Optimal finite impulse response approximate inverse of linear periodic filters

J.-Y. Wu and C.-A. Lin

Abstract: A method to construct an optimal finite impulse response (FIR) approximate inverse for discrete-time causal FIR periodic filters in the presence of measurement noise is proposed. The objective function to be minimised is the sum-of-mean-square errors over one period. On the basis of the matrix impulse response of the multi-input multi-output time-invariant representation of periodic filters, the optimisation problem is formulated as one that minimises the summed equation errors of a set of over-determined linear equations. It is shown that the problem is equivalent to a set of least-squares problems from which a simple, closed-form solution is obtained. Numerical examples are used to illustrate the performance of the proposed FIR approximate inverse.

1 Introduction

Periodic filters have found various applications in signal processing and communications, for example, in sub-band coding [1, 2], in modelling and design of transmultiplexers [3–5], in speech scrambling [6, 7], in spread spectrum multiple access communications [8, 9] and in blind channel identification and equalisation [10–12]. The inverse, or approximate inverse, of a periodic filter is used for recovering scrambled signals [7] and for equalisation of periodically modulated communication channels [12]. Inversion of periodic filters has been discussed by Kazlauskas [13], Lin and King [14] and Vetterli [15] for the noiseless case and by Wu and Lin [16], Wang *et al.* [17] and Zhou *et al.* [18] when measurement noise is present.

There are many different descriptions of single-input single-output (SISO) linear periodic digital filters [2, 19, 20], either in the time domain via periodic state equation and periodic difference equation or in the frequency domain using the poly-phase model. In terms of time-domain block signals, it is well known that associated with each SISO N-periodic filter, there is an N-input N-output linear time-invariant (LTI) system that exhibits an input–output relation identical to that of the filter [2, 20]. For general study of periodic systems, in particular, in the inverse filtering problem, this multi-input multi-output (MIMO) representation is often adopted because the LTI nature would allow considerable simplification in analysis and design. It is known that such an equivalent MIMO system must satisfy certain structural constraints due to causality [20]. As a result, the design of causal periodic inverse filter based on the MIMO LTI framework would amount to finding an appropriate inverse LTI system subject to this constraint. Lin and King [14] proposed a method for finding the inverse transfer matrix in the noiseless case. Recently, the MIMO LTI formulation was also used by Wu and Lin [16] and Wang *et al.* [17] for approximate inverse design in the presence of noise. For a given periodic filter, either infinite impulse reponse (IIR) or FIR, the solution reported by Wu and Lin [16] is in general IIR. In Wang *et al.* [17], the problem of FIR approximate inverse design for FIR periodic filters is investigated via the linear matrix inequality (LMI) framework. All the aforementioned works on inverse design use the periodic state equation as the filter model because there is a well-known formula for computing the transfer matrix of the associated MIMO LTI system [20].

This paper proposes a method to construct an FIR approximate inverse for a given FIR periodic filter in the presence of measurement noise. Unlike the previous studies [16, 17], we use the difference equation filter description. This allows a simple way of specifying the associated MIMO LTI system, in terms of the matrix impulse response. The cost function is the steady-state mean-square approximation errors summed over one period, as considered by Wu and Lin [16]. On the basis of the matrix impulse response of the MIMO LTI model, the problem is naturally formulated as minimisation of the summed equation errors of a set of over-determined linear equations. The causality constraint, in this case, is seen to impose a certain zero-padded structure in the unknown filter coefficient vector. There is a very simple way of resolving this constraint and the problem is shown to be reduced as a set of least-squares problems. An FIR approximate inverse, on the other hand, can be obtained by truncating the matrix impulse response of an IIR solution reported by Wu and Lin [16]. To tackle the causality condition in the general, possibly IIR, case, the problem formulation by Wu and Lin [16] builds on an infinite-dimensional space of matrix sequences and a related z-transform domain analysis; the computations involved are factorisations of rational matrices, followed by a QR decomposition for fulfilling the causality requirement (cf. [16, Section IV–V]). In the light of these points, the advantages of the current approach are 2-fold. First, the alternative formulation via difference equation, and hence the matrix impulse response in the MIMO LTI setting, leads to a simple linear equation analysis framework. Secondly, the

IEE Proceedings online no. 20045118

doi:10.1049/ip-vis:20045118

Paper first received 20th July 2004 and in revised form 28th July 2005

J.-Y. Wu is with the Department of Communication Engineering, National Chiao-Tung University, Hsinchu, Taiwan

C.-A. Lin is with the Department of Electrical and Control Engineering, National Chiao-Tung University, Hsinchu, Taiwan

E-mail: jywu@cc.nctu.edu.tw

resultant least squares based solutions would offer a reduction in algorithm complexity when compared with the truncation-based approach shown in Wu and Lin [16]. Also, in contrast with the iterative LMI method [17], in which essentially the same objective function is considered, the proposed approach can yield appealing, closed-form solutions.

Notation: We denote by $\Re^{l\times m}$ the set of all $l \times m$ real matrices. The notations $\mathbf{0}_{l\times m}$ and $\boldsymbol{I}_l$, respectively, stand for the $l \times m$ zero matrix and the $l \times l$ identity matrix. Denote by $\boldsymbol{\delta}$ the $N \times N$ matrix unit-impulse sequence, that is, $\boldsymbol{\delta}_n = \mathbf{0}_{N\times N}$ for all $n > 0$ and $\boldsymbol{\delta}_0 = \boldsymbol{I}_N$. Let z^{-k} be the k-step delay operator such that for any sequence s, $(z^{-k}s)_n = s_{n-k}$. Let $H^{l\times m}$ be the space of all causal sequences of matrices $\boldsymbol{X} = \{\boldsymbol{X}_n \in \Re^{l\times m}, n \geq 0\}$. Given a positive integer K, define the subspace of $H^{l\times m}$ as $H_K^{l\times m} := \{\boldsymbol{X} \in H^{l\times m}$: $\boldsymbol{X}_n = \mathbf{0}_{l\times m}$ for all $n \geq K\}$. The norm of $\boldsymbol{X} \in H_K^{l\times m}$ is defined by $\|\boldsymbol{X}\| := \left(\sum_{n=0}^{K-1} \|\boldsymbol{X}_n\|_{\mathrm{F}}^2\right)^{1/2}$, where $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm [21, p. 55]. For $\boldsymbol{X} \in H_{K_1}^{l\times m}$ and $\boldsymbol{Y} \in H_{K_2}^{l\times m}$, let the augmented sequence $[\boldsymbol{X} \quad \boldsymbol{Y}] \in H_{K_3}^{l\times 2m}$ be such that $[\boldsymbol{X} \quad \boldsymbol{Y}]_n = [\boldsymbol{X}_n \quad \boldsymbol{Y}_n]$ and $K_3 = \max\{K_1, K_2\}$. Denote by $\boldsymbol{X} * \boldsymbol{Y}$ the convolution of $\boldsymbol{X} \in H_{K_1}^{l\times m}$ and $\boldsymbol{Y} \in H^{m\times n}$ such that $(\boldsymbol{X} * \boldsymbol{Y})_n = \sum_{k=0}^{\min\{n, K_1-1\}} \boldsymbol{X}_k \boldsymbol{Y}_{n-k}$.

2 Problem statement and preliminary

2.1 Problem statement

Consider the discrete-time causal FIR N-periodic filter with input u and output z described by

$$z_n = \sum_{k=0}^{M} g_{n,k} u_{n-k}, \quad n \geq 0 \tag{1}$$

where u_n and z_n are, respectively, the input and output at time n, and the filter coefficient $g_{n,k}$ satisfies

$$g_{n,k} = g_{n+N,k}, \quad \forall n \geq 0, \quad 0 \leq k \leq M \tag{2}$$

Consider the block diagram shown in Fig. 1, where r is the observed signal, which is the sum of filter output z and a measurement noise v, that is

$$r = z + v \tag{3}$$

and $\hat{u}$ is the d-step delay of the input u to filter (1), that is

$$\hat{u}_n = \begin{cases} u_{n-d}, & n \geq d \\ 0, & 0 \leq n < d \end{cases} \tag{4}$$

An approximate inverse of filter (1) is a causal FIR N-periodic filter with input r and output y described by

$$y_n = \sum_{k=0}^{M_1} f_{n,k} r_{n-k} \tag{5}$$

where for each $0 \leq k \leq M_1$

$$f_{n,k} = f_{n+N,k}, \quad \forall n \geq 0 \tag{6}$$

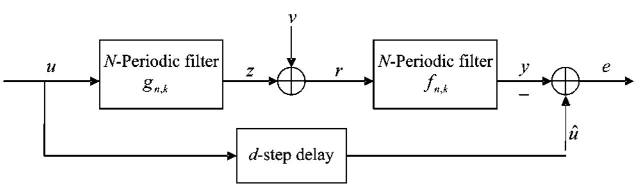

Fig. 1 *Schematic description of FIR periodic inverse filtering problem*

such that the output y is close to $\hat{u}$, that is, the error signal

$$e = \hat{u} - y \tag{7}$$

is small for the input signal u of interest.

The following assumptions are made in the sequel.

1. The input $u = \{u_n \in \Re, n \geq 0\}$ to filter (1) is a white sequence with zero mean and unit variance.
2. The noise $v = \{v_n \in \Re, n \geq 0\}$ is a white sequence with zero mean and variance σ_v^2 and is uncorrelated with the input u.

In this paper, we propose a method to construct an FIR approximate inverse of the form (5), with which the sum of error variances over one period is minimised.

2.2 Matrix impulse response of FIR periodic filters

Consider again the filter (1). Define the block input $\bar{\boldsymbol{u}}$ and output $\bar{\boldsymbol{z}}$ as

$$\bar{\boldsymbol{u}}_n := [u_{nN} \quad u_{nN+1} \quad \cdots \quad u_{nN+N-1}]^{\mathrm{T}} \in \Re^N, \quad n \geq 0 \tag{8}$$

and

$$\bar{\boldsymbol{z}}_n := [z_{nN} \quad z_{nN+1} \quad \cdots \quad z_{nN+N-1}]^{\mathrm{T}} \in \Re^N, \quad n \geq 0 \tag{9}$$

It is well known that associated with filter (1), there is an N-input N-output (FIR) time-invariant system, with input $\bar{\boldsymbol{u}}$ and output $\bar{\boldsymbol{z}}$, which exhibits an input–output relation identical to that of filter (1) [2, 20]. The matrix impulse response of the associated MIMO LTI system, which will be used in our subsequent discussions, can be directly determined from (1) as follows.

Write the ith component of $\bar{\boldsymbol{z}}_n$ as (see (1))

$$\begin{aligned} z_{nN+i} &= \sum_{k=0}^{M} g_{nN+i,k} u_{nN+i-k} \\ &= \sum_{k=0}^{M} g_{i,k} u_{nN+i-k}, \quad 0 \leq i \leq N-1 \end{aligned} \tag{10}$$

where the second equality follows from (2). Let $m := M$ modulo N, thus $0 \leq m \leq N-1$, and choose

$$L = \left\lceil \frac{M}{N} \right\rceil + 1 \tag{11}$$

where $\lceil M/N \rceil$ is the smallest integer that is greater than or equal to M/N. Collecting z_{nN+i} in (10), $0 \leq i \leq N-1$, into a vector and by rearrangement, we can express $\bar{\boldsymbol{z}}_n$ in (9) as the following product form

$$\bar{\boldsymbol{z}}_n = \boldsymbol{G}\boldsymbol{U} \tag{12}$$

where $\boldsymbol{G} \in \Re^{N\times LN}$ is the filter coefficient matrix whose ith row, $0 \leq i \leq N-1$, is

$$\begin{aligned} &[\mathbf{0}_{1\times i} \quad g_{i,M} \quad \cdots \quad g_{i,0} \quad \mathbf{0}_{1\times(N-1-i)}], \quad \text{if } m = 0 \\ &[\mathbf{0}_{1\times(N-m+i)} \quad g_{i,M} \quad \cdots \quad g_{i,0} \quad \mathbf{0}_{1\times(N-1-i)}], \\ &\quad \text{if } 1 \leq m \leq N-1 \end{aligned} \tag{13}$$

and $\boldsymbol{U} \in \Re^{LN}$ is the vector containing the input samples having contributions to $\bar{z}_n$ and is given as

$$\boldsymbol{U} = \begin{cases} [u_{nN-M} \ \cdots \ u_{uN-1} \ u_{nN} \ \cdots \ u_{nN+N-1}]^{\mathrm{T}}, & \text{if } m=0 \\ [u_{nN-M-(N-m)} \ \cdots \ u_{nN-M} \ \cdots \ u_{nN} \ \cdots \ u_{nN+N-1}]^{\mathrm{T}}, & \\ \quad \text{if } 1 \le m \le N-1 & \end{cases} \tag{14}$$

We note that the leading zero entries in the row vectors given in (13) result from the fact that filter (1) is FIR with order M; the tailing zero entries are due to the causality of filter (1). In terms of block input $\bar{\boldsymbol{u}}_n$ in (8), it can be checked that the vector $\boldsymbol{U}$ in (14), for each $0 \le m \le N-1$, is equal to

$$\boldsymbol{U} = \left[\bar{\boldsymbol{u}}_{n-(L-1)}^{\mathrm{T}} \ \cdots \ \bar{\boldsymbol{u}}_{n-1}^{\mathrm{T}} \ \bar{\boldsymbol{u}}_n^{\mathrm{T}}\right]^{\mathrm{T}} \tag{15}$$

Partition the matrix $\boldsymbol{G}$ in (13) as

$$\boldsymbol{G} = [\boldsymbol{G}_{L-1} \ \cdots \ \boldsymbol{G}_0] \tag{16}$$

where $\boldsymbol{G}_l \in \Re^{N\times N}$, $0 \le l \le L-1$. With (15) and (16), the product expression of $\bar{z}_n$ in (12) can be written in the convolutional form as

$$\bar{z}_n = \sum_{l=0}^{L-1} \boldsymbol{G}_l \bar{\boldsymbol{u}}_{n-l} \tag{17}$$

Hence, the MIMO LTI system associated with filter (1) is described by $\bar{z} = \boldsymbol{G} * \bar{\boldsymbol{u}}$, where the matrix impulse response $\boldsymbol{G}$ is given as

$$\boldsymbol{G} = \sum_{l=0}^{L-1} \boldsymbol{G}_l z^{-1} \boldsymbol{\delta} \in H_L^{N\times N} \tag{18}$$

As a result, an Mth order FIR N-periodic filter of the form (1) is represented by a $\boldsymbol{G} \in H_L^{N\times N}$ as in (18), where L is given in (11), and $\boldsymbol{G}_l \in \Re^{N\times N}$ is defined through the filter coefficient matrix $\boldsymbol{G}$ as in (16). Conversely, given a $\boldsymbol{G} \in H_L^{N\times N}$, if we form the $\boldsymbol{G}$ matrix according to (16) and if its rows are of the form (13), then $\boldsymbol{G}$ can be implemented as an SISO FIR N-periodic filter of the form (1). In particular, the $M+1$ non-zero entries in the ith row of $\boldsymbol{G}$, $0 \le i \le N-1$, yield the filter coefficients $\boldsymbol{g}_{i,k}$ for $0 \le k \le M$. In the sequel, we will simply call $\boldsymbol{G}$ the matrix impulse response of filter (1). The matrix impulse response of the d-step delay is given as follows.

Proposition 1 [14]: The matrix impulse response associated with the d-step delay, when regarded as an N-periodic system, is

$$\boldsymbol{D} = \sum_{n=q}^{q+1} D_n z^{-n} \boldsymbol{\delta} \in H_{q+2}^{N\times N} \tag{19}$$

where $d = p + qN$, p and q are non-negative integers with $0 \le p \le N-1$

$$\boldsymbol{D}_q = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I}_{N-p} & \boldsymbol{0} \end{bmatrix} \in \Re^{N\times N}$$

and

$$\boldsymbol{D}_{q+1} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_p \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \in \Re^{N\times N} \tag{20}$$

3 Optimisation problem

In this section, formulation of the optimisation problem is given. We first introduce the optimality criterion. The resultant cost function is then expressed via matrix impulse responses of periodic filters.

To proceed, we shall first characterise the error variance $E|e_n|^2$. As periodic filters will produce cyclostationary outputs when inputs are stationary [22], it is expected that $E|e_n|^2$ will exhibit certain periodic behaviour. Indeed, as both the input u and noise v considered in this paper are causal white sequences, $E|e_n|^2$ can be shown to be N-periodic for n large enough. To see this, we first note that the cascade connection of filters (1) and (5) is an FIR filter of order $M+M_1$: the reconstruction error e will be in the 'steady state' whenever $n \ge M+M_1$ [23, Chapter 4]. With such n, and by linking e_n with u_n and v_n via (1), (3) and (5), it can be directly verified that

$$E|e_n|^2 = E|e_{n+N}|^2, \quad \forall n \ge M+M_1 \tag{21}$$

From (21), the sum of $E|e_n|^2$ over an arbitrary block of N samples for $n \ge M+M_1$ is thus a constant independent of the blocks chosen for summation. This suggests the following objective function

$$J := \sum_{n=M+M_1}^{M+M_1+N-1} E|e_n|^2 \tag{22}$$

If J is small, then $E|e_n|^2$ is small for each $M+M_1 \le n \le M+M_1+N-1$. From (21), it follows that $E|e_n|^2$ is small for all $n \ge M+M_1$. The block invariant property of the steady-state block approximation error J will also enable us to analyse the optimisation problem by using the MIMO LTI representation of periodic filters. Moreover, as the objective function J is quadratic in nature, the optimisation problem, potentially, could be relatively easy to solve. Hence, we propose to find an approximate inverse by minimising the objective function J.

Before we proceed, we shall first express the objective function J in (22) in terms of the matrix impulse responses of filters (1) and (5). As we will see in the next section, this will allow a formulation of the optimisation problem in terms of a set of linear equations, based on which a closed-form optimal solution can be obtained. Let the matrix impulse response of filter (5) be

$$\boldsymbol{F} = \sum_{l=0}^{L_1-1} \boldsymbol{F}_l z^{-l} \boldsymbol{\delta} \in H_{L_1}^{N\times N} \tag{23}$$

where $L_1 = \lceil M_1/N \rceil + 1$ and $\boldsymbol{F}_l \in \Re^{N\times N}$, $0 \le l \le L_1 - 1$, contain the unknown filter coefficients $f_{i,k}$ for $0 \le i \le N-1$ and $0 \le k \le M_1$. Then, we have the following proposition.

Proposition 2: Let $\boldsymbol{G} \in H_L^{N\times N}$, $\boldsymbol{F} \in H_{L_1}^{N\times N}$ and $\boldsymbol{D} \in H_{q+2}^{N\times N}$ be, respectively, the matrix impulse responses of filters (1), (5) and the d-step delay. Define $L_2 := L + L_1 - 1$. Assume that L_1 is chosen so that $L_2 \ge q+2$. Then, the objective function J defined in (2) can be expressed as

$$J = \|\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G}\|^2 + \sigma_v^2 \|\boldsymbol{F}\|^2 \tag{24}$$

Proof: As $\boldsymbol{D} \in H_{q+2}^{N\times N} \subseteq H_{L_2}^{N\times N}$ and $\boldsymbol{F} * \boldsymbol{G} \in H_{L_2}^{N\times N}$, we have $\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G} \in H_{L_2}^{N\times N}$. Let $\bar{\boldsymbol{e}}$ and $\bar{\boldsymbol{v}}$ be the block error signal and noise, defined in an analogous way as (8). Then, in terms of block signals, we have

$$\begin{aligned} \bar{\boldsymbol{e}} &= \boldsymbol{D} * \bar{\boldsymbol{u}} - \boldsymbol{F} * (\boldsymbol{G} * \bar{\boldsymbol{u}} + \bar{\boldsymbol{v}}) \\ &= (\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G}) * \bar{\boldsymbol{u}} - \boldsymbol{F} * \bar{\boldsymbol{v}} \end{aligned} \tag{25}$$

Let k_0 be a positive integer such that

$$k_0 N \geq M + M_1 + 4N \tag{26}$$

From (21), we can thus express the objective function J as $J = \sum_{n=k_0 N}^{K_0 N+N-1} E|e_n|^2$. This implies that

$$\begin{aligned} J &= E\|\bar{\boldsymbol{e}}_{k_0}\|_{\mathrm{F}}^2 = E\,\bar{\boldsymbol{e}}_{k_0}^{\mathrm{T}}\bar{\boldsymbol{e}}_{k_0} = E\,\mathrm{Tr}\big[\bar{\boldsymbol{e}}_{k_0}^{\mathrm{T}}\bar{\boldsymbol{e}}_{k_0}\big] \\ &= E\,\mathrm{Tr}\big[\bar{\boldsymbol{e}}_{k_0}\bar{\boldsymbol{e}}_{k_0}^{\mathrm{T}}\big] = \mathrm{Tr}\big[E\,\bar{\boldsymbol{e}}_{k_0}\bar{\boldsymbol{e}}_{k_0}^{\mathrm{T}}\big] \end{aligned} \tag{27}$$

With (25), we have

$$\bar{\boldsymbol{e}}_{k_0} = \sum_{l=0}^{L_2-1}(\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G})_l \bar{\boldsymbol{u}}_{k_0-l} - \sum_{m=0}^{L_1-1} \boldsymbol{F}_m \bar{\boldsymbol{v}}_{k_0-m} \tag{28}$$

We note from (26) that k_0 thus chosen satisfies $k_0 \geq M/N + M_1/N + 4 \geq L + L_1 = L_2{+}1$ and hence no block input samples with negative time instants are involved in the summation in (28). Substituting $\bar{\boldsymbol{e}}_{k0}$ in (28) into (27) and as $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$ are uncorrelated white vector processes, it can be verified that

$$J = \sum_{l=0}^{L_2-1}\|(\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G})_l\|_{\mathrm{F}}^2 + \sigma_v^2 \sum_{l=0}^{L_1-1}\|\boldsymbol{F}_l\|_{\mathrm{F}}^2 \tag{29}$$

and the result follows. □

The optimisation problem, in terms of matrix impulse response, is formulated as follows: Given $\boldsymbol{G} \in H_L^{N\times N}$ and $\boldsymbol{D} \in H_{q+2}^{N\times N}$ as in (18) and (19), find an $\boldsymbol{F} \in H_{L_1}^{N\times N}$, which can be implemented as an SISO FIR N-periodic filter of the form (5), to minimise the objective function defined in (24).

Remarks:

(a) From (24), it appears that an optimal $\boldsymbol{F} \in H_{L_1}^{N\times N}$, if it exists, will tend to keep the quantity $\|\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G}\|^2$ small [i.e., $(\boldsymbol{F} * \boldsymbol{G})_n \simeq \boldsymbol{D}_n$ for $n = q$ and $n = q{+}1$ and $(\boldsymbol{F} * \boldsymbol{G})_n \simeq \boldsymbol{0}_{N\times N}$ for the other values of n] and, at the same time, $\|\boldsymbol{F}\|^2$ is maintained small (i.e. $\boldsymbol{F}_n \simeq \boldsymbol{0}_{N\times N}$) for $0 \leq n \leq L_1 - 1$).
(b) The assumption $L_2 \geq q + 2$, that is, the duration of the sequence $\boldsymbol{F} * \boldsymbol{G}$ is no less than that of $\boldsymbol{D}$, is necessary, since otherwise the quantity $\|\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G}\|^2$, in general, cannot be made small. This is because, if $L_2 < q + 2$, we have $(\boldsymbol{F} * \boldsymbol{G})_{q+1} = \boldsymbol{0}_{N\times N}$, which is impossible to be kept close to $\boldsymbol{D}_{q+1}$ by choosing any $\boldsymbol{F} \in H_{L_1}^{N\times N}$.
(c) As $\boldsymbol{F} * \boldsymbol{G}$ is the matrix impulse response of the cascade connection of filters (1) and (5) [14], the first term in the right-hand side of (24) thus measures the goodness of signal resolution (filter inversion), whereas the second term is the cost incurred by noise with respect to a desired signal resolution quality attained by $\boldsymbol{F}$.
(d) Note from (24) that large noise variance σ_v^2 tends to emphasise large noise reduction. However, this is done at the expense of the signal resolution quality. On the other hand, small σ_v^2 leads to better signal resolution quality but with smaller noise reduction. Hence, there is a trade-off between signal resolution quality and noise reduction.
(e) The problem of designing FIR approximate inverse for FIR periodic filter in the presence of noise is also addressed by Wang *et al.* [17]. By regarding the signals u and v as the input to the signal reconstruction system shown in Fig. 1 and the error e as the corresponding output, the optimality criterion adopted by Wang *et al.* [17] is to minimise the squared H_2-norm of the input–output map from the augmented signal $[u \quad v]^{\mathrm{T}}$ to error e. In terms of block signals, such an objective function is shown to be equal to [17, p. 2698]

$$\|[\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G} \quad \boldsymbol{F}]\|^2 \tag{30}$$

By definition of norm $\|\cdot\|$, it can be easily checked that the quantity in (30) is a special case of the proposed objective function (24) with noise variance fixed at $\sigma_v^2 = 1$. The resultant minimisation problem in Wang *et al.* [17] is formulated in terms of state equations of the augmented system $[\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G} \quad \boldsymbol{F}]$ and is solved by using the iterative LMI approach.

4 Optimal solution

If the approximate inverse (5) is allowed to be, in general, IIR, the objective function (24) then accounts for the asymptotic block mean-square error as considered by Wu and Lin [16]. The optimisation problem therein is formulated in the z-transform domain and is solved through factorisation of rational matrices. An FIR inverse, as a result, can be obtained by truncating the inverse z-transform of the constructed optimal IIR rational matrix. On the basis of the time-domain problem formulation in terms of the matrix impulse responses as in (24), this section presents a simple and elegant alternative to constructing an FIR inverse. As one will see, the problem amounts to computing a set of least-squares solutions: this is relatively simple when compared with the 'indirect' truncation-based approach.

4.1 Linear equations formulations

We shall first rewrite J in (24) as the 'matching error' of two augmented sequences. More precisely, by definition of norm $\|\cdot\|$, it follows immediately from (24) that

$$J = \|[\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G} \quad \sigma_v \boldsymbol{F}]\|^2 \tag{31}$$

Associated with the matrix impulse responses $\boldsymbol{G}$ and $\boldsymbol{D}$ [see (18) and (19)], we define the respective augmented sequences

$$\tilde{\boldsymbol{G}} := [\boldsymbol{G} \quad -\sigma_v \boldsymbol{\delta}] \in H_L^{N\times 2N} \tag{32}$$

and

$$\tilde{\boldsymbol{D}} := [\boldsymbol{D} \quad \boldsymbol{0}_{N\times N}] \in H_{L_2}^{N\times 2N} \tag{33}$$

In (33), it is noted that as $\boldsymbol{D} \in H_{q+2}^{N\times N}$, by definition of augmented sequence, we thus have $\tilde{\boldsymbol{D}} \in H_{q+2}^{N\times 2N} \subseteq H_{L_2}^{N\times 2N}$. As $[\boldsymbol{D} - \boldsymbol{F} * \boldsymbol{G} \quad \sigma_v \boldsymbol{F}] = \hat{\boldsymbol{D}} - \boldsymbol{F} * \hat{\boldsymbol{G}}$ and from (31), we have

$$J = \|\tilde{\boldsymbol{D}} - \boldsymbol{F} * \tilde{\boldsymbol{G}}\|^2 \tag{34}$$

On the basis of (34), the objective function J in (24) can be directly expressed in terms of the N rows of the filter coefficient matrix associated with filter (5). As we will see, this will lead to a very simple procedure for computing the optimal filter coefficients. From (34) and by definition of Frobenius norm, it follows that

$$\begin{aligned} J &= \sum_{n=0}^{L_2-1}\|\tilde{\boldsymbol{D}}_n - (\boldsymbol{F} * \tilde{\boldsymbol{G}})_n\|_{\mathrm{F}}^2 \\ &= \|\boldsymbol{D}^{\mathrm{T}} - [(\boldsymbol{F} * \tilde{\boldsymbol{G}})_0 \quad \cdots \quad (\boldsymbol{F} * \tilde{\boldsymbol{G}})_{L_2-1}]\|_{\mathrm{F}}^2 \end{aligned} \tag{35}$$

where

$$\boldsymbol{D}^{\mathrm{T}} := [\tilde{\boldsymbol{D}}_0 \quad \cdots \quad \tilde{\boldsymbol{D}}_{L_2-1}] \in \Re^{N\times 2L_2 N} \tag{36}$$

As $(\boldsymbol{F} * \tilde{\boldsymbol{G}})_n = \sum_{l=0}^{L_1-1} \boldsymbol{F}\tilde{\boldsymbol{G}}_{n-l}$, with some manipulations, it can be checked that

$$\left[(\boldsymbol{F} * \tilde{\boldsymbol{G}})_0 \quad \cdots \quad (\boldsymbol{F} * \tilde{\boldsymbol{G}})_{L_2-1}\right] = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} \tag{37}$$

where

$$\boldsymbol{X}^{\mathrm{T}} := [\boldsymbol{F}_{L_1-1} \quad \cdots \quad \boldsymbol{F}_0] \in \Re^{N \times L_1 N} \tag{38}$$

$\boldsymbol{A}^{\mathrm{T}} \in \Re^{L_1 N \times 2L_2 N}$ is the $N \times 2N$-block Hankel matrix with

$$\left[\boldsymbol{0}_{2N\times N} \quad \cdots \quad \boldsymbol{0}_{2N\times N} \quad \tilde{\boldsymbol{G}}_0^{\mathrm{T}}\right]^{\mathrm{T}} \in \Re^{L_1 N \times 2N} \tag{39}$$

as the first block column and

$$\left[\boldsymbol{0}_{N\times 2N} \quad \cdots \quad \boldsymbol{0}_{N\times 2N} \quad \tilde{\boldsymbol{G}}_0 \quad \cdots \quad \tilde{\boldsymbol{G}}_{L-1}\right] \in \Re^{N \times 2L_2 N} \tag{40}$$

as the first block row. We should note that as the sequence $\boldsymbol{F}$ in (23) is the matrix impulse response of filter (5), for each $0 \le i \le N-1$, the ith row of $\boldsymbol{X}^{\mathrm{T}}$ in (38), say, $\boldsymbol{X}_i^{\mathrm{T}}$, is thus of the form (13), viz., for $m_1 = M_1$ modulo N

$$\boldsymbol{X}_i^{\mathrm{T}} = \begin{cases} [\boldsymbol{0}_{1\times i} \;\; f_{i,M_1} \;\; \cdots \;\; f_{i,0} \;\; \boldsymbol{0}_{1\times(N-1-i)}], & \text{if } m_1 = 0 \\ [\boldsymbol{0}_{1\times(N-m_1+i)}, \;\; f_{i,M_1} \;\; \cdots \;\; f_{i,0} \;\; \boldsymbol{0}_{1\times(N-1-i)}], & \\ \quad \text{if } 1 \le m_1 \le N-1 & \end{cases} \tag{41}$$

With (35) and (37), we immediately have

$$J = \|\boldsymbol{D}^{\mathrm{T}} - \boldsymbol{X}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\|_{\mathrm{F}}^2 = \|\boldsymbol{D} - \boldsymbol{A}\boldsymbol{X}\|_{\mathrm{F}}^2 \tag{42}$$

where the last equality follows because the Frobenius norms of a matrix and its transpose are the same. The expression of the objective function J in (42), which involves the matrix $\boldsymbol{X}$ of the form (41) as unknown, can be further decomposed as a sum of equation errors of N groups of linear equations, each with one column of $\boldsymbol{X}$ as unknown. More precisely, write the matrix $\boldsymbol{D} - \boldsymbol{A}\boldsymbol{X}$ column by column as

$$\boldsymbol{D} - \boldsymbol{A}\boldsymbol{X} = [\boldsymbol{D}_0 - \boldsymbol{A}\boldsymbol{X}_0 \quad \cdots \quad \boldsymbol{D}_{N-1} - \boldsymbol{A}\boldsymbol{X}_{N-1}] \tag{43}$$

where $\boldsymbol{D}_i \in \Re^{2L_2 N}$ and $\boldsymbol{X}_i \in \Re^{L_1 N}$ are, respectively, the ith columns of the matrices $\boldsymbol{D}$ and $\boldsymbol{X}$ [in (36) and (38)]. From (42), (43) and by definition of the Frobenius norm, it follows that

$$J = \sum_{i=0}^{N-1} \|\boldsymbol{A}\boldsymbol{X}_i - \boldsymbol{D}_i\|_{\mathrm{F}}^2 \tag{44}$$

With (44), the objective function J is thus minimised if, for each $0 \le i \le N-1$, we can find an $\boldsymbol{X}_i$ of the form (41), or equivalently, $M_1 + 1$ unknown filter coefficients, $f_{i,k}$ $(0 \le k \le M_1)$ because the remaining entries in $\boldsymbol{X}_i$ are zero, which minimise $\|\boldsymbol{A}\boldsymbol{X}_i - \boldsymbol{D}_i\|_{\mathrm{F}}^2$. This is done in the next section.

4.2 Optimal solution

As $\boldsymbol{X}_i$ defined in (41) has only $M_1 + 1$ non-zero entries, the product $\boldsymbol{A}\boldsymbol{X}_i$ simplifies to a linear combination of $\boldsymbol{M}_1 + 1$ columns of $\boldsymbol{A}$. As a result, each group of equations $\boldsymbol{A}\boldsymbol{X}_i \simeq \boldsymbol{D}_i$ contains a set of $2L_2N$ scalar equations in $M_1 + 1$ unknowns. On the basis of this observation, the optimisation problem can be reduced to a set of N least-squares problems, whose solutions are very easy to compute.

To be specific, for $0 \le i \le N-1$, let

$$\boldsymbol{Y}_i := [f_{i,M_1} \quad \cdots \quad f_{i,0}]^{\mathrm{T}} \in \Re^{M_1+1} \tag{45}$$

be the ith filter coefficient vector. For $0 \le i \le N-1$, let $\boldsymbol{A}_i \in \Re^{2L_2N\times(M_1+1)}$ [Note 1] be the matrix obtained from $\boldsymbol{A}$ by deleting its first i (or first $N - m_1 + i$, if $m_1 \neq 0$) columns and last $N - 1 - i$ columns. Then, for any $\boldsymbol{X}_i$ of the form (41), it follows that

$$\boldsymbol{A}\boldsymbol{X}_i = \boldsymbol{A}_i\boldsymbol{Y}_i, \quad 0 \le i \le N-1 \tag{46}$$

With (46) and as $\boldsymbol{Y}_i$ is arbitrary, the optimisation problem is thus equivalent to

$$\min_{\boldsymbol{Y}_i} \|\boldsymbol{A}_i\boldsymbol{Y}_i - \boldsymbol{D}_i\|_{\mathrm{F}}^2, \quad 0 \le i \le N-1 \tag{47}$$

Assume that each $\boldsymbol{A}_i$ is of full column rank. The optimal filter coefficient vector $\tilde{\boldsymbol{Y}}_i$ is then computed as

$$\tilde{\boldsymbol{Y}}_i = \left(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{A}_i\right)^{-1}\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{D}_i, \quad 0 \le i \le N-1 \tag{48}$$

Remarks:

(a) We note that as each $\boldsymbol{A}_i$ is obtained from $\boldsymbol{A}$ by deleting its columns, a sufficient condition for each $\boldsymbol{A}_i$ to be of full column rank is that the matrix $\boldsymbol{A}$ itself is so. With the block Hankel structure of the matrix $\boldsymbol{A}$ [see (39) and (40)], it can be easily checked that the condition holds in general.
(b) When compared with the study [17] for FIR inverse design, in which the resultant solution is obtained via the iterative LMI method, the proposed approach leads to a relatively simple closed-form solution: computing N least-squares solutions as in (48).

4.3 Selection of the reconstruction delay

The proposed FIR solution (48) is optimal for an arbitrary, but fixed, reconstruction delay d. Different choices of delay, however, will lead to different approximation errors (42). Given an allowable inverse filter order M_1, the values of delay must be restricted to $0 \le d \le M + M_1$ for effective error reduction. Among the candidate choices, the optimal one yielding the smallest mean-square error can be determined as [cf. (42)]

$$d_{\mathrm{opt}} = \operatorname*{argmin}_d \|\boldsymbol{D} - \boldsymbol{A}\boldsymbol{X}_d\|_{\mathrm{F}}^2 \tag{49}$$

where $\boldsymbol{X}_d$ contains the computed filter coefficients in (48) for the prescribed d. We note that in constructing FIR inverse for FIR SISO LTI filters when there is noise, that is, the so-called spiking filter design, a similar strategy as (49) is suggested by Orfanidis [24, Section 5.14] for delay selection. On the basis of simulations (Simulation 3), there is a quite different solution tendency of d_{opt} in (49), depending on the zero location of the transfer matrix of filter (1). If all the associated zeros are inside the unit circle, that is, the minimum-phase case, the best choice seems to be $d_{\mathrm{opt}} = 0$; otherwise, a positive number of delay must be allowed to minimise the mean-square error. However, for non-minimum-phase transfer matrices, an unlimited increase in the delay d, and hence M_1, can hardly improve the performance; the error floor tends to converge towards a lower bound attained by the IIR solution [16] (Simulation 3).

Note 1: As $\boldsymbol{A} \in \Re^{2L_2N\times L_1N}$, we have $\boldsymbol{A}_i \in \Re^{2L_2N\times(L_1-1)N+1}$ (or $\boldsymbol{A}_i \in \Re^{2L_2N\times(L_1-2)N+m_1+1}$ if $m \neq 0$). For either case of m_1, it can be checked that the respective $\boldsymbol{A}_i$ matrix has a total number of $M_1 + 1$ columns.

Remark: When IIR approximate inverse is allowed, the selection of delay for improving the signal reconstruction performance is discussed in Wu and Lin [16, Section V].

4.4 Implementation complexity

The proposed method calls for computing N least-squares solutions in (48). Using the Housholder QR-based algorithm [21, p. 240], the number of flop counts, counting both addition and multiplication, is about $[4L_2N^2(M_1+1)^2 - 2N(M_1+1)^3]/[3+26L_2N^2(M_1+1) - 9(M_1+1)^2]/2$, where L_2 is defined in Proposition 2. For the truncation-based approach via the IIR solution [16], the computations required are mainly a QR decomposition of an $N \times N$ matrix and an inner–outer (or all pass and minimum phase) factorisation of a $2N \times N$ rational matrix. The number of flop counts of the QR stage is $4N^3/3$. A typical algorithm for an inner–outer factorisation is the Riccati-equation-based approach [25, p. 555]. The solution can be obtained by using the dare software packet in the Control System Toolbox of MATLAB, which relies on a QZ decomposition [21, p. 375] of a symplectic matrix pencil and the required flop cost is about $528M^3$ [M is the order of filter (1)]. It can be seen that the proposed solution (48) leads to less computational cost when M is large.

5 Simulation results

In this section, we use several numerical examples to illustrate the performance of the proposed optimal approximate inverse. In our simulations, we estimate the variance of the error signal at time n, viz., $E|e_n|^2$, via the time average

$$\hat{e}_n := \frac{1}{I}\sum_{i=1}^{I}|e_n^i|^2 \tag{50}$$

where I is the total number of independent Monte Carlo realisations and e_n^i is the nth sample of the computed error signal in the ith realisation. The estimated value of the objective function J in (22) is computed as

$$\hat{J} := \frac{1}{\lfloor S/N \rfloor}\sum_{n=M+M_1}^{M+M_1+S-1}\hat{e}_n \tag{51}$$

where S is the total number of data samples and $\lfloor\cdot\rfloor$ denotes the integer floor. For each Monte Carlo realisation, the input to filter (1) and the noise are uncorrelated white Gaussian sequences. In all simulations, the number of input samples is 100; the number of independent trials is $I = 1000$.

5.1 Simulation 1: approximation to optimal IIR solution [16]

In this simulation, we will see that an optimal FIR approximate inverse with M_1 large enough can achieve a performance very close to that obtained by the optimal IIR solution reported in [16]. Consider the following two-periodic filter

$$\begin{aligned} &g_{n,0} = 1.2, \quad g_{n,1} = 2, \quad g_{n,2} = -0.1555, \\ &\qquad g_{n,3} = 0.3318, \quad \text{for even } n \\ &g_{n,0} = 0.8, \quad g_{n,1} = -2.4, \quad g_{n,2} = -0.1037, \\ &\qquad g_{n,3} = 0.4976, \quad \text{for odd } n \end{aligned} \tag{52}$$

We fix reconstruction delay at $d = 6$ and consider the two cases SNR = 0 and 10 dB. We compute the optimal IIR approximate inverse based on Wu and Lin [16] for these two SNRs. The resultant minimal $\hat{J}$ for the two SNR levels are, respectively, -4.2 and -12.3 dB. For each $3 \le M_1 \le 20$, an FIR approximate inverse is designed using (48). Fig. 2 shows the computed $\hat{J}$ against M_1, with respect to the two SNRs. The respective theoretical values of the objective function J computed using (44) are also shown. It appears that the experimental values are almost identical to the theoretical values. Also, as long as $M_1 \ge 9$, the performances are almost identical to those obtained by the optimal IIR solutions.

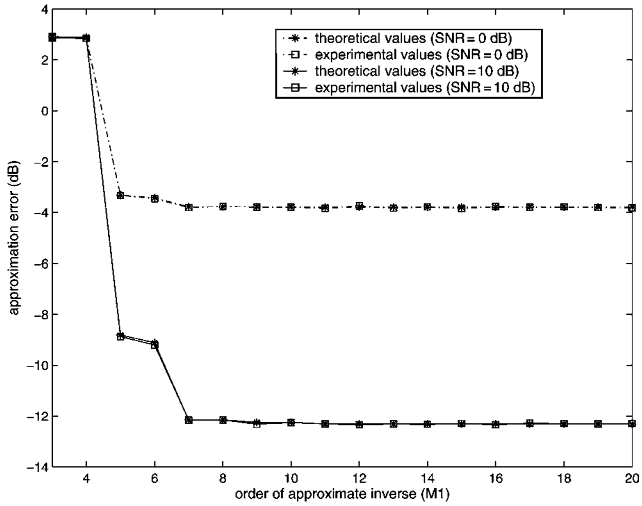

Fig. 2 *Approximation error against filter order, delay $d = 6$*

5.2 Simulation 2: comparison with previous works [16, 17]

In this simulation, we compare the proposed approach with the truncation-based solution reported in Wu and Lin [16] and the LMI method [17]. We consider the two-periodic filter in Wang *et al.* [17]

$$\begin{aligned} &g_{n,0} = 5, \quad g_{n,1} = 1, \quad g_{n,2} = 2, \quad g_{n,3} = -1, \quad \text{for even } n \\ &g_{n,0} = 3, \quad g_{n,1} = 2, \quad g_{n,2} = -2, \quad g_{n,3} = 1, \quad \text{for odd } n \end{aligned} \tag{53}$$

We set the reconstruction delay $d = 0$ as adopted in Wang *et al.* [17]. For each SNR level, the three comparative methods are respectively used for approximate inverse design (with $M_1 = 19$). It is noted that although the LMI method [17] considers the $\sigma_v^2 = 1$ case, it can directly incorporate different noise variance through scaling the

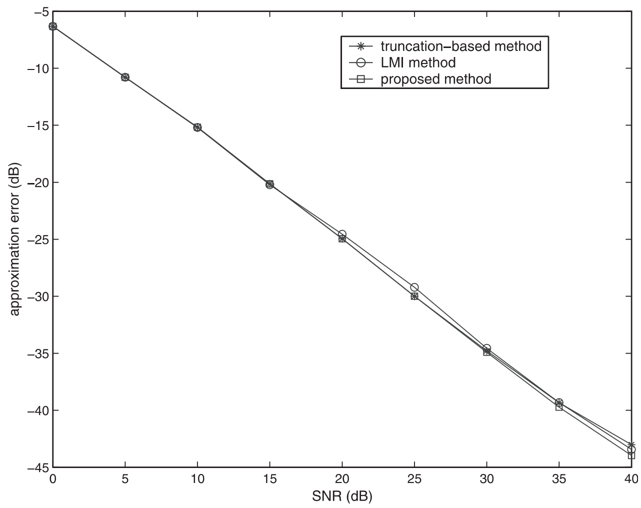

Fig. 3 *Comparisons of three methods*

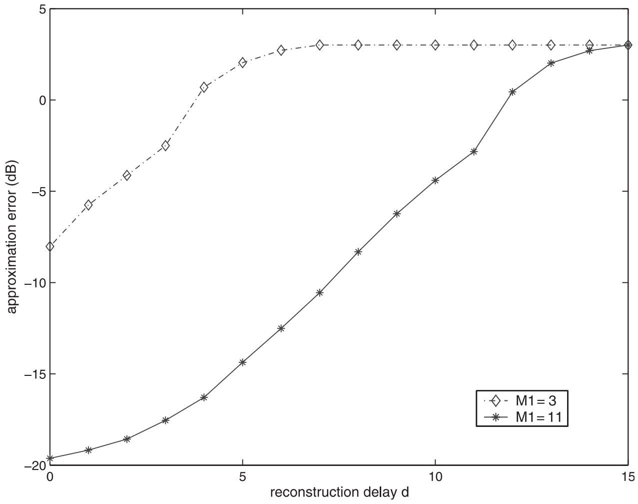

Fig. 4 *Approximation error against delay [filter (53)]*

parameter $\tilde{\boldsymbol{D}}_k$ in (11) in Wang *et al.* [17, p. 2698]; this strategy is adopted in Wu and Lin [16], and in our simulation as well, to reflect the actual noise levels. Fig. 3 shows the respective computed $\hat{J}$ at various SNRs, and the resultant performances are seen to be almost the same. This is not unexpected because all the three methods tend to minimise the same objective function, even though the underlying approaches are quite different.

5.3 Simulation 3: effect of reconstruction delay

In this simulation, we illustrate the effect of reconstruction delay on performance. In the first experiment, we consider filters (52) and (53), and fix SNR at 15 dB. Associated with each filter, the approximate inverse (48) is implemented with two prescribed orders $M_1 = 3$ and 11. Figs. 4 and 5 show the computed J at various selections of delay, respectively, for filters (53) and (52). It can be seen that for filter (53) (with a minimum-phase transfer matrix), $d = 0$ results in the best performance. For filter (52), whose transfer matrix has a zero at $z = -4.6937$, the minimal mean-square error is attained with positive delays: $d_{\mathrm{opt}} = 2$ for $M_1 = 3$ and $d_{\mathrm{opt}} = 6-8$ for $M_1 = 11$. In the SISO LTI spiking filter design, a similar phenomenon is also observed by Orfanidis [24, p. 297] and is believed to reflect the minimum-energy-delay property [26, Chapter 5] of the impulse responses of minimum-phase transfer

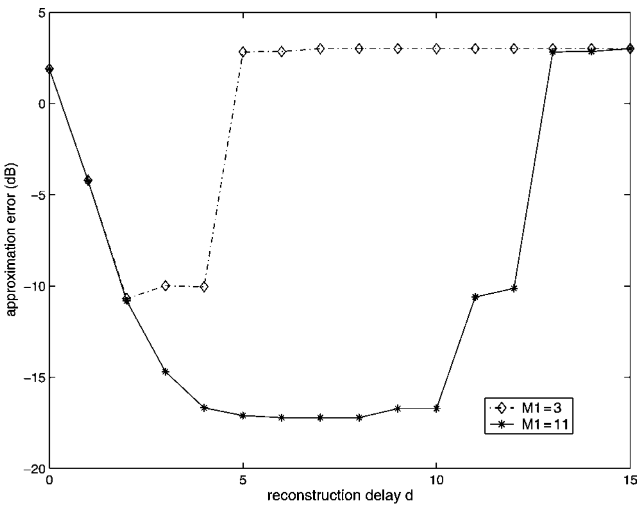

Fig. 5 *Approximation error against delay [filter (52)]*

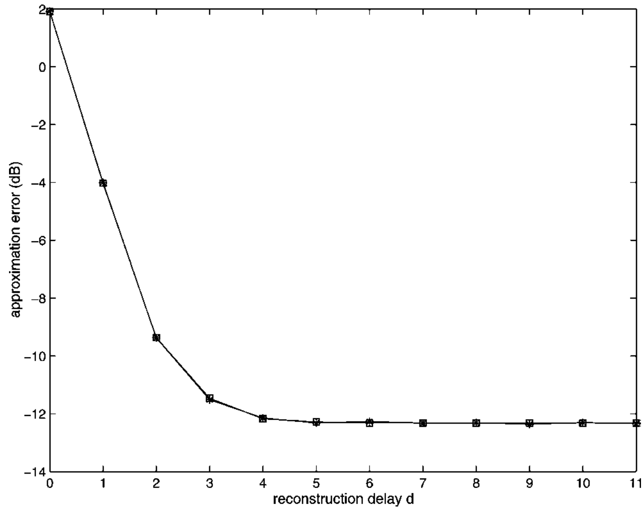

Fig. 6 *Approximation error against delay, SNR = 10 dB*

functions. It is noted that there is such an analogue characterisation in the MIMO case [27]: this would therefore account for our simulated outcome. In the second experiment, we consider filter (52) and the approximate inverse (48) is designed for each $0 \le d \le 11$ (SNR = 10 dB). The filter order is set to be $M_1 = d + 6$; through simulation, this turns out to be the smallest choice rendering the performance almost identical to that of the IIR solution [16] at each delay. Fig. 6 shows the resultant $\hat{J}$ and the theoretical solution J. As we can see, the performance is improved as d, and hence M_1, increases. However, there seems to be an error lower bound (-12.3 dB in our case), no matter how large d is used. Our simulation shows that this bound is very close to the performance achieved by the IIR solution [16].

6 Conclusions

We have proposed a method to construct an FIR approximate inverse for an FIR periodic filter in the presence of measurement noise. The presented study addresses the FIR case of our previous work [16]. The adopted optimality criterion, which minimises the sum of error variances over one period, allows us to formulate the problem in time domain in terms of the matrix impulse responses of MIMO time-invariant representation of periodic filters. There is a simple procedure for obtaining the matrix impulse response directly from the filter coefficients. The resultant optimisation problem is equivalent to solving a set of least-squares problems and a closed-form solution is obtained. The underlying computations are simply solving for a set of least-squares solutions but do not involve numerical optimisation as required in the LMI-based method [17]. The proposed method can also be used to obtain an FIR approximation to the optimal IIR solution in Wu and Lin [16].

7 Acknowledgment

Research sponsored by National Science Council under grant NSC-91-2219-E-009-049. This paper was presented in part at ICASSP 2004.

8 References

1 Min, X.W., and Ishii, R.: ‘An overall analysis of periodically time varying digital filters’, *IEICE Trans. Fundam.*, 1993, **E76-A**, (3), pp. 425–437

2 Vaidyanathan, P.P.: 'Multirate systems and filter banks' (Prentice-Hall, 1993)
3 Prater, J.S., and Loeffler, C.M.: 'Analysis and design of periodically time-varying IIR filters, with applications to transmultiplexing', *IEEE Trans. Signal Process.*, 1992, **40**, (11), pp. 2715–2725
4 Tanai, H., and Ishii, R.: 'A modem implementation using a periodically time varying digital filter', *IEICE Trans.*, 1991, **E74**, (11), pp. 3569–3574
5 Yang, X., Kawamata, M., and Hihuchi, T.: 'Implementation of TDM-FDM transmultiplexers by using periodically time-varying digital filters', *IEICE Trans.*, 1994, **J77-A**, (4), pp. 626–634
6 Ishii, R., and Kakishita, M.: 'A design method for periodically time varying digital filters for spectrum scrambling', *IEEE Trans. Acoust. Speech Signal Process.*, 1990, **38**, (7), pp. 1219–1222
7 King, C.W., and Lin, C.A.: 'A unified approach to scrambling filter design', *IEEE Trans. Signal Process.*, 1995, **43**, (8), pp. 1753–1765
8 Chauvet, W., Cristea, B., Lacaze, B., Roviras, D., and Duverdier, A.: 'Design of orthogonal LPTV filters: applications to spread spectrum multiple access', Proc. ICASSP, 2004, vol. II, pp. 645–648
9 Gelli, G., Paura, L., and Tulino, A.M.: 'Cyclostationarity-based filtering for narrowband interference suppression in direct-sequence spread-spectrum systems', *IEEE J. Sel. Areas Commun.*, 1998, **16**, (9), pp. 1747–1755
10 Gelli, G., and Verde, F.: 'Two-stage interference-resistant adaptive periodically time-varying CMA blind equalization', *IEEE Trans. Signal Process.*, 2002, **50**, (3), pp. 662–672
11 Orozco-Lugo, A.G., and Mclernon, D.C.: 'An application of linear periodically time-varying digital filters to blind equalization'. Proc. IEE Colloq. on Digital Filters: An Enabling Technology, no. 1998/252, London, 20 April 1998, pp. 11/1–11/6
12 Serpedin, E., and Giannakis, G.B.: 'Blind channel identification and equalization with modulation-induced cyclostationarity', *IEEE Trans. Signal Process.*, 1999, **47**, (7), pp. 1930–1944
13 Kazlauskas, K.: 'Inversions of periodically time-varying digital filters', *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, 1994, **41**, (2), pp. 173–175
14 Lin, C.A., and King, C.W.: 'Inverting periodic filters', *IEEE Trans. Signal Process.*, 1994, **42**, (1), pp. 196–200
15 Vetterli, M.: 'Invertibility of linear periodically time-varying filters', *IEEE Trans. Circuits Syst.*, 1989, **36**, (1), pp. 148–150
16 Wu, J.Y., and Lin, C.A.: 'Optimal approximate inverse of linear periodic filters', *IEEE Trans. Signal Process.*, 2004, **52**, (9), pp. 2371–2382
17 Wang, S., Xie, L., and Zhang, C.: 'H_2 optimal inverse of periodic FIR digital filters', *IEEE Trans. Signal Process.*, 2000, **48**, (9), pp. 2696–2700
18 Zhou, H., Xie, L., and Zhang, C.: 'A direct approach to H_2 optimal deconvolution of periodic digital channels', *IEEE Trans. Signal Process.*, 2002, **50**, (7), pp. 1685–1698
19 Mclernon, D.C.: 'One-dimensional linear periodically time-varying structures: derivations, interrelationships, and properties', *IEE Proc.-Vis. Image Signal Process.*, 1999, **146**, (5), pp. 245–252
20 Meyer, R.A., and Burrus, C.S.: 'A unified analysis of multirate and periodically time-varying digital filters', *IEEE Trans. Circuits Syst.*, 1975, **CAS-22**, pp. 162–168
21 Golub, G.H., and Van Loan, C.F.: 'Matrix computations' (Johns Hopkins University Press, 1996, 3rd edn.)
22 Akkarakaran, S., and Vaidyanathan, P.P.: 'Bifrequency and bispectrum maps: a new look at multirate systems with stochastic inputs', *IEEE Trans. Signal Process.*, 2000, **48**, (3), pp. 723–736
23 Orfanidis, S.J.: 'Introduction to signal processing' (Prentice-Hall, 1996)
24 Orfanidis, S.J.: 'Optimal signal processing: an introduction' (McGraw-Hill Book Company, 1990, 2nd edn.)
25 Zhou, K., Doyle, J.C., and Glover, K.: 'Robust and optimal control' (Prentice-Hall, Englewood Cliffs, NJ, 1996)
26 Oppenhiem, A.V., and Schafer, R.W.: 'Discrete-time signal processing' (Prentice-Hall, 1989)
27 Inouye, Y.: 'Notes on minimum-energy delay property of impulse response sequences of minimum-phase transfer functions', *IEEE Trans. Circuits Syst.*, 1987, **CAS-34**, pp. 189–190