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Continuous time Markov chains observed on an alternating renewal process with exponentially distributed durations

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Abstract

A continuous time Markov chain observed on a system following the dynamic behavior of an alternating renewal process is studied. This alternating renewal process is assumed to have exponentially distributed durations. The limiting behavior of the process is examined. Examples of applications are given.

Keywords: Alternating renewal process; Continuous time Markov chain; M/M/1 queue; G/M/1 queue

1. Introduction

Continuous time Markov chains are often applied in different areas of sciences including epidemiology, manufacturing systems and queueing networks. Alternating renewal processes are frequently observed natural phenomena, but do not appear often in the literature. Their practical applications include electric power system (Mortensen, 1990) and healthy and sick periods (Ramsay, 1984). Some of their special cases, such as machine breakdowns or servers on vacation, have been extensively studied with the queueing process (see, for example, Doshi, 1990). There have not been many studies on continuous time Markov chains observed from a system following an alternating renewal process.

In this paper we consider a stochastic process $X(t)$ resulted from a continuous time Markov chain having a discrete state space $\{0, 1, 2, \ldots\}$, observed on a system which consists of a sequence $Y = \{Y_1^{(1)}, Y_1^{(2)}, \ldots, Y_1^{(r)}, Y_2^{(1)}, Y_2^{(2)}, \ldots, Y_2^{(r)}, \ldots, Y_n^{(1)}, Y_n^{(2)}, \ldots, Y_n^{(r)}, Y_{n+1}^{(1)} \ldots\}$ of mutually independent, exponential random variables with $E(Y_n^{(i)}) = 1/\alpha_i$, for $i = 1, 2, \ldots, r$, and $n = 1, 2, \ldots$. This system is a special case of an alternating renewal process of r stages (see, for example, Karlin and Taylor, 1975, p. 207). In different stages we assume this continuous time Markov chain may have different parameters but otherwise preserves its properties as the original chain. In other words, for a realization of Y, $X(t)$ is a piecewise continuous time Markov chain. A typical example of this model is an M/M/1 queue with service breakdown as described in

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Avi-Itzhak and Noar (1963) or a more general model—a Markov modulated queue proposed by Regterschot and de Smit (1986). Note that the model discussed in this paper belongs to the family of switching processes, which can be found in Anisimov (1977) or the more recent related work, for example Anisimov (1995).

In Section 2, we study the limiting behaviors of this process and the embedded processes observed at the beginning of each stage. The limiting probability distribution of the whole process is derived in terms of stationary probabilities of the embedded processes. In Section 3, we apply these results in two models to calculate their limiting probabilities. One application leads to the result that has not been previously investigated. The other application confirms a well-known result.

2. Limiting probability distribution

Given a realization of Y, let $P_i(t)$ denote the transition probability matrices of $X(\cdot)$ for duration t when the system is on stage i, $i = 1, 2, \ldots, r$. We shall consider the embedded processes $X_n^{(i)}$ defined by $X_n^{(i)} = X(\sum_{l=1}^{n-1}\sum_{k=1}^{r} Y_l^{(k)} + \sum_{k=1}^{i-1} Y_n^{(k)})$ where $i = 1, 2, 3, \ldots, r$, and $n = 1, 2, 3, \ldots$. Thus, $X_n^{(i)}$ is the state of $X(t)$ at the beginning of stage i. Assume that the limiting probability vectors of $X_n^{(i)}$ exist, for $i = 1, 2, 3, \ldots, r$, and be denoted by the row vectors $\boldsymbol{\pi}^{(i)}$. Then a standard result on Markov chains gives, for $i = 1, 2, 3, \ldots, r$,

$$\boldsymbol{\pi}^{(i)} A_i A_{i+1} \cdots A_r A_1 \cdots A_{i-1} = \boldsymbol{\pi}^{(i)}, \tag{1}$$

where the matrices $A_i = \int_0^\infty P_i(t)\alpha_i \exp(-\alpha_i t)\,\mathrm{d}t$.

Lemma. *If the limiting probabilities of $X_n^{(1)}$ exist, then*

$$\boldsymbol{\pi}^{(j)} = \boldsymbol{\pi}^{(i)} \prod_{k=i}^{j-1} A_k \quad \textit{for } 1 \leqslant i \leqslant j \leqslant r$$

and

$$\boldsymbol{\pi}^{(j)} = \boldsymbol{\pi}^{(i)} \left(\prod_{k=i}^{r} A_k\right)\left(\prod_{k=1}^{j-1} A_k\right) \quad \textit{for } 1 \leqslant j < i \leqslant r.$$

Proof. Eq. (1) along with the existence of $\boldsymbol{\pi}^{(1)}$ guarantees existence $\boldsymbol{\pi}^{(j)}$. For the case when $i = j$, the result follows immediately from Eq. (1). Without loss of generality, we shall prove for the case when $1 \leqslant i < j \leqslant r$. Post-multiplying the row vector $\boldsymbol{\pi}^{(i)} A_i \cdots A_{j-1} - \boldsymbol{\pi}^{(j)}$ by $A_j \cdots A_r A_1 \cdots A_{j-1}$ and applying (1) to manipulate the algebra, one can show that $\boldsymbol{\pi}^{(i)} A_i \cdots A_{j-1} - \boldsymbol{\pi}^{(j)}$ is a row eigenvector of the matrix $A_j \cdots A_r A_1 \cdots A_{j-1}$ associated with the eigenvalue 1. Since, from (1) $\boldsymbol{\pi}^{(j)}$ is also a row eigenvector associated with the same eigenvalue, we have

$$\boldsymbol{\pi}^{(i)} A_i \cdots A_{j-1} - \boldsymbol{\pi}^{(j)} = c\boldsymbol{\pi}^{(j)} \tag{2}$$

for some constant c. Since $\boldsymbol{\pi}^{(i)} A_i \cdots A_{j-1}$ and $\boldsymbol{\pi}^{(j)}$ are both vectors of probability distributions, post-multiplying both sides of (2) by the column vector $\mathbf{1} = (1, 1, \ldots 1)'$, algebraic manipulations lead to $c = 0$. The proof of the Lemma is completed. □

If for $i = 1, 2, 3, \ldots, r$, the transition matrix $P_i(t)$ has an infinitesimal matrix Q_i, then we can calculate A_i in (1) as

$$A_i = \int_0^\infty \exp(Q_i t)\alpha_i \exp(-\alpha_i t)\,\mathrm{d}t. \tag{3}$$

Note that, if the absolute values of all eigenvalues of Q_i are less than α_i, then (3) can be further simplified as

$$A_i = \sum_{n=0}^{\infty} \left(\frac{Q_i}{\alpha_i}\right)^n = \left(I - \frac{Q_i}{\alpha_i}\right)^{-1}.$$

It is clear that the non-lattice times $\sum_{l=1}^{n-1}\sum_{k=1}^{r} Y_l^{(k)}, n = 1,2,3,\ldots$ at the beginning of a new cycle form multidimensional renewal times for $X(t)$. Furthermore, the cycle times $C_i = Y_i^{(1)} + Y_i^{(2)} + \cdots + Y_i^{(r)},\ i = 1,2,3,\ldots$ have a common finite expected length and are independent of the starting state. If their embedded Markov chain $X_n^{(1)},\ \ n = 1,2,3,\ldots$ is ergodic with stationary distribution $\boldsymbol{\pi}^{(1)}$, by the multidimensional renewal theorem, the limiting distribution, of $X(t)$ exists. Let $\boldsymbol{\rho} = (\rho_j)_{j=1}^{\infty}$ denote this limiting distribution. Then the stationary distribution of $X(t)$ can be expressed as

$$\rho_j = \frac{E(W_{1j})}{E(C_1)}, \tag{4}$$

where W_{1j} is the duration that the process $X(t)$ spent at j in the first cycle when it starts with $\boldsymbol{\pi}^{(1)}$.

Eq. (4) and algebraic manipulations lead to

$$\begin{aligned}
\boldsymbol{\rho} &= \frac{1}{\sum_{k=1}^{r}\frac{1}{\alpha_k}}\left[\sum_{k=1}^{r}\boldsymbol{\pi}^{(k)}\int_0^\infty \int_0^t P_k(s)\,\mathrm{d}s\alpha_k \mathrm{e}^{-\alpha_k t}\,\mathrm{d}t\right] \\
&= \frac{1}{\sum_{k=1}^{r}\frac{1}{\alpha_k}}\pi^{(1)}\left[\sum_{k=1}^{r}\left(\prod_{i=1}^{k-1}\int_0^\infty P_i(t)\alpha_i \mathrm{e}^{-\alpha_i t}\,\mathrm{d}t\right)\left(\int_0^\infty\left(\int_0^t P_k(s)\,\mathrm{d}s\right)\alpha_k \mathrm{e}^{-\alpha_k t}\,\mathrm{d}t\right)\right] \\
&= \frac{1}{\sum_{k=1}^{r}\frac{1}{\alpha_k}}\pi^{(1)}\left[\sum_{k=1}^{r}\left(\prod_{i=1}^{k-1}\int_0^\infty P_i(t)\alpha_i \mathrm{e}^{-\alpha_i t}\,\mathrm{d}t\right)\left(\int_0^\infty P_k(s)\mathrm{e}^{-\alpha_k s}\,\mathrm{d}s\right)\right] \\
&= \frac{1}{\sum_{k=1}^{r}\frac{1}{\alpha_k}}\pi^{(1)}\left[\sum_{k=1}^{r}\left(\frac{1}{\alpha_k}\prod_{i=1}^{k}A_i\right)\right],
\end{aligned} \tag{5}$$

where the matrices A_is are defined in (1). Note that, in (5) the first equality follows from (4) and the fact that the average time the process spent in j is equal to the weighted average of the average time the process spent in j on each stage of the system. The second equality is a direct application of the Lemma. The third and fourth equalities follow the algebraic calculations and the definition of A_is. Eq. (5) is derived when the process is considered to start at the beginning of stage 1.

Similarly, if we observe the process starting at the beginning of stage i, $i = 1,2,3,\ldots,r$, then we have

$$\boldsymbol{\rho} = \frac{1}{\sum_{k=1}^{r}\frac{1}{\alpha_k}}\boldsymbol{\pi}^{(i)}\left[\sum_{k=i}^{r}\left(\frac{1}{\alpha_k}\prod_{j=i}^{k}A_j\right) + \sum_{k=1}^{i-1}\frac{1}{\alpha_k}\left(\prod_{j=i}^{r}A_j\right)\left(\prod_{j=1}^{k}A_j\right)\right]. \tag{6}$$

From (5), (6) and the Lemma, we can state the following theorem.

Theorem 1. *If the limiting probabilities of* $X_n^{(1)}$ *exist, then*

$$\boldsymbol{\rho} = \frac{1}{\sum_{k=1}^{r}\frac{1}{\alpha_k}}\boldsymbol{\pi}^{(i)}\left[\sum_{k=i}^{r}\left(\frac{1}{\alpha_k}\prod_{j=i}^{k}A_j\right) + \sum_{k=1}^{i-1}\frac{1}{\alpha_k}\left(\prod_{j=i}^{r}A_j\right)\left(\prod_{j=1}^{k}A_j\right)\right] = \frac{1}{\sum_{k=1}^{r}\frac{1}{\alpha_k}}\left(\sum_{k=1}^{r}\frac{1}{\alpha_k}\boldsymbol{\pi}^{(k+1)}\right)$$

for $i = 1,2,3,\ldots,r$, *where, for notational convenience* $\boldsymbol{\pi}^{(r+1)} = \boldsymbol{\pi}^{(1)}$.

Remark 1. Under the conditions stated in Theorem 1, if $P_1(t) = P_2(t) = \cdots = P_r(t)$, then $\boldsymbol{\rho} = \boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^{(2)} = \cdots = \boldsymbol{\pi}^{(r)}$. This is also intuitively true, since the process in this case is independent of the underlying system.

Remark 2. Although $X(t)$ is Markov on any given stage and the sojourn time on each stage is exponentially distributed, the process may not preserve the Markov property.

3. Applications

In this section, we apply the results in Theorem 1 to two examples available in the literature.

Example 1. Consider an M/M/1 queue with server breakdown and customer discouragement. Thus, the system alternates between working and repair periods; each follows an exponential distribution. During the working periods, the system operates as an M/M/1 queue with arrival rate λ and service rate μ, and during the breakdown periods, no customers are served and the customers present will be discouraged and leave with a constant probability $\theta = 1 - \delta$. In addition, if a breakdown occurs at time T, then the number of customers left in the system at time T^+ is assumed to follow a binomial distribution $B(X(T), \delta)$. This example is to calculate the limiting probability distribution ρ. The necessary and sufficient conditions of its existence, along with other characteristics of this process have been proposed and examined by Chan et al. (1993). The limiting probability distribution of the whole process has not been previously investigated.

We shall first consider this problem as it were observed on a three-stage alternating renewal process, namely the working period, the occurrence of breakdown period and the repair period. The duration of each period is assumed to follow an exponential distribution with mean $1/\alpha_1$, $1/\alpha_2$ and $1/\alpha_3$, respectively. After applying Theorem 1, we shall approach this problem by letting $1/\alpha_2$ converge to 0.

For a fixed α_2, Theorem 1 leads to

$$\rho = \frac{1}{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}}\left(\frac{1}{\alpha_1}\boldsymbol{\pi}^{(1)}A_1 + \frac{1}{\alpha_2}\boldsymbol{\pi}^{(1)}A_1A_2 + \frac{1}{\alpha_3}\boldsymbol{\pi}^{(1)}\right),$$

where we adopt the corresponding notation as previously defined. In this example, $A_1 = [a_{ij}]$ and $A_2 = [b_{nk}]$ can be calculated as follows.

From Eq. (1), we have $a_{ij} = \int_0^\infty P_{ij}(t)\alpha_1 \exp(-\alpha_1 t)\,\mathrm{d}t$, where $P_{ij}(t)$ is the transition probability from state i to state j for an M/M/1 queue. For $i = 0$, Kleinrock (1975, p. 77) provides the explicit formula of $P_{0j}(t)$ and hence a_{0j}. Observing that

$$\begin{aligned} P_{00}(t) &= \int_0^t \lambda \exp(-\lambda\tau)P_{10}(t-\tau)\,\mathrm{d}\tau + \int_t^\infty \lambda\exp(-\lambda\tau)\,\mathrm{d}\tau \\ &= \int_0^t \lambda\exp(-\lambda(t-\tau))P_{10}(\tau)\,\mathrm{d}\tau + \mathrm{e}^{-\lambda t} \\ &= \lambda\mathrm{e}^{-\lambda t}\int_0^t \exp(\lambda\tau)P_{10}(\tau)\,\mathrm{d}\tau + \mathrm{e}^{-\lambda t} \end{aligned}$$

we can obtain $\mathrm{e}^{\lambda t}P_{00}(t) = \lambda\int_0^t \lambda\exp(\lambda\tau)P_{10}(\tau)\,\mathrm{d}\tau + 1$ and thus $P_{10}(t) = P_{00}(t) + (1/\lambda)P'_{00}(t)$. Similarly, one can calculate $P_{1j}(t) = P_{0j}(t) + (1/\lambda)P'_{0j}(t)$ and $P_{i+1,j}(t) = (1/\lambda)(P'_{ij}(t) + (\lambda+\mu)P_{ij}(t) - \mu P_{i-1,j}(t))$, for $i \geqslant 1$. From these iterative formulas of $P_{ij}(t)$ and the relationship

$$\int_0^\infty P'_{ij}(t)\alpha_1\exp(-\alpha_1 t)\,\mathrm{d}t = \alpha_1\mathrm{e}^{-\alpha_1 t}P_{ij}(t)\Big|_0^\infty + \int_0^\infty \alpha_1^2\exp(-\alpha_1 t)P_{ij}(t)\,\mathrm{d}t = \alpha_1\delta_{ij} + \alpha_1 a_{ij}, i \geqslant 1,$$

one can derive the iterative formulas for a_{ij} as $a_{1j} = a_{0j} + (1/\lambda)(\alpha_1\delta_{ij} + \alpha_1 a_{1j})$ and $a_{i+1,j} = (1/\lambda)(\alpha_1\delta_{ij} + \alpha_1 a_{ij} + (\lambda+\mu)a_{ij} - \mu a_{i-1,j})$, $i \geqslant 1$, where δ_{ij} is the Kronecker delta. For the computation of b_{nk}, we create a surrogate second stage renewal time, called $Y^{(2)}_{n,m}$ that satisfies the assumption of Theorem 1 and has the same limiting behavior as of our occurrence of breakdown period. Let

$$(Q_{2,m})_{ij} = \begin{cases} m\binom{i}{j}(1-\theta)^j\theta^{i-j}, & j<i \\ -m((1-(1-\theta)^i), & j=i \end{cases}$$

and $Y^{(2)}_{n,m} = \min$ (the first jump time after τ, V_m), where $\tau = \sum_{l=1}^{n-1}\sum_{k=1}^{3} Y_l^{(k)} + Y_n^{(1)}$ is the beginning of the second stage, $X(\tau) = i$ and V_m is exponentially distributed with rate $m(1-\theta)^i$. Then $Y^{(2)}_{n,m}$ has an exponential distribution with rate m. As $m \to \infty$, $Y^{(2)}_{n,m}$ converges to the degenerate distribution and behavior like our occurrence of breakdown period. For any m, Eq. (1) implies $b_{nk} = \binom{n}{k}\delta^k(1-\delta)^{n-k}$, for $k \leqslant n$ and $b_{nk} = 0$ otherwise. Although $Y^{(2)}_{n,m}$ depends on $X(t)$, the property of convergence to a degenerate distribution vanishes this dependence.

Now, letting $\alpha_2 = m \to \infty$, we have

$$\boldsymbol{\rho} = \frac{1}{\frac{1}{\alpha_1}+\frac{1}{\alpha_3}}\left(\frac{1}{\alpha_1}\boldsymbol{\pi}^{(1)}A_1 + \frac{1}{\alpha_3}\boldsymbol{\pi}^{(1)}\right). \tag{7}$$

Note that the probability generating function (p.g.f.) $H(s)$ of the limiting probability distribution $\boldsymbol{\pi}^{(1)}$ at the beginning of the working period can be obtained by using the results of Chan et al. (1993), in which the p.g.f. at the beginning of each working period is provided as

$$E(z^{X_n}\big|X_0) = \left[\prod_{j=1}^{n} b_j(z)\right][a_n(z)]^{X_0} + \sum_{i=0}^{n-1} d_n(i,z)\left[\prod_{j=1}^{i} b_j(\phi)\right][a_i(z)]^{X_0},$$

where

$$a_i(z) = \delta^i z + 1 - \delta^i, \quad b_i(z) = -\alpha_1\alpha_3 a_i(z)/\{\lambda[(\alpha_3 + \lambda\theta(1 - a_{i-1}(z)))(a_i(z) - \phi)(a_i(z) - \psi)]\}$$

and

$$d_n(n-i,z) = \left[\prod_{j=1}^{i-1} b_j(\phi)\right]c_i(\phi) + \sum_{m=1}^{i-1} d_n(n-i+m,z)\left[\prod_{j=1}^{m-1} b_j(\phi)\right]c_m(\phi).$$

In the last equation, $c_i(z) = -\alpha_1\alpha_3\phi(1 - a_i(z))/\{\lambda(1-\phi)[(\alpha_3 + \lambda\theta(1 - a_{i-1}(z)))(a_i(z) - \phi)(a_i(z) - \psi)]\}$. Hence, the p.g.f. $G(s)$ of the limiting probability distribution ρ of the whole process can be calculated as follows.

Let $\boldsymbol{s} = (s^0, s^1, \ldots, s^n, \ldots)$ then $G(s) = \sum_{k=0}^{\infty} \rho_k s^k = \boldsymbol{\rho s}'$ and $H(s) = \boldsymbol{\pi}^{(1)}\boldsymbol{s}'$. From (7) we can express $G(s)$ as

$$G(s) = \boldsymbol{\rho s}' = \frac{1}{\frac{1}{\alpha_1}+\frac{1}{\alpha_3}}\left(\frac{1}{\alpha_1}\pi^{(1)}A_1 s' + \frac{1}{\alpha_3}\pi^{(1)}s'\right). \tag{8}$$

Using the conventional notation $P^{(1)}(t) = (p_{jk}(t))$ for the transition matrix of the working period and assuming the arrival rate for the M/M/1 queue is λ and the service rate is μ, then we have

$$[A_1]_{jk} = a_{jk} = \left[\int_0^{\infty} p_{jk}(t)\alpha_1 \exp(-\alpha_1 t)\,\mathrm{d}t\right].$$

Furthermore,

$$\begin{aligned}
\boldsymbol{\pi}^{(1)}A_1\boldsymbol{s}' &= \sum_{j=0}^{\infty}\left[\int_0^{\infty}\left[\sum_{k=0}^{\infty} p_{jk}(t)s^k\right]\alpha_1\exp(-\alpha_1 t)\,\mathrm{d}t\right]\boldsymbol{\pi}_j^{(1)} \\
&= \alpha_1 \sum_{j=0}^{\infty}\left[\frac{s^{j+1} - (1-s)\varphi^{j+1}(1-\varphi)^{-1}}{-\lambda(s-\varphi)(s-\eta)}\right]\pi_j^{(1)} \\
&= \alpha_1 \frac{sH(s) - (1-s)\frac{\varphi}{1-\varphi}H(\varphi)}{-\lambda(s-\varphi)(s-\eta)},
\end{aligned} \tag{9}$$

where

$$\varphi = \frac{(\lambda+\mu+\alpha_1) - \sqrt{(\lambda+\mu+\alpha_1)^2 - 4\lambda\mu}}{2\lambda}$$

and

$$\eta = \frac{(\lambda+\mu+\alpha_1) + \sqrt{(\lambda+\mu+\alpha_1)^2 - 4\lambda\mu}}{2\lambda}$$

are from the Laplace transform of the p.g.f. of the M/M/1 queue size. The derivation of this Laplace transform can be found in, for example Bailey (1964).

Note that the first equality in (9) follows the definition of A_1 and the second equality is the result of Fubini's theorem. The third equality comes from the expression of the Laplace transform of the p.g.f. of the M/M/1 queue size and the last equality is a routine algebraic manipulation. Combining (8) and (9), we obtain the p.g.f. of the limiting probability distribution ρ of the whole process as

$$G(s) = \frac{1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_3}} \left(\frac{sH(s) - (1-s)\frac{\varphi}{1-\varphi}H(\varphi)}{-\lambda(s-\varphi)(s-\eta)} + \frac{1}{\alpha_3}H(s) \right).$$

Example 2. Consider a G/M/1 queue with the mean inter-arrival time $1/\lambda$, and the mean service time $1/\mu$. One can define a two-stage alternating renewal process embedded from this queue. The first stage is the period between two consecutive arrivals and the second stage is the instant that the new arriving customer joins the system. Mathematically, the second stage will be treated as an exponentially distributed period with mean converging to 0. Assuming the distribution of the inter-arrival time is $G(t)$, then, in this alternating process $F_1(t) = G(t)$ and $F_2(t) = 1 - e^{-\alpha_2 t}$ with $\alpha_2 \to \infty$. If the G/M/1 queue is observed on this alternating renewal process, then the vector of limiting probabilities of queue size seen by the new arrivals, will be equal to $\pi^{(1)}$. Similar to the techniques presented in Example 1, we can obtain the limiting probability distribution $\boldsymbol{\rho}$.

Let $\boldsymbol{\pi}^{(1)} = (\pi_0^{(1)}, \pi_1^{(1)}, \pi_2^{(1)}, \ldots)$, then it is known that $\pi_k^{(1)} = (1-\beta)\beta^k$, $k = 0, 1, 2, \ldots$ where $\beta = \int_0^\infty e^{-\mu t(1-\beta)}\, dG(t)$, (see, for example, Ross, 1996, p. 179–180). Hence, (5) implies

$$\boldsymbol{\rho} = \boldsymbol{\pi}^{(1)} \frac{\int_0^\infty \int_0^t P(s)\, ds\, dG(t)}{\frac{1}{\lambda}},$$

where

$$P(s) = (P_{ij}(s)) \quad \text{with } P_{i,i+1-j}(s) = e^{-\mu s}\frac{(\mu s)^j}{j!}, \quad j = 0, 1, 2, \ldots, i, \quad \text{and} \quad P_{i,0}(s) = \sum_{k=i+1}^{\infty} e^{-\mu s}\frac{(\mu s)^k}{k!}$$

(as presented by Ross (1996, p. 165)).

Thus, for $k \geqslant 1$ we have

$$\begin{aligned}
\rho_k &= \lambda \int_0^\infty \int_0^t \sum_{i=0}^\infty \pi_i^{(1)} P_{ik}(s)\, ds\, dG(t) = \lambda \int_0^\infty \int_0^t \sum_{i=k-1}^\infty (1-\beta)\beta^i \frac{e^{-\mu s}(\mu s)^{i+1-k}}{(i+1-k)!}\, ds\, dG(t) \\
&= \lambda \int_0^\infty \int_0^t \sum_{m=0}^\infty (1-\beta)\beta^{m+k-1} \frac{e^{-\mu s}(\mu s)^m}{m!}\, ds\, dG(t) \\
&= \lambda(1-\beta)\beta^{k-1} \int_0^\infty \int_0^t e^{-(\mu-\beta\mu)s}\, ds\, dG(t) \\
&= \lambda(1-\beta)\beta^{k-1} \int_0^\infty \frac{1 - e^{-(\mu-\beta\mu)t}}{\mu - \mu\beta}\, dG(t) \\
&= \lambda(1-\beta)\beta^{k-1} \frac{1-\beta}{\mu-\mu\beta} = \frac{\lambda}{\mu}(1-\beta)\beta^{k-1}
\end{aligned}$$

which coincides with the results in Kleinrock (1975, p. 251).

Note that the same limiting probabilities for the G/M/m queue can also be calculated using this method. The expressions for$P_{ij}(t)$ and $\pi^{(1)}$ are different from those in the G/M/1 queue. Their recursive form can be found in Kleinrock (1975, p. 254).

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References

Anisimov, V., 1977. Switching processes. Cybernetics 13, 590–595.

Anisimov, V., 1995. Switching processes: averaging principle, diffusion approximation and applications. Acta Appl. Math. 40, 95–141.

Avi-Itzhak, B., Naor, P., 1963. Some queueing problems with service station subject to breakdowns. Oper. Res. 10, 303–320.

Bailey, N., 1964. The Elements of Stochastic Processes with Applications to the Natural Sciences. Wiley, New York.

Chan, W., Bartoszyski, R., Pearl, D., 1993. Queues with breakdowns and customers discouragement. Probab. Math. Statist. 14, 77–87.

Doshi, B., 1990. Generalizations of the stochastic decomposition results for single server queues with vacations. Stoch. Models 6, 307–333.

Karlin, S., Taylor, H.M., 1975. A First Course in Stochastic Processes. Academic Press, New York.

Kleinrock, L., 1975. Queueing Systems, vol. 1: Theory. Wiley, New York.

Mortensen, R., 1990. Alternating renewal process models for electric power system loads. IEEE Trans. Automat. Control 35, 1245–1249.

Ramsay, C., 1984. The asymptotic ruin problem when the healthy and sick periods form an alternating renewal process. Insurance Math. Econom. 3, 139–143.

Regterschot, G.J.K., de Smit, J.H.A., 1986. The queue M/G/1 with Markov modulated arrivals and services, Math. Oper. Res. 11, 465–483.

Ross, S., 1996. Stochastic Processes, second ed. Wiley, New York.