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MULTISTABILITY IN RECURRENT NEURAL NETWORKS*

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Abstract. Stable stationary solutions correspond to memory capacity in the application of associative memory for neural networks. In this presentation, existence of multiple stable stationary solutions for Hopfield-type neural networks with delay and without delay is investigated. Basins of attraction for these stationary solutions are also estimated. Such a scenario of dynamics is established through formulating parameter conditions based on a geometrical setting. The present theory is demonstrated by two numerical simulations on the Hopfield neural networks with delays.

Key words. neural network, multistability, delay equations

AMS subject classifications. 34D20, 34D45, 92B20

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1. Introduction.

The studies of neural networks have attracted considerable multidisciplinary research interest in recent years. The developments for neural network models and the theory for the models are, on the one hand, driven by application motif or inspired by biological neuronal behaviors. On the other hand, the neural network theory has motivated and elicited further progress in dynamical system theory. For example, theory for existence of many stable patterns or chaotic dynamics for systems in phase space of large dimension is in strong demand for neural network applications. The progress in this direction of research has also enriched dynamical system theory [6, 17, 27].

The applications of neural networks range from classifications, associative memory, image processing, and pattern recognition to parallel computation and its ability to solve optimization problems. The theory on the dynamics of the networks has been developed according to the purposes of the applications. In the application to parallel computation and signal processing involving finding the solution of an optimization problem, the existence of a computable solution for all possible initial states is the best situation. Mathematically, this means that the network needs to have a unique equilibrium which is globally attractive. Such a convergent behavior is referred to as "monostability" of a network. On the other hand, when a neural network is employed as an associative memory storage or for pattern recognition, the existence of many equilibria is a necessary feature [7, 11, 16, 21]. The notion of "multistability" of a neural network is used to describe coexistence of multiple stable patterns such as equilibria or periodic orbits. In general, if the dynamics for a system are bounded, the existence of multiple stable patterns is accompanied with coexistence of stable and unstable equilibria or periodic orbits. The existence of unstable equilibria is essential in certain applications of neural network. For example, unstable equilibria are related to digital constraints on selection in winner-take-all problems [32, 33].

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Classical recurrent neural networks are usually systems of ordinary differential equations. Recently, neural network systems with delays have also been studied extensively, thanks to the need from practical applications and mathematical interests. In this presentation, we propose an approach to investigate existence of multiple stationary solutions and their stability for recurrent neural networks with delay and without delay. We shall illustrate our approach through the Hopfield-type model.

Hopfield-type neural networks and their various generalizations have been widely studied and applied in various scientific areas. A typical form for such a network is given by

$$C_i \frac{dx_i(t)}{dt} = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(x_j(t-\tau_{ij})) + I_i, \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where $C_i > 0$ and $R_i > 0$ are, respectively, the input capacitance and resistance associated with neuron i; I_i is the constant input; T_{ij} are the connection strengths between neurons; $\tau_{ij} > 0$ are the transmission delays; and $g_i, i = 1, 2, \ldots, n$, are neuron activation functions.

The classical Hopfield-type neural network [16] is system (1.1) without delay, that is, $\tau_{ij} = 0$ for all i, j. For the Hopfield-type neural networks, the theory of unique equilibrium and global convergence to the equilibrium has been extensively studied; cf. [9, 10] for the networks without delays and [5, 13, 19, 23, 24, 29, 30, 31, 34, 35] for the delay cases.

In contrast to these studies, we propose a treatment to explore the existence of multiple stationary solutions for (1.1) through a geometrical formulation on the parameter conditions. Stability of these equilibria for (1.1) with and without delay shall also be investigated. In addition, estimations of basins of attraction for these stable stationary solutions are derived. The stationary equations are identical for system (1.1) with delay and without delay. Thus, confirmation for the existence of equilibrium points is valid for both cases. However, stability of the equilibrium points and dynamical behaviors can be very different for the systems with delay and without delay. It is very interesting to explore such a difference as well as a possible coincidence of behaviors.

The theory for existence of multiple stable patterns has been developed for cellular neural networks [8, 17, 26, 27]. The neurons in such a system are locally connected and no time lags were considered therein. Our approach can be adopted to such a network with delays, as remarked in the later section. There are other interesting studies on delayed neural networks in [1, 2, 12, 22, 25].

This presentation is organized as follows. In section 2, we establish conditions for existence of 3^n equilibria for the Hopfield network. 2^n equilibria among them will be shown to be asymptotically stable for the system without delays, through a linearization analysis. In section 3, we shall verify that under the same conditions, there are 2^n regions in $\mathbb{R}^n$, each containing an equilibrium, which are positively invariant under the flow generated by the system with delays and without delays. Subsequently, it is argued that these 2^n equilibria are asymptotically stable, even in the presence of delays. We also formulate more sufficient conditions for stability of these 2^n equilibria. We extend our theory to more general activation functions, including those with saturations, in section 4. Two numerical simulations on the dynamics of two-neuron networks, which illustrate the present theory, are given in section 5. We summarize our results with a discussion (section 6).

2. Existence of multiple equilibria and their stability. In this section, we shall formulate sufficient conditions for the existence of multiple stationary solutions for Hopfield neural networks with and without delays. Our approach is based on a geometrical observation. The derived parameter conditions are concrete and can be examined easily. We also establish stability criteria of these equilibria for the system without delays, through estimations on the eigenvalues of the linearized system. Stability for the system with delays will be discussed in the next section. After rearranging the parameters, we consider system (1.1) in the following forms: for the network without delay,

$$\frac{dx_i(t)}{dt} = -b_i x_i(t) + \sum_{j=1}^{n} \omega_{ij} g_j(x_j(t)) + J_i, \quad i = 1, 2, \dots, n, \tag{2.1}$$

and for the network with delays,

$$\frac{dx_i(t)}{dt} = -b_i x_i(t) + \sum_{j=1}^{n} \omega_{ij} g_j(x_j(t-\tau_{ij})) + J_i, \quad i = 1, 2, \dots, n. \tag{2.2}$$

Herein, $b_i > 0$, $0 < \tau_{ij} \le \tau := \max_{1\le i,j\le n} \tau_{ij}$. While (2.1) is a system of ordinary differential equations, (2.2) is a system of functional differential equations. The initial condition for (2.2) is

$$x_i(\theta) = \phi_i(\theta), \quad -\tau \le \theta \le 0, \quad i = 1, 2, \dots, n,$$

and it is usually assumed that $\phi_i \in \mathcal{C}([-\tau, 0], \mathbb{R})$. Let $\ell > 0$. For $\mathbf{x} \in \mathcal{C}([-\tau, \ell], \mathbb{R}^n)$ and $t \in [0, \ell]$, we define

$$\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta), \quad \theta \in [-\tau, 0]. \tag{2.3}$$

Let us denote $\tilde{F} = (\tilde{F}_1, \dots, \tilde{F}_n)$, where $\tilde{F}_i$ is the right-hand side of (2.2),

$$\tilde{F}_i(\mathbf{x}_t) := -b_i x_i(t) + \sum_{j=1}^{n} \omega_{ij} g_j(x_j(t-\tau_{ij})) + J_i,$$

where $\mathbf{x} = (x_1, \dots, x_n)$. A function $\mathbf{x} = \mathbf{x}(t)$ is called a solution of (2.2) on $[-\tau, \ell)$ if $\mathbf{x} \in \mathcal{C}([-\tau, \ell), \mathbb{R}^n)$ and $\mathbf{x}_t$ defined as (2.3) lies in the domain of $\tilde{F}$ and satisfies (2.2) for $t \in [0, \ell)$. For a given $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, let us denote by $\mathbf{x}(t;\phi)$ the solution of (2.2) with $\mathbf{x}_0(\theta;\phi) := \mathbf{x}(0+\theta;\phi) = \phi(\theta)$ for $\theta \in [-\tau, 0]$.

The activation functions g_j usually have sigmoidal configuration or are nondecreasing with saturations. Herein, we consider the typical logistic or Fermi function: for all $j = 1, 2, \dots, n$,

$$g_j(\xi) = g(\xi) := \frac{1}{1+e^{-\xi/\varepsilon}}, \quad \varepsilon > 0. \tag{2.4}$$

One may also adopt $g_j(\xi) = 1/(1+e^{-\xi/\varepsilon_j})$, $\varepsilon_j > 0$, or other output functions, as discussed in section 4. Note that the stationary equations for systems (2.1) and (2.2) are identical; namely,

$$F_i(\mathbf{x}) := -b_i x_i + \sum_{j=1}^{n} \omega_{ij} g_j(x_j) + J_i = 0, \quad i = 1, 2, \dots, n, \tag{2.5}$$

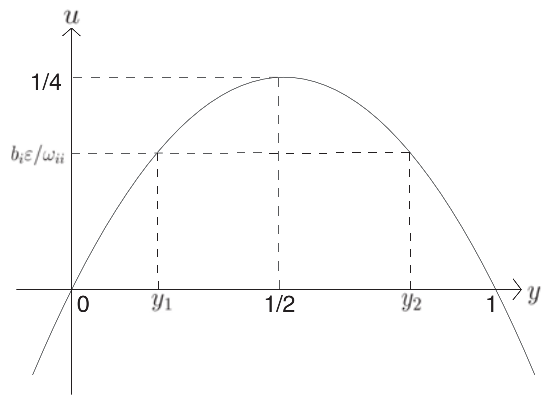

FIG. 1. *The graph for function $u(y) = y - y^2$ and $y_1 = g(p_i)$, $y_2 = g(q_i)$.*

where $\mathbf{x} = (x_1, \ldots, x_n)$. For our formulation in the following discussions, we introduce a single neuron analogue (no interaction among neurons),

$$\frac{d\xi}{dt} = f_i(\xi) := -b_i\xi + \omega_{ii}g(\xi) + J_i, \quad \xi \in \mathbb{R}.$$

Let us propose the first parameter condition:

$$(\mathrm{H}_1)\text{: } 0 < \frac{b_i\varepsilon}{\omega_{ii}} < \frac{1}{4}, \quad i = 1, 2, \ldots, n.$$

LEMMA 2.1. *Under condition* (H_1)*, there exist two points p_i and q_i with $p_i < 0 < q_i$ such that $f_i'(p_i) = 0$, $f_i'(q_i) = 0$ for $i = 1, 2, \ldots, n$.*

Proof. We compute that

$$g'(\xi) = \frac{1}{\varepsilon}(1 + e^{-\xi/\varepsilon})^{-2}e^{-\xi/\varepsilon}. \tag{2.6}$$

Note that g is strictly increasing and that the graph of function $g'(\xi)$ is concave down and has its maximal value at $\xi = 0$. We let $y = g(\xi)$, $\xi \in \mathbb{R}$. Then $y \in (0, 1)$ and $g(0) = 1/2$. It follows from (2.6) that

$$g'(\xi) = \frac{1}{\varepsilon}y^2\left(\frac{1}{y} - 1\right) = \frac{1}{\varepsilon}(y - y^2).$$

On the other hand, for each i, since $f_i'(\xi) = -b_i + \omega_{ii}g'(\xi)$, we have $f_i'(\xi) = 0$ if and only if $b_i = \omega_{ii}g'(\xi)$; equivalently,

$$\frac{b_i\varepsilon}{\omega_{ii}} = y - y^2.$$

From the configuration in Figure 1, it follows that, for each i, there exist two points p_i, q_i, $p_i < 0 < q_i$, such that $f_i'(p_i) = f_i'(q_i) = 0$ if the parameter condition $0 < b_i\varepsilon/\omega_{ii} < 1/4$ holds. This completes the proof. □

Note that condition (H_1) implies $\omega_{ii} > 0$ for all $i = 1, 2, \ldots, n$, since each b_i is already assumed to be a positive constant. We define, for $i = 1, 2, \ldots, n$,

$$\hat{f}_i(\xi) = -b_i\xi + \omega_{ii}g(\xi) + k_i^+,$$
$$\check{f}_i(\xi) = -b_i\xi + \omega_{ii}g(\xi) + k_i^-,$$

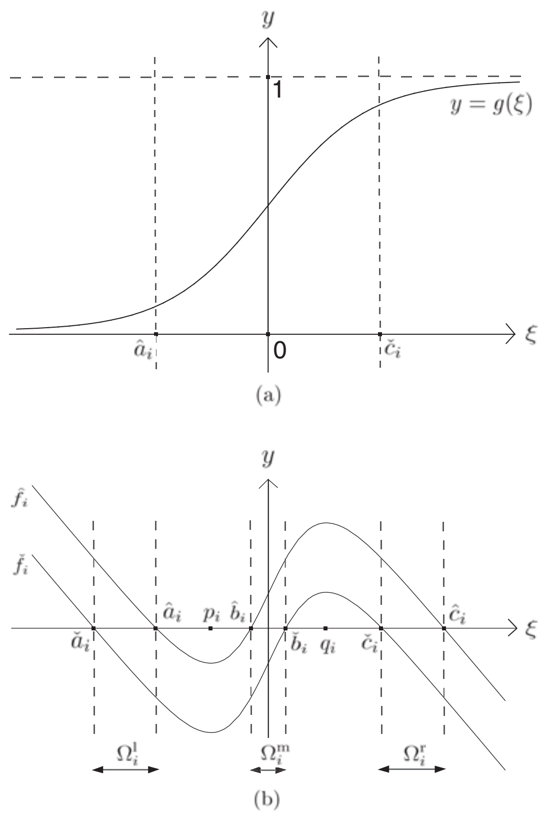

FIG. 2. (a) *The graph of* g *with* $\varepsilon = 0.5$; (b) *Configurations for* $\hat{f}_i$ *and* $\check{f}_i$.

where

$$k_i^+ := \sum_{j=1, j\neq i}^{n} |\omega_{ij}| + J_i, \; k_i^- := -\sum_{j=1, j\neq i}^{n} |\omega_{ij}| + J_i.$$

It follows that

$$\check{f}_i(x_i) \leq F_i(\mathbf{x}) \leq \hat{f}_i(x_i) \tag{2.7}$$

for all $\mathbf{x} = (x_1, \ldots, x_n)$ and $i = 1, 2, \ldots, n$, since $0 \leq g_j \leq 1$ for all j.

We consider the second parameter condition which is concerned with the existence of multiple equilibria for (2.1) and (2.2):

$$(\mathrm{H}_2)\text{: } \hat{f}_i(p_i) < 0, \quad \check{f}_i(q_i) > 0, \quad i = 1, 2, \ldots, n.$$

The configuration that motivates (H_2) is depicted in Figure 2. Such a configuration is due to the characteristics of the output function g. Under assumptions (H_1) and (H_2), there exist points $\hat{a}_i, \hat{b}_i, \hat{c}_i$ with $\hat{a}_i < \hat{b}_i < \hat{c}_i$ such that $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$ as well as points $\check{a}_i, \check{b}_i, \check{c}_i$ with $\check{a}_i < \check{b}_i < \check{c}_i$ such that $\check{f}_i(\check{a}_i) = \check{f}_i(\check{b}_i) = \check{f}_i(\check{c}_i) = 0$.

THEOREM 2.2. *Under* (H_1) *and* (H_2), *there exist* 3^n *equilibria for systems* (2.1) *and* (2.2).

Proof. The equilibria of systems (2.1) and (2.2) are zeros of (2.5). Under conditions (H_1) and (H_2), the graphs of $\hat{f}_i$ and $\check{f}_i$ defined above are as depicted in Figure

2. According to the configurations, there are 3^n disjoint closed regions in $\mathbb{R}^n$. Set $\Omega^\alpha = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in \Omega_i^{\alpha_i}\}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $\alpha_i =$ "l," "m," or "r," where

$$\begin{aligned} &\Omega_i^{\mathrm{l}} := \{x \in \mathbb{R} | \ \check{a}_i \le x \le \hat{a}_i\}, \quad \Omega_i^{\mathrm{m}} := \{x \in \mathbb{R} | \ \hat{b}_i \le x \le \check{b}_i\}, \\ &\Omega_i^{\mathrm{r}} := \{x \in \mathbb{R} | \ \check{c}_i \le x \le \hat{c}_i\}. \end{aligned} \tag{2.8}$$

Herein, "l," "m," and "r" mean, respectively, "left," "middle," and "right." Consider any fixed one of these regions Ω^α. For a given $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in \Omega^\alpha$, we solve

$$h_i(x_i) := -b_i x_i + \omega_{ii} g(x_i) + \sum_{j=1, j\neq i}^{n} \omega_{ij} g(\tilde{x}_j) + J_i = 0$$

for x_i, $i = 1, 2, \ldots, n$. According to an estimate similar to (2.7), the graph of h_i lies between the graphs of $\hat{f}_i$ and $\check{f}_i$. In fact, the graph of h_i is a vertical shift of the graph of $\hat{f}_i$ or $\check{f}_i$. Thus, one can always find three solutions, and each of them lies in one of the regions in (2.8) for each i. Let us pick the one lying in $\Omega_i^{\alpha_i}$ and set it as $\underline{\mathrm{x}}_i$ for each i. We define a mapping $\mathbf{H}_\alpha : \Omega^\alpha \to \Omega^\alpha$ by $\mathbf{H}_\alpha(\tilde{\mathbf{x}}) = \underline{\mathbf{x}} = (\underline{\mathrm{x}}_1, \underline{\mathrm{x}}_2, \ldots, \underline{\mathrm{x}}_n)$. Restated, we set

$$\begin{aligned} \underline{\mathrm{x}}_i &= (h_i|_{\Omega_i^{\mathrm{l}}})^{-1}(0) \text{ if } \alpha_i = \text{"l,"} \\ \underline{\mathrm{x}}_i &= (h_i|_{\Omega_i^{\mathrm{m}}})^{-1}(0) \text{ if } \alpha_i = \text{"m,"} \\ \underline{\mathrm{x}}_i &= (h_i|_{\Omega_i^{\mathrm{r}}})^{-1}(0) \text{ if } \alpha_i = \text{"r."} \end{aligned}$$

Since g is continuous and h_i is a vertical shift of function $\xi \mapsto -b_i\xi + \omega_{ii} g(\xi)$ by the quantity $\sum_{j=1, j\neq i}^{n} \omega_{ij} g(\tilde{x}_j) + J_i$, the map $\mathbf{H}_\alpha$ is continuous. It follows from Brouwer's fixed point theorem that there exists one fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ of $\mathbf{H}_\alpha$ in Ω^α which is also a zero of the function F, where $F = (F_1, F_2, \ldots, F_n)$. Consequently, there exist 3^n zeros of F, hence 3^n equilibria for systems (2.1) and (2.2), and each of them lies in one of the 3^n regions Ω^α. This completes the proof. ∎

We consider the following criterion concerning stability of the equilibria:

$$(\mathrm{H}_3)\text{: } -b_i + \sum_{j=1}^{n} |\omega_{ij}| g'(\eta_j) < 0, \quad g'(\eta_j) := \max\{g'(x_j) \mid x_j = \check{c}_j, \hat{a}_j\}, \quad i = 1, 2, \ldots, n. \tag{2.9}$$

A simplified yet more restrictive version for condition (H_3) is that for $i = 1, 2, \ldots, n$,

$$b_i > g'(\eta) \sum_{j=1}^{n} |\omega_{ij}| \text{ with } g'(\eta) := \max\{g'(x_j) \mid x_j = \check{c}_j, \hat{a}_j, j = 1, 2, \ldots, n\}. \tag{2.10}$$

THEOREM 2.3. *Under conditions* (H_1), (H_2), *and* (H_3), *there exist* 2^n *asymptotically stable equilibria for the Hopfield neural networks without delay* (2.1).

Proof. Among the 3^n equilibria in Theorem 2.2, we consider those $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ with $\bar{x}_i \in \Omega_i^{\mathrm{l}}$ or Ω_i^{r} for each i. The linearized system of (2.1) at equilibrium $\bar{\mathbf{x}}$ is

$$\frac{dy_i}{dt} = -b_i y_i + \sum_{j=1}^{n} \omega_{ij} g_j'(\overline{x}_j) y_j, \quad i = 1, 2, \ldots, n.$$

Restated, $\dot{\mathbf{y}} = A\mathbf{y}$, where $DF(\bar{\mathbf{x}}) =: A = [a_{ij}]_{n\times n}$ with

$$[a_{ij}] = \begin{pmatrix} -b_1 + \omega_{11}g'(\bar{x}_1) & \omega_{12}g'(\bar{x}_2) & \cdots & \omega_{1n}g'(\bar{x}_n) \\ \omega_{21}g'(\bar{x}_1) & -b_2 + \omega_{22}g'(\bar{x}_2) & & \omega_{2n}g'(\bar{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1}g'(\bar{x}_1) & \omega_{n2}g'(\bar{x}_2) & \cdots & -b_n + \omega_{nn}g'(\bar{x}_n) \end{pmatrix}.$$

Let

$$r_i = \sum_{j=1,j\neq i}^{n} |a_{ij}| = \sum_{j=1,j\neq i}^{n} |\omega_{ij}g'(\bar{x}_j)| = \sum_{j=1,j\neq i}^{n} |\omega_{ij}|g'(\bar{x}_j), \quad i = 1,2,\dots,n.$$

According to Gerschgorin's theorem,

$$\lambda_k \in \bigcup_{i=1}^{n} B(a_{ii}, r_i)$$

for all $k = 1,2,\dots,n$, where λ_k are the eigenvalues of A and $B(a_{ii}, r_i) := \{\zeta \in \mathbb{C} \mid |\zeta - a_{ii}| < r_i\}$. Hence, for each k, there exists some $i = i(k)$ such that

$$\mathrm{Re}(\lambda_k) < -b_i + \omega_{ii}g'(\bar{x}_i) + \sum_{j=1,j\neq i}^{n} |\omega_{ij}|g'(\bar{x}_j).$$

Notice that for each j, $g'(\xi) \leq g'(\check{c}_j)$ (resp., $g'(\xi) \leq g'(\hat{a}_j)$) if $\xi \geq \check{c}_j$ (resp., $\xi \leq \hat{a}_j$). Since $\bar{\mathbf{x}}$ is such that $\bar{x}_j \in \Omega^{\mathrm{l}}_j$ or Ω^{r}_j, we have $\bar{x}_j \geq \check{c}_j$ or $\bar{x}_j \leq \hat{a}_j$ for all $j = 1,2,\dots,n$. It follows that $\mathrm{Re}(\lambda_k) < 0$ by (2.9). Thus, under (H_3), all the eigenvalues of A have negative real parts. Therefore, there are 2^n asymptotically stable equilibria for system (2.1). The proof is completed. □

We certainly can replace condition (H_3) by weaker ones, such as an individual condition for each equilibrium. Let $\bar{\mathbf{x}}$ be an equilibrium lying in Ω^{α} with $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\alpha_i =$ "r" or $\alpha_i =$ "l," that is, $\bar{x}_i \in \Omega^{\mathrm{l}}_i$ or Ω^{r}_i, for each i. For such an equilibrium we consider, for $i = 1,2,\dots,n$,

$$b_i > \omega_{ii}g'(\xi_i) + \sum_{j=1,j\neq i}^{n} |\omega_{ij}|g'(\xi_j), \quad \xi_k = \check{c}_k \text{ if } \alpha_k = \text{“r,”} \quad \xi_k = \hat{a}_k \text{ if } \alpha_k = \text{“l,”} \quad k = 1,\dots,n.$$

Such conditions are obviously much more tedious than (H_3).

3. Stability of equilibria and the basins of attraction.

We plan to investigate the stability of equilibrium for system (2.2), that is, with delays. We shall also explore the basins of attraction for the asymptotically stable equilibria, for both systems (2.1) and (2.2), in this section.

Note that the function $\xi \mapsto [\omega_{ii} + \sum_{j=1,j\neq i}^{n} |\omega_{ij}|]g'(\xi)$ is continuous for all $i = 1,2,\dots,n$. From (2.9) and $\omega_{ii} > 0$, it follows that there exists a positive constant ϵ_0 such that

$$b_i > \max\left\{ \left[\omega_{ii} + \sum_{j=1,j\neq i}^{n} |\omega_{ij}|\right] g'(\xi) : \xi = \hat{a}_i + \epsilon_0, \check{c}_i - \epsilon_0 \right\}, \; i = 1,2,\dots,n. \tag{3.1}$$

Herein, we choose ϵ_0 such that $\epsilon_0 < \min\{|\hat{a}_i - p_i|, |\check{c}_i - q_i|\}$ for all $i = 1, 2, \ldots, n$. For system (2.1), we consider the following 2^n subsets of $\mathbb{R}^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i =$ "l" or "r," and set

$$\tilde{\Omega}^\alpha = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \tilde{\Omega}_i^{\mathrm{l}} \text{ if } \alpha_i = \text{"l,"}\ x_i \in \tilde{\Omega}_i^{\mathrm{r}} \text{ if } \alpha_i = \text{"r"}\}, \tag{3.2}$$

where $\tilde{\Omega}_i^{\mathrm{l}} := \{\xi \in \mathbb{R} \mid \xi \le \hat{a}_i + \epsilon_0\}$, $\tilde{\Omega}_i^{\mathrm{r}} := \{\xi \in \mathbb{R} \mid \xi \ge \check{c}_i - \epsilon_0\}$. For system (2.2), we consider the following 2^n subsets of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i =$ "l" or "r," and set

$$\Lambda^\alpha = \{\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \mid \varphi_i \in \Lambda_i^{\mathrm{l}} \text{ if } \alpha_i = \text{"l,"}\ \varphi_i \in \Lambda_i^{\mathrm{r}} \text{ if } \alpha_i = \text{"r"}\}, \tag{3.3}$$

where

$$\Lambda_i^{\mathrm{l}} := \{\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) \le \hat{a}_i + \epsilon_0 \text{ for all } \theta \in [-\tau, 0]\},$$
$$\Lambda_i^{\mathrm{r}} := \{\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) \ge \check{c}_i - \epsilon_0 \text{ for all } \theta \in [-\tau, 0]\}.$$

THEOREM 3.1. *Assume that* (H_1) *and* (H_2) *hold. Then each* $\tilde{\Omega}^\alpha$ *and each* Λ^α *are positively invariant with respect to the solution flow generated by systems* (2.1) *and* (2.2), *respectively.*

Proof. We prove only the delay case, i.e., system (2.2). Consider any one of the 2^n sets Λ^α. For any initial condition $\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in \Lambda^\alpha$, we claim that the solution $\mathbf{x}(t; \phi)$ remains in Λ^α for all $t \ge 0$. If this is not true, there exists a component $x_i(t)$ of $\mathbf{x}(t; \phi)$ which is the first (or one of the first) escaping from Λ_i^{l} or Λ_i^{r}. Restated, there exist some i and $t_1 > 0$ such that either $x_i(t_1) = \check{c}_i - \epsilon_0$, $\frac{dx_i}{dt}(t_1) \le 0$, and $x_i(t) \ge \check{c}_i - \epsilon_0$ for $-\tau \le t \le t_1$ or $x_i(t_1) = \hat{a}_i + \epsilon_0$, $\frac{dx_i}{dt}(t_1) \ge 0$, and $x_i(t) \le \hat{a}_i + \epsilon_0$ for $-\tau \le t \le t_1$. For the first case $x_i(t_1) = \check{c}_i - \epsilon_0$ and $\frac{dx_i}{dt}(t_1) \le 0$, we derive from (2.2) that

$$\frac{dx_i}{dt}(t_1) = -b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(x_i(t_1 - \tau_{ii})) + \sum_{j=1, j \ne i}^{n} \omega_{ij} g(x_j(t_1 - \tau_{ij})) + J_i \le 0. \tag{3.4}$$

On the other hand, recalling (H_2) and previous descriptions of $\check{c}_i$ and ϵ_0, we have $\check{f}_i(\check{c}_i - \epsilon_0) > 0$ which gives

$$\begin{aligned} &-b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(\check{c}_i - \epsilon_0) + k_i^- \\ &= -b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(\check{c}_i - \epsilon_0) - \sum_{j=1, j \ne i}^{n} |\omega_{ij}| + J_i > 0. \end{aligned} \tag{3.5}$$

Notice that t_1 is the first time for x_i to escape from Λ_i^{r}. We have $g(x_i(t_1 - \tau_{ii})) \ge g(\check{c}_i - \epsilon_0)$, by the monotonicity of function g. In addition, by $\omega_{ii} > 0$ and $|g(\cdot)| \le 1$, we obtain from (3.5) that

$$\begin{aligned} &-b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(x_i(t_1 - \tau_{ii})) + \sum_{j=1, j \ne i}^{n} \omega_{ij} g(x_j(t_1 - \tau_{ij})) + J_i \\ &\ge -b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(\check{c}_i - \epsilon_0) - \sum_{j=1, j \ne i}^{n} |\omega_{ij}| + J_i > 0, \end{aligned}$$

which contradicts (3.4). Hence, $x_i(t) \geq \check{c}_i - \epsilon_0$ for all $t > 0$. Similar arguments can be employed to show that $x_i(t) \leq \hat{a}_i + \epsilon_0$ for all $t > 0$ for the situation that $x_i(t_1) = \hat{a}_i + \epsilon_0$ and $\frac{dx_i}{dt}(t_1) \geq 0$. Therefore, Λ^α is positively invariant under the flow generated by system (2.2). The assertion for system (2.1) can be justified similarly. ∎

THEOREM 3.2. *Under conditions* (H_1), (H_2), *and* (H_3), *there exist* 2^n *exponentially stable equilibria for system* (2.2).

Proof. Consider an equilibrium $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in \Omega^\alpha$ for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, with α_i = "l" or "r," obtained in Theorem 2.2. We consider the single-variable functions $G_i(\cdot)$, defined by

$$G_i(\zeta) = b_i - \zeta - \sum_{j=1}^{n} |\omega_{ij}| g'(\xi_j) e^{\zeta \tau_{ij}},$$

where $\xi_j = \hat{a}_j + \epsilon_0$ (resp., $\check{c}_j - \epsilon_0$) if α_j = "l" (resp., "r"). Then, $G_i(0) > 0$ from (3.1) or (H_3). Moreover, there exists a constant $\mu > 0$ such that $G_i(\mu) > 0$ for $i = 1, 2, \ldots, n$, due to continuity of G_i. Let $\mathbf{x}(t) = \mathbf{x}(t; \phi)$ be the solution to (2.2) with initial condition $\phi \in \Lambda^\alpha$ defined in (3.3). Under the translation $\mathbf{y}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$, system (2.2) becomes

$$\frac{dy_i(t)}{dt} = -b_i y_i(t) + \sum_{j=1}^{n} \omega_{ij} [g(x_j(t - \tau_{ij})) - g(\overline{x}_j)], \tag{3.6}$$

where $\mathbf{y} = (y_1, \ldots, y_n)$. Now, consider functions $z_i(\cdot)$ defined by

$$z_i(t) = e^{\mu t} |y_i(t)|, \quad i = 1, 2, \ldots, n. \tag{3.7}$$

The domain of definition for $z_i(\cdot)$ is identical to the interval of existence for $y_i(\cdot)$. We shall see in the following computations that the domain can be extended to $[-\tau, \infty)$. Let $\delta > 1$ be an arbitrary real number and let

$$K := \max_{1 \leq i \leq n} \left\{ \sup_{\theta \in [-\tau, 0]} |x_i(\theta) - \bar{x}_i| \right\} > 0. \tag{3.8}$$

It follows from (3.7) and (3.8) that $z_i(t) < K\delta$ for $t \in [-\tau, 0]$ and all $i = 1, 2, \ldots, n$. Next, we claim that

$$z_i(t) < K\delta \quad \text{for all } t > 0, \ i = 1, 2, \ldots, n. \tag{3.9}$$

Suppose this is not the case. Then there are an $i \in \{1, 2, \ldots, n\}$ (say $i = k$) and a $t_1 > 0$ for the first time such that

$$\begin{aligned} z_i(t) &\leq K\delta, \quad t \in [-\tau, t_1], \quad i = 1, 2, \ldots, n, \ i \neq k, \\ z_k(t) &< K\delta, \quad t \in [-\tau, t_1), \\ z_k(t_1) &= K\delta \quad \text{ with } \frac{d}{dt} z_k(t_1) \geq 0. \end{aligned}$$

Note that $z_k(t_1) = K\delta > 0$ implies $y_k(t_1) \neq 0$. Hence $|y_k(t)|$ and $z_k(t)$ are differentiable at $t = t_1$. From (3.6), we derive that

$$\frac{d}{dt} |y_k(t_1)| \leq -b_k |y_k(t_1)| + \sum_{j=1}^{n} |\omega_{kj}| g'(\varsigma_j) |y_j(t_1 - \tau_{kj})| \tag{3.10}$$

for some ς_j between $x_j(t_1-\tau_{kj})$ and $\bar{x}_j$. Hence, from (3.7) and (3.10),

$$\begin{aligned}\frac{dz_k(t_1)}{dt} &\le \mu e^{\mu t_1}|y_k(t_1)| + e^{\mu t_1}\left[-b_k|y_k(t_1)| + \sum_{j=1}^n |\omega_{kj}|g'(\varsigma_j)|y_j(t_1-\tau_{kj})|\right]\\ &\le \mu z_k(t_1) - b_k z_k(t_1) + \sum_{j=1}^n |\omega_{kj}|g'(\varsigma_j)e^{\mu\tau_{kj}}z_j(t_1-\tau_{kj})\\ &\le -(b_k-\mu)z_k(t_1) + \sum_{j=1}^n |\omega_{kj}|g'(\xi_j)e^{\mu\tau_{kj}}\left[\sup_{\theta\in[t_1-\tau,t_1]} z_j(\theta)\right], \qquad (3.11)\end{aligned}$$

where $\xi_j = \hat{a}_j + \epsilon_0$ (resp., $\check{c}_j - \epsilon_0$) if $\alpha_j =$ "l" (resp., "r"). Herein, the invariance property of Λ^α in Theorem 3.1 has been applied. Due to $G_i(\mu) > 0$, we obtain

$$\begin{aligned}0 \le \frac{dz_k(t_1)}{dt} &\le -(b_k-\mu)z_k(t_1) + \sum_{j=1}^n |\omega_{kj}|g'(\xi_j)e^{\mu\tau_{kj}}\left[\sup_{\theta\in[t_1-\tau,t_1]} z_j(\theta)\right]\\ &< -\left\{b_i - \mu - \sum_{j=1}^n |\omega_{ij}|g'(\xi_j)e^{\mu\tau_{kj}}\right\}K\delta\\ &< 0, \qquad (3.12)\end{aligned}$$

which is a contradiction. Hence the claim (3.9) holds. Since $\delta > 1$ is arbitrary, by allowing $\delta \to 1^+$, we have $z_i(t) \le K$ for all $t > 0$, $i = 1, 2, \dots, n$. We then use (3.7) and (3.8) to obtain

$$|x_i(t) - \bar{x}_i| \le e^{-\mu t}\max_{1\le j\le n}\left(\sup_{\theta\in[-\tau,0]}|x_j(\theta)-\bar{x}_j|\right)$$

for $t > 0$ and all $i = 1, 2, \dots, n$. Therefore, $\mathbf{x}(t)$ is exponentially convergent to $\bar{\mathbf{x}}$. This completes the proof. □

In the following, we employ the theory of the local Lyapunov functional [15] and the Halanay-type inequality [4, 14] to establish other sufficient conditions for asymptotic stability and exponential stability for the equilibria of system (2.2).

THEOREM 3.3. *There exist 2^n asymptotically stable equilibria for system* (2.2) *under conditions* (H_1) *and* (H_2) *and one of the following conditions:*

$$(\mathrm{H}_4):\ 2b_i > \sum_{j=1}^n |\omega_{ij}| + [g'(\eta_i)]^2 \sum_{j=1}^n |\omega_{ji}| \quad \text{for } \eta_i = \hat{a}_i \text{ and } \check{c}_i,\ i = 1, 2, \dots, n,$$

$$(\mathrm{H}_5):\ \min_{1\le i\le n}\left[2b_i - \sum_{j=1}^n |\omega_{ij}|g'(\xi_j)\right] > \max_{1\le i\le n}\left[\sum_{j=1}^n |\omega_{ji}|g'(\eta_i)\right] \quad \text{for } \xi_j = \hat{a}_j \text{ and } \check{c}_j,\ \eta_i = \hat{a}_i \text{ and } \check{c}_i.$$

Proof. Similarly to (3.1), there exists $\epsilon_0 > 0$ such that (H_4) holds for $\eta_i = \hat{a}_i + \epsilon_0$, $\check{c}_i - \epsilon_0$, and (H_5) holds for $\xi_j = \hat{a}_j + \epsilon_0$, $\check{c}_j - \epsilon_0$, $\eta_i = \hat{a}_i + \epsilon_0$, $\check{c}_i - \epsilon_0$, $i = 1, 2, \dots, n$, by continuity of g'. We thus define Λ^α as in (3.3). The following computations are reserved for solutions lying entirely within each of the 2^n positively invariant regions Λ^α.

(i) We employ the following Lyapunov functional:

$$V(\mathbf{y})(t) = \sum_{i=1}^{n} y_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} |\omega_{ij}| \int_{t-\tau_{ij}}^{t} [g(x_j(s)) - g(\overline{x}_j)]^2 ds,$$

where $\mathbf{y}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$. By recalling (3.6) and using (H_4), we derive

$$\begin{aligned}
\frac{dV(\mathbf{y})(t)}{dt} &= 2\sum_{i=1}^{n} y_i(t)\left\{-b_i y_i(t) + \sum_{j=1}^{n} \omega_{ij}[g(x_j(t-\tau_{ij})) - g(\overline{x}_j)]\right\} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} |\omega_{ij}|[g(x_j(t)) - g(\overline{x}_j)]^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} |\omega_{ij}|[g(x_j(t-\tau_{ij})) - g(\overline{x}_j)]^2 \\
&\le -2\sum_{i=1}^{n} b_i y_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} |\omega_{ij}| y_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} |\omega_{ij}|[g'(\eta_j)]^2 y_j^2(t) \\
&= \sum_{i=1}^{n}\left\{-2b_i + \sum_{j=1}^{n}|\omega_{ij}| + [g'(\eta_i)]^2 \sum_{j=1}^{n} |\omega_{ji}|\right\} y_i^2(t) < 0.
\end{aligned}$$

We thus conclude the asymptotic stability for equilibrium $\bar{\mathbf{x}}$ via applying the theory of the local Lyapunov functional; cf. [15].

(ii) Recall (3.6), and let

$$W(\mathbf{y})(t) = \frac{1}{2}\sum_{i=1}^{n} y_i^2(t). \tag{3.13}$$

Then,

$$\begin{aligned}
\frac{dW(\mathbf{y})(t)}{dt} &= \sum_{i=1}^{n} y_i(t)\left\{-b_i y_i(t) + \sum_{j=1}^{n} \omega_{ij}[g(x_j(t-\tau_{ij})) - g(\overline{x}_j)]\right\} \\
&\le \sum_{i=1}^{n}\left\{-b_i y_i^2(t) + \frac{1}{2}\sum_{j=1}^{n} |\omega_{ij}| g'(\varsigma_j)[y_i^2(t) + y_j^2(t-\tau_{ij})]\right\} \\
&\le -\sum_{i=1}^{n}\left[b_i - \frac{1}{2}\sum_{j=1}^{n}|\omega_{ij}| g'(\xi_j)\right] y_i^2(t) \\
&\quad + \frac{1}{2}\left[\max_{1\le i\le n}\sum_{j=1}^{n}|\omega_{ji}| g'(\eta_i)\right]\sum_{i=1}^{n}\sup_{t-\tau\le s\le t} y_i^2(s) \\
&\le -\beta W(\mathbf{y})(t) + \zeta \sup_{t-\tau\le s\le t} W(\mathbf{y})(s),
\end{aligned}$$

where

$$\begin{aligned}
\beta &:= \min_{1\le i\le n}\left\{2b_i - \sum_{j=1}^{n}|\omega_{ij}| g'(\xi_j), \xi_j = \hat{a}_j + \epsilon_0,\ \check{c}_j - \epsilon_0\right\}, \\
\zeta &:= \max_{1\le i\le n}\left\{\sum_{j=1}^{n}|\omega_{ji}| g'(\eta_i), \eta_i = \hat{a}_i + \epsilon_0,\ \check{c}_i - \epsilon_0\right\}.
\end{aligned}$$

By (H_5), we have $\beta > \zeta > 0$. By using the Halanay inequality, we obtain that

$$W(\mathbf{y})(t) \le \left(\sup_{-\tau \le s \le 0} W(\mathbf{y})(s) \right) e^{-\gamma t} \tag{3.14}$$

for all $t \ge 0$, where γ is the unique solution of $\gamma = \beta - \zeta e^{\gamma\tau}$. It follows that

$$\frac{1}{2}\sum_{i=1}^{n} y_i^2(t) \le \left[\sup_{-\tau \le s \le 0} \left(\frac{1}{2}\sum_{i=1}^{n} y_i^2(s) \right) \right] e^{-\gamma t}. \tag{3.15}$$

Hence, the equilibrium $\bar{\mathbf{x}}$ is asymptotically stable. ∎

Corollary 3.4. *Under conditions* (H_1), (H_2), *and* (H_5), *there exist* 2^n *exponentially stable equilibria for system* (2.2).

We observe from (2.1) and (2.2) that for every i,

$$\begin{aligned} &F_i(\mathbf{x}),\ \tilde{F}_i(\mathbf{x}_t) < 0 \text{ whenever } x_i > 0 \text{ is sufficiently large,} \\ &F_i(\mathbf{x}),\ \tilde{F}_i(\mathbf{x}_t) > 0 \text{ whenever } x_i < 0 \text{ and } |x_i| \text{ is sufficiently large,} \end{aligned}$$

since $b_i > 0$ and $\sum_{j=1}^{n} \omega_{ij} g_j(x_j(t)) + J_i$ and $\sum_{j=1}^{n} \omega_{ij} g_j(x_j(t-\tau_{ij})) + J_i$ are bounded for any $\mathbf{x}$ and $\mathbf{x}_t$. Therefore, it can be concluded that every solution of (2.1) and (2.2) is bounded in forward time.

4. Further extension.

We shall extend our studies in sections 2 and 3 to more general activation functions in this section.

4.1. Activation functions in general form.

Let us consider the activation functions $\{g_i(\cdot)\}_1^n$ which are $\mathcal{C}^2$ and satisfy

$$(\mathrm{C}): \begin{cases} u_i \le g_i(\xi) \le v_i, & g_i'(\xi) > 0, \\ (\xi - \sigma_i) g_i''(\xi) < 0 & \text{for all } \xi \in \mathbb{R}, \end{cases}$$

$i = 1, 2, \ldots, n$. Herein, u_i, v_i, and σ_i are constants with $u_i < v_i$, $i = 1, 2, \ldots, n$. Under these circumstances, (H_1) can be modified to

$$(\mathrm{H}_1{}'):\ 0 = \inf_{\xi\in\mathbb{R}} g_i'(\xi) < \frac{b_i}{\omega_{ii}} < \max_{\xi\in\mathbb{R}} g_i'(\xi)\ (= g_i'(\sigma_i)),\ i = 1, 2, \ldots, n.$$

As in section 2, we define

$$f_i(\xi) = -b_i\xi + \omega_{ii} g_i(\xi) + J_i.$$

Lemma 4.1. *For* g_i *in the class* (C), *under condition* $(\mathrm{H}_1{}')$, *there exist constants* $\{p_i\}_1^n$ *and* $\{q_i\}_1^n$ *with* $p_i < \sigma_i < q_i$ *such that* $f_i'(p_i) = f_i'(q_i) = 0$ *for each* $i = 1, 2, \ldots, n$.

We define

$$\hat{f}_i(\xi) = -b_i\xi + \omega_{ii} g_i(\xi) + k_i^+, \quad \check{f}_i(\xi) = -b_i\xi + \omega_{ii} g_i(\xi) + k_i^-, \tag{4.1}$$

where

$$k_i^+ := \sum_{j=1, j\ne i}^{n} \rho_j |\omega_{ij}| + J_i, \quad k_i^- := -\sum_{j=1, j\ne i}^{n} \rho_j |\omega_{ij}| + J_i \tag{4.2}$$

with $\rho_j = \max\{|u_j|, |v_j|\}$. We locate the points $\hat{a}_i < \hat{b}_i < \hat{c}_i$ and $\check{a}_i < \check{b}_i < \check{c}_i$, where $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$ and $\check{f}_i(\check{a}_i) = \check{f}_i(\check{b}_i) = \check{f}_i(\check{c}_i) = 0$.

Let $\eta \in \mathbb{R}$ and $k \in \{1, \dots, n\}$ be such that $g_k'(\eta) = \max\{g_i'(\xi) : \xi = \hat{a}_i, \check{c}_i, i = 1, 2, \dots, n\}$. Consider

$$(\mathrm{H_3}'):\ b_i > g_k'(\eta) \left[\omega_{ii} + \sum_{j=1, j\neq i}^{n} |\omega_{ij}|\right], \quad i = 1, 2, \dots, n.$$

THEOREM 4.2. *Let g_i be in the class* (C). *Under conditions* $(\mathrm{H_1}')$, $(\mathrm{H_2})$, *and* $(\mathrm{H_3}')$, *there exist 3^n equilibria for systems* (2.1) *and* (2.2) *with 2^n among them being exponentially stable.*

4.2. Saturated activation functions. In this subsection, we investigate systems (2.1), (2.2) with saturated activation functions. In particular, we consider the following continuous functions:

$$g_i(\xi) = \begin{cases} u_i & \text{if } -\infty < \xi \leq p_i, \\ \text{increasing} & \text{if } p_i \leq \xi \leq q_i, \\ v_i & \text{if } q_i \leq \xi < \infty, \end{cases}$$

where p_i, q_i are constants with $p_i < q_i$ for $i = 1, 2, \dots, n$. Such a class of functions includes the piecewise linear function with saturations:

$$g_i(\xi) = \begin{cases} u_i & \text{if } -\infty < \xi \leq p_i, \\ u_i + \frac{v_i - u_i}{q_i - p_i}(\xi - p_i) & \text{if } p_i \leq \xi \leq q_i, \\ v_i & \text{if } q_i \leq \xi < \infty \end{cases}$$

for each i. Typical graphs for these functions are depicted in Figures 3(a) and (c). With such activation functions, existence of multiple equilibria for (2.1) and (2.2) can be obtained under condition

$$(\mathrm{H_s}):\ b_i > 0, -b_i p_i + \omega_{ii} u_i + k_i^+ < 0, \quad -b_i q_i + \omega_{ii} v_i + k_i^- > 0, \quad i = 1, 2, \dots, n,$$

where k_i^+, k_i^- are defined as in (4.2). We define $\hat{f}_i, \check{f}_i$ as in (4.1). The graphs of $\hat{f}_i$ and $\check{f}_i$ are depicted in Figures 3(b) and (d). Under condition $(\mathrm{H_s})$, we also locate the points $\hat{a}_i < \hat{b}_i < \hat{c}_i$ and $\check{a}_i < \check{b}_i < \check{c}_i$, where $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$ and $\check{f}_i(\check{a}_i) = \check{f}_i(\check{b}_i) = \check{f}_i(\check{c}_i) = 0$.

Note that we do not need differentiability at corner points p_i, q_i of g_i in our analysis; moreover, $g_i'(\xi) = 0$ for $\xi < p_i$ and $\xi > q_i$. Thus, $(\mathrm{H_3})$ is already satisfied if $b_i > 0$ for $i = 1, 2, \dots, n$. With these formulations, we can derive that there exist 3^n equilibria for systems (2.1) and (2.2), and that 2^n of them are exponentially stable under condition $(\mathrm{H_s})$.

4.3. Unbounded activation functions. Our theory can also be extended to certain unbounded activation functions with controlled slopes, for example, the activation functions g_i with bounded slopes in Figure 4. Herein, we require that the slopes m_i^{r} of the right- and m_i^{l} of the left-hand parts of g_i satisfy $b_i > \omega_{ii} m_i^{\mathrm{r}}$, $b_i > \omega_{ii} m_i^{\mathrm{l}}$ for $i = 1, \dots, n$.

5. Numerical illustrations. In this section, we present two examples (with delays) to illustrate our theory.

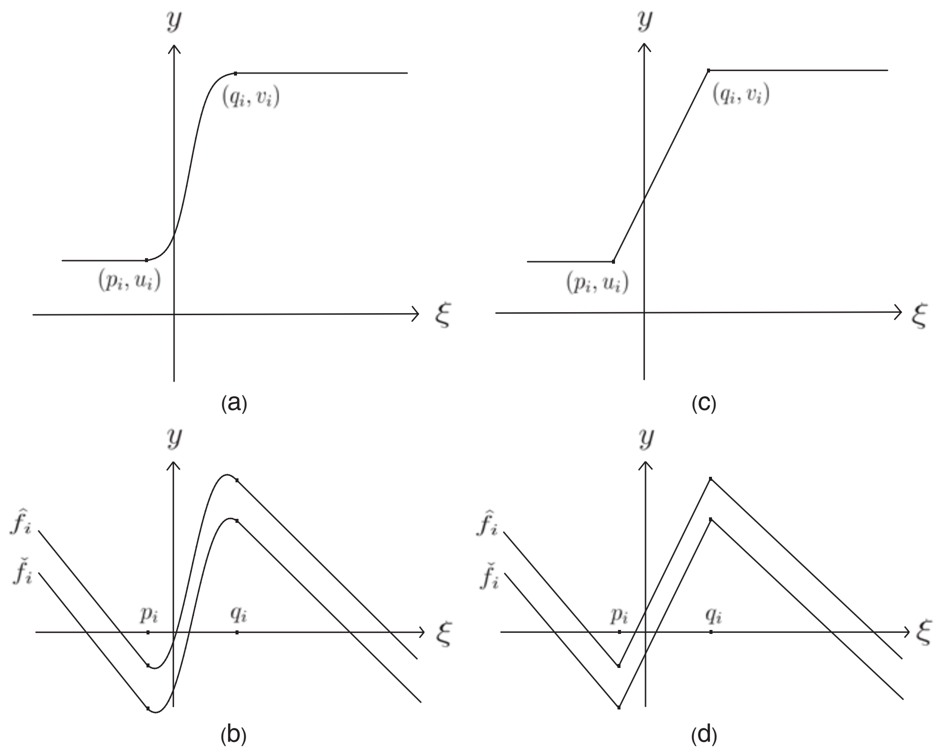

FIG. 3. (a) *The graph for a continuous activation function* g_i *with saturations.* (b) *The graphs for* $\hat{f}_i$ *and* $\check{f}_i$ *induced from the activation function in* (a). (c) *The graph for a piecewise linear activation function* g_i *with saturations.* (d) *The graphs for* $\hat{f}_i$ *and* $\check{f}_i$ *induced from the activation function in* (c).

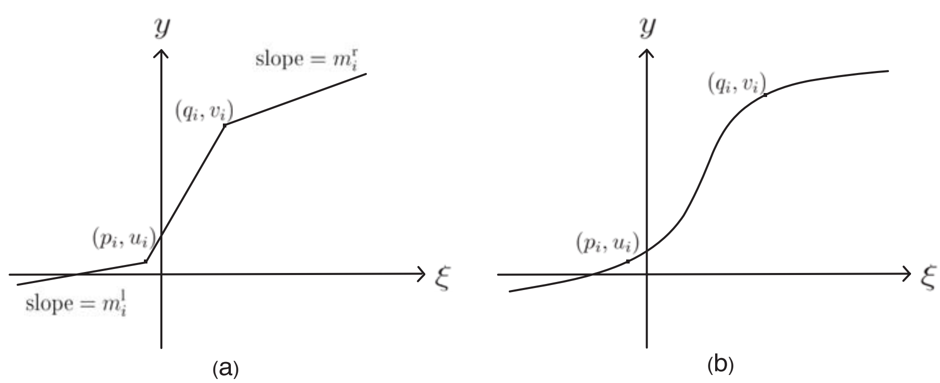

FIG. 4. (a) *The graph for an unbounded piecewise linear activation function.* (b) *The graph for an unbounded activation function with bounded slopes.*

Example 5.1. Consider the two-dimensional neural network

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -x_1(t) + 18g_1(x_1(t-10)) + 5g_2(x_2(t-10)) - 9,\\
\frac{dx_2(t)}{dt} &= -3x_2(t) + 5g_1(x_1(t-10)) + 30g_2(x_2(t-10)) - 15,
\end{aligned}$$

Table 1
Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$.

$\hat{a}_1 = -3.993889$	$p_1 = -1.762747$	$\hat{b}_1 = -0.757751$	q_1=1.762747	$\hat{c}_1$=14
$\check{a}_1 = -14$		$\check{b}_1$=0.757751		$\check{c}_1$=3.993889
$\hat{a}_2 = -3.320288$	$p_2 = -1.443635$	$\hat{b}_2 = -0.452309$	q_2=1.443635	$\hat{c}_2$=6.666650
$\check{a}_2 = -6.666650$		$\check{b}_2$=0.452309		$\check{c}_2$=3.320288

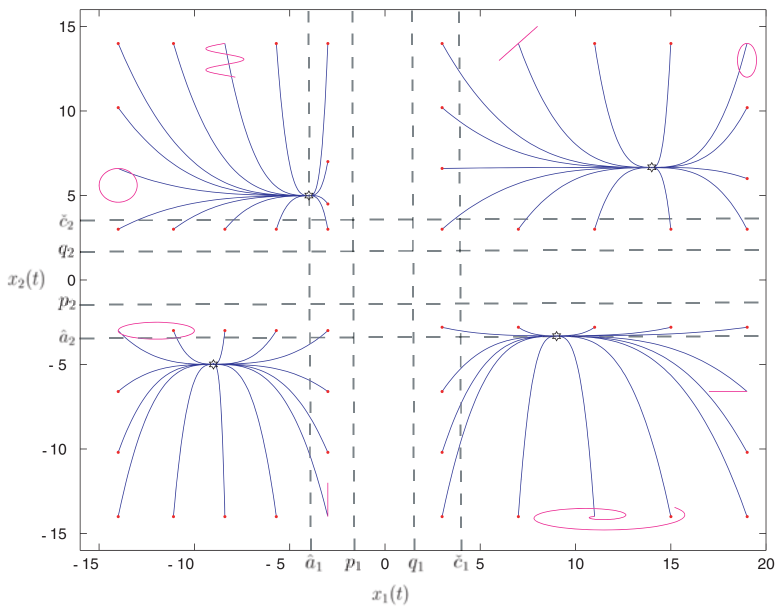

Fig. 5. *Illustrations for the dynamics in Example* 5.1.

where $g_1(x) = g_2(x) = g(x)$ in (2.4) with $\varepsilon = 0.5$. A computation gives

$$\hat{f}_1(x_1) = -x_1 + 18g(x_1) - 4, \quad \check{f}_1(x_1) = -x_1 + 18g(x_1) - 14,$$
$$\hat{f}_2(x_2) = -3x_2 + 30g(x_2) - 10, \quad \check{f}_2(x_2) = -3x_2 + 30g(x_2) - 20.$$

Herein, the parameters satisfy our conditions in Theorem 3.2:

$$\text{Condition (H}_1\text{): } 0 < \frac{b_1\varepsilon}{\omega_{11}} = \frac{1}{36} < \frac{1}{4}, \quad 0 < \frac{b_2\varepsilon}{\omega_{22}} = \frac{1}{20} < \frac{1}{4}.$$
$$\text{Condition (H}_2\text{): } \hat{f}_1(p_1) = -1.722534 < 0, \quad \check{f}_1(q_1) = 1.722534 > 0,$$
$$\hat{f}_2(p_2) = -4.085501 < 0, \quad \check{f}_2(q_2) = 4.085501 > 0.$$
$$\text{Condition (H}_3\text{): } b_1 = 1 > 0.025246 = \omega_{11}g'(\eta_1) + |\omega_{12}|g'(\eta_2),$$
$$b_2 = 3 > 0.081566 = |\omega_{21}|g'(\eta_1) + \omega_{22}g'(\eta_2),$$

where $\eta_1 = \pm 3.993889$, $\eta_2 = \pm 3.320288$ are defined in (2.9). Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$ are listed in Table 1. The dynamics of this system are illustrated in Figure 5, where evolutions of 56 initial conditions have been tracked. The constant initial conditions are plotted in red dots, and the time-dependent initial conditions are plotted in purple curves. The evolutions of components $x_1(t)$ and $x_2(t)$ are depicted

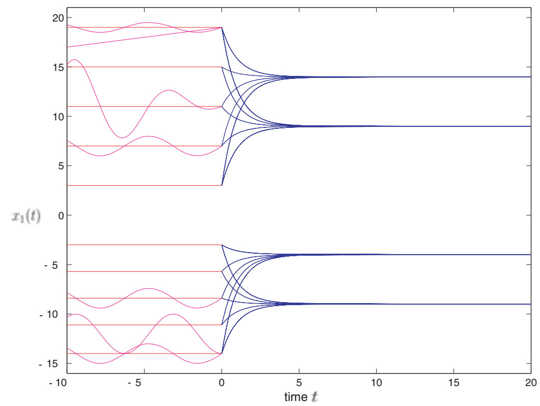

FIG. 6. *Evolution of state variable* $x_1(t)$ *in Example* 5.1.

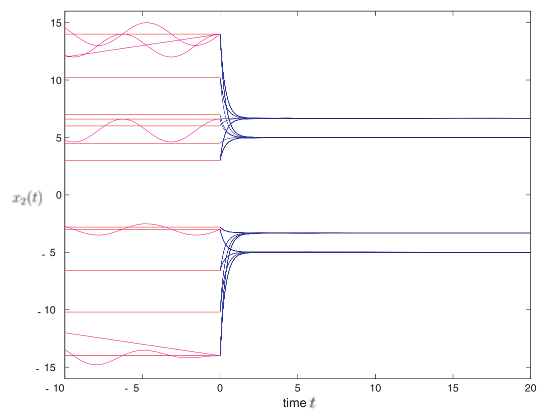

FIG. 7. *Evolution of state variable* $x_2(t)$ *in Example* 5.1.

in Figures 6 and 7, respectively. There are four exponentially stable equilibria in the system, as confirmed by our theory. The simulations demonstrate the convergence to these four equilibria from initial functions ϕ lying in the basin of the respective equilibrium.

Example 5.2. In this example, we simulate the neural network

$$\frac{dx_1(t)}{dt} = -x_1(t) + 18g_1(x_1(t-10)) + 11g_2(x_2(t-10)) + 1,$$
$$\frac{dx_2(t)}{dt} = -3x_2(t) + 11g_1(x_1(t-10)) + 30g_2(x_2(t-10)) + 4$$

with the output function $g_i(\xi) = h(\xi)$, where

$$h(\xi) = \frac{1}{2}(|\xi+1| - |\xi-1|), \tag{5.1}$$

for each i. The parameters also satisfy the conditions in our formulations with such an output function. We demonstrate the dynamics as well as evolutions of components $x_1(t)$, $x_2(t)$ for the system in Figures 8, 9, and 10, respectively.

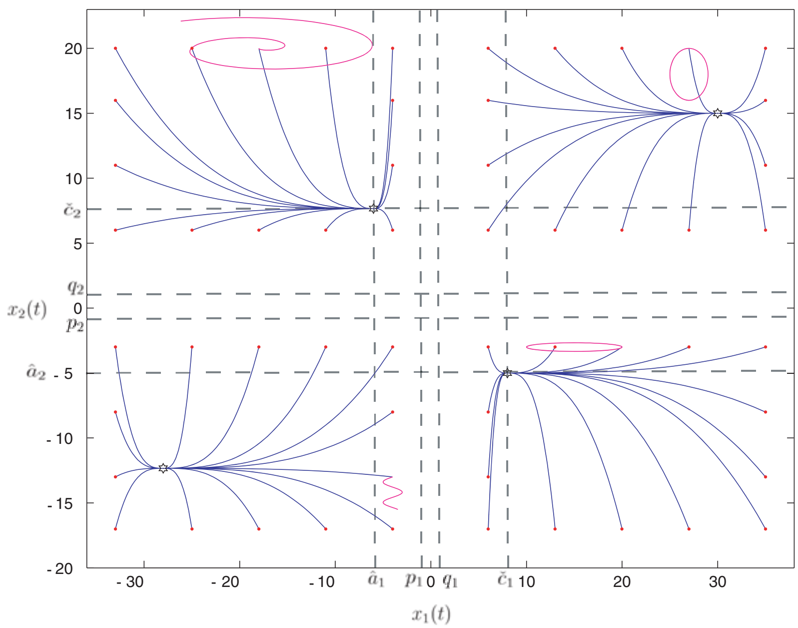

FIG. 8. *Illustrations for the dynamics in Example* 5.2.

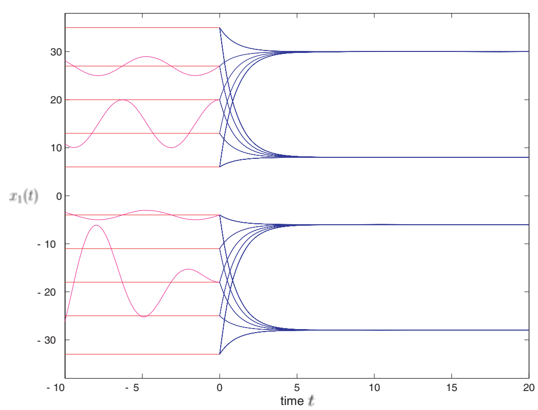

FIG. 9. *Evolution of state variable* $x_1(t)$ *in Example* 5.2.

6. Discussions. Our approach can also be adapted to the cellular neural networks with delays. The cellular neural networks (CNNs) were introduced by Chua and Yang [8] in 1988. A model called delayed cellular neural network [24] is given by

$$\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j \in N_r(i)} a_{ij} h(x_j(t)) + \sum_{j \in N_r(i)} b_{ij} h(x_j(t-\tau)) + J_i, \tag{6.1}$$

where $N_r(i) = \{i-1, i, i+1\}$ if $r = 1$. The standard activation function for such a network is the piecewise linear h defined in (5.1). Notably, (6.1) is a system of CNNs with cells coupled in the one-dimensional manner, and its local coupling structure is expressed in the equations. Global exponential stability of a single equilibrium for

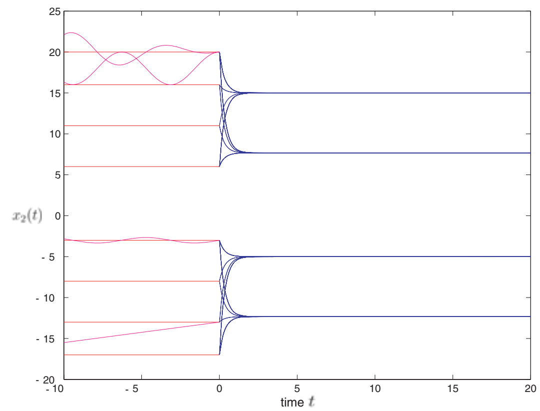

FIG. 10. *Evolution of state variable* $x_2(t)$ *in Example* 5.2.

(6.1) has been studied by many researchers, for instance, the authors of [3, 20]. The CNNs can be built by multidimensional couplings among cells. Since there are finitely many cells at most, the CNNs can always be rewritten in a one-dimensional coupling form by renaming the indices [28]. It can then be written in a form similar to (1.1). Such an arrangement, however, destroys the local connection representation. While previous studies on multistability for the CNNs without delays [17, 26, 27] employed the structure of local connections among cells of CNNs, our approach does not rely on such a structure. Moreover, our theory generalized the multistability to the CNNs with delays (6.1).

In this investigation, we have obtained existence of 2^n stable stationary solutions for recurrent neural networks comprised of n neurons, with delays and without delays. The theory is primarily based upon an observation on the structures of the equations. It is thus rather general and can be applied to at least the Hopfield-type neural networks and the cellular neural networks. The analysis is valid for the networks with various activation functions, including the typical sigmoidal ones and the saturated linear ones, as well as some unbounded activation functions. In fact, our formulation depends on the configuration of the activation functions instead of the precise form of the functions. The theorems thus developed are pertinent in neural network theory.

Stable periodic orbits and limit cycle attractors are also important for memory storage and other neural activities. By similar analysis, we can also establish existence of multiple limit cycles for systems (1.1) and (6.1) with periodic inputs $J_i = J_i(t) = J_i(t + T)$. The result will be reported in another article. The approach in this presentation can be adopted to discrete-time neural networks as well.

The major discussions on neural networks have been centered around monostability, in an abundance of articles in the areas of physics, information sciences, electrical engineering, and mathematics. Multistability in neural networks is, however, essential in numerous applications such as content-addressable memory storage and pattern recognition. Recently, further application potentials of multistability have been found in decision making, digital selection, and analogy amplification [18].

We have exploited further interesting structures of Hopfield-type neural networks in this study. Our investigations have provided computable parameter conditions for multistable dynamics in the recurrent neural networks and are expected to contribute toward practical applications.

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