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考慮壽險次級市場下的保單計價新思維：以變額壽險為例

The Alternative Pricing Approach for Variable Life Insurance Incorporating Secondary Life Insurance Market

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摘要：變額壽險保單主要特點為保單到期給付金額連結於投資標的資產之市場價格，因保單所附保證收益可視同賣權價值，典型評價模式即結合現代財務選擇權理論與傳統壽險精算等價原理，文獻上後續的相關研究皆沿用此計價原則。由於近年來美國壽險保單次級市場的快速發展與成長，使得壽險保單在市場交易的流動性大為增加，壽險保單已不單是為獲取保障的保險契約，亦是可交易的資產組合，此市場特徵提供了應用財務選擇權計價模式的重要條件，因此，有別於上述傳統典型的計價方法，本研究以純粹的財務選擇權定價觀點，特別納入壽險次級市場因素，針對變額壽險保單提出另一種計價模式，並以此觀點檢視變額壽險保單的傳統計價方法及其特性。在本研究的計價架構下，除証明了傳統計價方法所求得之價格將對應一組特定的風險中立測度外，數值分析結果亦說明了無套利的合理價格與傳統計價結果的關係，結果顯示，保單的無套利合理價格範圍將因連動資產價值的波動性、無風險利率及死亡機率型態之改變而呈不同方向的變動，尤其會隨著壽險保單在次級市場的流動性風險溢酬增加而擴大。

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Abstract: One distinguishing feature of variable life insurance policy is that the benefit payable at expiration depends on the market value of the linked reference portfolio as contrasted with traditional life insurance policies. The conventional pricing approach combines traditional law of large number considerations and financial mathematics. Subsequent relevant studies follow such the valuation approaches. Because recently secondary life insurance markets in America are developing and growing rapidly, liquidity of life insurance contracts has significantly improved. So life insurance contracts could not only be guarantees against losses, but also could be seen as tradable portfolio assets. This market characteristic could serve an extra condition for the application of option pricing model to the valuation of variable life insurance. In this article, in comparison with the conventional pricing approaches for variable life insurance, an alternative valuation method is developed with pure option pricing approach especially incorporating the secondary life insurance market. The conventional valuation approach and its properties are reviewed and its derived price is proved as a special one with respect to a specific risk-neutral probability measure in the present valuation framework. Numerical analysis illustrates the relationship between no-arbitrage price bounds and the conventional pricing approach as well. The results indicate that no-arbitrage bounds of the insurance contract would be influenced by asset price volatility, risk-free rate and mortality pattern in different directions, and particularly would be augmented with liquidity risk premium in the secondary life insurance market.

Keywords: Variable life insurance; Option pricing approach; Secondary life insurance market

1. Introduction

Secondary life insurance markets have been growing rapidly in America. A wide variety of similar products in secondary life insurance market have been developed, including viatical settlements, accelerated death benefits (ADBs) and life settlements. Secondary life insurance markets allow consumers to sell their

policies to independent financial companies or originally-issued insurance company for getting money back (Bhattacharya et. al. 2004). So life insurance contracts could not only serve as guarantees against losses, but also could be seen as tradable portfolio assets. This market characteristic could offer an extra condition for the application of option pricing model to the valuation of life insurance contract. This article focuses on the valuation of variable life insurance under a pure option pricing framework. The conventional approach for the valuation of variable life insurance combines traditional law of large number considerations and financial mathematics. It usually assumes the independence between the stochasticity of a reference fund and mortality distribution as well as the insurer's risk neutrality with respect to mortality. The logic behind those assumptions is that insurers usually suppose the policyholders with the same age will have the same death distribution (said to be homogenous) and each policyholder's death is independent of other's. Thus only when insurers can obtain a large number of independent homogenous insurance buyers, the conventional pricing approach could be applied. However, not all of insurance companies can satisfy completely the requirement for pooling arrangements. In comparison with the conventional pricing approach for variable life insurance, this study develops an alternative valuation method with the pure option pricing approach especially incorporating the secondary life insurance market. Without requiring the independence assumption in the conventional approach, the price process of a reference fund and the death process of an insured are considered jointly to create an underlying stochastic process. A typical option pricing approach usually begins with assuming an underlying asset following a specific stochastic process. Contingent payments at each time are determined by exercise price. The variable life insurance also could be treated as a contingent claim of the market structure we create. This proposed approach will lead to prices that coincide with those determined by the conventional pricing approach (CPA henceforth) suggested by Brennan and Schwartz (1976, 1977, 1979). A different insight into properties of the conventional pricing approach has been explored.

The distinguishing feature of variable life insurance is that benefit payable at expiration depends upon market value of some reference portfolios that may

consist of stocks, bonds or other financial assets with mutual funds as typical cases. Because policyholders have to bear more risk for this type of insurance product, insurers need to enhance the product attraction by posing additional guarantees. Thus these insurance products typically provide policyholder with a minimum guaranteed value on death of the insured or on maturity of the contract. This kind of insurance product is called the guaranteed variable life insurance policy. The benefit of the insurance contract thus can depend on the performance of the linked reference fund and the guaranteed values. The conventional pricing approach (CPA) initiated by Brennan and Schwartz basically starts by calculating the market price of the payoffs which occur at each time point within the contract term and then take account of the expectation on the mortality. In brief, the CPA integrates the option pricing theory and the principle of equivalence. For example, Brennan and Schwartz (1976, 1977) assumes that the price of the reference fund follows a geometric Brownian motion process and then the guarantees are treated as European-type put options which could be solved using Black-Scholes model (Black and Scholes, 1973); hence the fair price can be derived by specifying the market value of contingent payoffs times the mortality probability using independence assumption between market and mortality risks as well as the traditional law of large numbers. With the CPA, several relevant studies in the literature have been carried out. Some works consider other contract designs with different structures of benefits such as the caps (Eker and Persson, 1996) and the endogenous minimum guarantees (Bacinello and Ortu, 1993). stochastic interest rates are incorporated into pricing models in several studies (Nielsen and Sandmann, 1995; Bacinello and Persson, 2002; Gaillardetz, 2008). Besides, Bilodeau (1997) and Bacinallo (2003, 2005, 2008) consider different types of options embedded in the contract under the CPA framework. Following Brennan and Schwartz (1976, 1977), this study is also concentrated on endowment policy with guarantees which is primarily the combination of pure endowment policy and term insurance.

As suggested by Embrechts (2000), institutional issues such as the increasing collaboration between insurance companies and banks, and deregulation of insurance markets will be regarded as two further important

aspects. To search for combinations and unification of methodologies and traditional principle for the two fields of insurance and finance may deserve as a considerable issue. Obviously the variable life insurance products can involve both financial and insurance risks. For example, Melnikov and Romanyuk (2008) highlight the implications of efficient hedging for the management of financial and insurance risks of variable life insurance policies with numerical examples. As to the topic we concern here, one may wonder how the fair price of the variable life insurance with guarantees could be determined if both the independence assumptions and the insurer's risk neutrality are violated. Accordingly, we jointly consider the price process of a reference fund and the process mortality risk. We integrate the two risk processes into a new stochastic process. The variable insurance products could be seen as contingent claims of the new underlying process. To calculate the no-arbitrage price of the variable life insurance, life insurance portfolio is viewed as a tradable asset in secondary life insurance market. Investors then can have an additional basis asset to build the portfolio for duplicating the contingent claims. Under the market structure specified here, the complete market property cannot be preserved and insurers cannot replicate perfectly the contingent payment to policyholders at a future date. So the variable life insurance products couldn't be duplicated exactly with the portfolio consisting of the basis assets. Accordingly, the risk-neutral probability measure in this market structure is not unique and thus the corresponding no-arbitrage price composes an interval (see, Ch1 Pliska 1997). Hence this could leave an open pricing problem in the incompleteness. Even liquidity of life insurance portfolio cannot reach as high degree as general financial securities. In fact, the insurance contracts could be sold to get money back earlier under special conditions. For example, the senior life policies settlement or viatical settlement can be traded in the second life insurance market in USA. Although the assumption about the trading property of life insurance contracts don't meet fully real world, we can make efforts to reposition the CPA with this treatment in the option pricing framework. As compared with the Black-Scholes model, we won't render this model useless due to that the continuous self-financing strategy cannot be carried out completely in reality. Moller (2001) ever deals with pricing and

hedging problems for variable life insurance in an incomplete market. His main contribution is to obtain the optimal investment strategies that minimize the variance of the insurer's future cost based on the criterion of risk-minimization instituted by Follmer and Sondermann (1986). Nonetheless life insurance portfolio is not assumed as tradable assets in his seminal article.

This paper aims at developing an alternative pricing method and reviewing the CPA for variable life insurance using a pure no-arbitrage viewpoint. For simplicity, this work is restricted to the single premium case. The first task of this study is to present the general form of the no-arbitrage price bound for the insurance contracts. After that, we could verify that CPA will produce a no-arbitrage price with respect to a specific risk-neutral probability measure. The relationship between the present pricing approach and the CPA is explored. Through numerical analysis, we investigate and discuss how certain key financial factors can influence the relationship between the no-arbitrage price bound and the price derived by the CPA. If the reasonable price of the insurance contract would not be determined uniquely and could be affected by several financial parameters, those facts implies that insurers need to specify a pricing practice more sophisticatedly since they have more flexibility in pricing such insurance product.

The remaining of the paper flows as follows. The market structure for the variable life insurance contract is built firstly in Section 2. In Section 3, the underlying discrete process of the insurance contract, which is the consequence of combining the reference fund process and the mortality distribution, is established. We present the general form of the proposed approach incorporating the secondary life insurance market and explore the relationship between the CPA and the proposed approach as well. In Section 4, the numerical analysis is employed to illustrate the properties of fair price bound of the variable life insurance for various situations. Finally, conclusive remarks are provided in Section 5.

2. The Market Structure for the Insurance Contract

The variable life insurance contracts could be regarded as contingent claims

which can be affected by both the market risk and mortality risk. In this section, we first set up the market risk model for the insurance contract and then take the mortality risk into account. Here we consider the variable life insurance contract with guaranteed value that issues at the beginning of the contract term and matures T years later. The market risk associated with the insurance contract comes from a stochastic evolution of the return rates of the reference fund. To demonstrate a discrete-time model, each policy year is divided into n periods of equal length such that the total period is $N = T/\Delta$ with $\Delta = 1/n$. Hence there are totally N periods during T years. The t-th period is denoted Δ_t, for $t = 1, 2, \ldots N$-1, N. Following Bacinello (2003) and Moller (2001), this study also uses the CRR model proposed by Cox, Ross and Rubinstein (1979) to deal with pricing problems about the variable life insurance properly. This discrete model assumes that the risk-free interest rate r is constant and the financial market consists of two basis tradable asset, a reference stock (or fund) S and a risk free asset B. The CRR model may be viewed as an approximation of the Black-Scholes model due to its important properties of converging asymptotically to the later. The reference fund price follows a stochastic process: $S = \{S_t : t \in [0,T]\}$ or $S = \{S_0, S_1, ..., S_N\}$. The market price of the reference fund is set up as a binomial lattice. With a fixed volatility coefficient $\sigma > \Delta^{-0.5} \ ln(1+r)$, we can specify the accompanied upward-moving factor $u = exp(\sigma\Delta^{0.5})$, and downward-moving factor $d = 1/u$. The unit price of the fund at the end of the t-th period (S_t) would be either uS_t or dS_t in the next period for $t = 1,2,\ldots, N$. S_t is adapted to the filtration I_t of the binomial process. Let $B_t = B_0(1+r)^{t\Delta}$ with the constant annual interest rate $r > 0$ for $t = 1,2,\ldots, N$. Typically frictionless market is assumed to simplify the analyses. The financial treatments usually are based on the assumption of no-arbitrage opportunities.

During any trading time period (e.g., the t-th period), to each contingent claim $f(S_t, t)$, a unique self-financing trading strategy exists that can duplicate the payoff. With this strategy, a portfolio consisting of a certain number of reference fund and a certain amount of risk-free asset can be formed at any time to exactly meet the claim and there is no need to make additional inflow or outflow of capital. Such the financial market is called complete if the contingent claim can be

duplicated perfectly and hence can be priced uniquely. So the no-arbitrage condition could be satisfied. As is well known, the CRR model is a complete market model. Consequently, the no-arbitrage condition is equivalent to the existence of a risk-neutral probability measure under which all financial prices, discounted by the risk-free rate, are martingale. The unique risk-neutral probability measure, which is conditional on the information at time t, is

$$q = [(1+r)^{\Delta} - d]/(u-d) \quad \text{and} \quad 1-q = [u-(1+r)^{\Delta}]/(u-d) \tag{1}$$

for $\{S_{t+1} = uS_t\}$ and $\{S_{t+1} = dS_t\}$, respectively. Therefore, the risk-neutral probability measure is defined by the sequence

$$Q = \{Q_{t,j} \mid 1 \le t \le N, 0 \le j \le t\}, \tag{2}$$

for all possible paths $\{S_{t,j} = S_0 u^{t-j} d^j \mid 1 \le t \le N, 0 \le j \le t\}$

with $Q_{t,j} = \binom{t}{j} q^{t-j}(1-q)^j$

The arbitrage-free price of a derivative of the underlying asset with payoffs $f(S_t, t)$ at time t, for $t = 1, 2, \ldots, N$, , denoted by P_f, may be written by

$$P_f = E^Q[\sum_{t=1}^{N}(1+r)^{-t\Delta} f(S_t, t)], \tag{3}$$

where E^Q is the conditional expectation operator with respect to the risk-neutral probability measure in (2).

This insurance contract involves the risk associated with the future development of the reference fund as well as the uncertainty about the mortality of insured. Whereas the financial risk affects the amount of benefit for the policyholders, the mortality risk determines the times in which the benefit is due. For each period, there are two states for the insured's life status, i.e., alive and dead. Hence the death discrete process also could be set up as a binomial tree. It is assumed that a policyholder makes a single investment amount into the fund at the initial of the contract. Let m be the units invested in the fund at the initial time. Without any losses of generality, m is fixed to one. For an insured with age x, let

the mortality of the t-th period be denoted by q_{x+t-1} for $t = 1, 2,\ldots, N$. The mortality distribution could be extracted from a mortality table. Typically the guarantee asset value of the variable life insurance may be set to a function of time (t) and the market value of the fund at the purchase date (S_0). For simplicity, we suppose that the guarantee asset value is a constant, denoted by G (typically G is a percentage of S_0), in this study. That is, $f(S_t, t)$= Max (S_t, G). In other words, with the specification for the guarantees function, the benefit of this contract would be $max(S_t, G)$ at the end of the t-th period if the insured dies during the t-th period or $max(S_N, G)$ at the end of the N-th period if the insured survives to the maturity date. However, $f(S_t, t)$ could be set to a more complex form. For example, a guaranteed return is given by $f(S_t,t) = S_{t-1}\max(1+(S_t - S_{t-1})/S_{t-1}, 1+\kappa)$, where κ is the guaranteed return.

3. The Pricing Model

3.1 The Review of the CPA for Variable Life Insurance

Basically the CPA is derived by combining no-arbitrage argument and traditional large number principle from insurance. The contingent payoff at each time can be fairly priced with no-arbitrage argument as described in the precious section. The CPA is justified with the law of large number since insurers typically hold a portfolio of large number of contracts. The CPA usually assumes that the death process is stochastically independent of the reference fund. Hence the pricing problem can be resolved by specifying the payoff of Max (S_t, G) times the probability of mortality at time t. It is implied that the insurer is risk-neutral with respect to mortality in the CPA. Here we only concern about the core work for actuarial valuation, so any problem about expenses or other transaction cost is ignored. Based on the assumptions, the fair net premium can be derived with the CPA through first calculating the market values of all payoffs according to (3) and then taking account of the expectation on the mortality. Therefore with the financial set-up in the previous section, the fair price of the variable life insurance with guarantees, denoted by P^{CPA}, can be written as

$$P^{CPA} = \sum_{t=1}^{N} {}_{t-1|}q_x E^Q [(1+r)^{-t\Delta} max(S_t, G)] + {}_Np_x E^Q [(1+r)^{-N\Delta} max(S_N, G)], \quad (4)$$

where Q is defined in (2), ${}_{t-1|}q_x = (1- q_x)(1- q_{x+1}) \ldots (1- q_{x+t-2})\, q_{x+t-1}$, is the probability that the insured dies within the t-th period, and ${}_Np_x = (1 - q_x)(1 - q_{x+1}) \ldots (1 - q_{x+N-1})$ is the probability that the insured is still alive at the end of the N-th period.

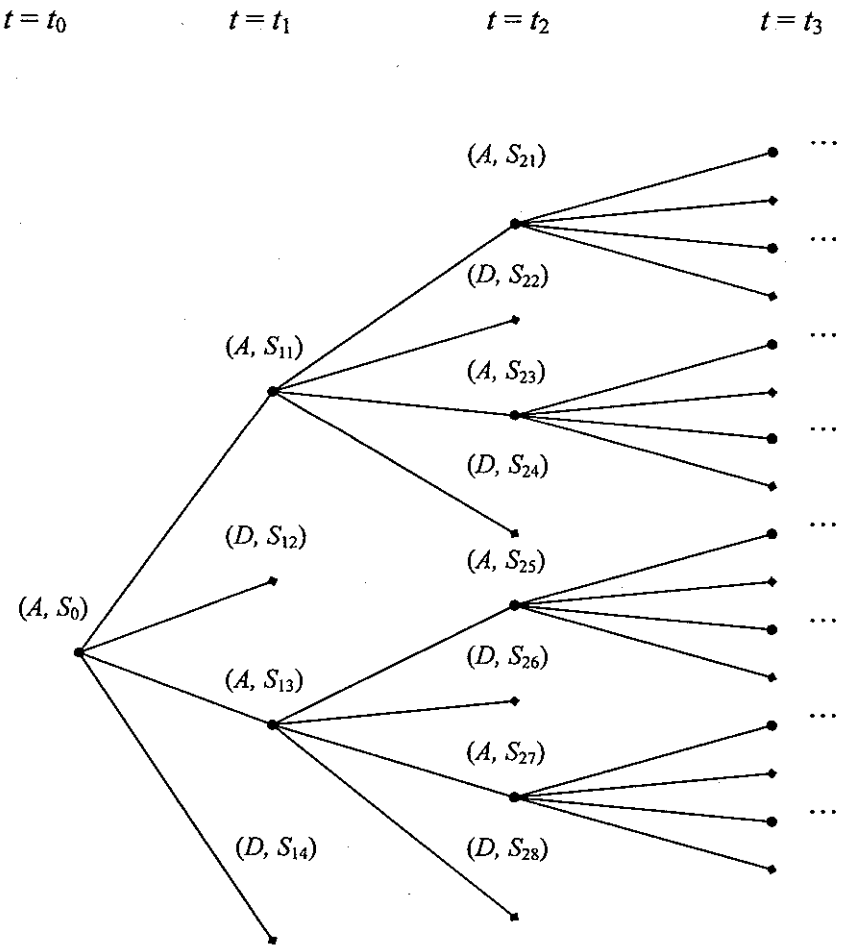
3.2 The Establishment of the Underlying Process for Variable Life Insurance

In order to establish an underlying process that can include all probabilities, the price process of the reference fund and the death process of an insured are considered jointly. By integrating the two binomial processes, the new underlying discrete process is created as shown in Figure 1. All states occurring possibly have been considered in the new underlying process since the stochastic processes for both mortality and reference fund price have been specified already. Whether the asset price process and the death process are mutually independent or not, Figure 1 actually covers all possible outcomes. So the independence assumption between market risk and mortality risk cannot be required.

The special feature for this model setting is to regard the life insurance portfolio as tradable assets such as the viatical settlements and the accelerated death benefits (ADBs). Under taking the term insurance into account, there thus are three basis assets in the market structure. The pure endowment policy is excluded because it may just be duplicated with a portfolio consisting of both the risk-free asset and the term insurance. It is also assumed that the market price of term insurance for a coverage period is determined by the traditional actuarial method, i.e., the principle of equivalence. This assumption implies that the net premium of term insurance portfolio under the principle of equivalence is regarded as a fair price accepted commonly by participants in the insurance market. In fact, to trade ordinary term insurance in the secondary markets is not illegal, e.g., the senior life policies settlement. Nevertheless, the liquidity of life insurance contracts actually cannot be as good as general financial securities. For this reason, we assume that investors will require a risk premium to compensate

for bearing more liquidity risk from the life insurance contracts. So a premium loading factor, denoted by λ, will exist for each trading period, which can be determined by the secondary life insurance market.

Figure 1
The Underlying Process Describing Jointly Market and Mortality Risks for the Variable Life Insurance Contract



As illustrated in Figure 1, there are four possible states, namely alive-up (A, uS_{t-1}), dead-up (D, uS_{t-1}), alive-down (A, dS_{t-1}) and dead-down (D, dS_{t-1}), in the

next period conditional on the state, (A, S_{t-1}). Thus our pricing model would be incomplete market since it involves four states but only three basis assets in each trading period. Based on the properties of an incomplete market, the risk-neutral probability measure would not be unique and the no-arbitrage price of a derivative asset would be an interval (see, e.g., Pliska 1997). The general form of the no-arbitrage price bounds will be calculated in the following subsection.

3.3 The Calculation of the No-Arbitrage Price Bounds

Before considering the multi-period model, the two-period model could be established firstly as an example with simpler calculation. We can refer the stochastic process of the two-period model to the first two periods in Figure 1. Each period in the process involves four states and three basis assets. For example, just consider the upper part of the second period in the two-period model. The all payoffs in every state for the three basis assets are exhibited in the Figure 2. Based on the no-arbitrage condition, the conditional risk-neutral probability measure, $q_2 = (q_{21}, q_{22}, q_{23}, q_{24})$, corresponding to the probability from (A, S_{11}) to (A, S_{21}), (D, S_{22}), (A, S_{23}), and (D, S_{24}) respectively, will satisfy the following equation system:

$$\begin{cases} (1+r)^{-\Delta}(u\,q_{21} + u\,q_{22} + d\,q_{23} + d\,q_{24}) = 1, \\ q_{22}/q_{x+1}(1+\lambda) + q_{24}/q_{x+1}(1+\lambda) = 1, \\ q_{21} + q_{22} + q_{23} + q_{24} = 1, \\ q_{21}, q_{22}, q_{23}, q_{24} \geqq 0 \end{cases} \tag{5}$$

The above first two equations are set respectively according to the risk-neutral property of the fund and the term insurance. The solution of the risk-neutral probability measure for the second period can be expressed as

$$q_2 = [q - q_{x+1}(1+\lambda) + \alpha_2,\ q_{x+1}(1+\lambda) - \alpha_2,\ 1 - q - \alpha_2,\ \alpha_2],$$

where $max(0,\ q_{x+1}(1+\lambda) - q) < \alpha_2 < min(1-q,\ q_{x+1}(1+\lambda))$ and q is defined in equation (1). By the same arguments, one can verify that another conditional measure of the second period from (A, S_{13}) to (A, S_{25}), (D, S_{26}), (A, S_{27}), and $(D,$

S_{28}), is the same as q_2, and the risk-neutral probability measure of the first period, $q_1 = (q_{11}, q_{12}, q_{13}, q_{14})$, is

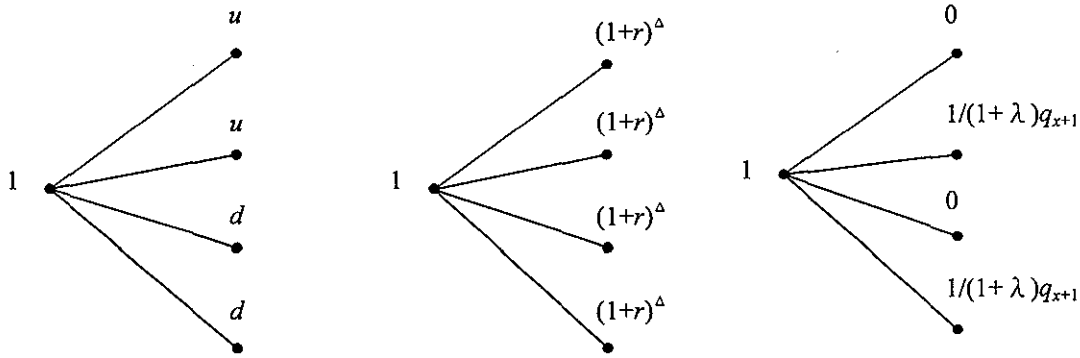
$$q_1 = [q - q_x(1+\lambda) + \alpha_1,\ q_x(1+\lambda) - \alpha_1,\ 1 - q - \alpha_1,\ \alpha_1],$$

where $max(0,\ q_x(1+\lambda) - q) < \alpha_1 < min(1-q,\ q_x(1+\lambda))$. The risk-neutral probability measure, denoted by Q_2^*, can be obtained through making up of q_1 and q_2. Therefore, no-arbitrage price of the variable life insurance with guarantees for the two-period model may be derived as

$$P_2 = E^{Q_2^*}\left(\sum_{t=1}^{2}(1+r)^{-t\Delta}\max(S_t, G)\right) \tag{6}$$

It is obvious that Q_2^* would not be unique and P_2 could serve as bounds. According to (6), the lower bound of P_2 can be calculated under the corresponding risk-neutral probability measure Q_2^* by setting $\alpha_1 = \min(1-q, q_x(1+\lambda))$ and $\alpha_2 = \min(1-q, q_{x+1}(1+\lambda))$, whereas the upper bound of P_2 can be obtained by setting $\alpha_1 = \max(0, q_x(1+\lambda)-q)$ and $\alpha_2 = \max(0, q_{x+1}(1+\lambda)-q)$. For example, if $q_x(1+\lambda) \le q$, $q_{x+1}(1+\lambda) \le q$, $q_x(1+\lambda) \le 1-q$, $q_{x+1}(1+\lambda) \le 1-q$, the lower and upper bounds could be derived by setting $\alpha_t = q_{x+t-1}(1+\lambda)$ and $\alpha_t = 0$ for $t = 1,2$ respectively, i.e.,

Figure 2
The Pavoffs in Four States for the Three Basis Assets



Reference Fund Risk-free Asset Term Insurance

$$P_2^L = (1+r)^{-2\Delta}\{q^2u^2 + q(1-q)G + [1-q-q_x(1+\lambda)]G\} + (1+r)^{-\Delta}(1+\lambda)q_xG$$

and

$$P_2^U = (1+r)^{-2\Delta}\{[1-q_x(1+\lambda)](qu^2 + G - qG) + (1-q)G\} + (1+r)^{-\Delta}(1+\lambda)q_xu$$

Extending the result to the multi-period case, the form of the no-arbitrage price bounds for the N-period model would be obtained in the same recursive solution. The conditional risk-neutral probability measure of the t-th period, $q_t = (q_{t1}, q_{t2}, q_{t3}, q_{t4})$, which is independent of the states of previous period, could be written as

$$q_t = (q - q_{x+t-1}(1+\lambda) + \alpha_t,\ q_{x+t-1}(1+\lambda) - \alpha_t,\ 1 - q - \alpha_t,\ \alpha_t) \tag{7}$$

with

$$max(0,\ q_{x+t-1}(1+\lambda) - q) < \alpha_t < min(1-q,\ q_{x+t-1}(1+\lambda)), \tag{8}$$

for $1 \leqq t \leqq N$. Then, the risk-neutral probability measure of the N-period model, denoted by Q^*, can be obtained by combining all q_t, for $1 \leqq t \leqq N$. The no-arbitrage price of the insurance contract could be expressed by

$$P = E^{Q^*}\left(\sum_{t=1}^{N}(1+r)^{-t\Delta}\max(S_t, G)\right) \tag{9}$$

Similarly, the lower and upper bounds of P, denoted by P^L and P^U, may be obtained with respect to the risk-neutral probability measures, Q^* by setting $\alpha_t = min(1-q,\ q_{x+t-1}(1+\lambda))$ and $\alpha_t = max(0,\ q_{x+t-1}(1+\lambda) - q)$ respectively, for $1 \leqq t \leqq N$. Separately, according to (9), the no-arbitrage price bound of the contract would be influenced by both mortality and loading factor. This result implies that the range of fair price will be larger for an elder insured than for a younger one. On the other hand, high loading factor then would amplify the range between the no-arbitrage price bounds.

3.4 The CPA as A Special Case

The CPA can be reviewed in the proposed valuation framework with the option pricing thinking. It is shown that the price obtained by the CPA is a

no-arbitrage one with respect to the specific risk-neutral probability measure Q^{**}, which is one of the risk-neutral probability measure Q^{*} defined with $\alpha_i = (1-q)q_{x+t-1}$, for $1 \leqq t \leqq N$. It is obvious that $max(0, q_{x+t-1}(1+\lambda) - q) < (1-q)q_{x+t-1} < min(1-q, q_{x+t-1}(1+\lambda))$ is held and thus the relationship satisfies (8). The price formula of the CPA in (4) could be derived with the risk-neutral probability measure Q^{**} and presented as follows:

$$P^{CPA} = \sum_{t=1}^{N} {}_{t-1|}q_x E^{Q}[(1+r)^{-t\Delta} max(S_t, G)] + {}_{N}p_x E^{Q}[(1+r)^{-N\Delta} max(S_N, G)]$$
$$= E^{Q^{**}}(\sum_{t=1}^{N}(1+r)^{-t\Delta}\max(S_t, G))$$

This means that the price obtained by the CPA (P^{CPA}) is one of the no-arbitrage prices under the new underlying process setting. Because P^{CPA} lies within the no-arbitrage price bounds, i.e., $P^{L} \leqq P^{CPA} \leqq P^{U}$, the properties of their relationship become an interesting issue. Thus, numerical analysis is conducted for this purpose in the next section.

Under the market structure specified in this study, we also can utilize the optimal portfolio pricing approach (see Ch 9, Luenberger 1997) to get a set of risk-neutral probability measure and determine the fair price for the insurance contract. However we need additionally to define a utility function and specify the optimal portfolio choice criterion for policyholders. As mentioned in the text, policyholders (or investors) have three basis assets to form the portfolio. The optimal portfolio pricing approach is based on the assumption that policyholders would make decision for allocating optimally their money among the alternatives. Similar to the discussion in Subsection 3.3, the multiple-period problem has the same solution as the sequence of one-period problem. Thus, N-period problem could also be reduced to one-period problem here. Denote the corresponding real probabilities for the four possible states of the t-th period conditional on the previous state, (A, S_{t-1}) as shown in Figure 1, including alive-up (A, uS_{t-1}), dead-up (D, uS_{t-1}), alive-down (A, dS_{t-1}) and dead-down (D, dS_{t-1}), p_t^{Au}, p_t^{Du}, p_t^{Ad} and p_t^{Dd} respectively for $t = 1, 2, \ldots, N$. We take the first period ($t = 1$) as an illustrative example. A portfolio of these basis assets is represented by a 3-dimentional vector $\beta = (\beta_1, \beta_2, \beta_3)$. The initial price of each asset is denoted by k_i, for i = 1~3.

Suppose that a policyholder has an initial wealth w_0. The future wealth would be governed by corresponding random variable. A utility function U provides a procedure for ranking random wealth levels. If w_1 is the random wealth at the end of the first period, we write $w_1 > 0$ to indicate that the variable is never less than zero and it is strictly positive with some positive probability. The random payoffs for the three asset are represented by d_i, i = 1~3. For simplicity, we ignore the liquidity problem about the insurance contract and let $\lambda = 0$. The investor wishes to form a portfolio to maximize the expected utility of the future wealth, i.e., w_1. Thus the policyholder's problem is:

$$\max E[U(w_1)], \textit{ subject to } \sum_{i=1}^{3}\beta_i d_i = w_1, w_1 > 0, \sum_{i-1}^{3}\beta_i k_i \le w_0.$$

The problem therefore becomes:

$$\max E[U(\sum_{i=1}^{3}\beta_i d_i)], \text{ subject to } w_1 > 1, \sum_{i=1}^{3}\beta_i k_i = w_0.$$

By introducing a Lagrange multiplier γ for the constraint, the necessary conditions are found by differentiating the Lagrangian:

$$L = E[U(\sum_{i=1}^{3}\beta_i d_i)] - \gamma(\sum_{i=1}^{3}\beta_i k_i - w_0)$$

with respect to each β_i. Using $w_1^* = \sum_{i=1}^{3}\beta_i^* d_i$ for the payoff of the optimal portfolio, this gives $E[U'(w_1^*)d_i] = \gamma k_i$ for i = 1~3. Since the risk-free asset (i = 1) has the total return of $(1+r)^{\Delta}$, it follows that if $k_1 = 1$, then $d_1 = (1+r)^{\Delta}$. Thus, we obtain $\gamma = E[U'(w_1^*)](1+r)^{\Delta}$. Substituting this value for γ would yield

$$k_i = \frac{E[U'(w_1^*)d_i]}{(1+r)^{\Delta}E[U'(w_1^*)]}.$$

Therefore the risk-neutral probabilities of the first period, $\tilde{q}_1 = (\tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{13}, \tilde{q}_{14})$, could be derived as

$$\tilde{q}_1 = (\frac{p_1^{Au}U_{11}'(w_1^*)}{E[U'(w_1^*)]}, \frac{p_1^{Ad}U_{12}'(w_1^*)}{E[U'(w_1^*)]}, \frac{p_1^{Du}U_{13}'(w_1^*)}{E[U'(w_1^*)]}, \frac{p_1^{Dd}U_{14}'(w_1^*)}{E[U'(w_1^*)]}).$$

With the same argument, the risk-neutral probabilities for the t-th period, $\tilde{q}_t = (\tilde{q}_{t1}, \tilde{q}_{t2}, \tilde{q}_{t3}, \tilde{q}_{t4})$, may be written by

$$\tilde{q}_t = \left(\frac{p_t^{Au} U'_{t1}(w_t^*)}{E[U'(w_t^*)]}, \frac{p_{t1}^{Ad} U'_{t2}(w_t^*)}{E[U'(w_t^*)]}, \frac{p_t^{Du} U'_{t3}(w_t^*)}{E[U'(w_t^*)]}, \frac{p_t^{Dd} U'_{t4}(w_t^*)}{E[U'(w_t^*)]}\right).$$

By doing so, a set of risk-neutral probability measure that depends on consumer's utility function could be derived as well.

Now the CPA is revisited in views of the optimal portfolio pricing approach. According to the *ration theory* (see Ch16, Luenberger 1997), the relationship $\tilde{q}_{t1}\tilde{q}_{t4} / \tilde{q}_{t2}\tilde{q}_{t3} = p_t^{Au} p_t^{Dd} / p_t^{Du} p_t^{Ad}$, for t =1, 2,..., N, would be held if each trading period Δ is enough small. Under the condition where the two types of risks are independent each other, it follows $\tilde{q}_{t1}\tilde{q}_{t4} / \tilde{q}_{t2}\tilde{q}_{t3} = p_t^{Au} p_t^{Dd} / p_t^{Du} p_t^{Ad} = 1$ such that $\tilde{q}_{t1}\tilde{q}_{t4} = \tilde{q}_{t2}\tilde{q}_{t3}$. As a result, the independence with respect to real probabilities is equivalent to the independence with respect to the risk-neutral probabilities. With this condition, the risk-neutral probabilities $\tilde{q}_t = (\tilde{q}_{t1}, \tilde{q}_{t2}, \tilde{q}_{t3}, \tilde{q}_{t4})$ derived from the optimal portfolio pricing approach would be equal to (7) with $\alpha_t = (1- q)q_{x+t-1}$, for $1 \leqq t \leq N$. Consequently, it can be verified that the optimal portfolio pricing approach could achieve the same result as the CPA does under certain conditions. So this pricing approach also could serve as another applicable valuation method for the variable life insurance using the created underlying stochastic process of this paper.

4. Numerical Results

In this section, the results of some numerical experiments for the comparison between the proposed pricing approach and the CPA are illustrated. It is attempted to understand how the no-arbitrage price bounds of the insurance contract is affected by some financial parameters. Since our numerical analysis is aimed to catch some comparative properties between the aforementioned two approaches, the real mortality is ignored here. Instead, we consider different patterns of the mortality distribution in which different mortality growth rates could be presented. For simplicity, we set T = 1, n =12 (N = 12) and S_0 =100,000. This setting won't get any losses of generality. Note that the choice for n implies a

monthly change in the unit price of the reference fund. Several numerical experiments are made with respect to five parameters, i.e., the volatility coefficient (σ), the interest rate (r), the guarantee value (G), the pattern of mortality distribution and the loading factor for liquidity risk premiums (λ). The results are reported in table 1 through table 4.

Table 1
Insurance Premiums Versus the Volatility Coefficient σ

σ	10%	20%	30%	40%	50%
Upper bound P^U	100041	101223	103366	106372	109454
Conventional approach P^{CPA}	100039	101193	103316	106314	109392
Lower bound P^L	100036	101158	103264	106255	109331
No-arbitrage interval P^U- P^L	5	65	102	117	123
$(P^U - P^{CPA})/(P^U - P^L)$	0.400	0.462	0.490	0.496	0.504

Table 2
Insurance Premiums Versus the Interest Rate *r*

σ	2%	4%	6%	8%	10%
Upper bound P^U	104256	103786	103366	102989	102652
Conventional approach P^{CPA}	104189	103729	103316	102947	102617
Lower bound P^L	104124	103670	103264	102901	102576
No-arbitrage interval P^U- P^L	132	116	102	88	76
$(P^U - P^{CPA})/(P^U - P^L)$	0.508	0.491	0.490	0.477	0.461

Table 3
Insurance Premiums Versus the Guarantee Value G (Percentage of S_0)

G	75%	80%	85%	90%	95%
Upper bound P^U	101555	102381	103366	105037	106742
Conventional approach P^{CPA}	101510	102329	103316	104997	106737
Lower bound P^L	101463	102275	103264	104954	106729
No-arbitrage interval P^U- P^L	92	106	102	83	13
$(P^U - P^{CPA})/(P^U - P^L)$	0.489	0.491	0.490	0.482	0.385

Table 4
Insurance Premiums Versus the Pattern of Mortality Distribution

Growth rate of mortality	-10%	-5%	0%	5%	10%
Upper bound P^U	103400	103385	103366	103343	103314
Conventional approach P^{CPA}	103358	103339	103316	103289	103256
Lower bound P^L	103314	103291	103264	103231	103193
No-arbitrage interval P^U- P^L	86	94	102	112	121
$(P^U - P^{CPA})/(P^U - P^L)$	0.488	0.489	0.490	0.482	0.479

Table 5
Insurance Premiums Versus the Loading factor λ

λ	0%	5%	10%	15%	20%
Upper bound P^U	103366	103366	103366	103366	103366
Conventional approach P^{CPA}	103316	103316	103316	103316	103316
Lower bound P^L	103264	103257	103253	103248	103243
No-arbitrage interval P^U- P^L	102	109	113	118	123
$(P^U - P^{CPA})/(P^U - P^L)$	0.490	0.459	0.442	0.424	0.407

First, the no-arbitrage price bound and P^{CPA} are calculated when the volatility coefficient (σ) varies between 10% and 50% with a step of 10%. We fix r = 6%, λ = 0 and K = 85% of S_0. A simple pattern of mortality distribution is given, where q_{x+t-1} is fixed to 0.1 % for $1 \leqq t \leqq N$. The results of this numerical experiment are presented in Table 1. It is obvious that all the premiums obtained, including the upper bound price P^U, the lower bound price P^L and the P^{CPA}, increase with the volatility coefficient (σ). This result agrees with the general properties of option pricing theory. Moreover, we notice that the no-arbitrage price interval, $P^U - P^L$, and the ratio, $(P^U - P^{CPA})/(P^U - P^L)$ increase with σ. In other words, the no-arbitrage interval becomes larger and P^{BS} becomes relatively closer to P^L as σ becomes larger. Then, as presented in table 2, the premiums are derived when the interest rate (r), varies between 2% and 10% with a step of 2% based on the conditions of σ = 30%, λ = 0, K = 85% of S_0 and the same mortality distribution. According to the results in Table 2, all the premiums decrease with the interest rate (r). Besides, both of no-arbitrage price interval and the ratio, $(P^U - P^{CPA})/(P^U - P^L)$, decrease with r. Furthermore, setting σ = 30%, r = 6%, λ = 0 and the same mortality distribution, the premiums are calculated when the guarantee value (G) varies between 75% and 95% of S_0 with a step of 5%, as presented in Table 3. We notice that all the premiums increase with the guarantee value (G). This obviously meets the prediction of option pricing theory. Moreover, it is also observed that the larger the guaranteed value, the narrower the no-arbitrage price interval (except in the situations of lower guaranteed values).

In addition, different patterns of mortality distribution are taken into account as well. Fixing σ =30%, r = 6%, λ = 0 and G = 85%, the premiums are calculated according to different growth rates of mortality, including -10%, -5%, 0%, 5%, and 10%. For example, the mortality growth rate of 5 % implies the relationship, q_{x+t} / q_{x+t-1} = 1+5%. The results are presented in Table 4. It is found that all the premiums decrease with the growth rate of mortality. However, the no-arbitrage interval increases with the growth rate of mortality. Finally, we test the effect of the loading factor λ on the no-arbitrage bounds. Using the same setting, i.e., σ =30%, r = 6%, K = 85% and zero growth rate of mortality, the premiums are obtained for various loading factors (λ) with $0 < \lambda < 8\%$. As

exhibited in Table 5, it is obvious that the no-arbitrage bound increases with the loading factor. This result implies high liquidity of life insurance contracts in secondary insurance market can decrease possible range of the no-arbitrage price.

In summary, according to the numerical results, all premiums of the variable life insurance policy increase with σ and G, but decrease with r and the growth rate of mortality. And the no-arbitrage price interval increases with σ, λ and the growth rate of mortality, but decrease with r and G. Additionally, almost values of the ratio, $(P^U - P^{CPA})/(P^U - P^L)$, would be between 0.4 and 0.6. This implies that P^{CPA} usually falls in the middle area of the no-arbitrage intervals. Accordingly, from the viewpoint of pure market-value based, the reasonable prices of the contracts couldn't be determined only by the traditional criterion while P^{CPA} could serve as a benchmark for pricing in practice. That is, the reasonable prices could depend on market situations and thus insurance companies could keep more cushions in making the pricing strategy.

5. Conclusive Remarks

In this article, life insurance contracts are seen as tradable portfolio assets since secondary life insurance markets allow consumers to cash out life insurance holding prior to death. Considering this characteristic of secondary life insurance markets, we propose alternative valuation methods for variable life insurance under a pure option pricing framework. Two proposed approaches, i.e., no-arbitrage pricing method and optimal portfolio pricing method, could lead to the results that coincide with the price determined by the conventional valuation principle. Actually the price obtained by the conventional principle would be verified to be a special case of our pricing framework. The result indicates that fair prices of the insurance contract may not be limited to those determined by the conventional pricing approach. It then implies that, from the market-value based perspective, insurers need to specify a more sophisticated pricing practice since they have more flexibility in pricing such insurance product. The optimal pricing policy may depend on market situations and consumer's utility function. The numerical analysis results show the properties of no-arbitrage price of the

insurance contract as well. A different insight into properties of the conventional pricing approach has been explored. Although this research is restricted to the single-premium case, the pricing model we propose could be extended to the annual-premium case for advanced applications. The valuation approach explored in this paper may be applied further to solve the analogous pricing problems related to other insurance contracts types.

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