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On embedding cycles into faulty twisted cubes ☆

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Abstract

The twisted cube TQ_n is an alternative to the popular hypercube network. Recently, some interesting properties of TQ_n were investigated. In this paper, we study the pancycle problem on faulty twisted cubes. Let f_e and f_v be the numbers of faulty edges and faulty vertices in TQ_n, respectively. We show that, with $f_e + f_v \leqslant n - 2$, a faulty TQ_n still contains a cycle of length l for every $4 \leqslant l \leqslant |V(TQ_n)| - f_v$ and odd integer $n \geqslant 3$.

Keywords: Cycle embedding; Twisted cube; Pancyclic; Hamiltonian; Fault tolerance

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1. Introduction

Parallel computing is important for speeding up computation. The design of an interconnection network, of course, is the first thing to be considered. In other words, network topology is an essential issue in parallel and distributed computing area. Many topologies have been proposed in the literature [2,3,8–10,13,16,19,26], and the desirable properties of an interconnection network include symmetry, relatively small degree, small diameter, embedding capabilities, scalability, robustness, and efficient routing. Among those proposed interconnection networks, the hypercube has been popular mainly because of its highly symmetric and easy routing structure.

The n-dimensional twisted cube TQ_n [13], as an alternative to the hypercube, has the same number of vertices and degree as the n-dimensional hypercube does. In fact, the twisted cube is derived from the hypercube with some edges twisted. Due to these twisted edges, the diameter of TQ_n is only about half of that of the hypercube. Some interesting studies on properties of TQ_n can be found in [1,6,18]. In particular, Huang et al. studied the fault-tolerant hamiltonicity of TQ_n in [18].

The embedding capabilities are important in evaluating an interconnection network. Given a host graph H and a guest graph G, an embedding of G into H is a mapping from each vertex of G to one vertex of H, and a mapping from each edge of G to one path of H. Graph embedding is useful because an algorithm designed for H can be applied to G directly. Therefore, the more we can embed guest graphs into a host graph, the better the host graph is. Popular guest graphs include cycles [11,15,16,18,19,25,26], paths [11,16], trees [22,23], etc. Four known parameters of an embedding are *dilation*, *congestion*, *load*, and *expansion*. The dilation is expressed by the maximum length of a path of the host graph which is mapped by an edge of the guest graph. The congestion is expressed by the maximum number of times that an edge of the host graph is mapped. The load is defined as the maximum number of vertices in the guest graph that are mapped to the same vertex in the host graph. The expansion is defined as the ratio of the number of vertices in the host graph to the number of vertices in the guest graph. If the embedding has dilation 1, congestion 1, and load 1, the guest graph is a subgraph of the host graph. In this paper, all embeddings have dilation 1, congestion 1, and load 1. Therefore, each of these embeddings is the best.

The ring structure is important for distributed computing, and its benefits can be found in [20]. Let us consider a problem about the cycle embeddings. The pancycle problem involves finding all possible lengths of cycles in a graph G. More precisely, letting $V(G)$ and $|V(G)|$ be the set and the number of vertices of G, respectively, the goal is to embed a cycle of length l into G for every l satisfying $b \leqslant l \leqslant |V(G)|$, where b is a specific positive integer. This problem has attracted a great deal of mathematicians [4,5,21,24] since it was brought up by Bondy [5] in 1971. Recently, many researchers studied this problem in the area of interconnection networks [2,7,11,12,14,17,27]. Most of the previous works did not consider the issue of the fault tolerance.

Since components of an interconnection network may malfuction and we want the network to keep working, the fault-tolerant capabilities of an interconnection network are essential. Moreover, nodes and edges may fail simultaneously when a network is put in use. Hence, we study the pancycle problem on faulty twisted cubes which can tolerate failures of nodes and edges at the same time in this paper. Let f_v and f_e be the numbers of faulty vertices and edges in TQ_n, respectively. We can embed a cycle of length l into faulty TQ_n if $f_v + f_e \leqslant n - 2$ for any integer $4 \leqslant l \leqslant |V(TQ_n)| - f_v$ and odd integer $n \geqslant 3$. Therefore, the expansion of these embeddings ranges between 1 and $\frac{|V(TQ_n)|-f_v}{4}$. In addition, this result is optimal since if there are $n - 1$ faulty elements around a single vertex of TQ_n, there is no hamiltonian cycle in faulty TQ_n.

This paper is organized as follows: We introduce some definitions and notation, including the definition of TQ_n, in Section 2. Then, Section 3 presents the main result, where we show the fault-tolerant pancyclicity of TQ_n. Finally, we give our conclusion in Section 4. To smooth the proof of Theorem 3 in Section 3, some details are left to Appendix A.

2. Definitions and notation

Letting G be a simple undirected graph, we use $V(G)$ and $E(G)$ to denote the sets of vertices and edges of G, respectively. Let $F_v \subseteq V(G)$, $F_e \subseteq E(G)$, and $F = F_v \cup F_e$. $G - F$ denotes the subgraph of $G - F_e$ induced by $V(G) - F_v$. $|S|$ denotes the number of elements in a set S. A *path*, denoted by $\langle u_1, u_2, \ldots, u_l \rangle$, is an ordered list of distinct vertices such that u_i and u_{i+1} are adjacent for $1 \leqslant i \leqslant l - 1$. Similarly, a *cycle*, denoted by $\langle u_1, u_2, \ldots, u_l, u_1 \rangle$, is an ordered list of distinct vertices except u_1, i.e., (u_l, u_1) is an edge. A *hamiltonian path* is a path that traverses every vertex of G exactly once. A graph G is *hamiltonian connected* if there is a hamiltonian path between any two vertices of G. We say that a graph G is *k-fault-tolerant hamiltonian connected* (abbreviated as *k-hamiltonian connected*) if $G - F$ is hamiltonian connected for any F with $|F| \leqslant k$. A pancyclic graph G, in its original definition [5], means that a cycle of length l can be embedded into G for every $3 \leqslant l \leqslant |V(G)|$. However, we note that the twisted cube TQ_n does not contain any cycle of length 3. For convenience of discussion, in this paper, we call a graph G *pancyclic* if, for every $4 \leqslant l \leqslant |V(G)|$, G has a cycle of length l. We say that a graph G is *k-fault-tolerant pancyclic* (abbreviated as *k-pancyclic*) if $G - F$ is pancyclic for any F with $|F| \leqslant k$. A *hamiltonian cycle* is defined as a cycle which traverses every vertex of G exactly once. A graph is called *hamiltonian* if it has a hamiltonian cycle. We

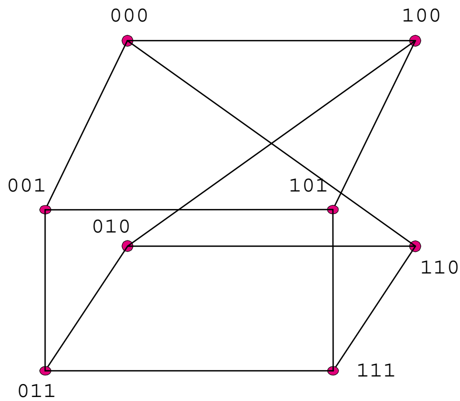


Fig. 1. TQ_3.

say that a graph G is *k-fault-tolerant hamiltonian* (abbreviated as *k-hamiltonian*) if $G - F$ is hamiltonian for any F with $|F| \leqslant k$.

The twisted cube was first proposed by Hilbers et al. in [13]. In the following, we give the recursive definition of the n-dimensional twisted cube TQ_n for any odd integer $n \geqslant 1$. TQ_n has 2^n vertices, and each of them is labeled by a binary string of length n. To define TQ_n, first of all, a parity function $P_i(x)$ is introduced. Let $u = u_{n-1}u_{n-2}\dots u_1u_0 \in V(TQ_n)$. For $0 \leqslant i \leqslant n-1$, $P_i(u) = u_i \oplus u_{i-1} \oplus \cdots u_1 \oplus u_0$, where $\oplus$ is the exclusive–or operation. TQ_1 is a complete graph with two vertices labeled by 0 and 1, respectively. For an odd integer $n \geqslant 3$, TQ_n is obtained by taking four copies of TQ_{n-2} and adding some additional edges to connect them. We use TQ^{ij}_{n-2} to denote an $(n-2)$-dimensional twisted cube which is a subgraph of TQ_n induced by the vertices labeled by $iju_{n-3}\dots u_0$, where $i,j \in \{0, 1\}$. Each vertex $u = u_{n-1}u_{n-2}\dots u_1u_0 \in V(TQ_n)$ is adjacent to $\bar{u}_{n-1}u_{n-2}\dots u_1u_0$ and $\bar{u}_{n-1}\bar{u}_{n-2}\dots u_1u_0$ if $P_{n-3}(u) = 0$; and to $\bar{u}_{n-1}u_{n-2}\dots u_1u_0$ and $u_{n-1}\bar{u}_{n-2}\dots u_1u_0$ if $P_{n-3}(u) = 1$. Fig. 1 illustrates TQ_3.

3. Main result

Lemma 1. *Let G be a graph. G is k-pancyclic if $G - F$ is pancyclic for every faulty set F with $|F| = k$.*

Proof. Suppose that $|F| \leqslant k$, and let $F' \subseteq E(G) - F_e$ with $|F| + |F'| = k$. So, $(G - F) - F'$ is a subgraph of $G - F$. Trivially, if $(G - F) - F'$ has a cycle C, $G - F$ contains C . This implies that if $(G - F) - F'$ is pancyclic, $G - F$ is also pancyclic. $\square$

Therefore, throughout this paper, whenever we prove that a graph G is k-pancyclic, we only consider the case $|F| = k$.

Theorem 1. *TQ_3 is 1-pancyclic.*

Proof. Fig. 2 is another layout of TQ_3, and it is vertex-transitive. We consider two cases (1) one faulty vertex and (2) one faulty edge as follows:

Case 1. *One faulty vertex.* Without loss of generality, we assume that vertex 000 is faulty. We list cycles of lengths from 4 to 7 as follows: $\langle 001, 101, 111, 011, 001\rangle$, $\langle 010, 100, 101, 111, 011, 010\rangle$, $\langle 001, 101, 111, 110, 010, 011, 001\rangle$, and $\langle 001, 101, 100, 010, 110, 111, 011, 001\rangle$.

Case 2. *One faulty edge.* We may assume that the faulty edge e is incident to 000 because of the symmetry of TQ_3. By Case 1, there are cycles of lengths from 4 to 7 in the faulty TQ_3. For a cycle of length 8, suppose that $e = (000, 100)$. Then $\langle 000, 001, 011, 010, 100, 101, 111, 110, 000\rangle$ is a desired one. Suppose that $e = (000, 001)$. Then $\langle 000, 110, 111, 101, 001, 011, 010, 100, 000\rangle$ is a cycle of length 8. If $e = (000, 100)$, this case is symmetric to the case $e = (000, 110)$. □

A *matching M* of a graph G is a set of pairwise disjoint edges. M is a *perfect matching* if each vertex of G belongs to some edge in M.

Lemma 2 [18]. *For $n \geqslant 1$, both of the subgraphs induced by $V(TQ_n^{00}) \bigcup V(TQ_n^{10})$ and $V(TQ_n^{01}) \bigcup V(TQ_n^{11})$ are isomorphic to $TQ_n \times K_2$. Furthermore, the edges joining $V(TQ_n^{00}) \bigcup V(TQ_n^{10})$ and $V(TQ_n^{01}) \bigcup V(TQ_n^{11})$ is a perfect matching of TQ_{n+2}.*

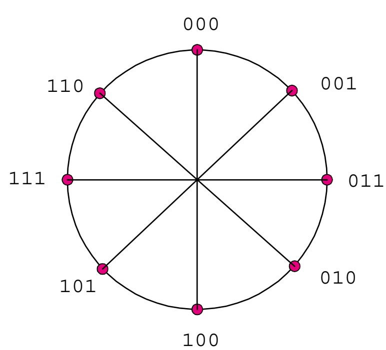


Fig. 2. Another layout of TQ_3.

Let G and H be two graphs having the same number of vertices. $G \oplus_M H$ denotes a graph which has copies of G and H connected by a matching M. Let G^0_{n+1} and G^1_{n+1} be the subgraphs induced by $V(TQ^{00}_n) \bigcup V(TQ^{10}_n)$ and $V(TQ^{01}_n) \bigcup V(TQ^{11}_n)$, respectively. Then by Lemma 2, both of G^0_{n+1} and G^1_{n+1} are isomorphic to $TQ_n \times K_2$, and $G^0_{n+1} \oplus_M G^1_{n+1}$ is isomorphic to TQ_{n+2} for a specific matching M. In addition, $TQ_n \times K_2$ has two copies of TQ_n, and we use TQ^0_n and TQ^1_n to denote them, respectively. For convenience of discussion, we add 0 to every vertex $v \in V(TQ^0_n)$ and 1 to every vertex $u \in V(TQ^1_n)$, respectively, as the leading bits. As a result, each vertex of $TQ_n \times K_2$ is represented by a binary string of length $n + 1$.

Let F be a set of faults in $TQ_n \times K_2$ (TQ_{n+2}, respectively). We say that a vertex u in TQ^0_n (G^0_{n+1}, respectively) is a *safe crossing-point* in $TQ_n \times K_2 - F$ ($TQ_{n+2} - F$, respectively) if u still connects to the neighbor v in TQ^1_n (G^1_{n+1}, respectively) in $TQ_n \times K_2 - F$ ($TQ_{n+2} - F$, respectively), i.e., vertices u, v and edge (u, v) are fault-free. If u is in TQ^1_n (G^1_{n+1}, respectively), we may define safe crossing-point in the same way.

Huang et al. [18] proved the following theorem concerning fault hamiltonicity and fault hamiltonian connectivity of TQ_n, and we shall use it in the proof of Theorem 3.

Theorem 2 [18]. *TQ_n is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected for any odd integer $n \geqslant 3$.*

Theorem 3. *Let $n \geqslant 3$ be an odd integer. If TQ_n is $(n-2)$-pancyclic, $TQ_n \times K_2$ is $(n-1)$-pancyclic.*

Proof. Suppose that TQ_n is $(n-2)$-pancyclic for some $n \geqslant 3$. We will show that $TQ_n \times K_2$ is $(n-1)$-pancyclic. Let $F \subseteq V(TQ_n \times K_2) \cup E(TQ_n \times K_2)$ be a set of faults. We divide F into five disjoint parts: $F^0_v = F \cap V(TQ^0_n)$, $F^0_e = F \cap E(TQ^0_n)$, $F^1_v = F \cap V(TQ^1_n)$, $F^1_e = F \cap E(TQ^1_n)$, and $F^c_e = F \cap \{(u, v) | (u, v)$ is an edge between TQ^0_n and $TQ^1_n\}$. Let $f = |F|$, $f^0_v = |F^0_v|$, $f^0_e = |F^0_e|$, $f^1_v = |F^1_v|$, $f^1_e = |F^1_e|$, and $f^c_e = |F^c_e|$. For convenience of discussion, we define the following subsets of F: $F_v = F \cap V(TQ_n \times K_2)$, $F_e = F \cap E(TQ_n \times K_2)$, $F^0 = F^0_v \cup F^0_e$, and $F^1 = F^1_v \cup F^1_e$. And let $f_v = |F_v|$, $f_e = |F_e|$, $f^0 = |F^0|$, and $f^1 = |F^1|$. Note that $f^0 + f^1 = f - f^c_e$.

For further discussion, we consider the following cases.

Case 1. There is a subcube containing all the $n-1$ faults.

Without loss of generality, we assume that TQ^0_n contains all the faults, i.e., $f^0 = n - 1$. Thus, $f^1 = f^c_e = 0$. We discuss the existence of cycles of all lengths from 4 to $2^{n+1} - f_v$ according to the following cases.

Case 1.1. Cycles of lengths from 4 to 2^n.

Since TQ_n is $(n-2)$-pancyclic , TQ_n^1 contains cycles of lengths from 4 to 2^n for $n \geqslant 3$. Thus, $TQ_n \times K_2 - F$ also contains cycles of these lengths.

Case 1.2. Cycles of lengths from $2^n + 2$ to $2^{n+1} - f_v$ (see Fig. 3(a)).

TQ_n^0 is $(n-2)$-pancyclic, and hence $(n-2)$-hamiltonian. Clearly, $TQ_n^0 - F^0$ still contains a hamiltonian path, say, $P = \langle u_1, u_2, \ldots, u_{2^n - f_v^0}\rangle$, where $f_v^0 = f_v$. Let $2 \leqslant l \leqslant 2^n - f_v$. We construct a cycle of length $2^n + l$ as follows: Suppose that v_1 and v_l are the neighbors in TQ_n^1 of u_1 and u_l, respectively. By Theorem 2, TQ_n is $(n-3)$-hamiltonian connected and $n \geqslant 3$. Therefore, there is a hamiltonian path Q in TQ_n^1 between v_1 and v_l containing 2^n vertices, and $\langle u_1, \ldots, u_l, v_l, Q, v_1, u_1\rangle$ forms a cycle of length $2^n + l$. Note that there are no faults outside TQ_n^0. Thus, all the vertices on P are safe crossing-points.

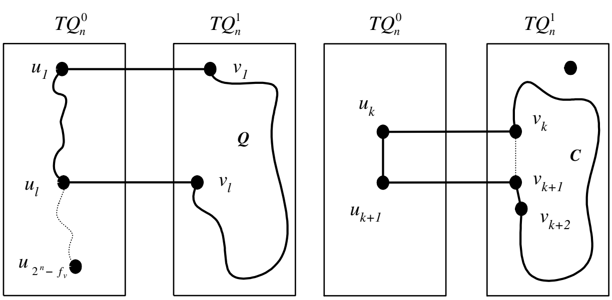
Case 1.3. A cycle of length $2^n + 1$ (see Fig. 3(b)).

Since TQ_n^1 is $(n-2)$-pancyclic and fault-free, we have a cycle $C = \langle v_1, v_2, \ldots, v_{2^n-1}, v_1\rangle$ of length $2^n - 1$ in TQ_n^1. There are $n-1$ faults in total, and $\frac{2^n-1}{2} > n-1$ for $n \geqslant 3$. So there exist two safe crossing-points v_k and v_{k+1} on C, and also their neighbors in TQ_n^0, say, u_k and u_{k+1}, respectively are connected in $TQ_n^0 - F^0$. $\langle v_{k+1}, v_{k+2}, \ldots, v_{2^n-1}, \ldots, v_k, u_k, u_{k+1}, v_{k+1}\rangle$ is a fault-free cycle of length $2^n + 1$.

Case 2. Both f^0 and f^1 are at most $n-2$.

Since $f^i \leqslant n-2$ for any $i \in \{0, 1\}$, $TQ_n^0 - F^0$ and $TQ_n^1 - F^1$ are still pancyclic. Without loss of generality, we assume that $f^0 \geqslant f^1$. We discuss the existence of cycles of all lengths from 4 to $2^{n+1} - f_v$ in the following cases.

Case 2.1. Cycles of lengths from 4 to $2^n - f_v^1$.



(a) case 1.2 **(b)** case 1.3

Fig. 3. Cases 1.2 and 1.3 of Theorem 3.

Since TQ_n^1 is $(n-2)$-pancyclic, we have cycles of lengths from 4 to $2^n - f_v^1$ in $TQ_n^1 - F^1$. Hence, $TQ_n \times K_2 - F$ also has cycles of these lengths.

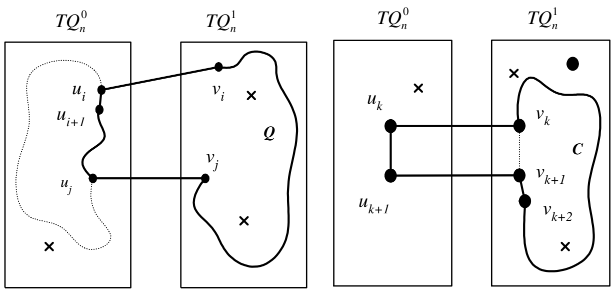
Case 2.2. Cycles of lengths from $2^n - f_v^1 + 2$ to $2^{n+1} - f_v$ (see Fig. 4(a)).

For the case $f^0 = f^1 = n-2$, we leave it to Appendix A because of its tediousness. For $f^1 \leqslant n-3$, the proof is as follows: $TQ_n^0 - F^0$ is pancyclic, and hence hamiltonian. We have a hamiltonian cycle $C = \langle u_1, u_2, \ldots, u_{2^n - f_v^0}, u_1 \rangle$ of length $2^n - f_v^0$ in $TQ_n^0 - F^0$. Let $2 \leqslant l \leqslant 2^n - f_v^0$. We construct a cycle of length $2^n - f_v^1 + l$ as follows: We claim that there exist two safe crossing-points u_i and u_j on C such that $(j-i) = l - 1(\mathbf{mod}\ 2^n - f_v^0)$. Suppose on the contrary that there do not exist such u_i and u_j. Then there are at least $\lceil \frac{2^n - f_v^0}{2} \rceil$ faults outside TQ_n^0. However, $\lceil \frac{2^n - f_v^0}{2} \rceil + f_v^0 \geqslant 2^{n-1} > n-1$ for $n \geqslant 1$. We obtain a contradiction. Thus, there exist such u_i and u_j. By Theorem 2, TQ_n^1 is $(n-3)$-hamiltonian connected and $f^1 \leqslant n-3$, so $TQ_n^1 - F^1$ is still hamiltonian connected. Let v_i and v_j be the neighbors in TQ_n^1 of u_i and u_j, respectively. There is a hamiltonian path Q in $TQ_n^1 - F^1$ between v_i and v_j. Clearly, Q contains $(2^n - f_v^1)$ vertices. Then $\langle u_i, u_{i+1}, \cdots, u_j, v_j, Q, v_i, u_i \rangle$ forms a cycle of length $2^n - f_v^1 + l$.

Case 2.3. A cycle of length $2^n - f_v^1 + 1$ (see Fig. 4(b)).

Since $TQ_n^1 - F^1$ is pancyclic, there is a cycle $\langle v_1, v_2, \ldots, v_{2^n - f_v^1 - 1}, v_1 \rangle$ of length $2^n - f_v^1 - 1$ in $TQ_n^1 - F^1$. Furthermore, there are $(n-1) - f^1$ faults outside TQ_n^1, and $\frac{2^n - f_v^1 - 1}{2} \geqslant \frac{2^n - 1}{2} - f^1 > (n-1) - f^1$ for $n \geqslant 3$. Thus there exist two safe crossing-points v_k and v_{k+1} on C, and also their neighbors in TQ_n^0, say, u_k and u_{k+1}, respectively are adjacent in $TQ_n^0 - F^0$. $\langle v_{k+1}, v_{k+2}, \ldots, v_{2^n - f_v^1 - 1}, \ldots, v_k, u_k, u_{k+1}, v_{k+1} \rangle$ is a fault-free cycle of length $2^n - f_v^1 + 1$ in $TQ_n \times K_2 - F$. This completes the proof of the theorem. □

For the following discussion, we recall that G_{n+1}^0 and G_{n+1}^1 are the subgraphs induced by $V(TQ_n^{00}) \bigcup V(TQ_n^{10})$ and $V(TQ_n^{01}) \bigcup V(TQ_n^{11})$, respectively. We say



(a) case 2.2 **(b)** case 2.3

Fig. 4. Cases 2.2 and 2.3 of Theorem 3.

that an edge is a *critical edge* of TQ_{n+2} if it is an edge in G^i_{n+1} with one **endpoint** in TQ^{i0}_n and the other in TQ^{i1}_n for $i \in \{0,1\}$.

Lemma 3. *Let $n \geqslant 3$ be an odd integer, and (u_1,u_2) be a critical edge of TQ_{n+2} which is in G^0_{n+1}, and v_1,v_2 be the neighbors in G^1_{n+1} of u_1 and u_2, respectively. Then (v_1,v_2) is also a critical edge of TQ_{n+2} in G^1_{n+1}.*

Proof. Without loss of generality, we assume that $u_1 = 00x_{n-3}x_{n-4}\ldots x_1x_0$. If $P_{n-3}(u_1) = 0$, $u_2 = 10x_{n-3}x_{n-4}\ldots x_1x_0$, $v_1 = 11x_{n-3}x_{n-4}\ldots x_1x_0$, and $v_2 = 01x_{n-3}x_{n-4}\ldots x_1x_0$. By definition, v_1 and v_2 are adjacent, and (v_1,v_2) is a critical edge in G^1_{n+1}. It can be checked that the statement is also true if $P_{n-3}(u_1) = 1$. □

It is observed that vertices u_1,u_2,v_1,v_2 in the above lemma form a 4-cycle. We call this cycle a *crossed* 4-*cycle* in TQ_{n+2}. It is clear that, for each vertex $00x_{n-3}\cdots x_0$, there is exactly one crossed 4-cycle corresponding to this vertex. Thus, there are 2^n disjoint crossed 4-cycles in TQ_{n+2}. We note that a crossed 4-cycle contains two critical edges.

Huang et al. [18] proved the following theorem.

Theorem 4 [18]. *$TQ_n \times K_2$ is $(n-1)$-hamiltonian and $(n-2)$-hamiltonian connected for any odd integer $n \geqslant 3$.*

Theorem 5. *Let $n \geqslant 3$ be an odd integer. If TQ_n is $(n-2)$-pancyclic, TQ_{n+2} is n-pancyclic.*

Proof. Suppose that TQ_n is $(n-2)$-pancyclic for some $n \geqslant 3$. By Theorem 3, $TQ_n \times K_2$ is $(n-1)$-pancyclic. That is, both G^0_{n+1} and G^1_{n+1} in TQ_{n+2} are $(n-1)$-pancyclic. We will show that TQ_{n+2} is n-pancyclic. Let $F \subseteq V(TQ_{n+2}) \cup E(TQ_{n+2})$ be a set of faults. We divide F into five disjoint parts: $F^0_v = F \cap V(G^0_{n+1})$, $F^0_e = F \cap E(G^0_{n+1})$, $F^1_v = F \cap V(G^1_{n+1})$, $F^1_e = F \cap E(G^1_{n+1})$, and $F^c_e = F \cap \{(u,v) \mid (u,v)$ is an edge between G^0_{n+1} and $G^1_{n+1}\}$. Let $f = |F|$, $f^0_v = |F^0_v|$, $f^0_e = |F^0_e|$, $f^1_v = |F^1_v|$, $f^1_e = |F^1_e|$, and $f^c_e = |F^c_e|$. For convenience of discussion, we define the following subsets of F: $F_v = F \cap V(TQ_{n+2})$, $F_e = F \cap E(TQ_{n+2})$, $F^0 = F^0_v \cup F^0_e$, and $F^1 = F^1_v \cup F^1_e$. And let $f_v = |F_v|$, $f_e = |F_e|$, $f^0 = |F^0|$, and $f^1 = |F^1|$. Note that $f^0 + f^1 = f - f^c_e$.

For further discussion, we consider the following cases.

Case 1. There is a subcube containing all the n faults.

Without loss of generality, we assume that $f^0 = n$. Thus, $f^1 = f^c_e = 0$. G^0_{n+1} is $(n-1)$-pancyclic, and hence $(n-1)$-hamiltonian. Clearly, $G^0_{n+1} - F^0$ still contains a hamiltonian path, say, $P = \langle u_1, u_2, \ldots, u_{2^{n+1}-f_v^0} \rangle$, where $f_v^0 = f_v$. We discuss the existence of cycles of all lengths from 4 to $2^{n+2} - f_v$ according to the following cases.

Case 1.1. Cycles of lengths from 4 to 2^{n+1}.

Since G_{n+1}^1 is $(n-1)$-pancyclic , G_{n+1}^1 contains cycles of lengths from 4 to 2^{n+1} for $n \geqslant 3$. So, $TQ_{n+2} - F$ also contains cycles of these lengths.

Case 1.2. Cycles of lengths from $2^{n+1}+2$ to $2^{n+2} - f_v$ (see Fig. 5(a)).

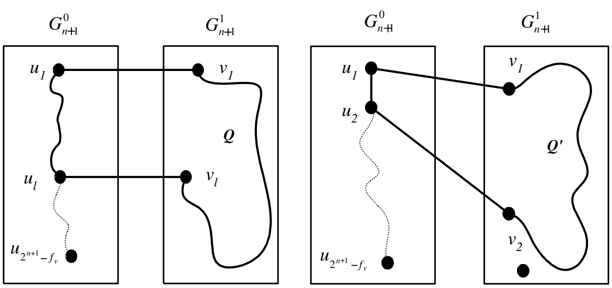
Let $2 \leqslant l \leqslant 2^{n+1} - f_v$. We construct a cycle of length $2^{n+1}+l$ as follows: Suppose that the neighbors in G_{n+1}^1 of u_1 and u_l are v_1 and v_l, respectively. By Theorem 4, G_{n+1}^1 is $(n-2)$-hamiltonian connected and $n \geqslant 3$. Hence there is a hamiltonian path Q in G_{n+1}^1 between v_1 and v_l containing 2^{n+1} vertices. $\langle u_1, \ldots, u_l, v_l, Q, v_1, u_1 \rangle$ forms a cycle of length $2^{n+1}+l$. Note that there are no faults outside G_{n+1}^0. So all the vertices on P are safe crossing-points.

Case 1.3. A cycle of length $2^{n+1}+1$ (see Fig. 5(b)).

Consider the vertices u_1 and u_2 on P and their neighbors in G_{n+1}^1, say v_1 and v_2, respectively. By Theorem 4, G_{n+1}^1 is $(n-2)$-hamiltonian connected for $n \geqslant 3$. Since $f^1 = 0$, we may find a path Q' between v_1 and v_2 containing $2^{n+1} - 1$ vertices in G_{n+1}^1. Then $\langle u_1, u_2, v_2, Q', v_1, u_1 \rangle$ forms a cycle of length $2^{n+1}+1$.

Case 2. Both f^0 and f^1 are at most $n-1$.

Since both of G_{n+1}^0 and G_{n+1}^1 are $(n-1)$-pancyclic for $n \geqslant 3$, both of $G_{n+1}^0 - F^0$ and $G_{n+1}^1 - F^1$ are still pancyclic. Without loss of generality, we assume that $f^0 \geqslant f^1$. We discuss the existence of cycles of all lengths from 4 to $2^n - f_v$ in the following cases.



(a) case 1.2 **(b)** case 1.3

Fig. 5. Cases 1.2 and 1.3 of Theorem 5.

Case 2.1. Cycles of lengths from 4 to $2^{n+1} - f_v^1$.

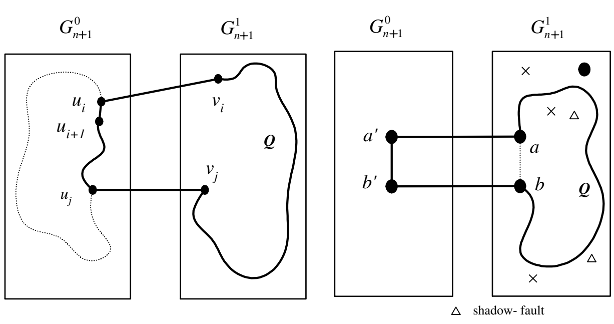
Since $G_{n+1}^1 - F^1$ is pancyclic for $n \geqslant 3$, we have cycles of lengths from 4 to $2^{n+1} - f_v^1$ in $G_{n+1}^1 - F^1$.

Case 2.2. Cycles of lengths from $2^{n+1} - f_v^1 + 2$ to $2^{n+2} - f_v$ (see Fig. 6(a)).

Since $G_{n+1}^0 - F^0$ is pancyclic, we have a hamiltonian cycle $C = \langle u_0, u_1, \ldots, u_{2^{n+1}-f_v^0-1}, u_0\rangle$ of length $2^{n+1} - f_v^0$ in $G_{n+1}^0 - F^0$. Let $2 \leqslant l \leqslant 2^{n+1} - f_v^0$. We construct a cycle of length $2^{n+1} - f_v^1 + l$ as follows: First, we claim that there exist two safe crossing-points u_i and u_j on C such that $(j - i) = l - 1(\textbf{mod}\ 2^{n+1} - f_v^0)$. Suppose on the contrary that there do not exist such u_i and u_j. Then there are at least $\lceil \frac{2^{n+1}-f_v^0}{2} \rceil$ faults outside G_{n+1}^0. However, $\lceil \frac{2^{n+1}-f_v^0}{2} \rceil + f_v^0 \geqslant 2^n > n$ for $n \geqslant 0$. We obtain a contradiction. Thus, there exist such u_i and u_j, and our claim is true. Secondly, we claim that $f^1 \leqslant n - 2$ for $n \geqslant 3$. Suppose for the sake of contradiction that $f^1 = n - 1$. Since $f^0 \geqslant f^1$, $f^0 = n - 1$. The total number of faults is at most n. Thus $(n - 1)+(n - 1) \leqslant n$. This implies that $n \leqslant 2$, which is a contradiction. This completes the proof of our second claim. By Theorem 4, G_{n+1}^1 is $(n - 2)$-hamiltonian connected and $f^1 \leqslant n - 2$ for $n \geqslant 3$. Hence $G_{n+1}^1 - F^1$ is hamiltonian connected. Let v_i and v_j be the neighbors in G_{n+1}^1 of u_i and u_j, respectively. There is a hamiltonian path Q in $G_{n+1}^1 - F^1$ between v_i and v_j. Clearly, Q contains $(2^{n+1} - f_v^1)$ vertices. Then $\langle u_i, u_{i+1}, \ldots, u_j, v_j, Q, v_i, u_i\rangle$ forms a cycle of length $2^{n+1} - f_v^1 + l$.

Case 2.3. A cycle of length $2^{n+1} - f_v^1 + 1$ (see Fig. 6(b)).

We want to construct a cycle containing $2^{n+1} - f_v^1 - 1$ vertices in $G_{n+1}^1 - F^1$ and two vertices in $G_{n+1}^0 - F^0$. To avoid faults in G_{n+1}^0, we introduce a term called shadows of faults. Let $\langle u_1, u_2, v_2, v_1, u_1\rangle$ be a crossed 4-cycle with u_1, u_2 in



(a) case 2.2 **(b)** case 2.3

Fig. 6. Cases 2.2 and 2.3 of Theorem 5.

G^0_{n+1} and v_1, v_2 in G^1_{n+1}, respectively. If there is a fault on this cycle but the fault is not in G^1_{n+1}, we call edge (v_1, v_2) a *shadow fault* of F on G^1_{n+1} (similarly, we may define a shadow fault on G^0_{n+1}). Let $F^s = \{e|$ edge e is a shadow fault of F on $G^1_{n+1}\}$. Then $|F^s \cup F^1| \leqslant n$. If $|F^s \cup F^1| = n$, we arbitrarily choose an edge e_1 in F^s, and let $F' = F^s \cup F^1 - e_1$, otherwise let $F' = F^s \cup F^1$. Then $|F'| \leqslant n-1$ and $G^1_{n+1} - F'$ is still pancyclic. Since $F' \cap V(G^1_{n+1}) = F^1_v$, there is a cycle C of length $2^{n+1} - f^1_v - 1$ in $G^1_{n+1} - F'$. Since $2^{n+1} - f^1_v - 1 > 2^n$ for $n \geqslant 3$, C contains two critical edges. Let $(a,b) \neq e_1$ be a critical edge on C, so $(a,b) \notin F^s$. Let a', b' be the neighbors of a and b in G^0_{n+1}, respectively. Then $\langle a, a', b', b, a\rangle$ is a fault-free crossed 4-cycle. Suppose that $C = \langle a, Q, b, a\rangle$. Then $\langle a', a, Q, b, b', a'\rangle$ forms a cycle of length $2^{n+1} - f^1_v + 1$ in $TQ_{n+2} - F$. $\square$

By Theorems 1, 5 and using the mathematical induction, we obtain the following theorem.

Theorem 6. *The twisted cube TQ_n is $(n-2)$-pancyclic for any odd integer $n \geqslant 3$.*

4. Conclusion

The twisted cube, proposed by Hilbers et al. [13], is an alternative to the hypercube architecture in parallel computing. We study a property called fault-tolerant pancyclicity on the twisted cube. We prove that TQ_n is $(n-2)$-pancyclic for any odd integer $n \geqslant 3$. That is, with maximum of $n-2$ faulty edges and/or vertices, TQ_n has cycles of all lengths from 4 to $|V(TQ_n)| - f_v$. Furthermore, if there exist $n-1$ faulty elements around a single vertex, then TQ_n cannot have a hamiltonian cycle. Hence, $n-2$ faults are the most that TQ_n can tolerant with respect to pancyclic property. The above result shows that the fault-tolerant capability of TQ_n is nice in terms of the cycle embeddings.

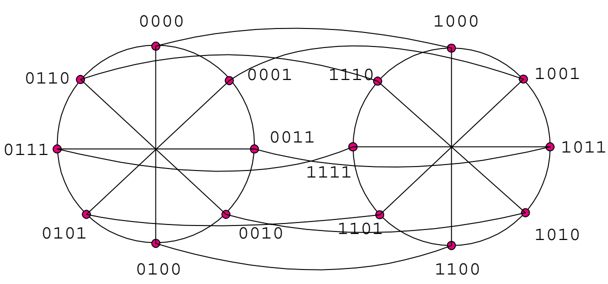


Fig. 7. $TQ_3 \times K_2$.

Appendix A

In the following, we construct cycles of lengths from $2^n - f_v^1 + 2$ to $2^{n+1} - f_v$ in $TQ_n \times K_2$ for the case $f^0 = f^1 = n - 2$. Since $f^0 + f^1 = 2n - 4 \leqslant n - 1$, $n \leqslant 3$. Thus, we need only to discuss the case $f^0 = f^1 = 1$ for $n = 3$ here.

First, we show the case $f_v^0 = f_v^1 = 1$, and thus find cycles of lengths from 9 to 14 in $TQ_3 \times K_2 - F$. Let $F = \{u, v\}$ for $u \in V(TQ_3^0)$ and $v \in V(TQ_3^1)$. We need

Table 1
Fault-free cycles of lengths from 9 to 14 in $TQ_3 \times K_2$ with two faulty vertices

Length	$u = 0000$, $v = 1000$
9	$\langle 0001, 0011, 0010, 0100, 1100, 1101, 1111, 1011, 1001, 0001 \rangle$
10	$\langle 0001, 0011, 0010, 0100, 0101, 1101, 1100, 1010, 1011, 1001, 0001 \rangle$
11	$\langle 0001, 0011, 0010, 0100, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 0001 \rangle$
12	$\langle 0001, 0011, 0010, 0110, 0111, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 0001 \rangle$
13	$\langle 0001, 0011, 0010, 0110, 0111, 0101, 1101, 1100, 1010, 1110, 1111, 1011, 1001, 0001 \rangle$
14	$\langle 0001, 0011, 0010, 0100, 0101, 0111, 0110, 1110, 1111, 1101, 1100, 1010, 1011, 1001, 0001 \rangle$
Length	$u = 0000$, $v = 1110$
9	$\langle 0111, 0110, 0010, 0011, 1011, 1010, 1100, 1101, 1111, 0111 \rangle$
10	$\langle 0111, 0101, 0100, 0010, 0011, 1011, 1010, 1100, 1101, 1111, 0111 \rangle$
11	$\langle 0111, 0101, 0100, 0010, 0011, 1011, 1001, 1000, 1100, 1101, 1111, 0111 \rangle$
12	$\langle 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1011, 1010, 1100, 1101, 1111, 0111 \rangle$
13	$\langle 0101, 0111, 0110, 0010, 0011, 0001, 1001, 1000, 1100, 1010, 1011, 1111, 1101, 0101 \rangle$
14	$\langle 0111, 0110, 0010, 0100, 0101, 0001, 0011, 1011, 1010, 1100, 1000, 1001, 1101, 1111, 0111 \rangle$
Length	$u = 0000$, $v = 1111$
9	$\langle 0110, 0111, 0101, 0100, 0010, 1010, 1100, 1000, 1110, 0110 \rangle$
10	$\langle 0110, 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1000, 1110, 0110 \rangle$
11	$\langle 0110, 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1011, 1010, 1110, 0110 \rangle$
12	$\langle 0110, 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1101, 1100, 1010, 1110, 0110 \rangle$
13	$\langle 0110, 0111, 0101, 0100, 0010, 0011, 0001, 1001, 1011, 1010, 1100, 1000, 1110, 0110 \rangle$
14	$\langle 0010, 0100, 0101, 0001, 0011, 0111, 0110, 1110, 1000, 1100, 1101, 1001, 1011, 1010, 0010 \rangle$
Length	$u = 0000$, $v = 1101$
9	$\langle 0001, 0011, 0010, 0100, 0101, 1101, 1111, 1011, 1001, 0001 \rangle$
10	$\langle 0001, 0011, 0010, 0110, 0111, 1111, 1110, 1010, 1011, 1001, 0001 \rangle$
11	$\langle 0001, 0011, 0010, 0110, 0111, 1111, 1011, 1010, 1100, 1000, 1001, 0001 \rangle$
12	$\langle 0001, 0011, 0010, 0110, 0111, 1111, 1110, 1000, 1100, 1010, 1011, 1001, 0001 \rangle$
13	$\langle 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1000, 1110, 1111, 1011, 1001, 0001 \rangle$
14	$\langle 0010, 0100, 0101, 0001, 0011, 0111, 0110, 1110, 1111, 1011, 1001, 1000, 1100, 1010, 0010 \rangle$
Length	$u = 0000$, $v = 1100$
9	$\langle 0101, 0001, 0011, 0111, 1111, 1110, 1000, 1001, 1101, 0101 \rangle$
10	$\langle 0001, 0011, 0111, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 0001 \rangle$
11	$\langle 0001, 0011, 0010, 0110, 0111, 0101, 1101, 1111, 1110, 1000, 1001, 0001 \rangle$
12	$\langle 0001, 0011, 0010, 0100, 0101, 1101, 1111, 1011, 1010, 1110, 1000, 1001, 0001 \rangle$
13	$\langle 0110, 0010, 0100, 0101, 0001, 0011, 0111, 1111, 1101, 1001, 1011, 1010, 1110, 0110 \rangle$
14	$\langle 0011, 0001, 0101, 0100, 0010, 0110, 0111, 1111, 1101, 1001, 1000, 1110, 1010, 1011, 0011 \rangle$

only discuss five cases due to the symmetry of $TQ_3 \times K_2$ (see Fig. 7): (1) $u = 0000$, $v = 1000$, (2) $u = 0000$, $v = 1110$, (3) $u = 0000$, $v = 1111$, (4) $u = 0000$, $v = 1101$, and (5) $u = 0000$, $v = 1100$. They are listed one by one in Table 1.

Second, consider that $f_v = 1$ and $f_e = 1$. We find cycles of lengths from 8 to 14 as follows. Let $F = \{u_1,(u_2,v_2)\}$ and $F' = \{u_1, u_2\}$. From the above discussion, there are cycles of lengths from 9 to 14 in $TQ_3 \times K_2 - F'$, which are also in $TQ_3 \times K_2 - F$. Furthermore, since $TQ_3 \times K_2$ is 2-hamiltonian, there is a cycle of length 15 in $TQ_3 \times K_2 - F$.

Finally, in the same way, we can deal with the case $f_e = 2$. In this case, cycles of lengths from 10 to 16 have to be found. Assume that $F = \{(u_1, v_1), (u_2, v_2)\}$. Then let $F' = \{u_1, (u_2, v_2)\}$. From the above discussion, there are cycles of lengths from 9 to 15 in $TQ_3 \times K_2 - F'$, which are also in $TQ_3 \times K_2 - F$. In addition, since $TQ_3 \times K_2$ is 2-hamiltonian, there is a cycle of length 16. This completes our proof. □

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