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Scattering Amplitudes for Multi-indexed Extensions of Soliton Potential and Extended KdV Integer Solitons

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Abstract. We calculate quantum mechanical scattering problems for multi-indexed extensions of soliton potential by Darboux transformations in terms of pseudo virtual wavefunctions. As an application, we calculate infinite set of higher integer KdV solitons by the inverse scattering transform method of KdV equation.

1. Introduction

The discovery of the exceptional orthogonal polynomials [1, 2, 3, 4, 5], and their applications to exactly solvable 1D quantum mechanical systems [3, 4, 5] have recently attracted many substantial researches in mathematical physics. In this talk, instead of the bound state problems, we will report on new development of 1D exactly solvable quantum mechanical scattering problems. In particular, we will calculate the scattering problems of the deformed soliton potential and apply the results to the KdV solitons.

Our calculation will be based on the multiple Darboux-Crum transformations [6, 7, 8]. Such transformations can generate new solvable quantum systems from the known ones by using certain polynomial type seed solutions. These seed functions are called the virtual and pseudo virtual state wavefunctions [5, 9, 10]. One way to obtain these seed solutions is to perform discrete symmetry operations on the eigenfunctions. The one-indexed [11] and more complete multi-indexed extensions [12] of the known quantum scattering problems [13] were recently calculated.

For the case of the deformed soliton potentials, one interesting application is to use the results of the scattering data to generate higher integer KdV solitons from the lower integer ones. Although the KdV solitons generated by this method are not new soliton solutions, we believe that the method we used is simple and effective, and is closely related to the recent developed multi-indexed extention of solvable quantum mechanical potentials. In this calculation, we will obtain an infinite number of reflectionless potentials, which can be served as the initial profiles of integer KdV solitons. We then use the scattering data to solve the Gel'fand-Levitan-Marchenko (GLM) equation [14, 15] in the inverse scattering transform (IST) method [16, 17], and obtain higher integer KdV solitons.

2. The Deformed Soliton Potential

We begin with a specific example of new solvable potential, namely, the deformed soliton potential under Darboux-Crum transformation. We will calculate both its bound state problem and scattering problem, and relate these two calculations. The bound state problem and scattering problem of the original soliton potential

$$U(x;\lambda) = -\frac{h(h+1)}{\cosh^2 x}, \lambda = h, h > 0, -\infty < x < \infty \tag{2.1}$$

were calculated in [18]. This potential contains finitely many bound states

$$\begin{aligned}\phi_n(x) &= \frac{1}{(\cosh x)^{h-n}} P_n^{(h-n,h-n)}(\tanh x) \\ &\sim \frac{1}{(\cosh x)^{h-n}}\ {}_2F_1\left(-n, 2h-n+1, h-n+1, \frac{1-\tanh x}{2}\right), \\ E_n &= -k^2 = -(h-n)^2; n = 0,1,2...[h]'\end{aligned} \tag{2.2}$$

where $P_n^{(h-n,h-n)}$ is the Jacobi polynomial and ${}_2F_1$ is the hypergeometric function. $[h]'$ denotes the greatest integer not exceeding and not equal to h. The soliton potential contains a discrete symmetry

$$h \to -(h+1), \tag{2.3}$$

which can be used to construct the seed function

$$\varphi_v(x) = (\cosh x)^{h+1+v} P_v^{(-h-1-v,-h-1-v)}(\tanh x), v = 0,1,2,3,4,.. \tag{2.4}$$

with energy $E_v = -(h+v+1)^2$. It turns out that for $v = 1,3,5....$, the deformed potential contains pole at $x=0$. For example, for $v=1$,

$$U_1 = U - 2\frac{d^2}{dx^2}\log\varphi_1(x) = U - \frac{2(h+1)}{\cosh^2 x} + \frac{2}{\sinh^2 x} \tag{2.5}$$

which contains pole at $x=0$. We note that although one can define the asymptotic forms of the scattering state for this potential, the corresponding bound state wavefunctions contain singularities. So for our purpose here, only $v = 2,4,6...$can be used to deform the soliton potential. For simplicity, we will use the seed function for $v=2$

$$\varphi_2(x) = \frac{h+1}{4}(\cosh x)^{h+3}[1+(2h+3)\tanh^2 x] \tag{2.6}$$

to illustrate the calculation. The deformed potential is

$$\begin{aligned}U_2 = U - 2\frac{d^2}{dx^2}\log\varphi_2(x) &= U - \frac{4(2h+3)(1-2\sinh^2 x)}{[1+(2h+3)\tanh^2 x](\cosh x)^4} \\ &- 8\left(\frac{(2h+3)\tanh x}{[1+(2h+3)\tanh^2 x](\cosh x)^2}\right)^2,\end{aligned} \tag{2.7}$$

which has no pole for the whole regime of x and approaches 0 asymptotically for $x \to \pm\infty$ as U does. Note that $U_2(x=0) - U(x=0) = -4(2h+3) < 0$, which suggests the existence of a lowest new bound state for the deformed potential U_2.

The bound state wavefunctions of the deformed potential Eq.(2.7) can be calculated through the Darboux-Crum transformation to be

$$\begin{aligned}\psi_b^{(1)}(x) &= \phi_n' - \frac{\varphi_2'}{\varphi_2}\phi_n \\ &= -\left(\frac{(2h-n+3)\tanh x}{(\cosh x)^{h-n}} + \frac{2(2h+3)\tanh x}{[1+(2h+3)\tanh^2 x](\cosh x)^{h-n-2}}\right) \\ &\cdot\ {}_2F_1\left(-n, 2h-n+1, h-n+1, \frac{1-\tanh x}{2}\right) \\ &+ \frac{n(2h-n+1)}{2(h-n-1)(\cosh x)^{h-n-2}} \\ &\cdot\ {}_2F_1\left(-n+1, 2h-n+2, h-n+2, \frac{1-\tanh x}{2}\right)\end{aligned} \tag{2.8}$$

with energy

$$E_n = -k^2 = -(h-n)^2; n = 0, 1, 2...[h]'. \tag{2.9}$$

In calculating Eq.(2.8), we have used the identity

$$\frac{d}{dz}\ {}_2F_1\left(a, b, c, z\right) = \ {}_2F_1\left(a+1, b+1, c+1, z\right). \tag{2.10}$$

It can be easily shown that there is another bound state

$$\frac{1}{\varphi_2} = \frac{4/(h+1)}{(\cosh x)^{h+3}[1+(2h+3)\tanh^2 x]} \tag{2.11}$$

with lowest energy

$$E = -(h+3)^2 \tag{2.12}$$

as was expected previously.

We next consider the scattering problem of the deformed potential. For this purpose, we introduce the wave vector K such that

$$E = K^2 > 0, K = ik. \tag{2.13}$$

In view of the bound state wavefunction in Eq.(2.8), the scattering state wavefunction for the undeformed potential is

$$\psi_+(x) = \frac{1}{(\cosh x)^{-iK}}\ {}_2F_1\left(-iK-h, -iK+h+1, -iK+1, \frac{1-\tanh x}{2}\right). \tag{2.14}$$

The asymptotic form of the scattering state for the deformed potential can be calculated to be

$$\psi_{+\infty}^{(1)}(x) = \psi_{+\infty}'(x) - \frac{\varphi_2'}{\varphi_2}\psi_{+\infty}'(x) = [iK-(h+3)]\exp iKx \tag{2.15}$$

as $x \to +\infty$. To calculate the asymptotic form of the scattering state as $x \to -\infty$, one needs to include the second solution of the Schrodinger equation. One way to achieve this is to use the identity [18]

$$\begin{aligned}{}_2F_1\left(a, b, c, z\right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1\left(a, b, a+b-c+1, 1-z\right) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} \\ &\cdot {}_2F_1\left(c-a, c-b, c-a-b+1, 1-z\right)\end{aligned} \tag{2.16}$$

in Eq.(2.14) to obtain

$$\begin{aligned}\psi_-(x) = &\frac{1}{(\cosh x)^{-iK}}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1\left(a,b,a+b-c+1,1-z\right)\\ &+\frac{1}{(\cosh x)^{-iK}}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\\ &\cdot {}_2F_1\left(c-a,c-b,c-a-b+1,1-z\right)\end{aligned} \tag{2.17}$$

where $a=-iK-h, b=-iK+h+1, c=-iK+1$ and $z=\frac{1-\tanh x}{2}$. One can now calculate the asymptotic form of Eq.(2.17)

$$\psi_{-\infty}(x) = \exp -iKx\ \frac{\Gamma(-iK+1)\Gamma(iK)}{\Gamma(1+h)\Gamma(-h)} + \exp iKx\ \frac{\Gamma(-iK+1)\Gamma(-iK)}{\Gamma(-iK-h)\Gamma(-iK+h+1)} \tag{2.18}$$

as $x\to-\infty$. The asymptotic form of the scattering state for the deformed potential can then be calculated to be

$$\begin{aligned}\psi^{(1)}_{-\infty}(x) = \psi'_{-\infty}(x) - \frac{\varphi_2'}{\varphi_2}\psi'_{-\infty}(x) = &[-iK+(h+3)]\exp -iKx\ \frac{\Gamma(-iK+1)\Gamma(iK)}{\Gamma(1+h)\Gamma(-h)}\\ &+[iK+(h+3)]\exp iKx\ \frac{\Gamma(-iK+1)\Gamma(-iK)}{\Gamma(-iK-h)\Gamma(-iK+h+1)}\end{aligned} \tag{2.19}$$

as $x\to-\infty$. Finally the transmission and reflection coefficients of the deformed soliton potential can be calculated to be

$$t_D(K) = \frac{iK-(h+3)}{iK+(h+3)}\cdot\frac{\Gamma(-iK-h)\Gamma(-iK+h+1)}{\Gamma(-iK+1)\Gamma(-iK)} = \frac{iK-(h+3)}{iK+(h+3)}\cdot t(K), \tag{2.20}$$

$$\begin{aligned}r_D(K) &= (-)\frac{K+i(h+n+1)}{K-i(h+n+1)}\cdot\frac{\Gamma(1+h-iK)\Gamma(-h-iK)\Gamma(ik)}{\Gamma(-h)\Gamma(1+h)\Gamma(-iK)}\\ &= (-)\frac{K+i(h+n+1)}{K-i(h+n+1)}\cdot r(K)\end{aligned} \tag{2.21}$$

where $t(K)$ and $r(K)$ are the transmission and reflection coefficients of the undeformed soliton potential. It is important to note that, in Eq.(2.20), in addition to the well known second factor for the undeformed soliton potential scattering, an additional ratio $\frac{iK-(h+3)}{iK+(h+3)}$ with a new pole $K=i(h+3)$ corresponding to the new bound state energy given by Eq.(2.12) is added to the transmission coefficient of the deformed potential. Moreover, the additional ratio is of module one so that the conservation of flux is preserved for the deformed potential. In general, It turns out that the M-step deformed transmission amplitude $t_D(K)$ and the reflection amplitude $r_D(K)$ are [12]

$$t_{\mathcal{D}}(K) = \prod_{j=1}^{M}\frac{K+i\Delta^+_{d_j}}{K+i\Delta^-_{d_j}}\cdot t(K),\quad r_{\mathcal{D}}(K) = (-1)^M\prod_{j=1}^{M}\frac{K-i\Delta^-_{d_j}}{K+i\Delta^-_{d_j}}\cdot r(K) \tag{2.22}$$

where $\Delta^+_{d_j}$ and $\Delta^-_{d_j}$ are the asymptotic exponents of seed solutions φ_{d_j}

$$\varphi_{d_j} \approx \begin{cases} e^{x\Delta^+_{d_j}} & x\to+\infty \\ e^{x\Delta^-_{d_j}} & x\to-\infty \end{cases}. \tag{2.23}$$

In view of the multiplicative form of $r_D(k)$ in Eq.(2.21), it is important to note that, for integer $h = 1, 2, 3....$, the scattering of the deformed potential remains reflectionless as the undeformed potential due to the factor $\Gamma(-h)$ in the denominator of $r(k)$. This key observation is important for generating higher integer soliton solutions in the calculation of the inverse transform method to be discussed in the next section.

3. Solvable Higher Integer KdV Solitons

The nontopological KdV solitons are described by the Korteweg-de Vries (KdV) equation, i.e.,

$$u_t - 6uu_x + u_{xxx} = 0 \tag{3.24}$$

in one space $x \in (-\infty, \infty)$ and one time $t > 0$ dimension. One of the important methods to solve nonlinear partial differential equation such as the KdV equation is the method of the inverse scattering transform(IST) [16, 17] invented in 1960's. According to the method of IST, the solution of the KdV equation is converted to the solution of two simpler linear equations, namely, the quantum mechanical Schrodinger equation and the Gel'fand-Levitan-Marchenko (GLM) equation [14, 15]. For KdV soliton solutions the related Schrödinger equation is connected with reflectionless potentials. For such reflectionless potentials, the reflection amplitudes vanish, and the corresponding GLM equation is easy to solve. For the general N-soliton solution, one gets $2N$ continuous parameters, N norming constants $c_n(0)$ and N energy parameters κ_n.

In the following [19], for simplicity and clarity of presentation, we present the result for 1-step deformation and take the seed function with $v = 2$, and parameter in the soliton potential $h = 1$

$$\varphi_2(x)_{h=1} = \frac{1}{2}\cosh^4 x(1 + 5\tanh^2 x) \tag{3.25}$$

to illustrate the calculation. The deformed potential is easily calculated to be

$$U_2(x)_{h=1} = U(x) - 2\frac{d^2}{dx^2}\log\varphi_2(x)_{h=1} = -\frac{30(4\cosh^4 x - 8\cosh^2 x + 5)}{\cosh^2 x(36\cosh^4 x - 60\cosh^2 x + 25)} \tag{3.26}$$

which has no pole and no zero for the whole regime of x and approaches 0 asymptotically for $x \to \pm\infty$ as $U(x)_{h=1}$ does. The bound state wavefunctions of the deformed potential Eq.(3.26) can be calculated through the Darboux-Crum transformation to be

$$\psi_0(x) = \phi_0' - \frac{\varphi_2'}{\varphi_2}\phi_0 = -5\,\mathrm{sech}x\tanh x\left(1 + \frac{2\,\mathrm{sech}^2 x}{(1 + 5\tanh^2 x)}\right) \tag{3.27}$$

with energy

$$E_0 = -\kappa_0^2 = -(h-n)^2 = -1. \tag{3.28}$$

It can be easily shown that there is another bound state of the deformed potential

$$\psi_1(x) \sim \frac{1}{\varphi_2} = \frac{2}{\cosh^4 x(1 + 5\tanh^2 x)} \tag{3.29}$$

with a lower energy

$$E_1 = -\kappa_1^2 = -(h+1+v)^2 = -4^2 \tag{3.30}$$

as was expected previously. The normalized wavefunctions and their asymptotic forms can be calculated to be

$$\psi_0(x) = \sqrt{\frac{15}{2}}\,\mathrm{sech}x\tanh x\left(1 + \frac{2\,\mathrm{sech}^2 x}{(1 + 5\tanh^2 x)}\right) \to \sqrt{\frac{10}{3}}e^{-x} \text{ as } x \to \infty, \tag{3.31}$$

$$\psi_1(x) = \sqrt{\frac{15}{8}} \frac{2}{\cosh^4 x(1 + 5\tanh^2 x)} \to \sqrt{\frac{40}{3}} e^{-x} \text{ as } x \to \infty. \tag{3.32}$$

The constants

$$c_0(0) = \sqrt{\frac{10}{3}}, \quad c_1(0) = \sqrt{\frac{40}{3}} \tag{3.33}$$

in equations Eq.(3.31) and Eq.(3.32) are called norming constants. The reflection amplitude of the scattering of the M-step ($M = 1$ for the present case) deformed soliton potential Eq.(3.26) was calculated in Eq.(2.22). In view of the multiplicative form of $r_D(k)$, it is important to note that, for integer $h = 1, 2, 3....$, the scattering of the deformed potential remains reflectionless as the undeformed potential due to the factor $\Gamma(-h)$ in the denominator of $r(k)$.

We can now use the scattering data $\{\kappa_n, c_n, r_D(k)\}$ to solve the KdV equation. For the reflectionless potential, $r_D(k) = 0$, the GLM equation is easy to solve, and the solution $u(x,t)$ is given by [17]

$$u(x,t) = -2\frac{d^2}{dx^2} \log(\det A), \tag{3.34}$$

where A is a $N \times N$ matrix ($N \equiv h + 1$) with elements A_{mn} given by

$$A_{mn} = \delta_{mn} + c_n^2(t) \frac{\exp -(\kappa_m + \kappa_n)x}{\kappa_m + \kappa_n}; \quad m, n = 0, 1, 2...., N - 1. \tag{3.35}$$

In Eq.(3.35) $c_n(t) = c_n(0) \exp(4\kappa_n^3 t)$ and is one of the Gardner-Greene-Kruskal-Miura (GGKM) equations [16].

For the present case, $N = h + 1 = 2$. The integer 2-soliton solution corresponding to $(\kappa_{0,}\kappa_1) = (1, 4)$ can be calculated to be [19]

$$u(x,t)_{(1,4)} = -\frac{120e^{8t+2x}(e^{1024t} + e^{16x} + 16e^{520t+6x} + 30e^{512t+8x} + 16e^{504t+10x})}{(3e^{520t} + 3e^{10x} + 5e^{512t+2x} + 5e^{8t+8x})^2}. \tag{3.36}$$

By taking $t = 0$ in Eq.(3.36), one reproduces the initial profile $u(x,0) = U_2(x)_{h=1}$ calculated in Eq.(3.26). The asymptotic form of the $(\kappa_{0,}\kappa_1) = (1, 4)$ solution is

$$u(x,t)_{(1,4)} \sim -2\sum_{n=0}^{N-1} \kappa_n^2 \sec h^2\{\kappa_n(x - 4\kappa_n^2 t) \pm \chi_n\}, t \to \pm\infty, \tag{3.37}$$

where

$$\exp(2\chi_n) = \prod_{\substack{m=0\\m\neq n}}^{N-1} \left|\frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m}\right|^{sgn(\kappa_n - \kappa_m)}. \tag{3.38}$$

Note that the previous integer 2-soliton solution corresponds to $(\kappa_{0,}\kappa_1) = (1, 2)$. It is interesting to see that the calculation of the $(1, 4)$ integer soliton is based on the recently developed multi-indexed extensions of the reflectionless soliton potential.

The calculation can be generalized to other higher integer solitons. The profile for $h = 2$, for example, is the extended solvable 3-soliton $(1, 2, 5)$ [19]

$$U_2(x)_{h=2} = u(x,0)_{(1,2,5)} = -\frac{4(144\cosh^4 x - 280\cosh^2 x + 147)}{\cosh^2 x(64\cosh^4 x - 112\cosh^2 x + 49)}, \tag{3.39}$$

and

$$\begin{aligned} u(x,t)_{(1,2,5)} &= -(16e^{8t+2x}(9e^{2128t}+9e^{28x}+1575e^{16(63t+x)}+882e^{16(66t+x)} \\ &+3252e^{14(76t+x)}+175e^{8(142t+x)}+49e^{8(250t+x)}+126e^{4(516t+x)} \\ &+56e^{2072t+2x}+126e^{2056t+6x}+1008e^{1128t+10x}+882e^{1072t+12x} \\ &+1575e^{1120t+12x}+1008e^{1000t+18x}+49e^{128t+20x} \\ &+175e^{992t+20x}+126e^{72t+22x}+126e^{64t+24x}+56e^{56t+26x}) \\ &/(2e^{1072t}+2e^{16x}+14e^{4(252t+x)}+9e^{2(532t+x)}+7e^{1000t+6x} \\ &+7e^{72t+10x}+14e^{64t+12x}+9e^{8t+14x})^2. \end{aligned} \tag{3.40}$$

4. Discussion

In this talk we report on the recent calculation of the scattering problems of multi-indexed extensions of the soliton potential. Similar calculation can be performed to other known solvable potentials [12]. As an application [19], we point out an infinite set of higher integer initial profiles of the KdV solitons, which are both exactly solvable for the Schrodinger equation and for the Gel'fand-Levitan-Marchenko equation in the inverse scattering transform method of KdV equation. The calculation of these solutions are based on the multi-indexed extensions of the reflectionless soliton potential based on the Darboux-Crum transformation.

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