PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 11, November 2006, Pages 3159–3162
S 0002-9939(06)08699-0
Article electronically published on June 5, 2006

ANDERSON'S THEOREM FOR COMPACT OPERATORS

HWA-LONG GAU AND PEI YUAN WU

(Communicated by Joseph A. Ball)

ABSTRACT. It is shown that if A is a compact operator on a Hilbert space with its numerical range $W(A)$ contained in the closed unit disc $\overline{\mathbb{D}}$ and with $\overline{W(A)}$ intersecting the unit circle at infinitely many points, then $W(A)$ is equal to $\overline{\mathbb{D}}$. This is an infinite-dimensional analogue of a result of Anderson for finite matrices.

The *numerical range* $W(A)$ of a bounded linear operator A on a complex Hilbert space H is the subset $\{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ of the complex plane, where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the inner product and norm in H, respectively. Basic properties of the numerical range can be found in [5, Chapter 22] or [4].

In the early 1970s, Joel Anderson proved an interesting result on the numerical ranges of finite matrices. Namely, if A is an n-by-n complex matrix, considered as an operator on $\mathbb{C}^n$ equipped with the standard inner product and norm, with its numerical range $W(A)$ contained in the closed unit disc $\overline{\mathbb{D}}$ ($\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$) and intersecting the unit circle $\partial\mathbb{D}$ at more than n points, then $W(A) = \overline{\mathbb{D}}$ (cf. [9, p. 507]). The purpose of this paper is to prove an infinite-dimensional analogue of Anderson's result for compact operators.

Theorem 1. *If A is a compact operator on a Hilbert space with $W(A)$ contained in $\overline{\mathbb{D}}$ and $\overline{W(A)}$ intersecting $\partial\mathbb{D}$ at infinitely many points,* then *$W(A) = \overline{\mathbb{D}}$.*

Anderson never published his proof of the above-mentioned result. As related by him many years later via an e-mail to the second author, his proof was based on the application of Bézout's theorem to the Kippenhahn curve of the matrix A. Generalizations of this result along this line can be found in [3]. In recent years, there appeared three more proofs. One is by Dritschel and Woerdeman [2, Theorem 5.8], based on the canonical decomposition and radial tuples for numerical contractions developed by them. (A *numerical contraction* is an operator A with $W(A) \subseteq \overline{\mathbb{D}}$.) The second one is due to the second author (cf. [12, Lemma 6]); it depends on the classical Riesz-Fejér theorem on nonnegative trigonometric polynomials. More recently, Hung gave another proof in his Ph.D. dissertation [6, Theorem 4.2.1] by making use of Ando's characterization of numerical contractions.

Received by the editors February 4, 2005 and, in revised form, March 23, 2005.
2000 *Mathematics Subject Classification.* Primary 47A12; Secondary 47B07.
Key words and phrases. Numerical range, compact operator.
This research was partially supported by the National Science Council of the Republic of China.

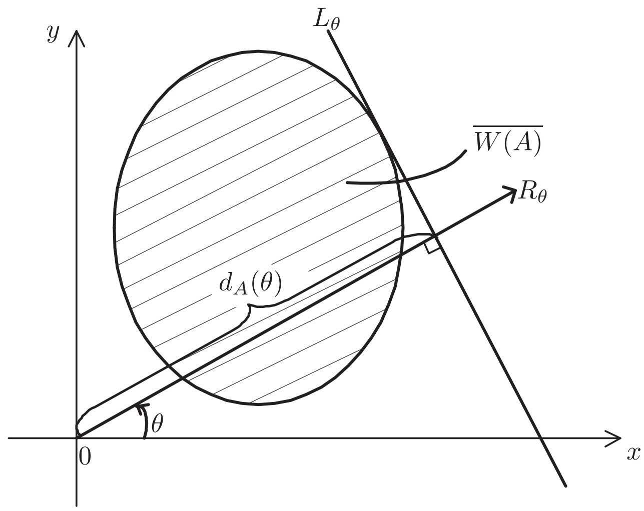

FIGURE 2.

We will prove Theorem 1 using the support function d_A of the compact convex set $\overline{W(A)}$ of an operator A:

$$\begin{aligned} d_A(\theta) &= \max \overline{W(\mathrm{Re}\,(e^{-i\theta}A))} \\ &= \max \overline{W(\cos\theta \mathrm{Re}\, A + \sin\theta \mathrm{Im}\, A)} \end{aligned}$$

for θ in $\mathbb{R}$, where $\mathrm{Re}\, A = (A + A^*)/2$ and $\mathrm{Im}\, A = (A - A^*)/(2i)$ are the real and imaginary parts of A. Note that $d_A(\theta)$ is simply the signed distance from the origin to the supporting line L_θ of $\overline{W(A)}$ which is perpendicular to the ray R_θ from the origin that forms angle θ from the positive x-axis (cf. Figure 2).

Our main tool is the next theorem, due to Rellich [10, p. 57], on the analytic perturbation for multiple eigenvalues of Hermitian operators; an elegant and elementary proof can be found in [11, p. 376]. The present form is from [8, Theorem 3.3].

Theorem 3. *Let $\theta \mapsto A_\theta$ be a real analytic function from an open interval I of $\mathbb{R}$ to Hermitian operators on a fixed Hilbert space, and let $d(\theta) = \max \overline{W(A_\theta)}$ for θ in I. Assume that for some θ_0 in I, $d(\theta_0)$ is an isolated eigenvalue of A_{θ_0} with finite multiplicity n. Then there is an open subinterval J of I which contains θ_0 and there are m, $1 \le m \le n$, real analytic functions $d_1, \ldots, d_m : J \to \mathbb{R}$ such that*

(a) $d_1(\theta_0) = \cdots = d_m(\theta_0) = d(\theta_0)$,

(b) *for every θ in $J \setminus \{\theta_0\}$, the $d_j(\theta)$'s are distinct isolated eigenvalues of A_θ with respective multiplicity n_j independent of θ which satisfies $\sum_{j=1}^m n_j = n$,*

(c) *there is some d_{j_1} (resp., d_{j_2}) such that $d(\theta) = d_{j_1}(\theta)$ (resp., $d(\theta) = d_{j_2}(\theta)$) for all θ, $\theta < \theta_0$ (resp., $\theta > \theta_0$) in J, and*

(d) *$d(\theta) = \max\{d_1(\theta), \ldots, d_m(\theta)\}$ for all θ in J.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. We first express our assumptions in terms of d_A. The condition $W(A) \subseteq \overline{\mathbb{D}}$ is obviously equivalent to $d_A(\theta) \le 1$ for all θ. Under this, we then have, for a fixed θ, the equivalence of $e^{i\theta} \in \overline{W(A)}$ and $d_A(\theta) = 1$. Indeed, $e^{i\theta}$ belonging to $\overline{W(A)}$ is equivalent to 1 belonging to $\overline{W(e^{-i\theta}A)}$, which is the same as 1 belonging to $\operatorname{Re} \overline{W(e^{-i\theta}A)} = \overline{W(\operatorname{Re}(e^{-i\theta}A))}$ (because $W(e^{-i\theta}A) \subseteq \overline{\mathbb{D}}$) or $d_A(\theta) = 1$.

Now let $e^{i\theta_n}$, $n \geq 1$, $\theta_n \in [0, 2\pi)$, be a sequence of distinct points in $\overline{W(A)} \cap \partial\mathbb{D}$. Passing to a subsequence, we may assume that θ_n converges to θ_0 in $[0, 2\pi]$. Since $d_A(\theta_n) = 1$ for all n and the function $\theta \mapsto \overline{W(\operatorname{Re}(e^{-i\theta}A))}$ is continuous (cf. [5, Solution 220]), we obtain $d_A(\theta_0) = 1$. Moreover, since $\overline{W(\operatorname{Re}(e^{-i\theta_0}A))}$ equals the convex hull of the spectrum of the compact operator $\operatorname{Re}(e^{-i\theta_0}A)$, we infer that $d_A(\theta_0)$ is an isolated eigenvalue of $\operatorname{Re}(e^{-i\theta_0}A)$ with finite multiplicity. Thus Theorem 3 may be applied to obtain two real analytic functions d_1 and d_2 on some neighborhood $J = (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ of θ_0 such that $d_A = d_1$ on $(\theta_0 - \varepsilon, \theta_0]$ and $d_A = d_2$ on $[\theta_0, \theta_0 + \varepsilon)$. Without loss of generality, we may assume that $(\theta_0 - \varepsilon, \theta_0]$ contains infinitely many θ_n's. Hence $d_1(\theta_n) = d_A(\theta_n) = 1$ for such θ_n's. Since θ_n converges to θ_0 and d_1 is analytic on J, we obtain $d_1 = 1$ on J. Therefore, $d_1 \leq d_A \leq 1$ implies that $d_A = 1$ on J. Let $\alpha = \{\theta \in \mathbb{R} : d_A(\theta) = 1\}$. The above arguments also show that if θ' is a limit point of α, then there is some neighborhood $(\theta' - \varepsilon', \theta' + \varepsilon')$ contained in α. Now let $a = \sup\{\theta \in \mathbb{R} : [\theta_0, \theta) \subseteq \alpha\}$ and $b = \inf\{\theta \in \mathbb{R} : (\theta, \theta_0] \subseteq \alpha\}$. We infer from the above that $a = \infty$ and $b = -\infty$, that is, $\alpha = \mathbb{R}$. This shows that $d_A = 1$ on $\mathbb{R}$ or, equivalently, $\partial\mathbb{D} \subseteq \overline{W(A)}$. As we have seen in the first paragraph of this proof, $d_A(\theta) = 1$ is equivalent to $1 \in \overline{W(\operatorname{Re}(e^{-i\theta}A))}$. Since this latter set equals the convex hull of the spectrum of the compact operator $\operatorname{Re}(e^{-i\theta}A)$, we infer that 1 is an eigenvalue of $\operatorname{Re}(e^{-i\theta}A)$. Hence 1 is in $W(\operatorname{Re}(e^{-i\theta}A))$ or in $W(e^{-i\theta}A)$ (since $W(e^{-i\theta}A) \subseteq \overline{\mathbb{D}}$), which is the same as $e^{i\theta}$ in $W(A)$. We conclude that $\partial\mathbb{D} \subseteq W(A)$. The convexity of $W(A)$ then implies that $W(A) = \overline{\mathbb{D}}$, completing the proof. □

An alternative proof for the last part of the preceding proof is, after obtaining $\overline{W(A)} = \overline{\mathbb{D}}$ from $\partial\mathbb{D} \subseteq \overline{W(A)}$ and the convexity of $\overline{W(A)}$, to invoke [5, Solution 213] that any compact operator A with $0 \in W(A)$ has $W(A)$ closed, concluding that $W(A) = \overline{\mathbb{D}}$.

We end this paper with some further remarks. First, any compact operator A with $W(A) = \overline{\mathbb{D}}$ must have norm bigger than one. This is because if $\|A\| \leq 1$, then from the equality case of the Cauchy-Schwarz inequality, we easily derive that $W(A) \cap \partial\mathbb{D} = \partial\mathbb{D}$ consists of eigenvalues of A, which is impossible for the compact A. Second, we note that in Theorem 1 the condition that $\overline{W(A)}$ intersects $\partial\mathbb{D}$ at infinitely many points cannot be weakened. For example, for each $n \geq 1$, if A_n is the finite-rank operator $\operatorname{diag}(1, \omega_n, \ldots, \omega_n^{n-1}, 0, 0, \ldots)$, where ω_n is the nth primitive root of 1, then $W(A_n) \subsetneqq \overline{\mathbb{D}}$ and $\overline{W(A_n)}$ intersects $\partial\mathbb{D}$ at the n points $1, \omega_n, \ldots, \omega_n^{n-1}$. Finally, Theorem 1 can be generalized from the unit disc to any elliptic disc centered at the origin: *if A is a compact operator with $W(A)$ contained in the closed elliptic disc*

$$E = \{x + iy \in \mathbb{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}, \quad a, b > 0,$$

and with $\overline{W(A)}$ intersecting ∂E at infinitely many points, *then* $W(A) = E$. This can be reduced to Theorem 1 by considering the affine transform

$$B = \frac{1}{a}\mathrm{Re}\, A + \frac{i}{b}\mathrm{Im}\, A$$

of A since the numerical range of B equals $\overline{\mathbb{D}}$.

[7] and [1] are the other papers which contain information on the numerical ranges of compact operators.

References

[1] J. Agler, Geometric and topological properties of the numerical range, *Indiana Univ. Math. J.* 31 (1982), 767–777. MR0674866 (84i:47004)

[2] M. Dritschel and H. J. Woerdeman, *Model Theory and Linear Extreme Points in the Numerical Radius Unit Ball*, Mem. Amer. Math. Soc. 129 (1997), no. 615. MR1401492 (98b:47007)

[3] H.-L. Gau and P. Y. Wu, Condition for the numerical range to contain an elliptic disc, *Linear Algebra Appl.* 364 (2003), 213–222. MR1971096 (2004b:15053)

[4] K. E. Gustafson and D. K. M. Rao, *Numerical Range, the Field of Values of Linear Operators and Matrices*, Springer, New York, 1997. MR1417493 (98b:47008)

[5] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982. MR0675952 (84e:47001)

[6] C.-N. Hung, *The Numerical Range and the Core of Hilbert-space Operators*, Ph.D. dissertation, Univ. of Toronto, 2004.

[7] J. S. Lancaster, The boundary of the numerical range, *Proc. Amer. Math. Soc.* 49 (1975), 393–398. MR0372644 (51:8851)

[8] F. J. Narcowich, Analytic properties of the boundary of the numerical range, *Indiana Univ. Math. J.* 29 (1980), 67–77. MR0554818 (81a:47005)

[9] M. Radjabalipour and H. Radjavi, On the geometry of numerical ranges, *Pacific J. Math.* 61 (1975), 507–511. MR0399891 (53:3732)

[10] F. Rellich, *Perturbation Theory of Eigenvalue Problems*, Gordon and Breach, New York, 1969. MR0240668 (39:2014)

[11] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar, New York, 1955. MR0071727 (17:175i)

[12] B.-S. Tam and S. Yang, On matrices whose numerical ranges have circular or weak circular symmetry, *Linear Algebra Appl.* 302/303 (1998), 193–221. MR1733531 (2000m:15040)

Department of Mathematics, National Central University, Chung-Li 32001, Taiwan
E-mail address: hlgau@math.ncu.edu.tw

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan
E-mail address: pywu@math.nctu.edu.tw