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Technical note

Formulations of composite parametric cubic curves and circle approximations

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The parametric cubic curve is a well-developed subject in the field of computer-aided design. Yet there are still some aspects of its nature that need to be more fully investigated. In this paper, formulations of composite PC curves are presented in an order of increasing rigorousness and order of accuracy. A unique PC curve formulation is developed for the practical use in circle approximation. This PC curve has a 3rd-order accuracy and merits of simplicity and ease of use. Copyright © 1996 Elsevier Science Ltd

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INTRODUCTION

Circle approximation is fundamental in the practice of computer-aided design. This is due to the fact that, besides the straight line, the circle is the most simple geometric entity with a well-known algebraic equation for its representation. A sound curve approximation scheme must be sound in representing the straight line and the circle in the first place. Then, the known algebraic equation of these simple curves can be used to check for the accuracy of the approximation.

Parametric curves and circle approximation have been addressed by many authors. The most systematic discussion of parametric curves and their properties can be found in the two well-known textbooks, one on computational geometry by Faux and Pratt[1], and the other on geometric modelling by Mortenson[2]. In Faux and Pratt, approximations of the straight line and the circle are treated as special cases of the Bézier cubic curve (Reference 1, p 134). It is mentioned there that, for the approximation of a quarter circle, the maximum deviation from the mean radius is $\pm 0.03\%$. In the book by Mortenson, the parametric cubic curve (the PC curve) is employed to approximate the general conic curve (Reference 2, pp 79–91). Then the conic curve approximation is extended to the approximation of circular arcs.

The accuracy study in Mortenson's circular arc approximation is based on two different approximation approaches. In the first approach, the mid-point of the approximating arc is placed exactly on the true arc. This yields an approximating arc with positive radial deviation which vanishes at the mid-point and the two ends of the arc. In the second approach, the mid-point of the approximating arc is placed slightly to the inside of the true arc, so that the negative radial deviation there is set balanced to the maximum positive derivations on the two half segments of the curve. It is mentioned that the accuracy of the second approach is better with a maximum deviation of $\pm 0.02\%$ for the quarter circle approximation and $\pm 0.0002\%$ for the octant circle approximation.

Other related articles can be found in journal and conference proceedings. In Reference 3, Peters converted circular arcs into parametric cubic curves with accuracies of $+0.03\%$ for the quarter circle and $+0.0004\%$ for the octant circle approximation. In Reference 4, DeBoor *et al.* developed an algorithm based on the geometric characterization of C^2 continuity with respect to arc length which approximates circular arcs with a 6th-order accuracy. Goldapp[5] gave a systematic approach in arc approximation by means of cubic polynomials. Besides the usual approach in which the two ends of the approximating curve are coincident with those of the arc, Goldapp improved the accuracy of the approximation by allowing the ends of the polynomial curve to miss those of the arc.

More related works can be found in the literature. It seems that all aforementioned works have one common feature. They all concentrate on the way to approximate circular arcs with the highest possible accuracy. While this has high academic credits, it is not convenient for the general practice of circle approximation. The reason for the inefficiency is due to the *ad hoc* nature of the approach that, for each circular arc of

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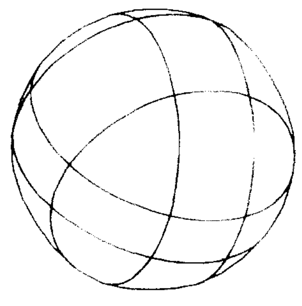

Figure 1 A sphere model in the framework of G^2 circles

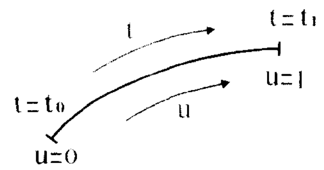

Figure 2 Reparametrization of a PC curve

different angular width, one has to search for the best end-tangent magnitudes for the most accurate fit. In the field of computer-aided design, situations may arise in which circular arcs of different angular spans are required (see, for instance, the sphere model shown in *Figure 1*). A simpler method for the construction of such general circles and spheres is deemed worthy of the effort.

In this article, the approximation of a circle is to be considered as the construction of composite PC splines with various compositions in curve segment lengths. Different methods of determining magnitudes of end-tangent vectors are demonstrated together with a comparison of the approximation accuracy of the different approaches. Finally, as a result of the investigation of geometric continuity employed in the approximation of the circle, a unique formulation for the determination of end-tangent magnitudes is derived. This formulation of end-tangent vectors for the construction of the parametric circle and sphere is believed to be most appropriate for the practice of computer-aided design.

Before going into the details of circle approximation, we lay down some fundamentals for the construction of PC curves and composite PC splines.

FUNDAMENTALS OF PC CURVES

In the practice of computer-aided design, a parametric cubic curve can be represented in matrix form as

$$\bar{\mathbf{p}}(u) = [1\ u\ u^2\ u^3]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} \bar{\mathbf{p}}(0) \\ \bar{\mathbf{p}}(1) \\ \bar{\mathbf{p}}'(0) \\ \bar{\mathbf{p}}'(1) \end{bmatrix}$$

$$= [\mathbf{U}]\,[\mathbf{C}]\,[\mathbf{B}] \qquad (1)$$

Here, $\bar{\mathbf{p}}(u)$ denotes a vector-valued function of the parameter u and $\bar{\mathbf{P}}'(u)$ denotes derivative of $\bar{\mathbf{p}}(u)$ with respect to the parameter. In the above formulation of the PC curve, the expressions in the boundary condition matrix **[B]** imply that we take the value $u = 0$ at the starting point and the value $u = 1$ at the ending point of the curve. This is the PC curve with normalized parameters commonly used in the field of computer-aided design.

There is one other, not so commonly used, form of the PC curve in which the parameter is taken as the arc length. Denoting the arc length by t, the same curve as formulated in Equation 1 can be reparametrized (see Reference 2, pp 53–56) and written as

$$\bar{\mathbf{q}}(t) = [1\ t\ t^2\ t^3]\,[\mathbf{C}]\begin{bmatrix} \bar{\mathbf{q}}(t_0) \\ \bar{\mathbf{q}}(t_1) \\ \dot{\bar{\mathbf{q}}}(t_0) \\ \dot{\bar{\mathbf{q}}}(t_1) \end{bmatrix}$$

$$= [\mathbf{T}]\,[\mathbf{C}]\begin{bmatrix} \bar{\mathbf{q}}(t_0) \\ \bar{\mathbf{q}}(t_1) \\ \hat{\mathbf{T}}_0 \\ \hat{\mathbf{T}}_1 \end{bmatrix} \qquad (2)$$

where, by the definition of a unit tangent vector, $\dot{\bar{\mathbf{q}}} = \mathrm{d}\bar{\mathbf{q}}/\mathrm{d}t = \hat{\mathbf{T}}$ and, $\hat{\mathbf{T}}_0$ and $\hat{\mathbf{T}}_1$ denote the unit tangent vectors at the start- and the end-point of the curve. This PC curve and the related notions are as shown in *Figure 2*. Notice that the end-tangent vectors are unit vectors only when the parameter is taken as the arc length. Here, for the construction of a single PC curve segment, the choice of the parameter is irrelevant to the result. For the construction of composite PC curves, it becomes more significant when data points are unevenly spaced. This will be further illustrated in the following sections.

CONSTRUCTION OF COMPOSITE PC CURVE

A general cubic spline curve consists of many cubic curve segments joined end-to-end satisfying certain continuity conditions at the joints. A common difficulty encountered in the construction of such composite curves is the appropriate assignment of parameter values to the joints, or the so-called data points, of the spline curve. In general practice, the number of and the locations of the joints in a composite curve are essentially determined by the curve designer by experience. The exact path of the resulting spline curve is not known before the computation of all unknown nodal

Figure 3 Notations of a composite PC spline

tangent vectors but, the computation of the unknowns tangent vectors requires the joint parameter values *a priori* (Reference 1, p 176). There are two generally adopted solutions to this computational dilemma.

Composite splines with normalized parametrization

The first solution of the aforementioned computational dilemma is the use of normalized parametrization. Continuing from last section, the kth curve segment in a general composite spline consisting of n curve segments as shown in *Figure 3*, with $k = 1, \ldots, n$ and $n + 1$ data points, with $i = 0, 1, \ldots, n$, can be expressed as

$$\bar{\mathbf{P}}_k(u) = [\mathbf{U}]\,[\mathbf{C}]\,\left[\bar{\mathbf{P}}_{i-1}\ \bar{\mathbf{P}}_i\ \bar{\mathbf{P}}'_{i-1}\ \bar{\mathbf{P}}'_i\right]^{\mathrm{T}} \tag{3}$$

Here, $\bar{\mathbf{P}}_i$ and $\bar{\mathbf{P}}'_i$ stand for the position and the first parametric derivative vectors of the ith data point and, the superscript T denotes a transpose. In more detail, we have

$$\bar{\mathbf{P}}_k(u) = (1 - 3u^2 + 2u^3)\bar{\mathbf{P}}_{i-1} + (3u^2 - 2u^3)\bar{\mathbf{P}}_i + (u - 2u^2 + u^3)\bar{\mathbf{P}}'_{i-1} + (-u^2 + u^3)\bar{\mathbf{P}}'_i \tag{4}$$

For a smooth joining between the curve segments, the following 2nd-order parametric continuity (C^2 continuity) conditions are applied:

$$\bar{\mathbf{P}}'_k(1) = \bar{\mathbf{P}}'_{k+1}(0) \tag{5}$$

$$\bar{\mathbf{P}}''_k(1) = \bar{P}''_{k+1}(0) \tag{6}$$

Here, the prime notation (′) denotes a derivative with respect to the parameter u. Substituting Equation 4 into Equation 6 leads to the following recursive equation for the computation of nodal tangent vectors:

$$\bar{\mathbf{P}}'_{j-1} + 4\bar{\mathbf{P}}'_j + \bar{\mathbf{P}}'_{j+1} = 3\left(\bar{\mathbf{P}}_{j+1} - \bar{\mathbf{P}}_{j-1}\right) \quad j = 1, \ldots, n-1 \tag{7}$$

Notice that the above continuity conditions are to be applied at the $n - 1$ internal joints and there are a total of $n + 1$ nodal tangent vectors to be determined. Two more tangent vector equations are needed and they can be picked up by selecting one of the following sets of end-point conditions:

(i) Fixed ends, whereof we have, with C_1 and C_2 predetermined constants

$$\bar{\mathbf{P}}_0 = C_1 \text{ and } \bar{\mathbf{P}}'_n = C_2 \tag{8}$$

(ii) Free ends, whereof we have

$$\bar{\mathbf{P}}''_0 = 0 \text{ and } \bar{\mathbf{P}}''_n = 0 \tag{9}$$

(iii) Cyclic ends, whereof we have

$$\bar{\mathbf{P}}_0 = \bar{\mathbf{P}}_n;\ \bar{\mathbf{P}}'_0 = \bar{\mathbf{P}}'_n;\ \bar{\mathbf{P}}''_0 = \bar{\mathbf{P}}''_n \tag{10}$$

For the completeness of the formulation and the purpose of illustration, we choose to use the cyclic-end conditions (10) which are for the case of circle construction. Together with the recursive Equation 7, we have for the computation of nodal tangent vectors,

$$\begin{bmatrix} 4 & 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 4 & 1 & 0 & & & \\ 0 & 1 & 4 & 1 & 0 & \cdots & \\ & & \ddots & & & & \\ & & & & \ddots & & \\ & & \cdots 0 & & 1 & 4 & 1 \\ 1 & 0 & \cdots\cdots & & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{P}}'_0 \\ \bar{\mathbf{P}}'_1 \\ \bar{\mathbf{P}}'_2 \\ \vdots \\ \vdots \\ \bar{\mathbf{P}}'_{n-2} \\ \bar{\mathbf{P}}'_{n-1} \end{bmatrix} = \begin{bmatrix} 3\left(\bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{n-1}\right) \\ 3\left(\bar{\mathbf{P}}_2 - \bar{\mathbf{P}}_0\right) \\ 3\left(\bar{\mathbf{P}}_3 - \bar{\mathbf{P}}_1\right) \\ \vdots \\ \vdots \\ 3\left(\bar{\mathbf{P}}_{n-1} - \bar{\mathbf{P}}_{n-3}\right) \\ 3\left(\bar{\mathbf{P}}_0 - \bar{\mathbf{P}}_{n-2}\right) \end{bmatrix} \tag{11}$$

Equations 11 indicate that for the construction of an enclosed composite spline one needs only to provide data point positions in an appropriately cyclic pattern. This formulation saves the trouble of assigning parameter values of the knots by adopting a uniform parameter range of zero to unity for each curve segment. While this feature may seem to be a merit of the formulation, undesirable effects may arise when data points are uncarefully arranged. An implication of the normalized parametrization is that all curve segments in the composite spline should have equal length or the data points should be evenly spaced. In the actual practice of curve and surface design, the requirement of evenly spaced data points may be excessively restrictive.

Composite splines with arc length as parameter

A second solution to the aforementioned computational dilemma is to strive for the curve length of the segments in the spline. As mentioned earlier, no curve length can be determined before the computation of nodal tangent vectors, and the computation of nodal tangent vectors requires appropriate length values at the nodes in the spline. A practical approach will be to start the computation with a rough estimate of the curve lengths and to arrive at more accurate curve lengths with appropriate iterations. The best estimate of curve length is the associated chord length of the curve segment. Therefore, with the assumption that the curve lengths of the segments are known, we have, for the typical kth segment in the spline as shown in *Figure 3*,

$$\bar{\mathbf{P}}_k(t) = \bar{\mathbf{P}}_{i-1}\left(1 - \frac{3t^2}{t_k^2} + \frac{2t^3}{t_k^3}\right) + \bar{\mathbf{P}}_i\left(\frac{3t^2}{t_k^2} - \frac{2t^3}{t_k^3}\right) + \dot{\bar{\mathbf{P}}}_{i-1}\left(t - \frac{2t^2}{t_k} + \frac{t^3}{t_k^2}\right) + \dot{\bar{\mathbf{P}}}_i\left(-\frac{t^2}{t_k} + \frac{t^3}{t_k^2}\right) \tag{12}$$

Here, the parameter t is taken as the arc length, and t_k stands for the arc length of the kth curve segment. And, the dot notation $(\cdot)$ denotes a derivative with respect to parameter t. A more detailed derivation which leads to Equation 12 can be found in Reference 7, pp 253–255.

For a smooth joining between two such curve segments, the kth segment and the $(k + 1)$th segment, the first and the second parametric derivative continuity (C^2 continuity) conditions now become

$$\dot{\bar{\mathbf{P}}}_k(t_k) = \dot{\bar{\mathbf{P}}}_{k+1}(0) \tag{13}$$

and

$$\ddot{\bar{\mathbf{P}}}_k(t_k) = \ddot{\bar{\mathbf{P}}}_{k+1}(0) \tag{14}$$

And, the recursive equations which relate the internal joint tangents and the node point positions become

$$\left(\frac{1}{t_k}\right)\dot{\bar{\mathbf{P}}}_{i-1} + \left(\frac{2}{t_k} + \frac{2}{t_{k+1}}\right)\dot{\bar{\mathbf{P}}}_i + \left(\frac{1}{t_{k+1}}\right)\dot{\bar{\mathbf{P}}}_{i+1} = \left(\frac{-3}{t_k^2}\right)\bar{\mathbf{P}}_{i-1} + \left(\frac{3}{t_k^2} - \frac{3}{t_{k+1}^2}\right)\bar{\mathbf{P}}_i + \left(\frac{3}{t_{k+1}^2}\right)\bar{\mathbf{P}}_{i+1} \tag{15}$$

It can be seen that Equations 15 are identical to Equation 7 when $t_k = t_{k+1} = 1$. Also, these equations reveal the fact that the determination of the unknown internal joint tangent vectors requires the segment curve lengths *a priori*. On the other hand, the following integral formula for the computation of the segment curve length

$$\bar{\bar{t}}_k = \int_0^{\bar{t}_k} \left|\dot{\bar{\mathbf{P}}}_k(t)\right| \mathrm{d}t \tag{16}$$

indicates that the computation of curve length requires the complete curve function *a priori*. The symbols $\bar{t}_k$ and $\bar{\bar{t}}_k$ used in Equation 16 also unveil the iterative nature of the computation. To initiate the iteration process, we approximate the segment curve length by the chord length which is actually the straight line distance between the two end-points of the curve segment, or

$$\bar{t}_k = |\bar{\mathbf{P}}_i - \bar{\mathbf{P}}_{i-1}| \tag{17}$$

Since $\bar{t}_k$ is always less than the true curve length, computational results indicate, after sufficient number of iterations, the computed curve length $\bar{\bar{t}}_k$ will approach the true curve length as an upper limit.

Composite splines with G^1 continuity

From the definition of unit tangent vector of a parametric curve, one can see that the two parametric derivative boundary conditions on the right-hand side of Equation 12 are actually the unit tangent vectors at the two ends of the kth curve segment or, since the parameter t is taken as the curve length in the present formulation

$$\dot{\bar{\mathbf{P}}}_{i-1} = \left.\frac{\mathrm{d}\bar{\mathbf{P}}_k(t)}{\mathrm{d}t}\right|_{t=0} = \hat{\mathbf{T}}_{i-1} \tag{18a}$$

and

$$\dot{\bar{\mathbf{P}}}_i = \left.\frac{\mathrm{d}\bar{\mathbf{P}}_k(t)}{\mathrm{d}t}\right|_{t=t_k} = \hat{\mathbf{T}}_i \tag{18b}$$

Rearranging terms in Equation 12 and replacing t/t_k by u, (again $0 \le u \le 1$), we have

$$\bar{\mathbf{P}}_k(u) = \bar{\mathbf{P}}_{i-1}(1 - 3u^2 + 2u^3) + \bar{\mathbf{P}}_i(3u^2 - 2u^3) + t_k\hat{\mathbf{T}}_{i-1}(u - 2u^2 + u^3) + t_k\hat{\mathbf{T}}_i(-u^2 + u^3) \tag{19}$$

A comparison between the two parametric cubic curve representations of Equations 4 and 19 reveals that they are form identical, since

$$\bar{\mathbf{P}}_i' = \left.\frac{\mathrm{d}\bar{P}_k(u)}{\mathrm{d}u}\right|_{u=1} = \left.\frac{\partial\bar{\mathbf{P}}_k(t)}{\partial t}\frac{\partial t}{\partial u}\right|_{t=t_k} = \hat{\mathbf{T}}_i t_k \tag{20}$$

Furthermore, from Equation 19, we have for the end at $u = 1$ of the kth segment

$$\bar{\mathbf{P}}_k'(1) = t_k\hat{\mathbf{T}}_i \tag{21}$$

and, for the end at $u = 0$ of the $(k + 1)$th segment

$$\bar{\mathbf{P}}_{k+1}'(0) = t_{k+1}\hat{\mathbf{T}}_i \tag{22}$$

Therefore, at the common joint between the two adjacent segments, we have

$$\bar{\mathbf{P}}_k'(1) = \frac{t_k}{t_{k+1}}\bar{\mathbf{P}}_{k+1}'(0) \tag{23}$$

which indicates that the joint is G^1 continuous (for geometric continuity, see Reference 6, p 149).

From a starting point with assumed known segment curve length, we arrive at the more general formulation of the PC curve, Equation 19. Now, one can see the difference between Equations 4 and 19. While the normalized parametrization associated with 4 implies an evenly-spaced data point spread, formulation associated with Equation 19 produces more accurate composite splines with the more general unevenly-spaced point spread. This will be illustrated by the circle approximations given in the following section.

Equation 19 further suggests: if one can determine the proper segment curve length t_k and the directions of the two end-point unit tangent vectors $\hat{\mathbf{T}}_{i-1}$ and $\hat{\mathbf{T}}_i$, then, with the two known end-point positions $\bar{\mathbf{P}}_{i-1}$ and

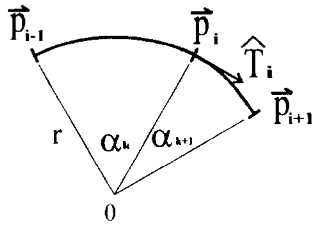

Figure 4 Joining of two circular arcs

$\bar{\mathbf{P}}_i$, the kth curve segment $\bar{\mathbf{P}}_k(u)$ is completely determined. This indeed is the case for the construction of circular arcs of arbitrary angular widths. Consider the joining of two arbitrary circular arc segments as shown in *Figure 4*. The arcs have radius r and angular width α_k and α_{k+1} in radians, respectively. It can easily be seen that the best arc lengths the approximation can achieve are the true arc lengths, or, respectively,

$$t_k = \alpha_k r \tag{24a}$$

and

$$t_{k+1} = \alpha_{k+1} r \tag{24b}$$

For the determination of the unit tangent vector $\hat{\mathbf{T}}_i$ at $\bar{\mathbf{P}}_i$, since $\vec{\mathbf{OP}} = x_i i + y_i j$,

$$\hat{\mathbf{T}}_i = \frac{1}{\sqrt{x_i^2 + y_i^2}}(y_i i - x_i j) \tag{25}$$

The other unit tangent vectors can be determined in a similar pattern. Generally, the kth arc segment in a general circle approximation can be expressed as

$$\bar{\mathbf{P}}_k(u) = \bar{\mathbf{P}}_{i-1}(1 - 3u^2 + 2u^3) + \bar{\mathbf{P}}_i(3u^2 - 2u^3) + r\alpha_k\hat{\mathbf{T}}_{i-1}(u - 2u^2 + u^3) + r\alpha_k\hat{\mathbf{T}}_i(-u^2 + u^3) \tag{26}$$

This is a unique formula for the construction of circular arcs since angular widths can always be determined by the known data point spread together with the known radius of the circle. The application and accuracy study of this circular arc formula will follow in the sequel.

CIRCLE APPROXIMATIONS

With the composite PC spline formulations developed in the previous section, we are to undertake the circle approximation with PC curves in the following settings.

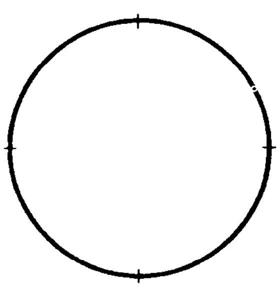

Figure 5 The C^2 circle with even spacing

The PC circle in even spacing and C^2 continuity

The circle as shown in *Figure 5* is a unit circle approximated by joining four PC curve segments. with the normalized parameter formulation, the Formulation I and the cyclic-end conditions for $n = 4$, the computation of Equations 11 yields the result of $|\bar{\mathbf{P}}_i'| = 1.50$ for $i = 0, 1, 2, 3$. An error estimation indicates that this approximated circle carries a maximal radial deviation $|\Delta r|/r = 0.02773$ where Δr is the difference between the radius of the approximating curve at the mid-point of its span and the corresponding radius r of the true circle. It can be seen by a visual inspection that this approximated circle is not very smooth. Further improvement on the approximation accuracy can be done either by increasing the number of curve segments or by raising the accuracy order of the curve formulation.

The PC circle in uneven spacing and C^2 continuity

The diagram in *Figure 6* shows a distorted circle with four curve segments of angular width $\pi/6$ and four segments of angular width $\pi/3$. The ticks on the circumference mark this unevenly spaced data point spread. For the illustration of the distortion effect of uneven spacing on the composite PC spline in normalized parametrization, this circle is purposefully constructed with the same conditions which produced the approximated circle in *Figure 5*.

The diagram in *Figure 7* shows a smoothly connected circle with the same data points as those used in the diagram of *Figure 6*. Now, in this circle, the component PC curves are of the arc-length-parameter formulation, Formulation II. These PC curves have end-tangent vectors with magnitudes equal to unity since the parameter is the arc length. Furthermore, the curve length t_k of each curve segment is obtained through

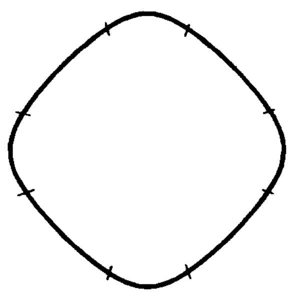

Figure 6 The C^2 circle with uneven spacing

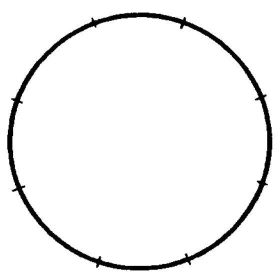

Figure 7 The G^2 circle

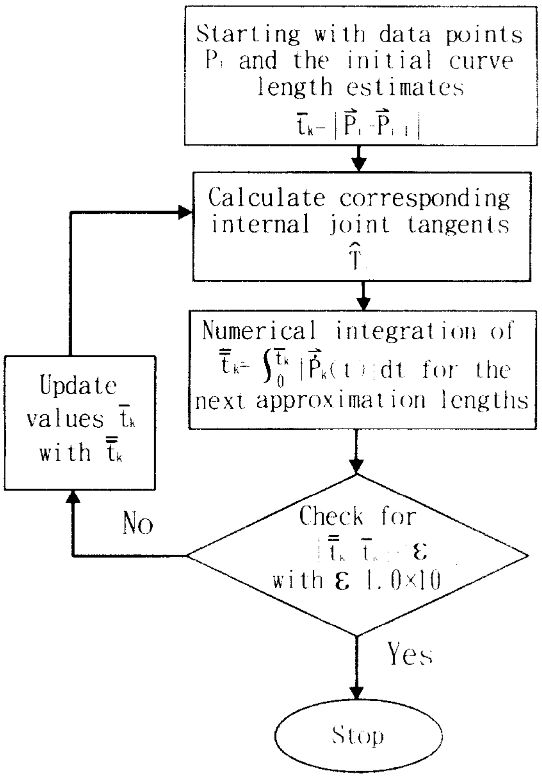

Figure 8 The flow chart for arc length iteration

Equation 16 with an initial length value given by Equation 17. The iteration cycles and the iteration ending criterion are given and depicted in *Figure 8*.

The PC circle in uneven spacing and G^2 continuity

The PC circle of *Figure 7* can also be constructed with the geometric continuity formulation, Formulation III. Generally, the curve segments in this formulation can be constructed in a piecewise independent manner. As one can see from the diagram, with the known arc radius r and the two end-point positions $\bar{\mathbf{P}}_{i-1}$ and $\bar{\mathbf{P}}_i$, the angular width α_k of the segment can be calculated with simple trigonometry. Together with Equations 24 and 25, PC curves expressed in Equation 19 are completely defined.

Furthermore, it can be established that the circle of this formulation also satisfies the G^2 continuity condition at the joints. From Reference 6, p 150, the G^2 continuity condition can be expressed as, with appropriate notations of the present paper

$$\bar{\mathbf{P}}''_{k+1}(0) = \beta_k^2 \bar{\mathbf{P}}''_k(1) + \upsilon_k \bar{\mathbf{P}}'_k(1) \qquad (27)$$

where β_k and υ_k are two constant parameters and are generally referred to as the bias parameter and the tension parameter for parametric curve shape adjustment (refer to Reference 8 for details). Here, for circle approximations, no tension adjustment is required and we have $\upsilon_k = 0$. Then, Equation 27 can be derived from Equation (23) with $\beta_k = t_{k+1}/t_k$. Therefore, circle approximations with curve segments by Equation 19 are actually G^2 continuous.

ACCURACY OF CIRCLE APPROXIMATIONS

The accuracies of the approximated circles formulated in this paper are tabulated in *Tables 1* and *2*. The common feature of the circles enlisted in *Table 1* is the angular width, or the number of segments, of PC curves in the approximation. The radial deviation $|\Delta r|/r$, indicates that the accuracy improves in the order of Formulations I, II, and III. This is a reasonable result from the standpoint of the rigorousness of the parametrization. Particularly worth mentioning is the accuracy improvement of Formulation III over that of Formulation II. In Formulation II, the improved accuracy is obtained by iterations. Through the iteration steps, the arc length of the PC curve segment approaches the best value that can be achieved by that formulation. Yet, it is still not the arc length of the true circle. In Formulation III, through Formulae 24, we adopt the true arc length for the use in the approximation, hence an even better result.

The common feature of the circles enlisted in *Table 2* is the formulation. Now by increasing the number of approximating curve segments, or by reducing the width of the arc span, we can reach the expected accuracy of design demands.

CONCLUSIONS

In this paper formulations of PC splines have been presented. They are presented in an order of increasing rigorousness and improved accuracy in circle approximation. Although the parametric curve is a common topic which has frequent appearance in the literature, some aspects of its nature still remain to be explored. In a loose sense, there are not strict standards for the performance judgment of free-form parametric curves. Their merits are based on the ease of use in practice and the smoothness in visual inspection, so that the normalization (Formulation I) evolves to be the most popular choice in the field without much concern to the effects of joint continuity and data point spread. Through the approximation of a circle, it has been shown here that the choice of joint continuity makes some significant improvement in the approximation accuracy and the spacing of data points will make a

Table 1 Approximation accuracy of arc with $\alpha = \pi/2$

Formulation	I	II	III
$\|\Delta r\|/r$	0.02773	0.01903	0.01521

Table 2 Approximation accuracy of Formulation III

Angular width	$\pi/2$	$\pi/3$	$\pi/6$
$\|\Delta r\|/r$	0.01521	0.00307	0.00019

striking difference in the appearance of the resulting spline curve.

Furthermore, with Formulation III, the G^2 continuous circle in uneven spacing, the construction of composite PC curves is rendered piecewise independent. In other words, unlike the conventional composite PC curve in which the end-tangent of curve segments at all internal joints are interlocked by Equations 7, the curve segments in this G^2-continuous PC circle can be constructed individually without influence from the other segments. This simplifies the construction of circles and spheres by the parametric cubic formation. Though the order of accuaracy of the approximated circle obtained by Formulation III is not quite comparable to some of those obtained in the referenced articles, it is accurate enough for the common practice of computer-aided design. Besides, its simple form and ease of use will certainly make it more attractive in the field.

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