\}$ . Other functional forms of  $L_0(X)$  are allowed; however, for our analysis we need  $L_0(X)$  to be a convex and differentiable function. Let  $\tau$  be the (deterministic) supply lead-time that is a non-negative integer and allowed to be larger than  $m$ , and  $L_i(X) \equiv E_{\xi} L_{i-1}(X - \xi)$  for any positive integer  $i \geq 1$ . Denote  $V_{n,0}(X)$  as the expected discounted cost with  $n$  cycles remaining until the end of the planning horizon when the starting inventory position (i.e., net inventory plus inventory on order) is  $X$  and an optimal ordering policy is used at that review epoch.  $V_{n,j}(X)$  for  $j \neq 0$  denotes the expected discounted cost with  $n$  cycles and  $j$  periods remaining when the inventory position is  $X$ . Note that  $V_{n,0}(X)$  excludes the fixed cost of ordering (if any), for an order is always placed at the beginning of a cycle. For simplicity of formulation, it also does not include the holding and shortage costs during the next  $\tau$  periods, because these costs

are not affected by the decision made at a review epoch. Consequently,  $V_{n,j}(X)$  for  $j \neq 0$  excludes these costs as well.  $V_{n,j}(X)$  satisfies the functional equations

$$V_{n,0}(X) = \min_{X \leq R} \{ \alpha^\tau cR + \alpha^\tau L_\tau(R) + \alpha E_\xi V_{n-1,m-1}(R - \xi) \} - \alpha^\tau cX, \tag{2}$$

$$V_{n,j}(X) = \alpha^\tau L_\tau(X) + \alpha E_\xi V_{n,j-1}(X - \xi), \quad j = 1, \dots, m - 1, \tag{3}$$

where  $V_{0,0}X \equiv 0$ ,  $\alpha$  is the discount factor ( $0 < \alpha \leq 1$ ) and  $R$  (the decision variable) is the inventory position after an order is placed at a review epoch. Note that the item cost  $c(R - X)$  (paid upon delivery) and the one-period holding and shortage costs  $L_\tau(R)$  in (2) and  $L_\tau(X)$  in (3) for the upcoming  $(\tau + 1)$ th period are discounted to the present time. As we see from (3), the holding and shortage costs are charged virtually on a continuous basis. The length of periods actually can be made arbitrarily small, if justified in practice. Then the number of periods in a cycle will increase.

Assume that  $V_{0,1}(X) = \alpha^\tau L_\tau(X)$  attains its minimum (this assumption is satisfied if  $h$  and  $p$  are both positive). Define the function  $G_{n,0}(R)$  as

$$G_{n,0}(R) = \alpha^\tau cR + \alpha^\tau L_\tau(R) + \alpha E_\xi V_{n-1,m-1}(R - \xi). \tag{4}$$

Then  $V_{n,0}(X)$  can be expressed by

$$V_{n,0}(X) = \min_{X \leq R} \{ G_{n,0}(R) \} - \alpha^\tau cX. \tag{5}$$

We show in the following lemma that  $V_{n,j}(X)$  is convex.

**Lemma 2.1.**  $V_{n,j}(X)$  for each  $(n,j)$  is a convex function.

**Proof.**  $V_{0,1}(X)$  is convex. Assuming that  $V_{0,j-1}(X)$  is convex, it follows from (3) that  $V_{0,j}(X)$  is convex.  $G_{1,0}(R)$  is thus convex. Hence  $V_{1,0}(X)$  is convex by Proposition B-4 of Heyman and Sobel [8]. Convexity is established by induction for the remaining  $V_{n,j}(X)$ .  $\square$

Denote by  $Df$  the first derivative of the function  $f$ . Let  $R_n$  be the (smallest) value of  $R$  that minimizes  $G_{n,0}(R)$ . It follows from (5) that the optimal policy with  $n$  cycles remaining is to order up to  $R_n$ . The following theorem shows that if two consecutive order-up-to levels are equal to each other, the sequence  $\{R_i\}$  has converged.

**Theorem 2.2.** If  $R_n = R_{n-1}$ , then  $R_i = R_n$  for  $i \geq n + 1$ .

**Proof.** We show that if  $R_n = R_{n-1}$ , then  $R_{n+1} = R_n$  and thus the convergence property holds for all  $i > n + 1$ . If  $R_n = R_{n-1}$ , it follows from (5) that  $DV_{n,0}(X) = DV_{n-1,0}(X) = -\alpha^\tau c$  for  $X \leq R_n$ . Hence, it can be seen from (3) that  $DV_{n,j}(X) = DV_{n-1,j}(X)$  for  $X \leq R_n$ ,  $j = 1, \dots, m - 1$ . It then follows from (4) that  $DG_{n+1,0}(R) = DG_{n,0}(R)$  for  $R \leq R_n$ . As  $R_n$  minimizes  $G_{n,0}(R)$ , it also minimizes  $G_{n+1,0}(R)$ , i.e.,  $R_{n+1} = R_n$ .  $\square$

As we see from Theorem 2.2, if for some  $n$ ,  $R_n = R_{n-1}$ , the sequence  $\{R_i\}$  converges to a level  $R^* = R_n$ , i.e.,  $R_i = R^*$  for all  $i \geq n + 1$ .  $R^*$  is then the optimal base-stock level for the infinite-horizon problem (see Chiang [2] for a similar result in the two-supply-mode setting). The condition of  $R_n = R_{n-1}$  for some  $n$  is expected to hold, if we rule out the bizarre “never to order” policy explained below. The reason is that in most cases in practice there exists a minimum divisible quantity and demand occurs in a multiple of this

quantity. Since demand in a period is non-negative and bounded, it follows that the state space for  $X$  is finite. Note that even if demand can occur in any finite non-negative real amount, the state space must be discretized when implemented on a digital computer. Moreover, the action space for  $R$  is also finite, since in practice the order quantity is also bounded and orders will be placed in a multiple of the above divisible quantity. Computational results indicate that it takes only 2 or 3 cycles for the sequence  $\{R_i\}$  to converge for a lot of problems we solve (some of which are described below).

We next derive a simple procedure of computing  $R^*$ . It follows from (5) that  $DV_{n,0}(X) = -\alpha^\tau c$  for  $X \leq R^*$ . Thus it is seen from (3) that  $DV_{n,1}(X) = \alpha^\tau DL_\tau(X) - \alpha^{\tau+1}c$  for  $X \leq R^*$  and  $DV_{n,2}(X) = \alpha^{\tau+1}DL_{\tau+1}(X) + \alpha^\tau DL_\tau(X) - \alpha^{\tau+2}c$  for  $X \leq R^*$ . By repeating this logic, we can show that for  $X \leq R^*$ ,

$$DV_{n,m-1}(X) = \alpha^{\tau+m-2}DL_{\tau+m-2}(X) + \dots + \alpha^{\tau+1}DL_{\tau+1}(X) + \alpha^\tau DL_\tau(X) - \alpha^{\tau+m-1}c.$$

Hence, it follows from (4) that for  $R \leq R^*$ ,

$$DG_{n+1,0}(R) = \alpha^{\tau+m-1}DL_{\tau+m-1}(R) + \dots + \alpha^{\tau+1}DL_{\tau+1}(R) + \alpha^\tau DL_\tau(R) + (1 - \alpha^m)\alpha^\tau c. \tag{6}$$

As  $R^*$  minimizes  $G_{n+1,0}(R)$ ,  $R^*$  can be obtained by solving  $DG_{n+1,0}(R) = 0$ . Let  $\Phi_k(\cdot)$  be the complement of the cumulative distribution function of  $k$ -period's demand. Noticing that  $DL_0(R) = h - (h + p)\Phi_1(R)$ ,  $DL_1(R) = h - (h + p)\Phi_2(R)$ , and in general  $DL_i(R) = h - (h + p)\Phi_{i+1}(R)$ , we can simplify  $DG_{n+1,0}(R) = 0$  to

$$(h + p)[\alpha^\tau \Phi_{\tau+1}(R) + \dots + \alpha^{\tau+m-1} \Phi_{\tau+m}(R)] = \alpha^\tau h(1 + \alpha + \dots + \alpha^{m-1}) + (1 - \alpha^m)\alpha^\tau c. \tag{7}$$

Note that for  $R \leq 0$ , the left-hand side of (7) is equal to  $\alpha^\tau(h + p)(1 + \alpha + \dots + \alpha^{m-1})$  which is greater than the right-hand side of (7). This is because  $p(1 + \alpha + \dots + \alpha^{m-1}) > (1 - \alpha^m)c$ ; otherwise if  $p(1 + \alpha + \dots + \alpha^{m-1}) \leq (1 - \alpha^m)c$  (which means that it is more economical to incur the shortage cost for  $m$  periods plus the discounted cost of a unit than to purchase a unit at cost  $c$ ), the optimal policy is never to order, i.e., go out of the business. For  $R > 0$ , the left-hand side of (7) is a decreasing function of  $R$ . As  $R$  approaches infinity, it approaches zero which is less than the right-hand side of (7). Hence, for continuous demand distributions,  $R^*$  can be obtained by a simple search procedure. For discrete demand distributions, we find the largest  $R$  such that the left-hand side of (7) is greater than or equal to the right-hand side of (7).

To illustrate, consider the base case (where periods are defined as days):  $c = \$10$ ,  $m = 10$  (i.e., a cycle consists of 10 days),  $\alpha = 0.999$ ,  $\tau = 6$  days,  $\mu = 2$  units per day (with Poisson demand),  $h = \$0.01$ , and  $p = \$20$  (i.e., the holding and shortage costs are charged at \$0.01 and \$20 per unit per day, respectively). By using (7), we find that  $R^* = 45$ , which implies a safety stock of 13 units. It can be verified from (7) that  $R^*$  is non-decreasing in  $p$  or  $\tau$  (other things being equal) and non-increasing in  $h$ .

Suppose that the period length is defined as half a day (4 working hours) and the data in the base case are changed as follows:  $c = \$10$ ,  $m = 20$ ,  $\alpha = (0.999)^{0.5}$ ,  $\tau = 12$ ,  $\mu = 1$  unit per half-a-day,  $h = \$0.005$ , and  $p = \$10$ . We find from (7) that  $R^* = 45$ . If the period length is further reduced to 2 hours (or even 1 hour) and the data are changed similarly,  $R^*$  is found to be 44. This result that  $R^*$  is the same or lower as the period length becomes smaller is explainable, though we cannot prove it rigorously. [However, if all periods in a cycle are associated with the same discount factor, i.e.,  $\alpha^\tau, \alpha^{\tau+1}, \dots$ , and  $\alpha^{\tau+m-1}$  respectively for  $L_\tau(R), L_{\tau+1}(R), \dots$ , and  $L_{\tau+m-1}(R)$  in (6), are all replaced by  $\alpha^\tau$ , the above result can be easily proved.] The proposed dynamic programming approach charges the holding and shortage costs based on the ending inventory of periods. The shorter the periods, the more *continuously in time* these two costs are computed. In the above example, any inventory held in the morning but sold in the afternoon of a day is charged at 0.005 per unit if the period length is half a day, while it will not be charged any holding costs for that day if the period length is one day. Also, if shortage occurs in the morning of the last day of lead-time, its cost is \$20 per unit whether the period length is one day or half a day. However, if shortage occurs in the afternoon, its cost is \$20 (respectively, only \$10) per unit if the period length is one day (respectively, half a day).

As discussed in Section 1, the period length can be defined to suit the needs of an application. It should be determined especially from the perspective of customers, as the satisfaction of customers has become one of the top-rated goals of a company. As a note, we do not compare the solutions above to those from the approximate model. This is because the shortage cost is assumed to be linear in both the amount and duration of shortage, while it is linear only in the amount of shortage in the approximate model (where choice of the unit shortage cost regardless of the period length, given the unit holding cost per cycle, decides the order-up-to level when applying expression (5–9) of Hadley and Whitin [7]).

Thus far we have assumed that the supply lead-time  $\tau$  is constant. If  $\tau$  is stochastic (integer-valued), all results obtained above are valid, provided that lead-times are generated by an exogenous, sequential supply process that is independent of demand and with the property that orders are received in the same sequence as they are placed, as in the ordinary periodic-review models where an order can be placed at each period and  $\tau$  is a multiple of a period (see, e.g., Zipkin [19, pp. 408–409]). Then  $\alpha^\tau c$  and  $\alpha^\tau L_\tau(\cdot)$  in (2) and (3) are replaced by  $E_\tau[\alpha^\tau c]$  and  $E_\tau[\alpha^\tau L_\tau(\cdot)]$ , respectively, and (7) would be replaced by its expectation taken over the lead-time distribution, i.e.,

$$(h + p)E_\tau[\alpha^\tau \Phi_{\tau+1}(R) + \dots + \alpha^{\tau+m-1} \Phi_{\tau+m}(R)] = E_\tau[\alpha^\tau] \{h(1 + \dots + \alpha^{m-1}) + (1 - \alpha^m)c\}. \tag{8}$$

The solution procedure of obtaining the optimal  $R^*$  described above for constant lead-time case holds here. For example, in the base case above (other things being equal) if  $\tau$  is stochastic and distributed as follows:  $\Pr(\tau = 4) = \Pr(\tau = 8) = 0.1$ ,  $\Pr(\tau = 5) = \Pr(\tau = 7) = 0.2$ , and  $\Pr(\tau = 6) = 0.4$  (the mean of  $\tau$  is still equal to 6 days), after solving we find that  $R^* = 46$ . Comparing this level to that of the base case, we see that introduction of the lead-time variability into the model makes the base-stock level higher. This agrees with our common knowledge that more uncertainty may lead to higher safety stocks.

### 3. The lost-sales model

Suppose now that all demand not immediately satisfied is lost. Use the notation in Section 2. Assume that the (deterministic) supply lead-time is less than or equal to the length of replenishment cycles, i.e.,  $\tau \leq m$ . Assume for the time being that  $\tau \geq 2$ . Let  $L(X)$  be the one-period's holding and shortage costs when the starting on-hand inventory is  $X$ .  $L(X)$  is expressed by

$$L(X) = \int_0^X h(X - \xi)\varphi(\xi) d\xi + \int_X^\infty p(\xi - X)\varphi(\xi) d\xi. \tag{9}$$

Note that  $p$  (the shortage cost per unit) has a different meaning in the lost-sales model. It should be larger here, for it now includes the sales price and also a lost sale is worse than a delayed one.

Let  $V_{n,0}(X, 0)$  denote the expected discounted cost with  $n$  cycles remaining until the end of the planning horizon when the starting on-hand inventory is  $X$  and an optimal ordering policy is used at that review epoch.  $V_{n,j}(X, Y)$  for  $j \neq 0$  denotes the expected discounted cost with  $n$  cycles and  $j$  periods remaining when the starting on-hand inventory is  $X$  and on-order inventory is  $Y$ .  $V_{n,j}(X, Y)$  is simply  $V_{n,j}(X, 0)$  for  $j = 1, \dots, m - \tau$ .  $V_{n,j}(X, Y)$  satisfies the functional equations

$$V_{n,0}(X, 0) = \min_{Z \geq 0} \left\{ \alpha^\tau cZ + L(X) + \alpha \int_0^X V_{n-1,m-1}(X - \xi, Z)\varphi(\xi) d\xi + \alpha V_{n-1,m-1}(0, Z) \int_X^\infty \varphi(\xi) d\xi \right\}, \tag{10}$$

$$V_{n,j}(X, Y) = L(X) + \alpha \int_0^X V_{n,j-1}(X - \xi, Y) \varphi(\xi) d\xi + \alpha V_{n,j-1}(0, Y) \int_X^\infty \varphi(\xi) d\xi, \\ j = 1, \dots, m-1 \quad \text{and} \quad j \neq m - \tau + 1, \quad (11)$$

$$V_{n,m-\tau+1}(X, Y) = L(X) + \alpha \int_0^X V_{n,m-\tau}(X - \xi + Y, 0) \varphi(\xi) d\xi + \alpha V_{n,m-\tau}(Y, 0) \int_X^\infty \varphi(\xi) d\xi, \quad (12)$$

where  $V_{0,0}(X, 0) \equiv 0$  and  $Z$  (the decision variable) is the quantity ordered at the beginning of a cycle which becomes inventory on order thereafter. As we see from (11) and (12), the holding and shortage costs are computed virtually on a continuous basis (as in the backorder model). If  $\tau = 1$ , there is only one state variable  $X$  and (10)–(12) reduce to

$$V_{n,0}(X) = \min_{Z \geq 0} \left\{ \alpha cZ + L(X) + \alpha \int_0^X V_{n-1,m-1}(X - \xi + Z) \varphi(\xi) d\xi + \alpha V_{n-1,m-1}(Z) \int_X^\infty \varphi(\xi) d\xi \right\}. \quad (13)$$

$$V_{n,j}(X) = L(X) + \alpha \int_0^X V_{n,j-1}(X - \xi) \varphi(\xi) d\xi + \alpha V_{n,j-1}(0) \int_X^\infty \varphi(\xi) d\xi, \quad j = 1, \dots, m-1. \quad (14)$$

In addition, if  $\tau = 0$ , Eq. (13) further simplifies to

$$V_{n,0}(X) = \min_{X \leq R} \left( cR + L(R) + \alpha \int_0^R V_{n-1,m-1}(R - \xi) \varphi(\xi) d\xi + \alpha V_{n-1,m-1}(0) \int_R^\infty \varphi(\xi) d\xi \right) - cX. \quad (15)$$

Using the same reasoning as in Section 2, we can show that a base-stock policy is optimal for the zero-lead-time case, and the optimal base-stock level  $R^*$  (for the infinite-horizon model) is obtained by solving the following equation:

$$(h + p) \{ \Phi_1(R) + \dots + \alpha^{m-1} \Phi_m(R) \} - p \{ \alpha \Phi_1(R) + \dots + \alpha^{m-1} \Phi_{m-1}(R) \} - \alpha^m c \Phi_m(R) \\ = h(1 + \alpha + \dots + \alpha^{m-1}) + (1 - \alpha^m)c. \quad (16)$$

We assume  $\tau \geq 2$  throughout the rest of this section (the analysis is similar and simpler when  $\tau = 1$ ). Define

$$J_n(X, Z) = \alpha^\tau cZ + L(X) + \alpha \int_0^X V_{n-1,m-1}(X - \xi, Z) \varphi(\xi) d\xi + \alpha V_{n-1,m-1}(0, Z) \int_X^\infty \varphi(\xi) d\xi. \quad (17)$$

Then (10) is expressed by

$$V_{n,0}(X, 0) = \min_{Z \geq 0} \{ J_n(X, Z) \}. \quad (18)$$

We show in the following lemma that the cost function  $V_{n,j}(X, Y)$  is convex. Denote by  $D_i f$  the first derivative of the function  $f$  with respect to its  $i$ th variable.

**Lemma 3.1.**  $V_{n,j}(X, Y)$  for each  $(n, j)$  is a convex function.

**Proof.** Please see Appendix A.



Let  $Z_n(X)$  be the (smallest) value of non-negative  $Z$  that minimizes  $J_n(X, Z)$  for a given  $X$ . Then it follows from (18) that the optimal policy at a review epoch with  $n$  cycles remaining is to order the amount  $Z_n(X)$ . The following property regarding  $V_{n,j}(X, Y)$ ,  $j = m - \tau + 1, \dots, m - 1$ , is useful for establishing Theorem 3.3. Let  $\Delta$  be a positive number.

**Lemma 3.2.** *The mixed second derivative of  $V_{n,j}(X, Y)$ ,  $j = m - \tau + 1, \dots, m - 1$ , is non-negative, i.e.,  $D_1 V_{n,j}(X, Y) \leq D_1 V_{n,j}(X, Y + \Delta)$ , or  $D_2 V_{n,j}(X - \Delta, Y) \leq D_2 V_{n,j}(X, Y)$ . Moreover,  $D_1 V_{n,j}(X - \Delta, Y + \Delta) \leq D_1 V_{n,j}(X, Y)$ , and  $D_2 V_{n,j}(X, Y) \leq D_2 V_{n,j}(X - \Delta, Y + \Delta)$ .*

**Proof.** Please see [Appendix A](#).

The following theorem states that the quantity ordered at a review epoch is non-increasing in the starting on-hand inventory; moreover, if the quantity ordered decreases as the starting on-hand inventory increases, it decreases by an amount that is less than or equal to the amount by which the starting on-hand inventory increases. This result generalizes Theorem 4 of Karlin and Scarf [10] for the lost-sales periodic-review problem with one-period lag (lead-time).

**Theorem 3.3.**  *$Z_n(X)$  is non-increasing in  $X$ , i.e.,  $Z_n(X) \leq Z_n(X - \Delta)$ . Moreover,  $Z_n(X - \Delta) - \Delta \leq Z_n(X)$ .*

**Proof.** Please see [Appendix A](#).

Let  $R_n$  be the minimum value of  $X$  for which  $Z_n(X) = 0$ . It follows from Theorem 3.3 that for  $X \leq R_n$ ,  $X + Z_n(X) \leq R_n + Z_n(R_n) = R_n$ . This implies that  $R_n$  is the maximum possible order-up-to level at a review epoch with  $n$  cycles remaining. We show in Theorem 3.4 that if two consecutive maximum order-up-to levels are equal to each other and the first derivatives of the two corresponding cost functions are equal, then the sequence  $\{Z_i(X)\}$  has converged.

**Theorem 3.4.** *If there exists some  $n$  such that*

- (a)  $R_n = R_{n-1}$
- (b)  $D_1 V_{n,0}(X, 0) = D_1 V_{n-1,0}(X, 0)$  for  $X \leq R_n$ , then  $Z_i(X) = Z_n(X)$ ,  $X \leq R_n$ , and  $R_i = R_n$ , for all  $i \geq n + 1$ .

**Proof.** Please see [Appendix A](#).

As we see from Theorem 3.4, if conditions (a) and (b) are satisfied, the sequence  $\{Z_i(X)\}$  converges to  $Z^*(X) = Z_n(X)$  ( $\{R_i\}$  also converges to  $R^* = R_n$ ), and thus the dynamic programming computation can be stopped (see Chiang and Gutierrez [5] for a similar result that is applied to a backorder model in the two-supply-mode setting). Condition (a) that is similar to the condition of Theorem 2.2 is expected to hold. To facilitate the computation, we say that the first derivatives of two consecutive cost functions (at review epochs) could be regarded as equal when

$$\max_{X \leq R_n} |D_1 V_{n,0}(X, 0) - D_1 V_{n-1,0}(X, 0)| \leq \varepsilon, \tag{19}$$

where  $\varepsilon$  is set to 0.02 for all the problems we solve, which are described below.

Consider the base case in Section 2:  $c = \$10$ ,  $m = 10$ ,  $\tau = 6$  days,  $\alpha = 0.999$ ,  $h = \$0.01$  (i.e., the holding cost is charged at \$0.01 per unit per day),  $p = \$20$  (i.e., the shortage cost is charged at \$20 per unit), and  $\mu = 2$  units per day (with Poisson demand). After solving the model expressed by (10)–(12), we find that the



sequence  $\{Z_i(X)\}$  converges to  $Z^*(X) = Z_3(X)$ , i.e., it converges after only 3 cycles, and  $Z^*(X) = 29$  for  $X \leq 12$ ,  $Z^*(13) = Z^*(14) = 28$ ,  $Z^*(15) = Z^*(16) = 27$ ,  $Z^*(17) = 26$ ,  $Z^*(18) = 25$ ,  $Z^*(X) = 44 - X$  for  $19 \leq X \leq 44$  (and thus  $R^* = 44$ ).

In addition, we vary the value of  $p$  and  $\tau$  in the base case to investigate the speed of convergence of the sequence  $\{Z_i(X)\}$  and their effect on the optimal solutions. There are a total of 25 problems solved. Computational results indicate that it takes no more than 5 cycles for convergence to occur. Also, as we see from Table 1, that  $R^*$  is non-decreasing in  $p$  or  $\tau$  (other things being equal). This result is intuitively reasonable, though we cannot prove it rigorously. Let  $X^*$  be the minimum starting on-hand inventory for which the order-up-to level is  $R^*$ . For example,  $X^* = 19$  in the base case above. The level of  $X^*$  for each problem is also recorded in Table 1.

Moreover, we compute the optimal order-up-to level, denoted by  $S$ , of the approximate model for each of the 25 problems (note that the holding cost per unit per cycle is \$0.1 when applying expression (5–13) of Hadley and Whitin [7]). As we see from Table 1,  $S$  is, in general, higher than  $R^*$ . This is because the approximate model ignores the effect of lost sales occurring between the time an order is placed and the time it arrives [7, p. 241]. A larger order placed now by the approximate model cannot save possible lost sales in the near future. The difference between  $S$  and  $R^*$  tends to be larger as  $p$  decreases (other things being equal), i.e., a much smaller order than that yielded by the approximate model is actually enough as  $p$  be-

Table 1

A comparison of solutions of the proposed lost-sales model and the approximate model

Parameters		The proposed model		The approximate model
$\tau$	$p$	$X^*$	$R^*$	$S$
4	\$12	13	35	41
	16	13	38	42
	20	12	39	42
	24	13	40	42
	28	16	41	43
5	12	15	37	44
	16	15	40	44
	20	20	42	45
	24	15	42	45
	28	16	43	45
6	12	17	39	46
	16	17	42	47
	20	19	44	47
	24	20	45	47
	28	18	45	48
7	12	19	41	48
	16	23	45	49
	20	21	46	50
	24	21	47	50
	28	23	48	50
8	12	21	43	51
	16	24	47	51
	20	29	49	52
	24	23	49	52
	28	24	50	53

Data:  $c = \$10$ ,  $h = \$0.01$ ,  $\alpha = 0.999$ ,  $m = 10$ , and  $\mu = 2$  (Poisson demand).

comes lower. Hence, we recommend the use of the proposed lost-sales model, especially for the items with low shortage costs.

#### 4. Conclusions

This paper considers inventory models for periodic-review systems with replenishment cycles (which consist of a number of periods). We use dynamic programming to formulate both the backorder and lost-sales models, and propose to charge the holding and shortage costs based on the ending inventory of periods. Since periods can be made any time units to suit the needs of an application, this approach in fact computes the holding cost based on the average inventory of a cycle and the shortage cost in proportion to the duration of shortage (for the backorder model), and remedies the shortcomings of the approximate models found in many textbooks. We show that a base-stock policy is optimal for the backorder model, while in general it is not for the lost-sales model. Moreover, for the backorder model, we develop a simple expression for computing the optimal base-stock level; for the lost-sales model, we derive convergence conditions for stopping computation and obtaining the optimal operational parameters.

There are several possible directions for future research. One is to allow the lead-time to be greater than the length of a replenishment cycle for the lost-sales model. The state space would then increase from two to multiple dimensions. This conceivably reduces the feasibility of the optimal policy and makes the approximate (lost-sales) model more attractive. Another is to consider stochastic lead-times for the lost-sales model. The dynamic program then seems difficult to formulate, making the approximate model again more practical.

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#### Appendix A

**Proof of Lemma 3.1.**  $V_{0,1}(X,0) = L(X)$  is convex and  $D_1V_{0,1}(X,0) \geq -p$ . By examining the second derivative of  $V_{0,2}(X,0)$ , the convexity of  $V_{0,2}(X,0)$  is established and  $D_1V_{0,2}(X,0) \geq -p$ , and similarly for  $V_{0,j}(X,0)$ ,  $j = 3, \dots, m - \tau$ . Thus  $V_{0,m-\tau+1}(X, Y)$  is jointly convex in  $X$  and  $Y$  by examining its Hessian. Noting that the second derivative of  $V_{0,m-\tau+1}(X, Y)$  with respect to either  $X$  or  $Y$  is greater than the mixed second derivative of  $V_{0,m-\tau+1}(X, Y)$ , the convexity of  $V_{0,m-\tau+2}(X, Y)$  can be established. Likewise,  $V_{0,j}(X, Y)$  is jointly convex in  $X$  and  $Y$ ,  $j = m - \tau + 3, \dots, m - 1$ , and  $J_1(X, Z)$  is jointly convex in  $X$  and  $Z$ . Hence,  $V_{1,0}(X,0)$  is convex by Proposition B-4 of Heyman and Sobel [8]. Convexity is established by induction for the remaining  $V_{n,j}(X, Y)$ .

**Proof of Lemma 3.2.** We show only  $D_1V_{n,m-\tau+1}(X - \Delta, Y + \Delta) \leq D_1V_{n,m-\tau+1}(X, Y) \leq D_1V_{n,m-\tau+1}(X, Y + \Delta)$  and  $D_2V_{n,m-\tau+1}(X - \Delta, Y) \leq D_2V_{n,m-\tau+1}(X, Y) \leq D_2V_{n,m-\tau+1}(X - \Delta, Y + \Delta)$ ; similarly, the properties hold for  $V_{n,j}(X, Y)$ ,  $j = m - \tau + 2, \dots, m - 1$ . It follows from (12) that

$$D_1V_{n,m-\tau+1}(X, Y) = DL(X) + \alpha \int_0^X D_1V_{n,m-\tau}(X - \xi + Y, 0)\varphi(\xi) d\xi. \quad (\text{A.1})$$

Hence, it is easily seen by Lemma 3.1 and convexity of  $L(X)$  that  $D_1V_{n,m-\tau+1}(X - \Delta, Y + \Delta) \leq D_1V_{n,m-\tau+1}(X, Y) \leq D_1V_{n,m-\tau+1}(X, Y + \Delta)$ . In addition,

$$\begin{aligned}
D_2V_{n,m-\tau+1}(X, Y) &= \alpha \left( \int_0^X D_1V_{n,m-\tau}(X - \xi + Y, 0)\varphi(\xi) d\xi + D_1V_{n,m-\tau}Y, 0 \int_X^\infty \varphi(\xi) d\xi \right) \\
&= \alpha \left( \int_0^{X-\Delta} D_1V_{n,m-\tau}(X - \xi + Y, 0)\varphi(\xi) d\xi + \int_{X-\Delta}^X D_1V_{n,m-\tau}(X - \xi + Y, 0)\varphi(\xi) d\xi \right. \\
&\quad \left. + D_1V_{n,m-\tau}(Y, 0) \int_X^\infty \varphi(\xi) d\xi \right) \leq \alpha \left( \int_0^{X-\Delta} D_1V_{n,m-\tau}(X - \xi + Y, 0)\varphi(\xi) d\xi \right. \\
&\quad \left. + \int_{X-\Delta}^X D_1V_{n,m-\tau}(Y + \Delta, 0)\varphi(\xi) d\xi + D_1V_{n,m-\tau}(Y + \Delta, 0) \int_X^\infty \varphi(\xi) d\xi \right) \\
&= \alpha \left( \int_0^{X-\Delta} D_1V_{n,m-\tau}(X - \xi + Y, 0)\varphi(\xi) d\xi + D_1V_{n,m-\tau}(Y + \Delta, 0) \int_{X-\Delta}^\infty \varphi(\xi) d\xi \right) \\
&= D_2V_{n,m-\tau+1}(X - \Delta, Y + \Delta). \tag{A.2}
\end{aligned}$$

Similarly, it can be verified that  $D_2V_{n,m-\tau+1}(X - \Delta, Y) \leq D_2V_{n,m-\tau+1}(X, Y)$ .

**Proof of Theorem 3.3.** It follows from (17) and Lemma 3.2 that

$$\begin{aligned}
D_2J_n(X - \Delta, Z) &= \alpha^\tau c + \alpha \int_0^{X-\Delta} D_2V_{n-1,m-1}(X - \Delta - \xi, Z)\varphi(\xi) d\xi + \alpha D_2V_{n-1,m-1}(0, Z) \int_{X-\Delta}^\infty \varphi(\xi) d\xi \\
&= \alpha^\tau c + \alpha \int_0^{X-\Delta} D_2V_{n-1,m-1}(X - \Delta - \xi, Z)\varphi(\xi) d\xi + \alpha D_2V_{n-1,m-1}(0, Z) \left( \int_{X-\Delta}^X \varphi(\xi) d\xi \right. \\
&\quad \left. + \int_X^\infty \varphi(\xi) d\xi \right) \leq \alpha^\tau c + \alpha \int_0^{X-\Delta} D_2V_{n-1,m-1}(X - \xi, Z)\varphi(\xi) d\xi \\
&\quad + \alpha \int_{X-\Delta}^X D_2V_{n-1,m-1}(X - \xi, Z)\varphi(\xi) d\xi + \alpha D_2V_{n-1,m-1}(0, Z) \int_X^\infty \varphi(\xi) d\xi \\
&= \alpha^\tau c + \alpha \int_0^X D_2V_{n-1,m-1}(X - \xi, Z)\varphi(\xi) d\xi + \alpha D_2V_{n-1,m-1}(0, Z) \int_X^\infty \varphi(\xi) d\xi \\
&= D_2J_n(X, Z).
\end{aligned}$$

This implies that  $Z_n(X) \leq Z_n(X - \Delta)$ . Moreover, by Lemma 3.2,

$$\begin{aligned}
D_2J_n(X, Z) &= \alpha^\tau c + \alpha \int_0^{X-\Delta} D_2V_{n-1,m-1}(X - \xi, Z)\varphi(\xi) d\xi + \alpha \int_{X-\Delta}^X D_2V_{n-1,m-1}(X - \xi, Z)\varphi(\xi) d\xi \\
&\quad + \alpha D_2V_{n-1,m-1}(0, Z) \int_X^\infty \varphi(\xi) d\xi \leq \alpha^\tau c + \alpha \int_0^{X-\Delta} D_2V_{n-1,m-1}(X - \Delta - \xi, Z + \Delta)\varphi(\xi) d\xi \\
&\quad + \alpha \int_{X-\Delta}^X D_2V_{n-1,m-1}(\Delta, Z)\varphi(\xi) d\xi + \alpha D_2V_{n-1,m-1}(0, Z + \Delta) \int_X^\infty \varphi(\xi) d\xi \leq \alpha^\tau c \\
&\quad + \alpha \int_0^{X-\Delta} D_2V_{n-1,m-1}(X - \Delta - \xi, Z + \Delta)\varphi(\xi) d\xi + \alpha \int_{X-\Delta}^X D_2V_{n-1,m-1}(0, Z + \Delta)\varphi(\xi) d\xi \\
&\quad + \alpha D_2V_{n-1,m-1}(0, Z + \Delta) \int_X^\infty \varphi(\xi) d\xi \\
&= \alpha^\tau c + \alpha \int_0^{X-\Delta} D_2V_{n-1,m-1}(X - \Delta - \xi, Z + \Delta)\varphi(\xi) d\xi + \alpha D_2V_{n-1,m-1}(0, Z + \Delta) \int_{X-\Delta}^\infty \varphi(\xi) d\xi \\
&= D_2J_n(X - \Delta, Z + \Delta).
\end{aligned}$$

As  $D_2J_n(X, Z) \geq 0$  for  $Z \geq Z_n(X)$ ,  $D_2J_n(X - \Delta, Z + \Delta) \geq 0$  for  $Z \geq Z_n(X)$  or equivalently  $Z + \Delta \geq Z_n(X) + \Delta$ . This implies that  $Z_n(X - \Delta) \leq Z_n(X) + \Delta$ .

**Proof of Theorem 3.4.** We show that if for some  $n$ , conditions (a) and (b) are satisfied, then  $Z_{n+1}(X) = Z_n(X)$  for  $X \leq R_{n+1} = R_n$ , and  $D_1V_{n+1,0}(X, 0) = D_1V_{n,0}(X, 0)$  for  $X \leq R_{n+1}$ . Thus by induction the convergence property holds for all  $i \geq n + 2$ .

If  $R_n = R_{n-1}$  and  $D_1V_{n,0}(X, 0) = D_1V_{n-1,0}(X, 0)$  for  $X \leq R_n$ , then it follows from (11) that  $D_1V_{n,1}(X, 0) = D_1V_{n-1,1}(X, 0)$  for  $X \leq R_n, \dots, D_1V_{n,m-\tau}(X, 0) = D_1V_{n-1,m-\tau}(X, 0)$  for  $X \leq R_n$ . Also, due to (A.1) and (A.2),

$$\begin{aligned} D_1V_{n,m-\tau+1}(X, Y) &= D_1V_{n-1,m-\tau+1}(X, Y) \quad \text{for } X \leq R_n - Y, \\ D_2V_{n,m-\tau+1}(X, Y) &= D_2V_{n-1,m-\tau+1}(X, Y) \quad \text{for } Y \leq R_n - X. \end{aligned}$$

Hence, it is seen from (11) again that

$$D_1V_{n,j}(X, Y) = D_1V_{n-1,j}(X, Y) \quad \text{for } X \leq R_n - Y, \quad j = m - \tau + 2, \dots, m - 1, \tag{A.3}$$

$$D_2V_{n,j}(X, Y) = D_2V_{n-1,j}(X, Y) \quad \text{for } Y \leq R_n - X, \quad j = m - \tau + 2, \dots, m - 1. \tag{A.4}$$

Hence, it follows from (17) that  $D_2J_{n+1}(X, Z) = D_2J_n(X, Z)$  for all  $Z \leq R_n - X$ . Also, for  $X \leq R_n$ ,  $Z_n(X) \leq R_n - X$ . As  $Z_n(X)$ ,  $X \leq R_n$ , minimizes  $J_n(X, Z)$ , it also minimizes  $J_{n+1}(X, Z)$ , i.e.,  $Z_{n+1}(X) = Z_n(X)$  for  $X \leq R_n$ . Since  $R_n$  is the minimum value of  $X$  for which  $Z_n(X) = 0$ , it is also the minimum value of  $X$  for which  $Z_{n+1}(X) = 0$ , i.e.,  $R_{n+1} = R_n$ .

Finally, it follows from (17) and (18) that for  $X \leq R_n$ ,

$$\begin{aligned} D_1V_{n,0}(X, 0) &= D_1J_n(X, Z_n(X)) + D_2J_n(X, Z_n(X))DZ_n(X) = D_1J_n(X, Z_n(X)) \\ &= DL(X) + \alpha \int_0^X D_1V_{n-1,m-1}(X - \xi, Z_n(X))\varphi(\xi) d\xi. \end{aligned}$$

The second equality is due to the fact that for all  $X < R_n$ ,  $Z_n(X) > 0$  and  $D_2J_n(X, Z_n(X)) = 0$ . Similarly, for  $X \leq R_{n+1}$

$$D_1V_{n+1,0}(X, 0) = DL(X) + \alpha \int_0^X D_1V_{n,m-1}(X - \xi, Z_{n+1}(X))\varphi(\xi) d\xi.$$

Hence, it follows from (A.3) that for  $X \leq R_{n+1}$  [thus  $X + Z_{n+1}(X) \leq R_{n+1}$ ],  $D_1V_{n+1,0}(X, 0) = D_1V_{n,0}(X, 0)$ .

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