Transform Domain Approach for Sequence Design and Its Applications

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***Abstract*—Many communication and radar systems necessitate the use of sequences with desired autocorrelation (AC) and cross-correlation (CC) properties. This paper presents a systematic method based on the transform domain characterization of the AC and CC constraints to generate new families of sequences that meet the requirements. We demonstrate that some existing families can easily be generated by our approach. Our approach, however, renders new families of sequences with less constraints. The proposed approach is elementary and can easily be extended to synthesize two-dimensional arrays or even higher dimensional waveforms that possess the desired multidimensional correlation properties. A preamble structure based on our new sequence family is suggested and performance of frequency offset and channel estimation algorithms for a multiantenna orthogonal frequency-division multiplexing (OFDM) system that uses such a preamble is given.**

***Index Terms*—Autocorrelation (AC), cross-correlation (CC), orthogonal waveform design, polyphase sequence.**

I. Introduction

SETS OF periodic sequences with good correlation properties are desired in many communication and radar applications. In a communication system, such sequences are used either in the preamble such that a receiver can easily perform pilot-assisted synchronization and/or channel estimation or as the signature codes for a spread spectrum multiple access network.

Oftentimes, we hope to have a family of sequences whose autocorrelation (AC) function has a single peak at the zero delay and whose cross-correlation (CC) values are identically zero at all delays. Such sequences can be used to avoid or minimize: 1) the interference from other users or other antennas if multiple transmit antennas were in place and 2) self-interference [e.g., intersymbol interference (ISI)] due to multiple propagation paths. Practical considerations also require that the sequence length be arbitrary and the family size be as large as possible, while maintaining the desired AC and CC properties.

Unfortunately, such an optimal family of sequences does exist for one cannot have both the ideal AC and CC. In fact, the bounds on CC and AC of sequences discussed in [1] indicate that there is a tradeoff between AC and CC when designing sequences. An alternate design approach is to either give up one of the desired correlation properties which has less impact on the system performance or to loosen the requirements and grant certain degree of nonzero correlations. For example, since the AC at time-lags (delays) larger than the transmission channel's maximum multipath delay do not contribute to self-interference, we might just require zero AC values at those nonzero lags less than the maximum delay, allowing arbitrary AC values outside the range of concern. The families of PS sequences [2] do have similar suboptimal correlation properties but its length is limited to squares of integers.

Recently, Park *et al.* [3] proposed a sequence generation method based on perfect reconstruction quadrature mirror filter banks. The sequences produced by this method have zero AC and CC at some delays; the remaining nonzero correlation values are in general lower than those of the well-known Gold sequences. However, their AC and CC are defined as the real parts of the conventional complex AC and CC functions and the length of the sequences must be of the form $N = \lfloor \{L(M-1)+1\}/2 \rfloor$. Tropp *et al.* [4] formulated the sequence design problem as an inverse singular value problem. But they considered symbol-synchronous code-division multiple-access (CDMA) systems, thus, require zero CC at periodic delays only.

In this paper, we present a transform domain approach for generating families of sequences whose periodic AC functions have nonzero values only at some subperiodic delays and whose periodic CC functions are identically zero. Although the PS sequences have similar correlation properties, our approach for constructing the desired sequences is elementary and simpler. It transforms the correlation requirements into transform domain identities, imposes almost no constraint on the sequence duration and, moreover, can be used to generate a large number of families. Our approach also has the benefit of interpreting the so-called modulatable orthogonal sequences [5] from the frequency (transform) domain's viewpoint.

The rest of this paper is organized as follows. In Section II, we derive the basic transform domain requirements on AC and CC. Section III presents the main theorem and a systematic transform domain process for constructing families of sequences with the desired correlation properties. Section IV provides a transform domain derivation of the class of modulatable orthogonal sequences. We then proceed to show that the class of PS sequences can be easily generated by our method (Section V). A transform domain approach for constructing multidimensional arrays with desired AC and CC is presented in Section VI. A preamble structure for multiple-input–multiple-output (MIMO)

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orthogonal frequency-division multiplexing (OFDM) wideband local area network (WLAN) systems is suggested in Section VII in which related frequency and channel estimator performance is also provided. Finally, we give some concluding remarks in Section VIII.

II. Definitions and Fundamental Properties

Let $\mathbf{X}$ denote a set of K complex-valued sequences of period N, i.e., for every sequence $\{u(n)\} \in \mathbf{X}$, $u(i) = u(i+N)$, for all $i \in \mathbb{Z}$, $\mathbb{Z}$ being the set of integers.

Definition 1: The *periodic* CC function of two period-N sequences $u \overset{\text{def}}{=} \{u(n)\}$, $v \overset{def}{=} \{v(n)\} \in \mathbf{X}$ is defined by

$$\theta_{u,v}(l) = \sum_{i=0}^{N-1} u(i)v^*(i+l), \quad l \in \mathbb{Z}. \tag{1}$$

The periodic AC function for the sequence $\{u(n)\}$ is just $\theta_{u,u}(l)$. We assume that $\theta_{u,u}(0) = N$ for all $\{u(n)\} \in \mathbf{X}$, then it is obvious that $|\theta_{u,u}(l)| \le N$ and $|\theta_{u,v}(l)| \le N$ for all u, $v \in \mathbf{X}$.

To facilitate the subsequent discourse, we need some more definitions; some of them are adopted from [2] for convenience of reference.

Definition 2: The $N \times N$ discrete Fourier transform (DFT) matrix with index m is given by

$$F^{(N,m)}(k,l) = \left[W_N^{-klm}\right] = \left(W_N^m\right)^{-kl} \tag{2}$$

where m is a natural number, $k, l = 0, 1, \ldots, N-1$, $W_N = e^{j2\pi/N}$ and $j = \sqrt{-1}$.

Definition 3: The *diagonalized matrix* $D(\{x_l\})$ associated with the sequence $\{x_l\}$ is defined as

$$D(\{x_l\}) = \operatorname{diag}(\{x_l\}). \tag{3}$$

Definition 4: A period-N sequence $\{u(n)\}$ is called an *orthogonal* sequence if its AC $\theta_{u,u}(l)$ satisfies

$$|\theta_{u,u}(l)| = C \sum_k \delta(l - kN), \qquad \forall\, l \tag{4}$$

where $\delta(n)$ is the Kronecker delta function and $C = \sum_{n=0}^{N-1} |u(n)|^2$.

It is *periodic orthogonal* with period N_c if

$$|\theta_{u,u}(l)| = C \sum_k \delta(l - kN_c), \qquad \forall\, l \tag{5}$$

where N_c divides N.

Definition 5: A family of period-N sequences $\{c_i(n)\}$ is called *near-optimal* if all its member sequences are periodic orthogonal and each pair has zero CC, i.e., for $0 \le l < N$

$$\begin{aligned} |\theta_{c_i,c_i}(l)| &= C \sum_{k=0}^{K-1} \delta(l - kN_c), \quad \forall\, i \\ \theta_{c_i,c_j}(l) &= 0, \qquad \text{if } i \ne j \end{aligned} \tag{6}$$

where $N = KN_c$.

Definition 6: A set of N-dimensional vectors $\{\vec{g}_i\}$ is said to be an *orthogonal tone set* if $\Omega_i \cap \Omega_j = \emptyset, \forall\, i \ne j$, where Ω_i is the support of the vector $\vec{g}_i$. The set is *complete* if $\sum_i |\Omega_i| = N$, i.e., if $\{\Omega_i\}$ is a partition of the set $\{0, 1, \cdots, N-1\}$. If the set has only two vectors ($|\{\vec{g}_i\}| = 2$), we say they are an *orthogonal tone pair*.

Definition 7: Let D^{-1} be the *delay operator* that cyclicly shifts the components of a vector to the right by one place and $D^{-i} = \underbrace{D^{-1} \circ D^{-1} \circ \cdots \circ D^{-1}}_{i\text{-fold}}$ denote the operator that shifts the components of a vector to the right cyclicly by i places.

Definition 8: The *L-fold expander* $\mathbb{E}_L$ converts a length-N sequence $\{x(n)\}$ into a length-LN sequence $\{x_E(n)\}$ by

$$\mathbb{E}_L\{x(n)\} = x_E(n) = \begin{cases} x\left(\frac{n}{L}\right), & \text{if } L|n \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

L is referred to as the expanding rate, while $\mathbb{E}_L$ is a rate-expanding mapping [6].

For convenience of reference we list the following well known property concerning the last definition.

Property 1: $X_E(k) = \text{DFT}\{x_E(n)\}$ is an L-period extension of $X(k)$, i.e., $X_E(k) = X(|k|_N)$, $0 \le k \le LN-1$, and $\text{IDFT}\{\mathbb{E}_L\{X(k)\}\}$ is an L-period extension of $x(n)$, where $|k|_N \overset{def}{=} k$ modulo N.

We shall refer to an N-dimensional vector $\vec{z}$ and its periodic extension $\{z(n)\}$ interchangeably, making no distinction between an N-dimensional vector and its periodic extension when there is no danger of confusing. The N-point DFT of $\vec{z}$, $\vec{Z}$ (or $\{Z(n)\}$), is called its spectral vector (representation).

We also need the following fundamental lemma and its corollaries in our subsequent discourse.

Lemma 1: The DFT of the periodic CC function $\theta_{x,y}(n)$ of two period-N sequences, $\{x(n)\}$ and $\{y(n)\}$, is equal to $X(k)Y^*(k)$, where $X(k)$ and $Y(k)$ are the DFTs of $\{x(n)\}$ and $\{y(n)\}$, respectively.

This lemma is well known and has appeared in the literature in various forms; e.g., see [10] and [11]. It follows immediately that.

Corollary 1: The AC function $\theta_{x,x}(n)$ is equivalent to $x(n) \circledast x^*(-n)$ and $\Theta_{x,x}(k) = \text{DFT}[\theta_{x,x}(n)] = |X(k)|^2$. Hence, a sequence $\{x(n)\}$ is orthogonal, i.e., $\theta_{x,x}(n) = C\delta(n)$, iff $|X(k)|^2$ is a constant for all k.

Corollary 2: The periodic CC $\theta_{x,y}(n)$ of the two sequences $\{x(n)\}$, $\{y(n)\}$ is identically zero iff their DFT's satisfy $X(k)Y^*(k) = 0, \forall k$.

Corollary 3: Members of the set of sequences $\{x_i(n)\}$, $i = 0, \cdots, L-1$, have zero CC if their spectral representations $\{X_i(k)\}$ form a set of orthogonal tones. Moreover, the "combined" sequence $\{\sum_i x_i(n)\}$ has impulse-like AC (i.e., orthogonal) if $\{X_i(k)\}$ is a complete set of orthogonal tones and $|X_i(k)|$ is a constant for all k.

III. Frequency-Domain Synthesis

A. Main Theorem

Based on the transform domain characterization of the AC and CC correlation functions, we now present a theorem that suggests a frequency-domain approach for generating *near-optimal* families of sequences.

Lemma 2: Let $\{\vec{c}_0, \vec{c}_1, \ldots, \vec{c}_{K-1}\} \stackrel{def}{=} \mathbf{C}$ be a set of K period-N sequences, where $N = KN_c$ and denote the N-point DFT of $\vec{c}_i$ by $\{C_i(\lambda), 0 \le \lambda < N\}$, then $\mathbf{C}$ is a *near-optimal* family if

$$C_i(\lambda) = K \cdot D^{-i} \circ \mathbb{E}_K \{X(k)\} \tag{8}$$

where $\{X(k)\}$ is the DFT of a length-N_c perfect AC sequence.

Proof: Let $\Theta_{c_i,c_i}(\lambda)$ be the N-point DFT of the AC function of $\vec{c}_i$. Then

$$\Theta_{c_i,c_i}(\lambda) = C_i(\lambda) C_i^*(\lambda), \quad 0 \le \lambda < N. \tag{9}$$

The establishment of the perfect CC property is straightforward since the spectral vectors $\{C_i(\lambda), 0 \le i < K\}$ form a complete set of orthogonal tones.

Substituting (8) into (9), we obtain

$$\Theta_{c_i,c_i}(\lambda) = K^2 \cdot D^{-i} \circ \mathbb{E}_K \{X(k)X^*(k)\}. \tag{10}$$

If $\{x(n)\} = \mathrm{IDFT}\{X(k)\}$ is a period-N_c sequence with perfect AC function, then

$$\Theta_{x,x}(k) = X(k)X^*(k) = N_c. \tag{11}$$

Therefore, for $i = 0$

$$\Theta_{c_0,c_0}(\lambda) = \begin{cases} K^2 \Theta_{x,x}(k) = K^2 N_c, & \lambda = Kk \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

and according to *Property 1*, the AC function of $\vec{c}_0$ is the K-period extension of that of $\{x(n)\}$, i.e.,

$$\theta_{c_0,c_0}(n) = N\delta\left(|n|_{N_c}\right). \tag{13}$$

The other $\Theta_{c_i,c_i}(\lambda)$'s for $0 < i < K$ are simply frequency-shifted versions of $\Theta_{c_0,c_0}(\lambda)$, i.e.,

$$\Theta_{c_i,c_i}(\lambda) = \Theta_{c_0,c_0}(\lambda - i). \tag{14}$$

The frequency-shifting operation induces a phase rotation in the time-domain

$$\theta_{c_i,c_i}(n) = W_N^{ni} \theta_{c_0,c_0}(n) = N W_N^{ni} \delta\left(|n|_{N_c}\right) \tag{15}$$

i.e., the AC function has a nonzero value only when $|n|_{N_c} = 0$. $x(n) = \mathrm{IDFT}\{X(k)\}$ is referred to as the generating sequence of $\mathbf{C}$ since its spectrum determines those of all members of $\mathbf{C}$. ■

In proving the above lemma, we make use of the facts that $X(k)$ has constant modulus and the spectral vectors $C_i(\lambda)$ form a complete set of orthogonal tones to derive the near-optimal AC and CC properties. It can easily be generalized to the following.

Theorem 1: Let $\vec{X}_i$, $i = 0, 1, \cdots, L-1$ be N-dimensional vectors, not necessarily distinct and define the LN-dimensional vectors, $\vec{C}_i = KD^{-i} \circ \mathbb{E}_L\{\vec{X}_i\}$. Then, the sequences, $\{c_i(n)\} = \mathrm{IDFT}\{\vec{C}_i\}$, $i = 0, 1, \cdots, L-1$, have zero CC and the corresponding AC functions in one period are L-period extensions of those of $\{x(n)\} = \mathrm{IDFT}\{\vec{X}_i\}$. Furthermore, if $\vec{X}_i$ are constant modulus vectors, then the family $\{c_i(n)\}_{i=0}^{L-1}$ is near-optimal.

B. Synthesis Process

The construction procedure suggested by *Lemma 2* requires that a sequence with constant modulus spectrum (or perfect AC) be found to begin with. There are many such sequences for use as the generating sequence. In fact, *Corollary 1* tells us that a perfect AC sequence can be generated by specifying a constant modulus spectral sequence and there are practically infinite many such sequences. Practical consideration, however, prefers to use a sequence whose components are drawn from a finite constellation such as phase-shift keying (PSK) or quadrature amplitude modulation (QAM). One known candidate sequence is the Frank–Zadoff–Chu (FZC) sequences [7], [8]—a class of unity-modulus polyphase sequences with impulse-like periodic AC functions.

Theorem 1 also suggests that if we let $\vec{X}_i$, $i = 0, 1, \cdots, L-1$ be the spectral vectors of m-sequences $\vec{x}_i$ (not necessarily distinct) of period $N_c = 2^m - 1$, then we can generate a family with zero CC and AC function the same as a periodic extension of that of $\vec{x}_i$. We summarize the synthesis procedure and the major attributes of the resulting sequences as follows.

(S.1) Given an orthogonal sequence $\{x(n)\}$ of period N_c and let $X(k)$ be the corresponding spectral sequence of constant modulus. Rate-expanding $K \cdot X(k)$ by K-fold and taking KN_c-point inverse discrete Fourier transform (IDFT) on the K cyclic-shifted versions $D^{-i} \circ \mathbb{E}_K\{KX(k)\}$, $i = 0, 1, \cdots, K-1$, we obtain a near-optimal family of K length-KN_c sequences.

(S.2) The generating sequence $\{x(n)\}$ determines the AC function in one period, and the CC properties between sequences are determined by their DFTs.

(S.3) A member sequence will consists of complex numbers with unity magnitude if $\{x(n)\}$ is an FZC sequence (see *Example 1* below) or a polyphase sequence generated by the method suggested in *Lemma 3*. We can also achieve the same result by selecting an m-sequence as the generating sequence. However, in most cases an fast Fourier transform (FFT)-like transform for a sequence with period $2^n - 1$ is not as fast as the conventional FFT on sequences with a period of powers of 2 [11].

Example 1: To generate two near-optimal sequences of length 12, one has to choose a perfect AC sequence first. Let an FZC sequence of length 6 be chosen and pick $M = 5$ which is relative prime to $N = 6$. According to the generating method in [8], the elements of the FZC sequence $\{a_k\}$ would be $\{W_{2N}^{\phi_k}\}$, where $\phi_k = Mk^2$. Following the above synthesizing procedure, we obtain two orthogonal polyphase sequences $\{\vec{c}_i\} = \{W_{2N}^{\phi_k^{(i)}}\}$, $i = 0, 1$, where $\{\phi_k^{(0)}\} = \{0,5,8,9,8,5,0,5,8,9,8,5\}$ and $\{\phi_k^{(1)}\} = \{0,6,10,0,0,10,6,0,4,6,6,4\}$. It can easily be proved that the AC functions of them are $|\theta_{c_i,c_i}(n)| = 12 \cdot [\delta(n) + \delta(n-6)]$ for $0 \le n < 12$, and the CC function $\theta_{c_0,c_1}(n)$ is identically zero for all n.

IV. Class of Orthogonal Sequences

Although a sequence with perfect AC can be immediately obtained by taking IDFT on a vector of constant modulus, it is always preferred that the components of the sequence belong to a small set of special symbols. A method to construct a class of polyphase sequences that have perfect AC was proposed in [5]. This section provides a transform domain approach and interpretation for generating the same sequences and their generalizations.

For a set of basic symbols $\{b_i\}$ located on the circle of radius d in the complex plane ($|b_i| = d$) and $0 \le m \le N_b - 1$, we define the basic orthogonal sequence matrix G of size $N_b \times N_b$ as

$$G^{(m)} = F^{(N_b,-m)} D\left(\{b_i\}\right). \tag{16}$$

Note that for the case $m = 1$, $F^{(N,1)}\vec{x} = \mathrm{DFT}\{\vec{x}\}$, $F^{(N,-1)}\vec{y} = N \cdot \mathrm{IDFT}\{\vec{y}\}$. As the factor N does not affect the AC property, it will be omitted in the remaining discussion. Consider the sequence

$$\vec{g} = \mathrm{vec}\left(\left[G^{(m)}\right]^T\right) \tag{17}$$

where $\mathrm{vec}(\cdot)$ denotes the (column vectors) stacking operator. Without loss of generality, we take the case $N_b = 3$, $m = 1$ as an example and let $G^{(1)} \stackrel{\mathrm{def}}{=} G$, then

$$\begin{aligned} G &= F^{(N_b,-1)} D\left(\{b_i\}\right) = F^{(N_b,-1)} \left[\vec{b}_0 \quad \vec{b}_1 \quad \vec{b}_2\right] \\ &= \left[\vec{g}_0 \quad \vec{g}_1 \quad \vec{g}_2\right] = \begin{bmatrix} g_0 & g_3 & g_6 \\ g_1 & g_4 & g_7 \\ g_2 & g_5 & g_8 \end{bmatrix} \end{aligned} \tag{18}$$

where $\vec{b}_0 = [b_0 \;\, 0 \;\, 0]^T$, $\vec{b}_1 = [0 \;\, b_1 \;\, 0]^T$, and $\vec{b}_2 = [0 \;\, 0 \;\, b_2]^T$. The ith column vector $\vec{g}_i$ of the matrix G is equal to the N_b-point IDFT of $\vec{b}_i$. Since there is only one nonzero component in $\vec{b}_i$ and all the components in F have unit magnitude, the components of $\vec{g}_i$ must have identical magnitude.

Employing a procedure similar to (8), we first expand the time-domain vectors $\vec{g}_i$ by threefold, and then cyclic-shift the rate-expanded vectors $\vec{g}_{e_i} = \mathbb{E}_3\{\vec{g}_i\}$ to obtain the set of orthogonal tones $\vec{g}_{es_i} = D^{-i} \circ \vec{g}_{e_i}$

$$\begin{aligned} \vec{g}_{es_0} &= [g_0 \quad 0 \quad 0 \quad g_1 \quad 0 \quad 0 \quad g_2 \quad 0 \quad 0]^T \\ \vec{g}_{es_1} &= [0 \quad g_3 \quad 0 \quad 0 \quad g_4 \quad 0 \quad 0 \quad g_5 \quad 0]^T \\ \vec{g}_{es_2} &= [0 \quad 0 \quad g_6 \quad 0 \quad 0 \quad g_7 \quad 0 \quad 0 \quad g_8]^T. \end{aligned} \tag{19}$$

The N_b^2-point DFT of $\vec{g}_{e_i}$ is the N_b-period extension of $\vec{b}_i$, while that of $\vec{g}_{es_i}$ is a linearly phase-shifted N_b-period extension of $\vec{b}_i$. Define the new vector

$$\vec{g} = \vec{g}_{es_0} + \vec{g}_{es_1} + \vec{g}_{es_2} = \mathrm{vec}(G^T). \tag{20}$$

The N_b^2-point DFT of $\vec{g}$ is a constant modulus vector (see Fig. 1) and, according to *Corollary 1*, $\mathrm{vec}(G^T)$ has the perfect AC property, irrespective of the linear phase rotation induced by the time-shifting operation. The reason why we obtain a constant modulus spectrum is obviously due to the fact that $\{\mathrm{DFT}\{\vec{g}_{es_i}\}\}$ is a set of orthogonal tones which resulted from the tone-orthogonality of the set $\{\vec{b}_i\}$. Generalizing the above argument, we can easily verify.

Lemma 3: Let $G = F^{(N_b,-1)}B$, where B is an $N_b \times L$ matrix whose column vectors $\{\vec{b}_i = (b_{i0}, b_{i1}, \cdots, b_{i(N_b-1)})\}$ have constant modulus components, i.e., $|b_{ij}| = c$, $\forall i$ and j, and form a complete set of orthogonal tones. Then

$$\mathrm{vec}(G^T) = N_b \cdot \sum_{i=0}^{L-1} D^{-i} \circ \mathbb{E}_L \left\{\mathrm{IDFT}\{\vec{b}_i\}\right\}$$

is an orthogonal sequence of period $N_b L$. Moreover, if each $\vec{b}_i$ has only one nonzero unit-modulus component, then $\mathrm{vec}(G^T)$ is a polyphase sequence.

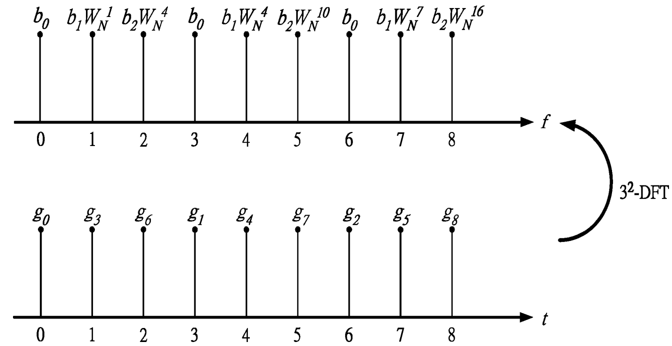

Fig. 1. Generating a sequence with perfect periodic AC property. All time and spectral components are of the same magnitude.

Note that considering $\vec{g}_{es_i}$ of (20) not as components of $\mathrm{vec}(G^T)$ but as individual transform domain vectors, *Corollary 3* then implies that the sequences $\mathrm{IDFT}\{\vec{g}_{es_i}\}$ do form a near-optimal family.

Extending to the cases $m \neq 1$ now, we first note that $\vec{b}_i = [X_i(0), X_i(1), \ldots, X_i(N_b - 1)]^T$, where $X_i(k) = b_i \delta(k - i)$, which is the case that $B = D\{(b_i)\}$. Taking N_b-point IDFT on $\vec{b}_i$, we obtain the vector $[x_i(0), x_i(1), \ldots, x_i(N_b - 1)]^T \stackrel{\mathrm{def}}{=} \vec{g}_i^{(1)} \stackrel{\mathrm{def}}{=} \vec{g}_i$. In general, the ith column of $G^{(m)}$, $m \neq 1$ can be written as

$$\vec{g}_i^{(m)} = \left[x_i^{(m)}(0),\, x_i^{(m)}(1), \ldots,\, x_i^{(m)}(N_b - 1)\right]^T \tag{21}$$

where

$$x_i^{(m)}(n) = \sum_{k=0}^{N_b-1} X_i(k) W_{N_b}^{kmn} = b_i W_{N_b}^{imn} \tag{22}$$

is a phase-rotated version of $x_i(n)$, for $x_i^{(1)}(n) = x_i(n) = b_i W_{N_b}^{in}$. Hence, we have

$$\begin{aligned} & x_i(n) \stackrel{\mathrm{DFT}}{\longrightarrow} X_i(k) = b_i \delta(k - i) \\ \Longrightarrow \quad & x_i^{(m)}(n) \stackrel{\mathrm{DFT}}{\longrightarrow} X_i\left(|k - i(m-1)|_{N_b}\right) \\ & \qquad = b_i \delta\left(|k - im|_{N_b}\right). \end{aligned} \tag{23}$$

The above relation implies that the parameter m determines the order of the frequency-domain index cyclic-shifting. As long as $\{\vec{b}_i\}$ form a complete set of orthogonal tones, the sequence $\mathrm{vec}([G^{(1)}]^T)$ will preserve the perfect periodic AC property. On the other hand, if $\exists m$ such that the sequence $\mathrm{vec}([G^{(m)}]^T)$ has perfect AC, then we can find a column-permuted matrix P of the diagonal matrix $D(\{b_i\}) = [\vec{b}_0, \ldots, \vec{b}_{N_b-1}]$ such that $F^{(N_b,-m)}D(\{b_i\}) = F^{(N_b,-1)}P$. We, thus, claim the following generalization of the results reported in [5].

Theorem 2: Let $G = F^{(N_b,-m)}B$, where B is an $N \times L$ matrix whose column vectors $\{\vec{b}_i\}$ form a complete set of orthogonal tones. Then, $\text{vec}(G^T)$ is an orthogonal sequence of period N_bL iff $g.c.d.(m, N_b) = 1$.

Proof: We shall start with the special case $B = D\{(b_i)\}$, then proceed to prove the general case using (23) directly.

1) $g.c.d.(m, N_b) = 1 \Rightarrow \Omega_i^{(m)} \cap \Omega_j^{(m)} = \emptyset, \forall i \neq j$, where $\Omega_i^{(m)}$ is the support of the ith column vector of the matrix P.

For if $\exists i \neq j \ni \Omega_i^{(m)} \cap \Omega_j^{(m)} \neq \emptyset$, then

$$\begin{aligned} & im \bmod N_b = jm \bmod N_b, \;\; 0 \le i,j < N_b. \\ \Rightarrow \;\; & (i-j)m \bmod N_b = 0 \\ \Rightarrow \;\; & (i-j)m = pN_b, \;\; p \in \mathbb{Z}. \end{aligned} \tag{24}$$

The assumption $g.c.d.(m, N_b) = 1$ then implies that N_b divides $(i-j)$ and so $i = j$. Therefore, we must have $g.c.d.(m, N_b) \neq 1$.

2) $g.c.d.(m, N_b) = d > 1 \Rightarrow \Omega_i^{(m)} \cap \Omega_j^{(m)} \neq \emptyset$.

Let $m = hd$ and $N_b = kd$, then $g.c.d.(h,k) = 1$. It can easily be established that $im(\text{mod } N_b) = jm(\text{mod } N_b) \Rightarrow (i-j)hd(\text{mod } kd) = 0$, $\Rightarrow (i-j)h(\text{mod } k) = 0 \Rightarrow (i-j)h = qk, q \in \mathbb{Z}$. But $g.c.d.(h,k) = 1$ implies that $k|(i-j)$ and $i = j \bmod k$. Hence, $|\Omega_i^{(m)} \cap \Omega_j^{(m)}| = d$. The equivalence class $\{j, j+k, j+2k, \ldots\}$ under mod k addition has d elements and there are N/d different equivalence classes.

For the general case that the $N_b \times L$ matrix $B = [\vec{B}_0 \cdots \vec{B}_{L-1}]$ is composed of column vectors $\{\vec{B}_l\}$ that form a complete set of orthogonal tones, we can treat each column vector $\vec{B}_l$ as the sum of vectors $\vec{b}_j$ having only one nonzero component so that $\vec{B}_l = \sum_{j \in \Omega_{B_l}} \vec{b}_j$, where Ω_{B_l} is the support of the vector $\vec{B}_l$. Obviously, we have $\bigcap_l \Omega_{B_l}^{(m)} = \emptyset$. Following a similar argument in proving for the special case $B = D\{(b_i)\}$, we conclude that $\text{vec}(G^T)$ is an orthogonal sequence of period N_bL iff $g.c.d.(m, N_b) = 1$. ■

The following example checks a special case of this theorem; other examples can be found in [9].

Example 2: Consider the case, $N_b = 4$, $m = 3$, $W_4 = e^{j2\pi/4}$, and $\vec{b} = \{W_4^1, W_4^2, W_4^3, W_4^4\}$. In this case, we have $g.c.d.(m, N_b) = 1$. The basic orthogonal sequence matrix $G^{(m)}$ is $F^{(N_b,-m)}D(\{b_i\})$. Taking DFT on column vectors of $G^{(m)}$ and using the resulting frequency-domain vectors as the column vectors of P, we then obtain

$$P = 4 \begin{bmatrix} W_4^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_4^4 \\ 0 & 0 & W_4^3 & 0 \\ 0 & W_4^2 & 0 & 0 \end{bmatrix}. \tag{25}$$

It can be proved that the AC function of the sequence $\text{vec}(G^{(m)^T})$ is $|R(\tau)| = 16 \cdot \delta(|\tau|_{16})$. ■

As b_i's are selected from the unit circle, the generated sequence will be composed of complex numbers with the same magnitude. Fig. 1 plots an example in which the entries of the sequence have the same magnitude in both time and frequency domains. As mentioned before, a sequence has perfect AC function if all of its frequency components have the same magnitude. The time-frequency duality property of Fourier transform pairs then implies that if one exchanges the roles of the "time-domain" sequence and the "frequency-domain" sequence, the AC property can still be maintained. This property is shared by the sequence generated here and the FZC sequences, therefore, (8) can be modified as

$$C_i(\lambda) = \begin{cases} Kx_k, & \lambda = Kk + i \\ 0, & \text{otherwise} \end{cases}. \tag{26}$$

This sequence generation procedure is a much simpler than but equivalent to that of the PS sequences.

V. Generation of the PS Sequences

The so-called PS sequences [2] is also a near-optimal family that consists of K polyphase sequences of period N_s, where $N_s = KN_b^2$. The value N_b^2 determines the period of the AC function of the member PS sequences. We now provide a transform domain derivation of the PS sequences and their AC and CC properties. The derivation is much simpler and more instructive than that given in [2].

Let the basic orthogonal sequence $\{g(p)\}$ of length N_b^2 [5] be generated by (16) with $m = 1$ and $|b_i| = 1$, i.e.,

$$\vec{g}_b = \left[g(0), g(1), \ldots, g\left(N_b^2 - 1\right)\right]^T = \text{vec}(G^T). \tag{27}$$

Again, following a process similar to (8), we first rate-expand the basic orthogonal sequence $\vec{g}_b$ with a rate of K, and then use the K cyclic-shifted versions $D^{-i} \circ \vec{g}_{bs} = \vec{h}_i$, $i = 0, 1, \cdots, K-1$ of the rate-expanded vector $\vec{g}_{bs}$ as column vectors of the $KN_b^2 (= N_s) \times K$ matrix

$$H = [\vec{h}_0, \vec{h}_1, \cdots, \vec{h}_{K-1}]. \tag{28}$$

Since $\{\vec{h}_i\}$ is a complete set of orthogonal tones, their time-domain representations $\{\text{IDFT}\{\vec{h}_i\}\}$ must have zero CC. The column vectors $\vec{c}_k = (c_{0,k}, c_{1,k}, \cdots, c_{(N_s-1),k})^T$ of the $N_s \times K$ matrix

$$C = [c_{l,k}] = \frac{1}{N_b} F^{(N_s,-1)} H \tag{29}$$

form a near-optimal family whose member sequences are called *PS sequences*. As $\vec{h}_i$ is a K-fold-expanded version of a constant modulus vector, the AC function of a PS sequence is

$$\theta_{c,c}(\tau) = N_s W_{N_s}^{\tau k} \delta\left(|\tau|_{N_b^2}\right). \tag{30}$$

A careful examination of the above construction procedure reveals that it is the same as that suggested by *Lemma 2* except for: 1) the choice of the generating sequence and its period and 2) instead of using the Fourier transform $X(k)$ of a perfect AC sequence $x(n)$, a perfect AC sequence $g(p)$ is used for rate-expanding and cyclic-shifting. Despite 2), we obtain a near-optimal family since $g(p)$ happens to be a constant modulus sequence. Replacing $g(p)$ by any other perfect AC N_b-phase sequence of the same period, we obtain another set of PS-like sequences. Note, however, that our proposed construction imposes no constraint on the period of the sequences except that it cannot be a prime. For example, one cannot generate PS sequences having AC function of period 38; since $38 \neq KN_b^2$, for any natural numbers K and N_b.

VI. Multidimensional Arrays

Like the one-dimensional (1-D) case, two-dimensional (2-D) arrays that possess some desired AC or CC properties are useful in sonar/radar and communication applications. Similarly, higher dimensional array signals are needed in some multidimensional search and detection applications. In this section, we extend the method we developed for 1-D sequences to two or higher dimensions arrays.

A. Preliminary

For convenience of reference, we follow the notations and definitions used by [10].

Definition 9: Let an array sequence $A = \{a_{i,j}\}$ be denoted by

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N_2-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N_2-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N_1-1,0} & a_{N_1-1,1} & \cdots & a_{N_1-1,N_2-1} \end{bmatrix}. \quad (31)$$

The 2-D periodic AC function between two array sequences A and B having the same dimensions is defined as

$$R_{A,B}(\phi,\omega) = \sum_{p=0}^{N_1-1}\sum_{q=0}^{N_2-1} a_{p,q} b^{*}_{|p+\phi|_{N_1},|q+\omega|_{N_2}}. \quad (32)$$

Definition 10: An array is called a perfect array if its periodic AC function satisfies

$$R_{A,A}(\phi,\omega) = R_A(\phi,\omega) = \begin{cases} E, & (\phi,\omega) = (0,0) \\ 0, & (\phi,\omega) \neq (0,0) \end{cases} \quad (33)$$

where $E = \sum_{p=0}^{N_1-1}\sum_{q=0}^{N_2-1} |a_{p,q}|^2$.

There are many published works on the syntheses of perfect arrays; one of them is based on [10]

Theorem 3: (Folding method.) Let $\{b_l\}$ be an orthogonal sequence of length $N = N_1N_2$. Then, the array $\{a_{m,n}\}$ defined by

$$a_{m,n} = b_l,\ m = l \bmod N_1,\ n = l \bmod N_2 \quad (34)$$

is perfect if $g.c.d.(N_1, N_2) = 1$.

B. Generation of New 2-D Arrays

Our approach for generating a family of 2-D arrays consists of three steps.

Step 1) Applying the folding method to the FZC sequence or any other orthogonal sequence of length N_1N_2, where $gcd(N_1, N_2) = 1$, we obtain an $N_1 \times N_2$ perfect array $A = [a_{p,q}]$. Denote the 2-D DFT of this perfect array by $F(u, v)$, i.e.,

$$F(u,v) = \sum_{p=0}^{N_1-1}\sum_{q=0}^{N_2-1} a_{p,q} W_{N_1}^{-pu} W_{N_2}^{-qv}. \quad (35)$$

Step 2) Using the $F(u, v)$ as the basic building spectral array, we construct 2-D spectral arrays $F^{(i)}(U, V)$, $i = K_2\alpha + \beta$, $0 \leq \alpha < K_1$, and $0 \leq \beta < K_2$, according to

$$F^{(i)}(U,V) = \begin{cases} K_1K_2F(u,v), & U = K_1u+\alpha, \\ & V = K_2v+\beta \\ 0, & \text{otherwise} \end{cases}. \quad (36)$$

This assignment rule is illustrated in Fig. 2.

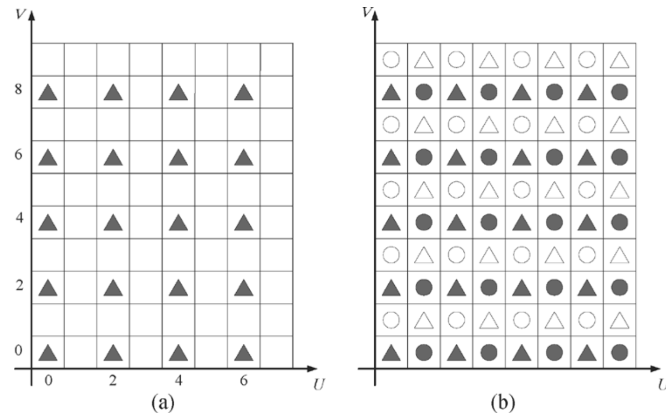

Fig. 2. (a) Constructing $F^{(0)}(U, V)$ from the 2-D DFT of the basis array. (b) Composite spectral representations of $F^{(i)}(U, V)$; different symbols are used to represent the nonzero positions of different spectral arrays ($K_1 = 2$, $K_2 = 2$, $N_1 = 4$, and $N_2 = 5$).

Step 3) Taking the 2-D IDFT on $F^{(i)}(U, V)$, we obtain an array sequence $C^{(i)} = [c^{(i)}_{m,n}]$ of dimension $K_1N_1 \times K_2N_2$, where the 2-D IDFT is defined by

$$C^{(i)}(m,n) = \frac{1}{K_1N_1K_2N_2} \cdot \sum_{U=0}^{K_1N_1-1}\sum_{V=0}^{K_2N_2-1} F^{(i)}(U,V) W_{K_1N_1}^{mU} W_{K_2N_2}^{nV}. \quad (37)$$

The elements of the array $C^{(i)}$ are in the form of $W_Q^{\phi^{(i)}_{m,n}}$, where

$$Q = \begin{cases} l.c.m.(2N_1N_2, K_1K_2N_1N_2), & \text{if } N_1N_2 \text{ is even} \\ l.c.m.(N_1N_2, K_1K_2N_1N_2), & \\ \quad = K_1K_2N_1N_2 & \text{if } N_1N_2, \text{ is odd} \end{cases} \quad (38)$$

and $\phi^{(i)}_{m,n}$'s are integers. The new array sequences possess some desired properties similar to those in the 1-D case. It can be easily proved that

$$|R_{C^{(i)}}(\phi,\omega)| = K_1K_2N_1N_2 \cdot \delta\left(|m|_{N_1}\right) \cdot \delta\left(|n|_{N_2}\right). \quad (39)$$

The AC function $|R_{C^{(i)}}(\phi, \omega)|$ is periodic in both arguments—the period in ϕ is N_1, while the period in ω is N_2. Moreover, the periodic CC function between any two arrays of $C^{(i)}$'s is zero and the number of the member arrays in the family is $K_1 \times K_2$.

Example 3: Suppose we have a perfect array of dimension $N_1 \times N_2$ already and want to generate K_1K_2 near-optimal arrays of dimension $K_1N_1 \times K_2N_2$ with $N_1 = 4$, $N_2 = 5$, and $K_1 = K_2 = 2$. Applying the folding method to the FZC sequence of length 20, we immediately have a 4 × 5 perfect array. By following steps 2–3, we obtain $K_1K_2 = 4$ near-optimal arrays. The magnitude of the AC function for these arrays, $|R_{C^{(i)}}(\phi, \omega)|$, $i = 0, \ldots, 3$, is periodic in both ϕ and ω and is plotted in Fig. 3, where the period along ϕ is 4, while the period along ω is 5.

C. Extension to Multidimensional Arrays

It is straightforward to extend the above concept to higher dimensional arrays. One of the key steps in generalizing the technique presented in the previous subsection for synthesizing 2-D arrays is to find a 2-D perfect array. Similarly, to construct a

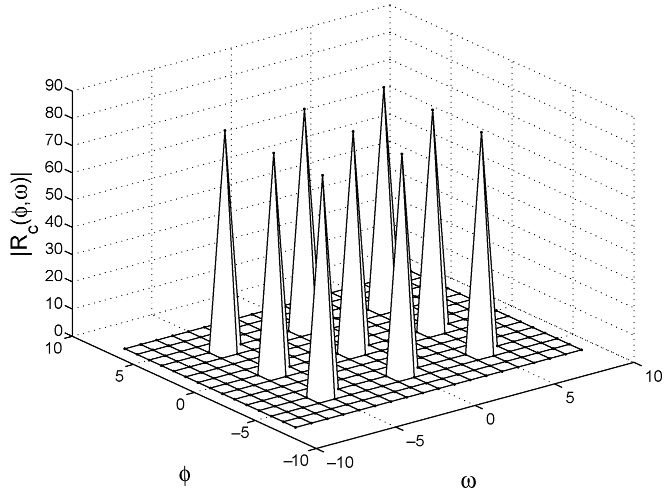

Fig. 3. Magnitude $|R_{C^{(i)}}|$ of the 2-D periodic AC function associated with the proposed array. ($K_1 = 2, K_2 = 2, N_1 = 4$, and $N_2 = 5$).

family of multidimensional arrays with the near-optimal property, one has to have a perfect multidimensional array (i.e., one whose multidimensional AC function is nonzero only at the origin) to begin with. This task is made easier by noting that *Corollary 1* can be extended to higher dimensional domains. Define the n-dimensional DFT of an array $A = \{a_{p_1,p_2,\ldots,p_n}\}$ of dimension $N_1 \times N_2 \times \cdots \times N_n$ by

$$\begin{aligned} F_A(u_1, u_2, \ldots, u_n) &= F_A(\vec{u}) \\ &= \sum_{p_1=0}^{N_1-1} \cdots \sum_{p_n=0}^{N_n-1} a_{u_1,\ldots,u_n} W_{N_1}^{-p_1 u_1} \cdots W_{N_n}^{-p_n u_n}. \end{aligned} \quad (40)$$

Then, we have as follows.

Corollary 4: An array A is perfect if $F_A(\vec{u})$ is a constant for all $\vec{u}$.

In *Theorem 3*, we mention a method to construct a perfect 2-D array from an orthogonal sequence. It can be shown that this method does satisfy the above corollary for $n = 2$.

Given an perfect array $\{a_{p_1,p_2,\ldots,p_n}\}$ of dimension $N_1 \times N_2 \times \cdots \times N_n$, we generate a family of multidimensional arrays by a three-step procedure similar to that of Section VI-B.

(C.1) Taking n-dimensional DFT on this basis (perfect) array.

(C.2) Let $C^{(i)}$ be $K_1N_1 \times K_2N_2 \times \cdots \times K_nN_n$ arrays to be generated and denote the corresponding n-dimensional DFT's by $F^{(i)}(U_1, U_2, \ldots, U_n) \stackrel{def}{=} F^{(i)}(\vec{U})$, i.e.,

$$F^{(i)}(\vec{U}) = \sum_{p_0=0}^{K_1N_1-1} \sum_{p_1=0}^{K_2N_2-1} \cdots \sum_{p_n=0}^{K_nN_n-1} C^{(i)}_{p_1,p_2,\ldots,p_n} W_{K_1N_1}^{-p_1U_1} W_{K_2N_2}^{-p_2U_2} \cdots W_{K_nN_n}^{-p_nU_n}, \quad (41)$$

where $i = 0, \ldots, (K_1K_2 \cdots K_n - 1)$. Then, $F^{(i)}(\vec{U})$ is constructed according to the assignment rule

$$F^{(i)}(\vec{U}) = \begin{cases} K_1 \cdots K_n F(u_1, \ldots, u_n), & \vec{U} = f(\vec{u}, i) \\ 0, & \text{otherwise} \end{cases} \quad (42)$$

where $\{f(\vec{u}, i)\}$ defines the support (nonzero positions) in transform domain for the ith newly-generated array. The distribution of $f(\vec{u}, i)$ is a multidimensional generalization of that of Fig. 2(b); for a given i, they are equally spaced along all axes.

(C.3) Taking n-dimensional IDFT on $F^{(i)}(\vec{U})$

$$\begin{aligned} C^{(i)}(\vec{p}) &= C^{(i)}(p_1, \ldots, p_n) \\ &= \frac{1}{K_1N_1 \cdots K_nN_n} \sum_{U_1=0}^{K_1N_1-1} \cdots \\ &\quad \sum_{U_n=0}^{K_nN_n-1} F^{(i)}(\vec{U}) W_{K_1N_1}^{p_1U_1} \cdots W_{K_nN_n}^{p_nU_n}. \end{aligned} \quad (43)$$

We then obtain an array sequence $C^{(i)}$ of dimension $K_1N_1 \times K_2N_2 \times \ldots \times K_nN_n$. It can be proved that the CC function between any two array sequences so generated is identically zero.

VII. Preamble Design for Multiantenna OFDM Systems

Applications of the near-optimal family of sequences to CDMA or similar systems are well known [2]. This section provides another application example of such sequences in designing preambles for multiantenna OFDM systems. Note that, as mentioned in (S.3), we can generate finite-constellation polyphase sequences in both domains, hence, at least in the preamble part, the peak-to-average power ratio is not a concern.

We assume that the OFDM guard interval is of length L and is larger than the maximum delay spread of the channel of concern. For such a bounded maximum delay spread scenario, we use the proposed near-optimal polyphase family, choosing a sequence length such that the unwanted AC peak values occurs at delays larger than L. Using members of a near-optimal family with period $N = KN_c = KL$ as preamble sequences, where K is the number of transmit antennas, we then have all the self interference due to multipath propagation eliminated. For applications to IEEE 802.11a-compatible multiantenna OFDM systems with K transmit antennas, we can use a family of near-optimal sequences with period 64 or 32 to replace the two long training symbols of the 802.11a standard.

In our simulation, we use an exponentially decayed Rayleigh-fading channel with the impulse response given by $\{h_t = \alpha_t + j\beta_t\}$. α_t and β_t are independent zero-mean complex Gaussian random variables with the same variance $\sigma_t^2 = \sigma_0^2 e^{-tT_s/T_{\max}}$, where $\sigma_0^2 = 1 - e^{-T_s/T_{\max}}$, T_s is the sampling period, and $T_{\max}$ is the maximum delay spread. Table I summarizes system and sequence parameters used in our simulation. We also assume that channels between different transmit and receive antennas are independent.

A. Frequency Synchronization

Moose [12] proposed a correlation-based technique that uses two consecutive identical pilot symbols to estimate CFO. Schmidl and Cox (SC) [13] suggested a differentially encoded preamble and applied the correlation metric to obtain both integer and fractional parts of CFO. Subsequent techniques use multiple identical or differentially encoded pilots with a smaller

TABLE I
SOME SYSTEM AND SEQUENCE PARAMETERS USED IN SIMULATION

Sample period (T_s)	50 ns	Sequence parameters	
Guard interval width	16 samples	K	4
Training symbol duration	64 samples	N_c (AC period)	16

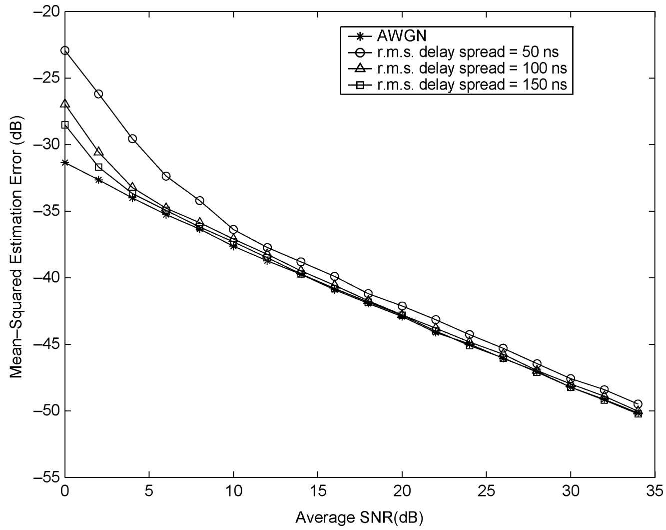

Fig. 4. Effect of the channel delay spread on the CFO estimator's MSEE performance; 2Txs and 1 Rx with frequency offset = 0.3 subcarrier spacings.

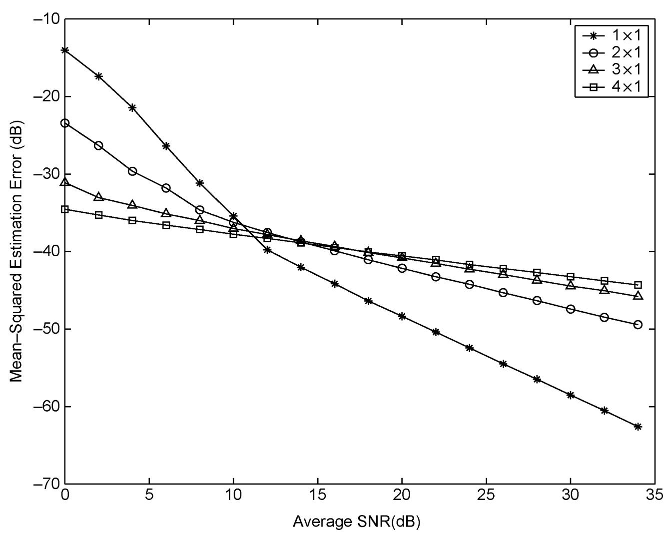

Fig. 5. Effect of the transmit antenna numbers on the MSEE performance of the SC frequency estimator; CFO $=0.3$ subcarrier spacings and r.m.s. channel delay spread $=50$ ns.

symbol period to increase the estimation range of CFO (see [14] and the references therein). For the multiple antenna systems with two identical pilots, the correlation metric of [12] is still applicable. A proper weighted version would give the optimal estimate, however. In Figs. 4 and 5, we respectively examine the influences of the channel's delay spread and the number of transmit antennas on the mean-squared estimation error (MSEE) performance of the SC estimate. We notice that the estimator is insensitive to the r.m.s. delay spread of the channel except for low average SNRs, where a larger delay spread helps reducing MSEE, if the maximum channel delay spread does not exceed the length of the guard interval. In Fig. 5, for fairness of comparison, we assume that the total transmit power is fixed. These performance curves indicate that there is a threshold (around 11.5 dB) beyond which the noncoherent combining loss associated with the SC frequency estimate outweighs the corresponding diversity combining gain.

B. Channel Estimation

With the use of members of a near-optimal family as pilot sequences, the receiving end of a MIMO system can separate the signals originated from different transmit antennas. Hence, one can easily modify any pilot-assisted, correlation-based channel estimator designed for conventional single antenna systems to serve as a MIMO channel estimator. The structure of the proposed pilot sequences, however, allows a very simple least-squares (LS) channel estimator to achieve the optimal performance. We give a sketch of the proof for a 2 × 1 system in the following paragraph. Extension to the general MIMO systems is straightforward [15].

Let $\vec{r}(t) = [r_t, r_{t+1}, \ldots, r_{t+N_s-1}]^T$ be the received data vector and denote the transmit pilot vector from the kth antenna by $\vec{s}_k = [s_{k0}, s_{k1}, \ldots, s_{k(N_s-1)}]^T$. Assuming perfect frame timing and frequency offset compensation, we have, for a 2 × 1 system

$$\vec{r} = \mathbf{S}\mathbf{h} + \vec{n} = [\mathbf{S}_1\mathbf{S}_2]\begin{bmatrix}\vec{h}_1\\ \vec{h}_2\end{bmatrix} + \vec{n} \tag{44}$$

where $\vec{h}_i$ is the channel impulse response vector associated with the ith transmit antenna and

$$\mathbf{S}_i = \begin{bmatrix} s_{i0} & s_{i(-1)} & \cdots & s_{i(-L+1)} \\ s_{i1} & s_{i0} & \cdots & s_{i(-L+2)} \\ \vdots & \vdots & & \vdots \\ s_{i(N_s-1)} & s_{i(N_s-2)} & \cdots & s_{i(-L+N_s)} \end{bmatrix}_{N_s\times L}. \tag{45}$$

Notice that $s_{i(-t)} = s_{i(N_s-t)}$ for $1 \le t < L$ since cyclic prefix is added. It can be shown that if $\vec{n}$ is a zero-mean complex white Gaussian vector, the corresponding MSEE of the LS channel estimator $\hat{\mathbf{h}} = (\mathbf{S}^H\mathbf{S})^{-1}\mathbf{S}^H\vec{r}$ is lower-bounded by

$$E\left[(\mathbf{h}-\hat{\mathbf{h}})^H(\mathbf{h}-\hat{\mathbf{h}})\right] = \sigma_n^2 \,\mathrm{trace}\left\{(\mathbf{S}^H\mathbf{S})^{-1}\right\} \ge \frac{2L\sigma_n^2}{E_s} \tag{46}$$

where $E_s = \|\vec{s}_k\|^2$ is the energy of the polyphase pilot vector. The lower bound is achieved by a pilot vector that satisfies $\mathbf{S}^H\mathbf{S} = E_s\mathbf{I}$ and the minimum MSEE is independent of the channel response. It is clear that our pilot sequences satisfy this criterion with $E_s = N_s$ and the LS estimator can be further simplified to $\hat{\mathbf{h}} = (1/N_s)\mathbf{S}^H\vec{r}$, which means it is to be derived from a simple matched filtering.

As the two long training symbols associated with each antenna contain two periods of the corresponding pilot sequence, an improved channel estimator can be obtained by

$$\hat{\mathbf{h}} = \begin{bmatrix}\hat{\vec{h}}_1\\ \hat{\vec{h}}_2\end{bmatrix} = \frac{1}{2N_s}\mathbf{S}^H\left[\vec{r}(0) + \vec{r}(N_s)\right]. \tag{47}$$

The minimum MSEE is guaranteed so long as the channel memory is less the guard interval, irrespective of the true channel response. When the channel memory is longer than the guard interval, the performance deteriorates accordingly. Nevertheless, it is highly likely that the strengths for those paths

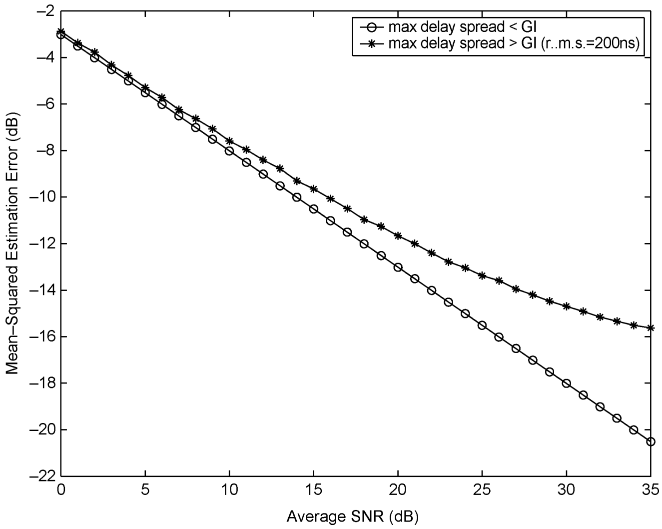

Fig. 6. MS channel estimation error performance for channels with different memory lengths; guard interval width $= 16$ samples.

whose relative delays are larger than the guard interval are very small. Fig. 6 compares the per-channel MSEE performance for both cases.

VIII. Conclusion

We have presented systematic methods for generating 1-D sequences and multidimensional arrays with near-optimal AC and CC properties. The well-known fact that the AC and CC functions are closely related to the DFTs of the desired sequences enable our approach to render a natural interpretation, a much simpler derivation and a generalization of the orthogonal sequences proposed in [5]. We also show that the class of PS sequences [2] is a special family within the sequence families generated by our new synthesis method. Our approach is elementary and it is believed that our approach can provide an avenue for discovering new sequences with other desired properties.

These families of sequences or arrays have many interesting applications. We present application examples for frequency synchronization and channel estimation in MIMO-OFDM systems. Numerical simulation indicates that, as the preamble sequences used possess the desired properties, both estimators yield excellent performance.

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