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Wide-sense nonblocking for multi-$\log_d N$ networks under various routing strategies

F.H. Chang, J.Y. Guo*, F.K. Hwang

Department of Applied Mathematics, National Chiaotung University, Hsinchu, ROC Taiwan, 300

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Abstract

Chang et al. showed that the number of middle switches required for WSNB under strategies: save the unused, packing, minimum index, cyclic dynamic, and cyclic static, for the 3-stage Clos network $C(n, m, r)$ with $r \geqslant 3$ is the same as required for SNB. In this paper, we prove the same conclusion for the multi-$\log_d N$ network. We also extend our results, except for the minimum index strategy, to a general class of networks including the 3-stage Clos network and the multi-$\log_d N$ network as special cases.

1. Introduction

The symmetric 3-stage Clos network $C(n, m, r)$ which has r switches of size $n \times m$ in the first stage, m switches of size $r \times r$ in the second(middle) stage, and r switches of size $n \times m$ in the third stage (see Fig. 1).

The multi-$\log_d N$ network, first proposed by Lea [7], is composed of p copies of $\log_d N$ network connected in parallel (see Fig. 2). Each copy of the $\log_d N$ network, also called banyan-type networks, is constructed of $d \times d$ switches arranged in n stages, $N = d^n$, labeled $1, 2, \ldots, n$ from left to right. Each stage has $d^{n-1} d \times d$ switches. In each copy, there is exactly one path between an arbitrary input and an arbitrary output. There are many varieties of $\log_d N$ networks, such as banyan, Omega, baseline, ..., but they are all equivalent in the sense that the connection property is invariant under a permutation of switches in the same stage.

A request is an (input, output) pair seeking connection. A set of requests can be routed if there exists connecting paths not intersecting each other in a node.

A multi-$\log_d N$ network is said to be strictly nonblocking (SNB) if a request can always be routed regardless of how the previous pairs are routed. It is said to be wide-sense nonblocking (WSNB) with respect to a routing strategy A if every request is routable under A. It is said to be rearrangeable nonblocking (RNB) if every request can be connected provided routing paths of existing connections can be rearranged (rerouted).

For convenience of analysis, we transform a $\log_d N$ network to a digraph by converting each link, including the inputs and the outputs, to a node, while a crosspoint connecting two links in the network becomes an arc in the digraph

* Corresponding author. Tel.: +886 937699688.
E-mail address: davidguo.am90g@nctu.edu.tw (J.Y. Guo).

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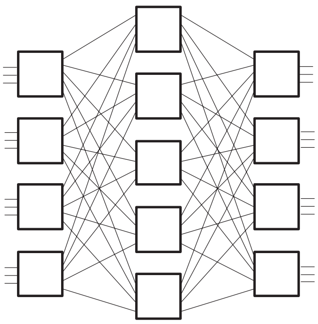


Fig. 1. $C(3, 5, 4)$.

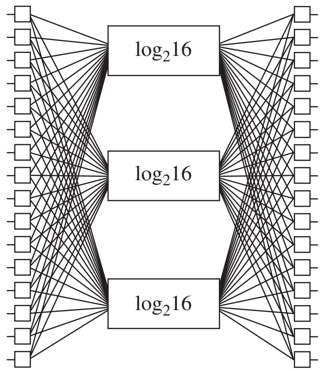
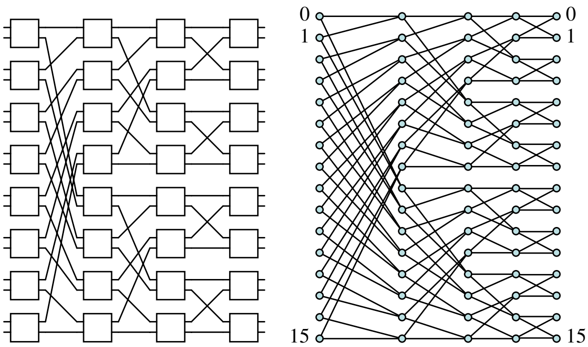


Fig. 2. A multi-$\log_2 16$ network with 3 copies of $\log_2 16$ networks.



(a) (b)

Fig. 3. (a) A 16 × 16 binary baseline network and (b) its graph model.

(see Fig. 3). Nodes are arranged in $n + 1$ stages labeled $0, 1, \ldots, n$ from left to right. The nodes in stage 0 correspond to inputs and the nodes in stage n correspond to outputs. The restraint that no two paths in the original network competes for the same link is translated to that no two paths in the derived network(digraph) competes for the same node.

For the 3-stage Clos network, a routing strategy deals with the choice of a middle switch to route the request when many are available. Five routing strategies have been proposed in the literature (see [2] for a survey):

(i) Save the unused (STU). Do not route through an empty middle switch unless there is no choice.
(ii) Packing (P). Choose a busiest, yet available, middle switch.
(iii) Minimum index (MI). Label all middle switches from M_1 to M_p. For each request, route in the order $M_1, M_2, \ldots,$ until the first available one emerges.
(iv) Cyclic dynamic (CD). If M_k was used last, try $M_{k+1}, M_{k+2}, \ldots,$ until the first available one emerges.
(v) Cyclic static (CS). If M_k was used last, try copy $M_k, M_{k+1}, \ldots,$ until the first available one emerges.

The existence of a WSNB network was first demonstrated by Beneš [1] for the symmetric 3-stage Clos network. He proved that $C(n, m, 2)$ is WSNB under packing if and only if $m \geqslant \lfloor 3n/2 \rfloor$ which is the only positive result. Smith [9] proved that $C(n, m, r)$ is not WSNB under P or MI if $m < \lfloor 2n - n/r \rfloor$, which was improved to $\lfloor 2n - (n/(2^r - 1)) \rfloor$ in Du et al. [3] and extended to all five strategies. For P, Yang and Wang [11] gave a linear programming formulation of the problem and ingeniously found the closed-form solution $m \geqslant \lfloor 2n - n/F_{2r-1} \rfloor$ where F_{2r-1} is the $2r - 1$st Fibonacci number, as a necessary condition for $C(n, m, r)$ to be WSNB. Note that for r large, all the above negative results show that $2n - 1$ middle switches are required for WSNB. Tsai et al. [10] culminated this line of results by giving a unifying proof for all possible strategies, not just the listed five.

For finite r, Du et al. [3] proved that for $r \geqslant 3$ $C(n, m, r)$ is WSNB for P or STU if and only if it is SNB, namely, $m \geqslant 2n - 1$, with a complicated proof. Chang et al. [2] simplified the proof and extended it to the other three strategies for $r \geqslant 2$; thus severely dashing the hope that a clever strategy can save hardware from SNB networks and yet preserve nonblockingness. We can translate the five routing strategies to the multi-$\log_d N$ network by replacing "choosing a middle switch" to "choosing a copy (of $\log_d N$)". In Section 2, we prove a similar conclusion that these five strategies require the same number of copies as SNB does. In Section 3, we extend our results to a general class.

Presumably, one can ask the same question for RNB, namely, how many middle switches are required for RNB if a certain routing strategy is followed. This has not been studied in the literature, not even for $C(n, m, r)$. The reason is because RNB can also be interpreted as nonblocking if all requests are to be routed simultaneously [1,6]. Then there exist better routing strategies yielding the requirements of n middle switches for the 3-stage Clos network [4] and $d^{\lceil n/2 \rceil}$ copies for the multi-$\log_d$ network [8]; showing that the cost of RNB is much less than that of SNB.

2. Main result

Shyy and Lea [8] proved the following theorem for $d = 2$ and Hwang [5] extended it to the d-nary version.

Theorem 1. *Multi-*$\log_d N$ *network is strictly nonblocking if* $p \geqslant p(n)$, *where*

$$p(n) = \begin{cases} (d+1) \times d^{\frac{n}{2}-1} - 1 & \text{for } n \text{ even}, \\ 2 \times d^{\frac{n-1}{2}} - 1 & \text{for } n \text{ odd}. \end{cases}$$

A request from input x to output y, represented by (x, y), has a unique path in a $\log_d N$ network. Hence two intersecting paths must be routed through different copies of $\log_d N$ network.

Theorem 1 was stated in [5] only as a sufficient condition. We need prove that it is also necessary.

Theorem 2. *Multi-*$\log_d N$ *network is strictly nonblocking only if* $p \geqslant p(n)$.

Proof. For any request $\gamma = (x, y)$, assume that the path of γ consists of links $L_0, L_1, \ldots, L_n$. For n odd, let $I_1(O_2)$ be the set of inputs(outputs), except $x(y)$, which can reach $L_{(n-1)/2}$, then $|I_1| = d^{(n-1)/2} - 1$ and $|O_2| = d^{(n+1)/2} - 1$. Let $O_1(I_2)$ be the set of outputs(inputs), except $y(x)$, which can reach $L_{(n+1)/2}$. Then $|O_1| = d^{(n-1)/2} - 1$ and $|I_2| = d^{(n+1)/2} - 1$. Note that γ cannot be routed through the same copy with any request from I_1 to O_2 or I_2 to O_1. Suppose $p = p(n) - 1$ while $|I_1|$ requests from I_1 to $O_2 \setminus O_1$ and $|O_1|$ requests from O_1 to $I_2 \setminus I_1$ have already been connected in different copies. In this case, they can occupy $|I_1| + |O_1| = p(n) - 1 = p$ copies, with no copy left for γ. For n even, let $I_1(O_2)$ be the set of inputs(outputs), except $x(y)$, which can reach $L_{n/2-1}$, then $|I_1| = d^{n/2-1} - 1$ and $|O_2| = d^{n/2+1} - 1$. Let $O_1(I_2)$ be the set of outputs(inputs), except $y(x)$, which can reach $L_{n/2+1}$. Then $|O_1| = d^{n/2-1} - 1$ and $|I_2| = d^{n/2+1} - 1$. Let $I_3(O_3)$ be the set of inputs(outputs), except $x(y)$, which can reach $L_{n/2}$. Then $|I_3| = |O_3| = d^{n/2} - 1$. Note that γ cannot be routed through the same copy with any request from I_1 to O_2, I_2 to O_1, or I_3 to O_3. Suppose $p = p(n) - 1$ while $|I_1|$ requests from I_1 to $O_2 \setminus O_3$, $|O_1|$ requests from O_1 to $I_2 \setminus I_3$, and $|I_3 \setminus I_1|$ requests from $I_3 \setminus I_1$ to $O_3 \setminus O_1$ have already been connected in different copies. In this case, they can occupy $|I_1| + |O_1| + |I_3 \setminus I_1| = |I_1| + |O_1| + |O_3 \setminus O_1| = p(n) - 1 = p$ copies, with no copy left for γ. Hence p must be greater than or equal to $p(n)$. □

We call such a set of $p(n) - 1$ requests blocking γ the maximal blocking configuration (MBC), denote by $M(n, \gamma)$.

Note that if a network is SNB, then it is also WSNB. i.e. multi-$\log_d N$ is WSNB if $p \geqslant p(n)$. Therefore, we only need to prove necessity in the following proofs. In all these proofs, we assume that the network carries no traffic at the beginning.

We consider strategy CD first.

Theorem 3. *Multi-*$\log_d N$ *network is WSNB under CD if and only if* $p \geqslant p(n)$.

Proof. Suppose $p < p(n)$. Consider a sequence of $p+1$ requests with p requests from $M(n, \gamma)$ followed by the request γ. By the property of strategy CD, these p requests will be routed in p copies. Then we cannot route γ any more. Hence p must be greater than or equal to $p(n)$. □

For strategy CS,

Theorem 4. *Multi-*$\log_d N$ *network is WSNB under CS if and only if* $p \geqslant p(n)$.

Proof. Suppose $p < p(n)$. For a request γ and any p requests of $M(n, \gamma)$, say $\gamma_1, \gamma_2, \ldots, \gamma_p$, route γ_1 in copy 1, then route γ in copy 2(because γ_1 blocks γ in copy 1). Then disconnect γ and route γ_2 in copy 2. Then route γ in copy 3. Again disconnect it and route γ_3 in copy 3. Doing this iteratively until γ_p is routed in copy p. Then γ cannot be routed any more. Hence p must be greater than or equal to $p(n)$. □

For strategies P or STU, we introduce a lemma.

Lemma 5. *For any request* γ *and* $M(n, \gamma)$, *there exists a request* γ' *which does not block* γ *or any request in* $M(n, \gamma)$ *in the* $\log_d N$ *network.*

Proof. Use the graph model of the baseline network as an example. Without loss of generality, let $\gamma = (0, 0)$. For all requests (i, j) in $M(n, \gamma)$, we obtain $i < N/d$ and $j < N/d$. Hence $\gamma' = (N - 1, N - 1)$ will satisfy our claim. □

Theorem 6. *Multi-*$\log_d N$ *network is WSNB under P or STU if and only if* $p \geqslant p(n)$.

Proof. Suppose to the contrary, $p < p(n)$. For any request γ and any p requests of $M(n, \gamma)$, say $\gamma_1, \gamma_2, \ldots, \gamma_p$, we route γ_1 in copy 1 first. Then route γ in copy 2 and route γ' in copy 2 (because copy 1 are as busy as copy 2, we can choose copy 2). Now, we disconnect γ and route γ_2 in copy 2. Then disconnect γ'. Similarly, we route γ in copy 3 and γ' in copy 3, then disconnect γ and route γ_3 in copy 2. Finally, we route γ_p in copy p. Then γ cannot be routed any more. Hence p must be greater than or equal to $p(n)$. □

MI is more complicated. We first introduce a result in [2].

Theorem 7. *The* 3*-stage Clos network* $C(n, m, r)$ *for* $r \geqslant 2$ *is WSNB under MI if and only if* $m \geqslant 2n - 1$.

In the following theorem, only the baseline architecture will be considered. However, the theorem is also true for other equivalent $\log_d N$ network.

Theorem 8. *Multi-*$\log_d N$ *network is WSNB under MI if and only if* $p \geqslant p(n)$.

Proof. We discuss two cases:

(i) n is odd. Select two subset I_1 and I_2 of inputs and two subset O_1 and O_2 of outputs. Set $I_1 = O_1 = \{0, 1, 2, \ldots, d^{(n-1)/2} - 1\}$, $I_2 = O_2 = \{d^{(n-1)/2}, \ldots, 2 \times d^{(n-1)/2} - 1\}$. See Fig. 4. By the configuration of baseline network, every request from I_1 to $O_1 \cup O_2$ must intersect node 0 in stage $(n - 1)/2$ and every request from I_2 to $O_1 \cup O_2$ must intersect node 1 in stage $(n - 1)/2$. Therefore, for $i = 1$ or 2, all requests from I_i to $O_1 \cup O_2$ must use different copies. Similarly, every request from $I_1 \cup I_2$ to O_1 must intersect node 0 in stage $(n + 1)/2$ and every request from $I_1 \cup I_2$ to O_2 must intersect node $d^{(n-1)/2}$ in stage $(n + 1)/2$. Therefore, for $i = 1$ or 2, all requests from $I_1 \cup I_2$ to O_i must use different copies. Now, we match this to a 3-stage Clos network $C(d^{(n-1)/2}, 1, 2)$, where I_i is the ith input switch, O_i is the ith output switch, for $i = 1$ or 2, and the complete bipartite graph induced by nodes 0 and 1 of stage $(n - 1)/2$ and nodes 0 and $d^{(n-1)/2}$ of stage $(n + 1)/2$ is the middle switch. Then a request (i, j) in $C(d^{(n-1)/2}, p, 2)$ routed through the kth middle switch under MI corresponds to a request (i, j) in the multi-$\log_d N$ using copy k. Therefore, by Theorem 7, the network is not WSNB if

$$p < 2 \cdot (d^{\frac{n-1}{2}}) - 1 = 2 \times d^{\frac{n-1}{2}} - 1 = p(n).$$

(ii) n is even. Select four subset I_1, I_1', I_2 and I_2' of inputs and four subset O_1, O_1', O_2, and O_2' of outputs. Set $I_1 = O_1 = \{0, 1, 2, \ldots, d^{n/2-1} - 1\}$, $I_1' = O_1' = \{d^{n/2-1}, \ldots, d^{n/2} - 1\}$, $I_2 = O_2 = \{d^{n/2}, \ldots, (d + 1)d^{n/2-1} - 1\}$, and $I_2' = O_2' = \{(d + 1)d^{n/2-1}, \ldots, 2 \times d^{n/2} - 1\}$. See Fig. 5. Then every request from I_1 to $O_1 \cup O_2$ must

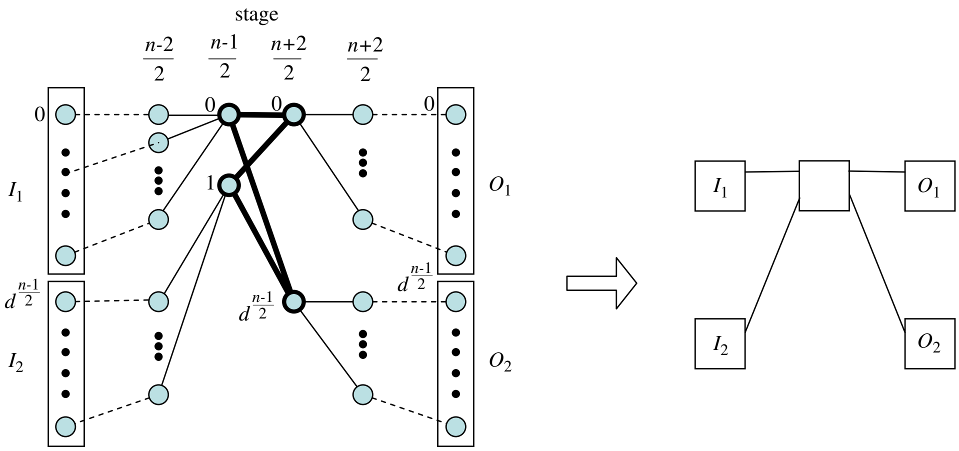


Fig. 4. The left figure is an induced graph of the graph model of a multi-$\log_d N$ network, for n odd. And the right figure is its correspondence to a 3-stage Clos network.

intersect node 0 in stage $n/2-1$, every request from I_2 to $O_1 \cup O_2$ must intersect node d in stage $n/2-1$, every request from $I_1 \cup I_2$ to O_1 must intersect node 0 in stage $n/2+1$, and every request from $I_1 \cup I_2$ to O_2 must intersect node $d^{n/2}$ in stage $n/2+1$. Similar to case (i), we can treat I_1, I_2, O_1, O_2 as the inputs and outputs of $C(d^{n/2-1}, 1, 2)$, and the subgraph sketch in bold line in Fig. 5 is the middle switch. Therefore, by Theorem 7, the network is not WSNB if

$$p < 2 \cdot (d^{\frac{n}{2}-1}) - 1. \tag{1}$$

Besides, we observe that, for $i, j = 1, 2$, every request from I_i to O_j must block every request from I_i' to O_j' in the same node in the stage $n/2$. Therefore, if we connect all $(d-1)d^{n/2-1}$ requests in I_i' to O_j' in copy 0 to copy $(d-1)d^{n/2-1}-1$ before every time we connect a request γ from I_i to O_j and disconnect them after γ connected, then we can force the copy chosen to route γ begin at least $(d-1)d^{n/2-1}$th copy. Hence (1) can be enlarged to

$$p < 2 \times (d^{\frac{n}{2}-1}) - 1 + (d-1)d^{\frac{n}{2}-1} = p(n). \qquad \square$$

Note that, in Theorem 8, it does not need to consider all inputs and outputs, because $I_1 \cup I_2$ and $O_1 \cup O_2$ are enough to force $p \geqslant p(n)$ which is the bound of SNB.

3. Some generalizations

We extend our results to a class of networks including the 3-stage Clos networks, the multi-$\log_d N$ and the $\log_d(N, k, m)$ networks as special cases.

A vertical-copy network V consists of an input stage of r_1 $(n_1 \times m)$-crossbars, an output stage of $r_2(m \times n_2)$-crossbars and a middle stage of m copies of a network v with r_1 inputs and r_2 outputs. There exists exactly one link between each input(output) crossbar and each copy of v. When v is the $r_1 \times r_2$ crossbar, V is a 3-stage Clos network. When $n_1 = n_2 = 1$ and v is the $\log_d N$ network, V is a multi-$\log_d N$ network. When $n_1 = n_2 = 1$ and v is the k-extra-stage $\log_d N$ network, then V is the $\log_d(N, k, m)$ network. In particular, if $k = n-1$, then V is the Cantor network.

Suppose that the necessary and sufficient condition for v to be SNB is known. Consider $p = p(n) - 1$. For any request γ, there must be a state s such that γ is blocked in each of the $p(n)-1$ copies $v_1, v_2, \ldots, v_{p(n)-1}$. Let R_i be the set of all requests routing through v_i in s and $M(v, \gamma) = \{R_i \mid i = 1, 2, \ldots, p(n)-1\}$. i.e., V is SNB if and only if the number of copies is larger than $|M(v, \gamma)|$. Let "Route R_i in v_j" mean "Route all requests in R_i in v_j consecutively".

Theorem 9. *A vertical-copy network V is WSNB under the CS routing if and only if V is SNB.*

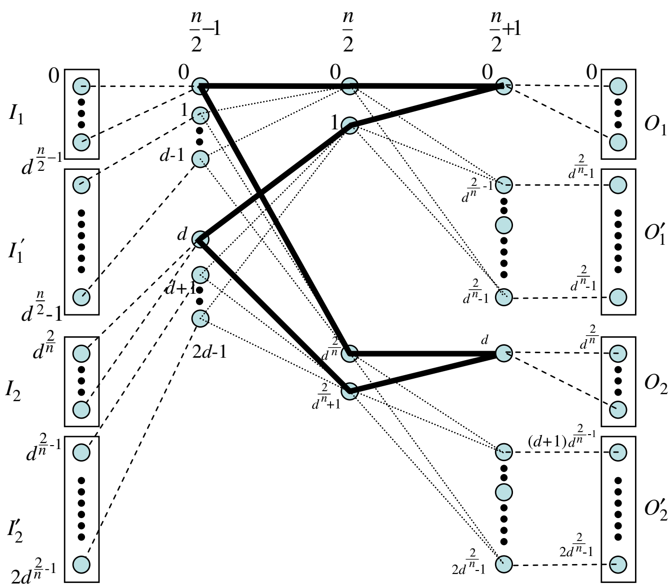


Fig. 5. This is an induced graph of the graph model of a multi-$\log_d N$ network, for n is even.

Proof. Suppose there are $p < p(n)$ copies $v_1, v_2, \ldots, v_p$ in V. For a request γ, we route R_1 in v_1, then route γ in v_2 (γ is blocked in v_1). Then disconnect γ and route R_2 in v_2. Then route γ in v_3. Again disconnect it and route R_3 in v_3. Doing this iteratively until R_p is routed in v_p. Then γ cannot be routed in any copy. Hence p must be greater than or equal to $p(n)$. $\square$

For CD, we use another argument.

Theorem 10. *A vertical-copy network V is WSNB under the CD routing if and only if V is SNB.*

Proof. First, we claim every request γ can be routed in v_k for a given k. Route γ in v_i. If $i \neq k$, then disconnect γ and route it again in v_{i+1}. Similarly, if $i+1 \neq k$, then disconnect γ and route it again in v_{i+2} until γ is routed in v_k. Note that if $i = p$, then we let $i+1$ be 1. Therefore, if $p < p(n)$, then we can route R_i in v_i for $i = 1$ to p as we want. Then γ cannot be routed in any copy. Hence p must be greater than or equal to $p(n)$. $\square$

For STU, if there exists a request γ'_i which does not block $\{\gamma\} \cup R_i$ for all i, Theorem 6 remains true if $M(n, \gamma)$ is replaced by $M(v, \gamma)$ and γ_i is replaced by R_i. But we use a different argument for P.

Theorem 11. *Suppose there exists a request γ'_i which does not block $\{\gamma\} \cup R_i$ for all i. A vertical-copy network V is WSNB under the P routing if and only if V is SNB.*

Proof. It suffices to prove the "only if" part. Suppose there are only $p = p(n) - 1$ copies $v_1, v_2, \ldots, v_p$ in V. For the request $\gamma = (0, 0)$, without loss of generality, suppose $R_i = \{\gamma_{i,j} \mid j = 1, \ldots, \lambda_i\}$ and $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_p$. Let $|v_i|$ denote the number of connections in v_i. For a given k, let $s(k, \mathbb{B})$ be a state satisfying the following conditions:

(i) $|v_k| < \lambda_k$,
(ii) Connections in v_i are those from R_i,
(iii) $|v_i| = |v_k| + 1$ or $|v_i| = \lambda_i$ if $i \in \mathbb{B} \equiv \{i \mid |v_i| > |v_k|\}$

Let $S(k)$ denote the state that v_i contains R_i for all $1 \leqslant i \leqslant k$. We make two claims:

Claim A. *We can add another connection δ of R_k in v_k in state $s(k, \mathbb{B})$.*

Claim B. *$S(k)$ can be realized.*

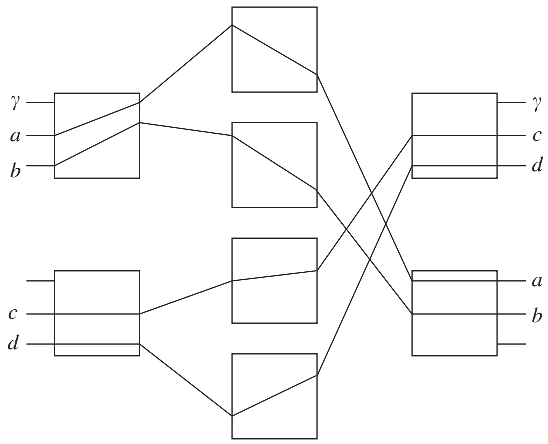


Fig. 6. γ and $M(V,\gamma)=\{a,b,c,d\}$ in $C(3,4,2)$.

We prove both claims by induction on k. For $k=1$, then $\mathbb{B}=\emptyset$. Clearly, we can add δ to v_1, and keep on adding other connections until v_i contains R_i. So consider general $k>1$. From $s(k,\mathbb{B})$ we can obtain the state $s^*(k,\mathbb{B})$, which differs from $s(k,\mathbb{B})$ by having v_i containing R_i for all $1\leqslant i\leqslant k-1$, by applying induction to claim B(with $k=k-1$). In state $s^*(k,\mathbb{B})$, γ must be routed in v_k. Now delete all connections in $s^*(k,\mathbb{B})\setminus s(k,\mathbb{B})$ so that $|v_k|\geqslant|v_i|$ for all i. Then γ'_k can be routed in v_k. Delete γ and route δ in v_k. Delete γ'_k and Claim A is proved. Also, we can keep on adding all remaining connections of R_k to v_k to prove Claim B.

Setting $k=p$ in Claim B, then γ cannot be routed in any of the p copies. Hence at least $p(n)$ copies are needed. □

Example 1. For simplicity, we will represent a state by its $|v|$-sequence. To help clarify the state, let $|v_i|^*$ denote the fact that γ is in the v_i, $|v_i|'$ the fact that γ' is and $|v_i|''$ the fact that both are. Suppose $p=3$ and we want to reach the state $S(3)=(\lambda_1,\lambda_2,\lambda_3)=(2,3,4)$. The the $|v|$-sequence of our construction in Theorem 11 would be:

$$(0,0,0)\Rightarrow(1,0,0)\Rightarrow(2,0,0)\Rightarrow(2,1^*,0)\Rightarrow(1,1^*,0)\Rightarrow(1,2'',0)\Rightarrow(1,1',0)\Rightarrow$$
$$(1,2',0)\Rightarrow(1,1,0)\Rightarrow(2,1,0)\Rightarrow(2,2^*,0)\Rightarrow(2,3'',0)\Rightarrow(2,2',0)\Rightarrow(2,3',0)\Rightarrow$$
$$(2,2,0)\Rightarrow(2,3,0)\Rightarrow(2,3,1^*)\Rightarrow(1,1,1^*)\Rightarrow(1,1,2'')\Rightarrow(1,1,1')\Rightarrow(1,1,2')\Rightarrow$$
$$(1,1,1)\Rightarrow(2,1,1)\Rightarrow(2,2^*,1)\Rightarrow(2,3'',1)\Rightarrow(2,2',1)\Rightarrow(2,3',1)\Rightarrow(2,2,1)\Rightarrow$$
$$(2,3,1)\Rightarrow(2,3,2^*)\Rightarrow(2,2,2^*)\Rightarrow(2,2,3'')\Rightarrow(2,2,2')\Rightarrow(2,2,3')\Rightarrow(2,2,2)\Rightarrow$$
$$(2,3,2)\Rightarrow(2,3,3^*)\Rightarrow(2,3,4'')\Rightarrow(2,3,3')\Rightarrow(2,3,4')\Rightarrow(2,3,3)\Rightarrow(2,3,4)$$

Therefore, we obtain the state $S(3)$.

Corollary 12. $\log_d(N,k,m)$ *is WSNB under any of CS, CD, STU, and P if and only if it is SNB, i.e.,* [5],

$$m>\begin{cases}k+3\cdot 2^{\frac{n-k}{2}-1}-2 & \text{for } n-k \text{ even},\\ k+2^{\frac{n-k+1}{2}}-2 & \text{for } n-k \text{ odd}.\end{cases}$$

Proof. Note that $\log_d(N,k,m)$ is a vertical copy network. Then the results for CS and CD follow from Theorems 9 and 10. For P and STU, it is easily verified that $\gamma'_i=(N-1,N-1)$ does not block any request in $\{\gamma\}\cup R_i$ for all i. Then the results follow from Theorems 11. □

What packing is a good routing strategy has been a folklore for a long time and documented in literature [1]. One motivation for that folklore is that $C(n,m,2)$ is WSNB under P if and only if $m\geqslant\lfloor 3n/2\rfloor$ [1], while it is SNB if and only if $m\geqslant 2n-1$. The seemingly discrepancy between the $m\geqslant\lfloor 3n/2\rfloor$ result and Theorem 11 is explained by the fact that γ' does not exist in $C(n,m,2)$ since $M(V,\gamma)$ occupies both input switches (see Fig. 6).

For $r \geqslant 3$, it was proved [2] that $C(n, m, r)$ is WSNB under P if and only if it is SNB. Thus the saving of $C(n, m, 2)$ under P seems to be a fluke rather than a testimony of its goodness. In this paper, again we showed that in the worst-case scenario, P does not help. Instead, MI is the only routing strategy which is still not ruled out to be useful.

References

[1] V.E. Beneš, Mathematical Theory of Connecting Networks and Telephone Traffic, Academic Press, New York, 1965.

[2] F.H. Chang, J.Y. Guo, F.K. Hwang, J.K. Lin, Wide-sense nonblocking for symmetric or asymmetric 3-stage Clos networks under various routing strategies, Theoret. Comput. Sci. 314 (2004) 375–386.

[3] D.Z. Du, P.C. Fishburn, B. Gao, F.K. Hwang, Wide-sense nonblocking for 3-stage Clos networks, in: D.Z. Du, H.Q. Ngo (Eds.), Switching Networks: Recent Advances, Kluwer, Boston, 2001, pp. 89–100.

[4] A.M. Duguid, Structural properties of switching networks, Progress Report BTL-7, Brown University, 1959.

[5] F.K. Hwang, Choosing the best $\log_k(N, m, P)$ strictly nonblocking networks, IEEE Trans. Comm. 46 (4) (1998) 454–455.

[6] F.K. Hwang, The Mathematical Theory of Nonblocking Switching Networks, World Scientific, Singapore, first ed., 1998; second ed., 2004.

[7] C.-T. Lea, Multi-$\log_2 N$ networks and their applications in high-speed electronic and photonic switching systems, IEEE Trans. Comm. 38 (10) (1990) 1740–1749.

[8] D.-J. Shyy, C.-T. Lea, $\log_2(N, m, p)$ strictly nonblocking networks, IEEE Trans. Comm. 39 (10) (1991) 1502–1510.

[9] D.G. Smith, Lower bound in the size of a 3-stage wide-sense nonblocking network, Elec. Lett. 13 (1977) 215–216.

[10] K.H. Tsai, D.W. Wang, F.K. Hwang, Lower bounds of wide-sense nonblocking Clos networks, Theoret. Comput. Sci. 261 (2001) 323–328.

[11] Y. Yang, W. Wang, Wide-sense nonblocking Clos networks under packing strategy, IEEE Trans. Comput. 45 (1999) 265–284.