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Optical beams in sub-strongly non-local nonlinear media: A variational solution

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Abstract

Discussed is the propagation of optical beams in non-local nonlinear media modelled by 1 + 1D non-local nonlinear Schrödinger equation (NNLSE). In the sub-strongly non-local case, an approximate analytical solution is obtained for an arbitrary response function by a variational approach. Described by a combination of the Jacobian elliptic functions, the solution is periodic, and its period depends on not only the input power but also the initial beam width, which is confirmed by the numerical simulation of the NNLSE.

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Keywords: Optical beams; Non-local nonlinear media; Sub-strong non-locality; Spatial optical solitons; Variational approach

1. Introduction

Snyder and Mitchell's work [1], in which an exact Gaussian-shaped stationary solution called an accessible soliton was found in a linear model, called Snyder–Mitchell model [2], of the non-local nonlinearity with infinite characteristic length, has stimulated a strong interest, an omen of "a new surge of soliton activities" predicted by Shen [2], on spatial solitons in non-local nonlinear media modelled by the non-local nonlinear Schrödinger equation (NNLSE) that the nonlinear term assumes a non-local form (convolution integral) with a symmetric and real-valued response kernel.

So far, more properties of such spatial solitons and the related phenomena have been opened out. Given were the reviews of the most advances on experiments [3], on both theories and experiments [4], as well as on theories with a little description of experiments [5], which presented general surveys of the subject from different angles. A study by a variational approach was carried out with respect to the specific power-law response kernel [6], and a tractable model of the logarithmic non-local nonlinear media with the Gaussian response kernel was presented [7]. Sub-wavelength non-local spatial solitons were also studied [8]. Exact solutions in the limit of weak non-locality were obtained [9], modulational instabilities were analyzed [10,11], and the properties of soliton stabilization with arbitrary degree of non-locality were investigated [12]. The equivalence of quadratic solitons to the solitons in non-local cubic media was presented [13], which provides new physical insight into the properties of quadratic solitons. A novel phenomenon was revealed that the phase shift of a strongly non-local spatial soliton can be very large [14]. It was pointed out that [4] the specimen length for the π phase shift of the soliton has an order less than 0.1 mm in the visible spectrum region, hence the phenomenon might have its great potential applications in integrated photonic (all-optical) signal processing devices. In a probe-pump (weak-strong) geometry, a probe beam is predicted to experience π nonlinear phase shift, which can be

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modulated by a pump beam via the probe-pump strongly non-local nonlinear interaction, within a rather short distance [15]. Unusual attraction of dark solitons due to non-locality was explored [16]. Both non-local incoherent solitons [17] and non-local vortex solitons [18] were also predicted. In experiments, a single non-local spatial soliton [19], the interaction of such a soliton pair [20], and the related phenomenon of a modulational instability [21] have been observed in a nematic liquid crystal. Principle experiments of all-optical switching and logic gates were implemented in the nematic liquid crystal thanks to the phase-independently attractive collisional behavior of non-locality [22]. The most exciting progress after Snyder and Mitchell's work, in author's point of view, is the latest finished works by Assanto's group [23,24]. They demonstrated both theoretically [23] and experimentally [24] that the nematic liquid crystal is indeed one of strongly non-local nonlinear media. Before their work, it would be considered that the strongly non-local nonlinear media had not been discovered [2]. Both the experiment and the theory of the propagation of a non-local anisotropic spatial soliton in the nematic liquid crystal were reported [25]. The nematic liquid crystal has been considered to be promising for novel generations of all-optically addressable interconnects for computing and communications [3].

According to the degree of the non-locality determined by the ratio of the extent of a material non-local nonlinear response to the width of an optical beam, it is generally considered that there are four categories of the non-locality [10,12]: local, weakly non-local, generally non-local, and strongly non-local. The way to control the degree of the non-locality was suggested for the cases of both the quadratic medium [13] and the nematic liquid crystal [23], and was experimentally demonstrated in the nematic liquid crystal very recently [26]. A local one is the limit situation when the response function is a delta function. In this case, the NNLSE is simplified into the standard NLSE that has a stable sech-form soliton solution for $1+1$D case, which there have been an immense collection of literatures to have dealt with [27]. A weak non-locality is when the characteristic length of the response function is much narrower than the width of the optical beam, and is modeled by a modified NLSE [9]. The analytical bright and dark soliton solutions in self-focusing and self-defocusing media, respectively, were found for $(1+1)$-dimensional weakly non-local nonlinear media [9]. Another extreme is a strong non-locality [10,12–15], referred as a high non-locality in some papers [1,23,24], that the characteristic length of nonlinear response is much broader than the width of the optical beam. By use of Taylor expansion of the response function [10,12] to the second-order, one can deduce the strongly non-local model [14,15], and further transform it into Snyder–Mitchell (linear) model [15]. Snyder–Mitchell model has an exact Gaussian-shaped single beam solution [1], whose beam width is stationary (soliton state) when the beam power equals exactly a critical power, but generally changes in the way by a combination of sine and cosine functions. The other situation except the three limit cases mentioned above is a generally non-local. In our point of view, however, the general non-locality can further be divided into two categories: sub-strong non-locality and sub-weak non-locality. The former is the situation when the beam width is about but less than the characteristic length of the response function, while the latter is the reverse case.

In this paper, we present an approximate variational solution of the $1+1$D NNLSE, which models the beam behavior in a planar non-local cubic nonlinear medium waveguide, for the sub-strongly non-local case. The solution also has a Gaussian-shaped form, but the beam width is a combination of Jacobian elliptic functions, rather than trigonometrical functions. Our result is independent of the concrete form of response kernels.

2. A variational solution for sub-strong non-locality

The $1+1$D NNLSE in non-local cubic nonlinear media has the form [7,10,14]

$$\mathrm{i}\frac{\partial\psi}{\partial z}+\mu\frac{\partial^2\psi}{\partial x^2}+\rho\psi\int_{-\infty}^{\infty}R(x-\xi)|\psi(\xi,z)|^2\,\mathrm{d}\xi=0, \tag{1}$$

where $\psi(x,z)$ is a paraxial beam, $\mu=1/2k$, $\rho=k\eta$, k is the wave number in the media without nonlinearity (that is, $k=\omega n_0/c$, and n_0 is the linear refractive index of the media), η is a material constant ($\eta>0$ or <0 corresponds to a focusing or defocusing material.), z and x are the longitudinal and transverse coordinates, respectively. Here, R is normalized symmetrical real spatial response of the media such that $\int_{-\infty}^{\infty}R(\xi)\,\mathrm{d}\xi=1$. In the limit case that the characteristic length of the material tends to zero, $R(x)\to\delta(x)$ (Dirac delta function), then Eq. (1) will become standard NLSE.

By the variational approach [28], Eq. (1) can be interpreted as an Euler–Lagrange equation corresponding to a vanishing variation

$$\delta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathscr{L}(\psi,\psi^*,\psi_z,\psi_z^*,\psi_x,\psi_x^*)\,\mathrm{d}x\,\mathrm{d}z=0 \tag{2}$$

of Lagrangian density

$$\mathscr{L}=\frac{\mathrm{i}}{2}\left(\psi^*\frac{\partial\psi}{\partial z}-\psi\frac{\partial\psi^*}{\partial z}\right)-\mu\left|\frac{\partial\psi}{\partial x}\right|^2+\frac{1}{2}\rho|\psi|^2\int_{-\infty}^{\infty}R(x-\xi)|\psi(\xi,z)|^2\,\mathrm{d}\xi. \tag{3}$$

We optimize the function (2) by employing the Gaussian ansatz for the trial solution

$$\psi(x,z)=A(z)\exp[\mathrm{i}\alpha(z)]\exp\left[-\frac{x^2}{2w(z)^2}+\mathrm{i}c(z)x^2\right], \tag{4}$$

where A and α are the amplitude and phase of the complex amplitude of the solution, respectively, w is beam width, c is the phase-front curvature, and they are all allowed to vary with propagation distance z. Inserting the trial function above into the variational principle, Eq. (2), we obtain the reduced variational problem

$$\delta\int_{-\infty}^{\infty}\mathscr{L}_{\rm r}\,{\rm d}z=0, \tag{5}$$

where $\mathscr{L}_{\rm r}=\int_{-\infty}^{\infty}\mathscr{L}_{\rm g}\,{\rm d}x$, and $\mathscr{L}_{\rm g}$ denotes the result of inserting the Gaussian ansatz (4) into the Lagrangian (3). Generally, it is difficult to analytically determine $\mathscr{L}_{\rm r}$ for an arbitrary response function $R(x)$. However, for the sub-strongly non-local case that $w/w_{\rm m}<1$, where $w_{\rm m}$ is the characteristic length of the material, the approximate integration can analytically be obtained because $R(x)$ can be expanded in Taylor series like the case of the strong non-locality [10,12,14]. Concretely, we first expand the response $R(x-\xi)$ with respect to ξ about $\xi=0$ to the fourth-order in the first integral over ξ, then expand again the functions $R^{(j)}(x)$ in the result of the first integrand, where $R^{(j)}(x)$ denotes $\partial^j R(x)/\partial x^j$, and $j=0,\ldots,4$, with respect to x about $x=0$ till all of the $R^{(4)}(0)$ terms appear before evaluation of the second integral over x [29]. In this way, $\mathscr{L}_{\rm r}$ can be analytically determined

$$\mathscr{L}_{\rm r}=-\frac{\sqrt{\pi}A^2}{2}\left[\frac{\mu}{w}+w^3\left(4\mu c^2+\frac{{\rm d}c}{{\rm d}z}\right)+2w\frac{{\rm d}\alpha}{{\rm d}z}\right]+\frac{1}{16}\pi\rho w^2A^4(8R_0-4\gamma w^2+R_0^{(4)}w^4), \tag{6}$$

where $R_0^{(j)}=R^{(j)}(0)$, $R_0=R_0^{(0)}$, $\gamma=-R_0^{(2)}>0(R_0^{(2)}<0$ because R_0 is a maximum of $R(x)$.

Following the standard procedures of the variational approach [28], we have, respectively

$$\frac{\delta\mathscr{L}_{\rm r}}{\delta\alpha}=0\Rightarrow\frac{{\rm d}(A^2w)}{{\rm d}z}=0, \tag{7}$$

$$\frac{\delta\mathscr{L}_{\rm r}}{\delta c}=0\Rightarrow-3A\frac{{\rm d}w}{{\rm d}z}-2w\frac{{\rm d}A}{{\rm d}z}+8\mu Awc=0, \tag{8}$$

$$\frac{\delta\mathscr{L}_{\rm r}}{\delta w}=0\Rightarrow\frac{3w^2}{2}\frac{{\rm d}c}{{\rm d}z}+\frac{{\rm d}\alpha}{{\rm d}z}=\frac{\mu}{2w^2}-6\mu w^2c^2+\sqrt{\pi}\rho wA^2\left(R_0-\gamma w^2+\frac{3}{8}R_0^{(4)}w^4\right), \tag{9}$$

$$\frac{\delta\mathscr{L}_{\rm r}}{\delta A}=0\Rightarrow\frac{{\rm d}c}{{\rm d}z}+\frac{2}{w^2}\frac{{\rm d}\alpha}{{\rm d}z}=-\frac{\mu}{w^4}-4\mu c^2+\frac{\sqrt{\pi}\rho A^2}{4w}(8R_0-4\gamma w^2+R_0^{(4)}w^4). \tag{10}$$

Eq. (7) represents the fact that the beam power is conservative, from which it can be obtained

$$A^2=\frac{P_0}{\sqrt{\pi}w}, \tag{11}$$

where P_0 is the input power at $z=0$, and the power is understood to be $P=\int_{-\infty}^{\infty}|\psi|^2\,{\rm d}x$. Substitution of Eq. (7) into Eq. (8) can determine a relation

$$\frac{{\rm d}w}{{\rm d}z}=4\mu wc. \tag{12}$$

Solving Eqs. (9) and (10) gives out two ordinary differential equations for the parameter c and α

$$\frac{{\rm d}c}{{\rm d}z}=\frac{\mu}{a^4}-4\mu c^2+\frac{1}{4}\rho P_0(-2\gamma+R_0^{(4)}w^2) \tag{13}$$

and

$$\frac{{\rm d}\alpha}{{\rm d}z}=-\frac{\mu}{w^2}+\frac{1}{4}\rho P_0(4R_0-\gamma w^2). \tag{14}$$

Substitution of Eqs. (12) and (13) into the first derivative of Eq. (12) with respect to z reads

$$\frac{1}{\mu}\frac{{\rm d}^2y}{{\rm d}z^2}-\frac{4\mu}{w_0^4y^3}+2\rho\gamma P_0y(1-2\epsilon y^2)=0, \tag{15}$$

where $y(z)=w(z)/w_0$, $w_0=w(z)|_{z=0}$, and $\epsilon=w_0^2R_0^{(4)}/4\gamma$. It should be paid attention that ϵ is a parameter determined by the initial value of the beam and the material property, similar to the parameter σ $(=R_0/\gamma w_0^2)$ in [14]. The difference between the two parameters is that σ is positive, while ϵ can be either positive or negative, determined by the sign of the forth derivative of the response function $R(x)$ at the symmetrical (original) point, $R_0^{(4)}$ [30]. Observing two relations $\gamma=-R_0^{(2)}\sim R_0/w_{\rm m}^2$, and $|R_0^{(4)}|\sim R_0/w_{\rm m}^4$, we then have that

$$\epsilon=\mathscr{Q}\frac{w_0^2}{w_{\rm m}^2}, \tag{16}$$

where the proportional coefficient $\mathscr{Q}$ is determined only by the material property. The sign of $\mathscr{Q}$ is the same with the sign of $R_0^{(4)}$, and $|\mathscr{Q}|$ has a order of 1. For the limit case that $w_{\rm m}$ tends to infinite, we have ϵ tends to zero.

Like the case discussed in [14], Eq. (15) is equivalent to Newton's second law in classical mechanics for the motion of an one-dimensional particle acted by an equivalent force $F=4\mu/w_0^4y^3-2\rho\gamma P_0y(1-2\epsilon y^2)$. The first term of F is a so-called diffractive force, and its second term is a refractive force provided that $1-2\epsilon y^2>0$. Following the treatment in [14], letting the two forces equal and $y=1$, we obtain the critical (input) power for the soliton propagation

$$P_{\rm c}=\frac{P_{\rm c0}}{1-2\epsilon}, \tag{17}$$

where $P_{\rm c0}=1/(\gamma w_0^4k^2\eta)$.

Assuming that the beam at $z=0$ has ${\rm d}w(z)/{\rm d}z|_{z=0}=0$, the one-time integration of Eq. (15) yields the first-order ordinary differential equation

$$\left(\frac{{\rm d}y}{{\rm d}z}\right)^2+\frac{\eta\gamma P_0(1-y^2)[P_{\rm c0}/P_0+y^2(\epsilon-1+\epsilon y^2)]}{y^2}=0. \tag{18}$$

Eq. (18) is analytically integrable. In the condition that $\epsilon<1/3$ and $\eta>0$ (for self-focusing media), the result of the integral reads

$$w=w_0[{\rm cn}^2(\beta z,k_{\rm m})+\Lambda_\epsilon{\rm sn}^2(\beta z,k_{\rm m})]^{1/2}, \tag{19}$$

where ${\rm sn}(z,k_{\rm m})$ and ${\rm cn}(z,k_{\rm m})$ are the sine-amplitude and the cosine-amplitude of Jacobian elliptic functions [31], respectively, $k_{\rm m}=\sqrt{\varrho_-/\varrho_+}$, $\beta=\beta_0\sqrt{\varrho_+/2}$, $\varrho_\pm=1-3\epsilon\pm[(\epsilon-1)^2-4\epsilon P_{\rm c0}/P_0]^{1/2}$, $\beta_0=(\gamma\eta P_0)^{1/2}$, and

$$\Lambda_\epsilon=\frac{1-\epsilon-\sqrt{(\epsilon-1)^2-4\epsilon P_{\rm c0}/P_0}}{2\epsilon}. \tag{20}$$

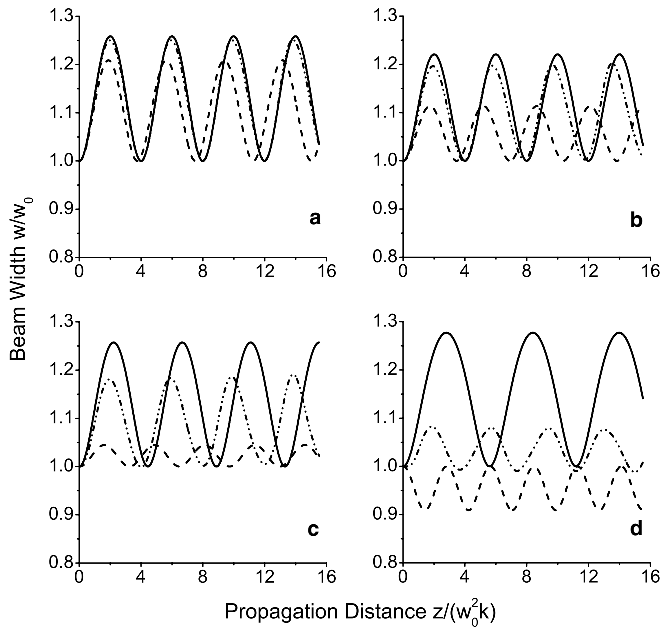

Fig. 1. Comparison of the analytical variational beam widths (solid curves) with the exact numerical ones (dash-dot curves) and these from Snyder–Mitchell model (dashed curves) for the beam evolution in the Gaussian-shaped response material in the case that $P_0 < P_c$. (a) $w_0/w_m = 0.2$ and $P_0/P_c = 0.643$, (b) $w_0/w_m = 0.3$ and $P_0/P_c = 0.698$, (c) $w_0/w_m = 0.4$ and $P_0/P_c = 0.696$, as well as (d) $w_0/w_m = 0.5$ and $P_0/P_c = 0.756$.

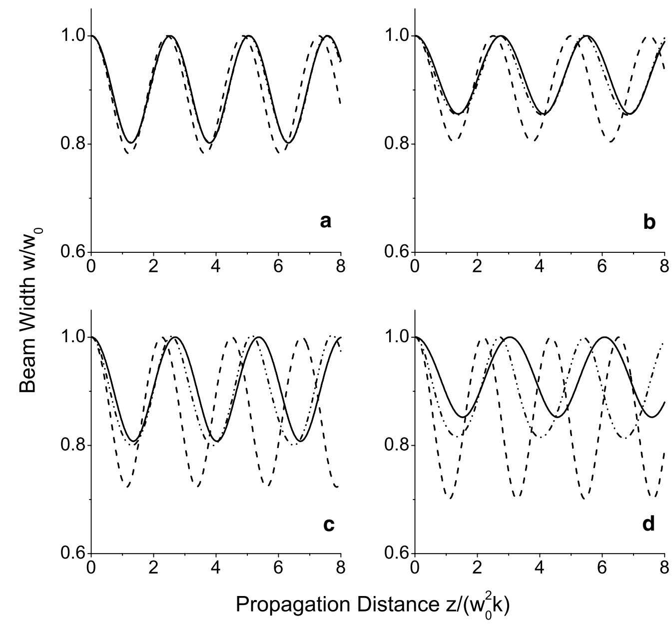

Fig. 2. Comparison of the analytical variational beam widths (solid curves) with the exact numerical ones (dash-dot curves) and these from Snyder–Mitchell model (dashed curves) for the beam evolution in the Gaussian-shaped response material in the case that $P_0 > P_c$. (a) $w_0/w_m = 0.2$ and $P_0/P_c = 1.537$ (The dash-dot and solid curves are so close that they almost can not be distinguished in this case.), (b) $w_0/w_m = 0.3$ and $P_0/P_c = 1.338$, (c) $w_0/w_m = 0.4$ and $P_0/P_c = 1.453$, as well as (d) $w_0/w_m = 0.5$ and $P_0/P_c = 1.273$.

The solution (19) is periodic, and its period is

$$T_0 = 2\mathbf{K}(k_m)/\beta, \tag{21}$$

where **K** is the complete elliptic integral of the first kind [31].

The phase-front curvature of the beam, c, can be found by direct substitution of Eq. (19) into (12), and its phase delay, α, can be obtained by a integration of Eq. (14).

By using the definition of the critical power, we can observe that when $P_0 = P_c$, Λ_ϵ becomes 1, then Eq. (19) gives $w(z) = w_0$ because $\mathrm{cn}^2(z, k_m) + \mathrm{sn}^2(z, k_m) = 1$ [31]. This is a stationary solution, i.e., a spatial soliton. In such a situation, the beam diffraction is exactly balanced by the beam-induced refraction. When $P_0 < P_c$, the diffraction initially overcomes the refraction, and the beam initially expands, with w^2/w_0^2 vibrating between the maximum Λ_ϵ ($\Lambda_\epsilon > 1$ in this case) and the minimum 1, which is shown by the solid curves in Fig. 1; whereas when $P_0 > P_c$, the reverse happens and the beam initially contracts, with w^2/w_0^2 breathing between the maximum 1 and the minimum Λ_ϵ ($\Lambda_\epsilon < 1$ now), shown by the solid curves in Fig. 2.

3. Discussion

3.1. Comparison with numerical simulation of NNLSE

Figs. 1 and 2 show the comparison of the analytical variational solutions for the sub-strong non-locality with the exact results of numerical simulation of Eq. (1) together with the input condition of a Gaussian function

$$\psi(x,z)|_{z=0} = \frac{\sqrt{P_0}}{(\sqrt{\pi}w_0)^{1/2}} \exp\left(-\frac{x^2}{2w_0^2}\right). \tag{22}$$

The analytical variational predictions are found to be close approximation to the simulations.

To simulate the propagation, we assume the material response is the Gaussian function [10,14] $R(x) = \exp(-x^2/2w_m^2)/(\sqrt{2\pi}w_m)$, and use the normalized transform [15] $X = x/w_0, Z = z/w_0^2k, \Psi = kw_0\eta^{1/2}\psi$, where X, Z, and Ψ are the variables in the normalized coordinate system. In such a normalized coordinate system, the evolution of the optical beams described by the NNLSE (1) and the initial condition (22) is completely determined by the two free parameters P_0/P_c and w_0/w_m [15].

3.2. Comparison with strong non-locality

In the limit case that ϵ tends to zero ($w_m \to \infty$), by use of the related properties of Jacobian elliptic functions that read [31] $\mathrm{cn}(z,0) = \cos(z)$, $\mathrm{sn}(z,0) = \sin(z)$, $\mathbf{K}(0) = \pi/2$, we can have that

$$\beta \to \beta_0, \quad k_m \to 0, \quad \Lambda_\epsilon \to P_{c0}/P_0,$$
$$T_0 \to \pi/\beta_0, \quad P_c \to P_{c0} \tag{23}$$

then Eq. (19) degenerates as

$$w = w_0\left[\cos^2(\beta_0 z) + \frac{P_{c0}}{P_0}\sin^2(\beta_0 z)\right]^{1/2}. \tag{24}$$

This is exactly the accessible soliton obtained by Snyder and Mitchell [1], and P_{c0}, the critical power of the limit case, does be the critical power of the accessible soliton [By using the Table in [4] (see, its reference [17]), one can convert the quantities (symbols) in [1] to the corresponding ones in this paper and vice versa.].

We can observe some differences between the two cases. For the strong non-locality the width of the optical beam with the Gaussian profile changes sinusoidally [1,14], while it does in the way given by Eq. (19), a combination of Jacobian elliptic functions, in the sub-strong non-locality. Although the evolution of the beam for both of the two cases is periodic, the dependence of the periods upon the parameters is different. The period, given by Eq. (21), depends on not only the input power, but also the initial beam width (via ϵ) for the sub-strong non-locality, but in the case of the strong non-locality, the period (π/β_0) is independent of the initial width.

In order to show which of the results, the Jacobian elliptic function or the trigonometrical function, is more approximate to the exact solution, we compare the beam evolution from the variational approach for the sub-strong non-locality, the Snyder–Mitchell model for the strong non-locality, and the numerical simulation in Figs. 1 and 2. We also compare the on-axis amplitude $|\psi(0,z)|$ obtained from the three different ways in Table 1. Through the comparison we can conclude that when $w_0/w_m \leqslant 0.1$, both of the variational approach and the Snyder–Mitchell model have an excellent approximation to the NNLSE, but as the non-locality becomes weaker (w_0/w_m becomes bigger), the variational approach gives a more exact quantitative description of the beam propagation in the non-local nonlinear media than the Snyder–Mitchell model and the precision of the former is almost one order higher than the latter with a few exceptions. In the case that $w_0/w_m = 0.5$ and $P_0/P_c = 0.756$ (Fig. 1(d)), Snyder–Mitchell model even fails to give a right prediction. As a result, the Jacobian elliptic function give more exact prediction than the trigonometrical function in the sub-strong non-locality.

3.3. Conditions for the variational solution

In this section, we discuss the limits of the parameters where the solution (19) is valid.

First, the condition for solution (19) is $\epsilon < 1/3$. This limit come from the process of the integral of Eq. (18), and it turns out that

$$\frac{w_0}{w_m} < \sqrt{\frac{1}{3\mathscr{Q}}}. \tag{25}$$

This equation tells us that $(3\mathscr{Q})^{-1/2}$ is a up limit of the ratio w_0/w_m for the variational solution (19). For Gaussian response media where $\mathscr{Q} = 3/4$, the up limit of the ratio $(w_0/w_m)_{\text{up limit}} = 2/3$.

Second, as mentioned in the foregoing, $R(x)$ is expanded to the fourth-order to analytically determine $\mathscr{L}_r$, and this demands that $w/w_m \leqslant \Delta \approx 1$, where Δ is the up limit for the expansion holding. On the other hand, when $P_0 < P_c$, the beam width will initially diffracts to its maximum w_{max}. If nonlinearity is not strong enough, w_{max}/w_m will exceed Δ, and the much more error of the approximate solution (19) to the exact one will be introduced. From Eq. (19), we observe that $w_{max}^2 = w_0^2 \Lambda_\epsilon$, then it is obtained that $\Lambda_\epsilon \epsilon/\mathscr{Q} (= w_{max}^2/w_m^2) \leqslant \Delta^2$, which turns out to be

$$P_0 \geqslant P_{vc} = \frac{(1-2\epsilon)\epsilon P_c}{\Delta^2 \mathscr{Q}(1-\epsilon-\Delta^2\mathscr{Q})}. \tag{26}$$

Here P_{vc} gives out another critical value, than which input power should be larger to insure that Eq. (19) yields a right

Table 1
The exact numerical results, the analytical Jacobian elliptic solutions (the variational approach), the analytical trigonometrical solutions (Snyder–Mitchell model), and their relative errors for the maximums (or minimums) of the on-axis amplitude $|\psi(0,z)|/\psi(0,0)$

w_0/w_m $\epsilon(<1/3)$	0.1 0.0075	0.2 0.03	0.3 0.0675	0.4 0.12	0.5 0.1875
The case that $P_0 < P_c$ ($|\psi(0,z)|$ has the minimum)					
P_0/P_c	0.713	0.643	0.698	0.696	0.756
ER[a]	0.919	0.894	0.912	0.916	0.961
AVR[b]	0.918	0.891	0.905	0.892	0.885
ATR[c]	0.922	0.910	0.948	0.978	1.049[d]
RE_V (%)[e]	0.1	0.3	0.8	2.6	7.9
RE_T (%)[e]	−0.3	−1.8	−3.9	−6.8	X[d]
The case that $P_0 > P_c$ ($|\psi(0,z)|$ has the maximum)					
P_0/P_c	1.348	1.537	1.338	1.453	1.273
ER	1.078	1.117	1.082	1.118	1.111
AVR	1.078	1.117	1.081	1.113	1.083
ATR	1.082	1.131	1.115	1.176	1.195
RE_V (%)	0.0	0.0	0.1	0.4	2.5
RE_T (%)	−0.4	−1.3	−3.0	−5.2	−7.6

[a] ER: exact numerical results to Eq. (1) along with the initial condition of the Gaussian function.
[b] AVR: analytical variational results to Eq. (1), $|\psi(0,z)|_{max}[|\psi(0,z)|_{min}]/\psi(0,0) = \Lambda_\epsilon^{-1/4}$.
[c] ATR: analytical trigonometrical results to Eq. (1), $|\psi(0,z)|_{max}[|\psi(0,z)|_{min}]/\psi(0,0) = (P_0/P_{c0})^{1/4}$.
[d] This is the case that the analytical result fails to give right prediction.
[e] RE: relative errors, $RE_V = (ER - AVR)/ER$, and $RE_T = (ER - ATR)/ER$.

prediction. Taking example for Gaussian response, we have $P_{vc} \approx 0.8P_c$ if taking $\epsilon = 1/5(w_0/w_m \approx 1/2)$ and $\Delta = 0.8$.

4. Conclusion

We discuss the evolution of the optical beam in the planar non-local cubic nonlinear medium waveguide, modelled by the $1+1$D NNLSE with a symmetric and real-valued response kernel. The approximate variational solution is obtained for the sub-strong non-locality case, which is confirmed by the numerical simulation of the NNLSE.

The strong non-locality is the case that the characteristic length of nonlinear response is much larger than the spatial extent occupied by light beams, while the sub-strong non-locality the situation when the extent occupied by the light beams is about but less than the characteristic length of the response function. The evolution of the beam width of a single Gaussian-shaped beam is qualitatively periodic for both cases, but the quantitative description is different. It is described by a combination of the Jacobian elliptic functions for the sub-strong non-locality, while by the trigonometric functions for the strong non-locality. The period depends on not only the input power, but also the initial beam width for the sub-strong non-locality, which is different from that in the strong non-locality where the period is independent of the initial width.

It is reasonable for us to believe that the extension of the variational approach from the $1+1$D case to the $1+2$D case should be feasible.

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