Cyclic m-Cycle Systems with $m \leq 32$ *or* $m = 2_q$ *with q a Prime Power*

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Abstract: In this paper, the necessary and sufficient conditions for the existence of cyclic $2q$-cycle and m-cycle systems of the complete graph with q a prime power and $3 \leq m \leq 32$ are given. © 2005 Wiley Periodicals, Inc. J Combin Designs 14: 66–81, 2006.

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1. INTRODUCTION

For a graph G, let $V(G)$ and $E(G)$ be respectively the vertex set and edge set of G and let $\boldsymbol{S}$ be a collection of cycles of length m (namely, m-cycles) such that each edge in $E(G)$ belongs to exactly one cycle in $\boldsymbol{S}$. Then the pair $(V(G), \boldsymbol{S})$ is called an *m-cycle system* of G. An m-cycle system of K_v is also referred to as an m-cycle system of order v. Here, K_v is the complete graph on v vertices. An obvious necessary condition for the existence of an m-cycle system of K_v is that $m \leq v$, v is odd, and m divides the number of edges in K_v.

Alspach and Gavlas [1] and Šajna [14] have completely settled the existence problem of m-cycle systems of K_v and $K_v - I$, where I is a 1-factor.

Let a pair $(V, \boldsymbol{S})$ be an m-cycle system of K_v and let Π be an automorphism group of the m-cycle system $(V, \boldsymbol{S})$ (i.e., a group of permutations on v vertices leaving the collection $\boldsymbol{S}$ of cycles invariant). If there is an automorphism $\pi \in \Pi$ of order v, then

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the m-cycle system $(V, \boldsymbol{S})$ is said to be *cyclic*. For an m-cycle system of K_v, the vertex set V can be identified with Z_v. So, the automorphism can be represented by

$$\pi : i \to i + 1 (\text{mod } v) \text{ or } \pi : (0, 1, \ldots, v - 1)$$

on the vertex set $V = Z_v$.

In 1938, Peltesohn [10] proved that there exists a cyclic 3-cycle system for each admissible value of $v \neq 9$. Kotzig [9] and Rosa [11,13] showed that for even m, there exists a cyclic m-cycle system of order $2km + 1$. Moreover, Rosa [12] also proved that there exist cyclic m-cycle systems where $m = 3, 5, 7$. Buratti and Del Fra [5], Bryant et al. [7], and the present authors [8] independently proved that for any integer m with $m \geq 3$, there exists a cyclic m-cycle system of order $2km + 1$. Recently, Buratti and Del Fra [6] present the result that if m is an odd integer with $m \neq 15$ and $m \neq p^{\alpha}$ where p is prime and $\alpha > 1$, then there exists a cyclic m-cycle system of order $2km + m$ with exception: $(m, k) = (3, 1)$. More recently, Vietri [16] has completely filled in the gap created by Buratti and Del Fra [6]. So we have the following results.

Theorem 1.1 [5,7,8]. *For any integer m with* $m \geq 3$, *there exists a cyclic m-cycle system of order* $2km + 1$.

Theorem 1.2 [6,16]. *Given an odd integer* $m \geq 3$, *there exists a cyclic m-cycle system of order* $2km + m$ *for any admissible value of k with the only definite exceptions of* $(m, k) = (3,1)$, $(15,0)$, and $(p^{\alpha}, 0)$ *with p a prime and* $\alpha > 1$.

The above theorem gives, in particular, a complete answer to the existence question for cyclic q-cycle systems with q a prime power.

With the joint effort of a number of researches [5–13,16], the existence question for cyclic q-cycle systems has been settled for q a prime power. When q is not a prime power, the problem becomes much more difficult and is far from being solved.

In this paper, we settle the existence questions for cyclic $2q$-cycle systems with q a prime power and for cyclic m-cycle systems with $m \leq 32$.

2. DEFINITIONS AND PRELIMINARIES

Throughout this paper, we shall assume that the vertex set of K_v is Z_v and use $\pm(a - b)$ to denote the *difference* of the edge $\{a, b\}$ in K_v. Given an m-cycle $C = (c_0, c_1, \ldots, c_{m-1})$ on K_v, let $C + i = (c_0 + i, c_1 + i, \ldots, c_{m-1} + i)$ (mod v), where $i \in Z_v$.

The *cycle orbit* of C is the set of distinct m-cycles in the collection $\{C + i | i \in Z_v\}$. The *length* of a cycle orbit is its cardinality, i.e., the minimum positive integer k such that $C + k = C$. A *base cycle* of a cycle orbit Ò is a cycle $C \in$ Ò that is chosen arbitrarily. Any cyclic m-cycle system of order v is generated from base cycles. For the convenience of notation, we write a cycle k-orbit for a cycle orbit of length k.

A cycle v-orbit of C on K_v is said to be *full* and otherwise *short*; and for convenience, the cycle C is called *full* or *short*, respectively.

A cycle C with vertices in Z_v is of *type d* if its stabilizer under the natural action of Z_v has order d. The type of an m-cycle in Z_v is a common divisor of m and v. It is obvious that a cycle of type 1 $(d > 1)$ is a full (short) cycle.

The following lemma (see [3,4]) is a crucial tool for constructing a cycle of a prescribed type $d(>1)$ in a cyclic m-cycle system.

Lemma 2.1. *Let* $C=(c_0,c_1,\ldots,c_{m-1})$ *be an* m*-cycle on* K_{qt} *satisfying the following conditions*:

(1) *For* $0\le i\ne j\le r-1, c_i\not\equiv c_j$ (mod t).
(2) *The differences between edges* $\{c_i,c_{i-1}\}$ $(1\le i\le r)$ *are all distinct*.
(3) $c_r-c_0=\alpha t$ *with* α *coprime with* q.
(4) $c_{ir+j}=i\alpha t+c_j$ (mod qt), *where* $0\le j\le r-1$ *and* $1\le i\le q-1$.

Then C is a cycle of type q and the set $\{C+i|0\le i<t\}$ *forms a cycle t-orbit of C. Consequently, C can be viewed as a base cycle of the cycle t-orbit*.

To simplify, the m-cycle $C=(c_0,c_1,\ldots,c_{r-1},\alpha t,\alpha t+c_1,\ldots,\alpha t+c_{r-1},\ldots,(q-1)\alpha t,(q-1)\alpha t+c_1,\ldots,(q-1)\alpha t+c_{r-1})$ in Lemma 2.1 is denoted by $C=[c_0,c_1,\ldots,c_{r-1}]_{\alpha t}$ in accordance with [6].

For example, the 10-cycle $C=(0,14,13,27,26,40,39,53,52,1)=[0,14]_{13}$ is of type 5 on K_{65} and the set $\{C,C+1,\ldots,C+12\}$ forms the cycle 13-orbit of C.

Proposition 2.2. *If* $m<v<2m+1$ *and gcd*(m,v) *is an odd prime power, then no cyclic m-cycle system of order v exists*.

Proof. Suppose, on the contrary, that there exists such a cyclic m-cycle system of order v, $(V,\boldsymbol{S})$, set gcd$(m,v)=p^\alpha$ (where p is a prime), and let C be the m-cycle of $\boldsymbol{S}$ containing the edge $\{0,v/p\}$. The hypothesis $v<2m$ implies that $|\boldsymbol{S}|=v(v-1)/2m<v$ so that the orbit of C has length smaller than v. Equivalently, the stabilizer of C is not trivial. On the other hand, the type of C is a divisor of p^α so that the subgroup of Z_v of order p (that is, $<v/p>$) is certainly contained in the stabilizer of C. This means that $C+iv/p=C$ for $i=0,1,\ldots,p-1$ and hence the edges

$$\{0,v/p\},\{v/p,2v/p\},\ldots,\{(p-1)v/p,0\}$$

belong to C. Moreover, it is immediate to see that these edges form the p-cycle $(0,v/p,2v/p,\ldots,(p-1)v/p)$. This is possible only if $m=p$ but, in this case, m would be a divisor of v so that we would have $v=m$ or $v>2m$, a contradiction. □

It is worthwhile to note that a cyclic m-cycle system of order less than $2m+1$ may exist. As stated previously, Buratti and Del Fra in [6] proved that if m is odd with $m\ne 15$ and $m\ne p^\alpha$, where p is prime and $\alpha>1$, then there exists a cyclic m-cycle system of order m.

Throughout this paper, we shall use ∂C to denote the multiset of partial differences $\{\pm(c_i-c_{i-1})|i=1,2,\ldots,m/d\}$ of an m-cycle $C=(c_0,c_1,\ldots,c_{m-1})$ of type d where $c_m=c_0$. Given a set $D=\{C_1,C_2,\ldots,C_p\}$ of m-cycles with vertices in Z_v, the list of partial differences from D is the union of the multisets $\partial C_1,\ldots,\partial C_p$, i.e., $\partial D=\cup_{i=1}^{p}\partial C_i$.

As a special case of general results concerning graph decompositions with a sharply vertex transitive automorphism group [2], we have:

Lemma 2.3. *A set D of m-cycles with vertices in* Z_v *is a set of base cycles of a cyclic m-cycle system of order v if and only if* $\partial D=Z_v-\{0\}$.

For each integer $m \geq 3$, let $Spec(m)$ be the set of v for which there exists an m-cycle system of order v. By [1] and [14] we have $Spec(m) = \{2mt + w | t \in N; w \in W(m)\}$ where $W(m)$ is the set of odd integers w in the open interval (1, 2m) such that $w(w-1)/2 \equiv 0 \pmod{2m}$. We have:

Proposition 2.4.

(1) *If m is an odd prime power, then* $W(m) = \{1, m\}$.
(2) $W(15) = \{1, 15, 21, 25\}$.
(3) $W(21) = \{1, 7, 15, 21\}$.
(4) *If m is an odd prime power and* $m \equiv 1 \pmod 4$, *then* $W(2m) = \{1, m\}$.
(5) *If m is an odd prime power and* $m \equiv 3 \pmod 4$, *then* $W(2m) = \{1, 3m\}$.
(6) $W(2^k) = \{1\}$ *for* $k \geq 2$.
(7) $W(12) = \{1, 9\}$.
(8) $W(20) = \{1, 25\}$.
(9) $W(24) = \{1, 33\}$.
(10) $W(28) = \{1, 49\}$.
(11) $W(30) = \{1, 21, 25, 45\}$.

Proof. Applying the Chinese Remainder Theorem, we have that $|W(m)| = 2^n$ where n is the number of odd prime factors of m. This allows us to check immediately all equalities (1–11).

For instance, it is immediate to check that $W(15) \supseteq \{1, 15, 21, 25\}$. On the other hand, from the above paragraph, $W(15)$ has size 4 so that (2) follows. □

Throughout we shall assume C_i and C_j^* to be respectively full and short m-cycles on K_v, and each set of values of the form $\{\pm c_1, \pm c_2, \ldots, \pm c_n\}$ will be denoted by $\pm\{c_1, c_2, \ldots, c_n\}$. In particular, the short m-cycle has the form stated in Lemma 2.1.

Proposition 2.5. *Let* $m \equiv 2 \pmod 8$. *Then there exists a cyclic m-cycle system of order v with* $v \equiv m/2 \pmod{2m}$.

Proof. Set $v = 2pm + m/2$ for $p \geq 1$. We claim that C_i $(1 \leq i \leq p)$ are full base cycles and $C_j^* (1 \leq j \leq k)$ are short base cycles.

For $i = 1, 2, \ldots, p$, let $C_i = (c_{i,0}, c_{i,1}, \ldots, c_{i,m-1})$ be $(8k+2)$-cycles defined as

$$c_{i,2j} = \begin{cases} 2j, & \text{for } 0 \leq j \leq 2k, \\ 8k - 2j + 1, & \text{for } 2k+1 \leq j \leq 4k; \text{and} \end{cases}$$

$$c_{i,2j+1} = \begin{cases} 2k(4p+1) + i + 2\lfloor (i+1)/2 \rfloor, & \text{for } j = 0, \\ (2k-1-j)(4p-1) + 4k + 4i - 1, & \text{for } 1 \leq j \leq 2k-1, \\ (2k-1)(4p+3) + 4i, & \text{for } j = 2k, \\ (j - 2k - 1)(4p-1) + 4k + 4i - 3, & \text{for } 2k+1 \leq j \leq 4k-1, \\ (2k-1)(4p+1) + 4i + 1, & \text{for } j = 4k. \end{cases}$$

We have $\bigcup_{i=1}^{p} \partial C_i = \pm\{2 + j(4p+1), \ldots, (j+1)(4p+1), 2k(4p+1) + 1, \ldots, 2k(4p+1) + 2p | 0 \leq j \leq 2k-1\}$.

For $j = 1, 2, \ldots, k$, let $C_j^* = [0, (2j-1)(4p+1) + 1]_{4p+1}$ and so $\partial C_j^* = \pm\{(2j-2)(4p+1) + 1, (2j-1)(4p+1)1\}$.

Since $(\bigcup_{i=1}^{p} \partial C_i) \cup (\bigcup_{j=1}^{k} \partial C_j{}^*) = Z_v - \{0\}$, the desired result follows from Lemma 2.3. □

For clearness, we give an example to demonstrate the construction of full even cycles. Let C_1 and C_2 be full cycles in a cyclic 18-cycle system of order 81. The construction of C_1 and C_2 is shown in Figure 1. Note that the vertices with label 0 stand for the same one. By easy computation, we have $\partial C_1 \cup \partial C_2 = \pm\{2, \ldots, 9, 11, \ldots, 18, 20, \ldots, 27, 29, \ldots, 40\}$.

Proposition 2.6. *Let* $m \equiv 6 \pmod 8$. *Then there exists a cyclic m-cycle system of order v with* $v \equiv 3m/2 \pmod{2m}$ *and* $v > 3m/2$.

Proof. Similarly, set $v = 2pm + 3m/2$ $(p \geq 1)$ and $m = 8k + 6$ $(k \geq 0)$. The proof is divided into two cases, depending on whether $k = 0$ or $k > 0$.

Case 1. $k = 0$.

For $i = 1, 2, \ldots, p$, let C_i *be 6-cycles defined as*

$$C_i = \left(0, 4(p+1) + i + 2\left\lfloor \frac{(i+1)}{2} \right\rfloor, 2, 2p + 3 + 2i, 1, 2i + 2\right), \text{ if } 1 \leq i \leq p - 1 \text{ and}$$

$$C_p = \left(0, 5p + 4 + 2\left\lfloor \frac{(p+1)}{2} \right\rfloor, 2, 4p + 5, 1, 2p + 2\right).$$

We have $\bigcup_{i=1}^{p} \partial C_i = \pm\{3, \ldots, 4p, 4p + 3, \ldots, 6p + 4\}$.

The short 6-cycles are: $C_0{}^* = [0, 4p + 1]_{4p+3}$ and $C_1{}^* = [0, 4p + 2]_{4p+3}$; and it follows that $\partial C_0{}^* \cup \partial C_1{}^* = \pm\{1, 2, 4p + 1, 4p + 2\}$.

Case 2. $k > 0$.

For $i = 1, 2, \ldots, p$, let $C_i = (c_{i,0}, c_{i,1}, \ldots, c_{i,8k+5})$ be $(8k + 6)$-cycles given by

$$c_{i,2j} = \begin{cases} 2j, & \text{for } 0 \leq j \leq 2k + 1, \\ 8k - 2j + 5, & \text{for } 2k + 2 \leq j \leq 4k + 2; \text{and} \end{cases}$$

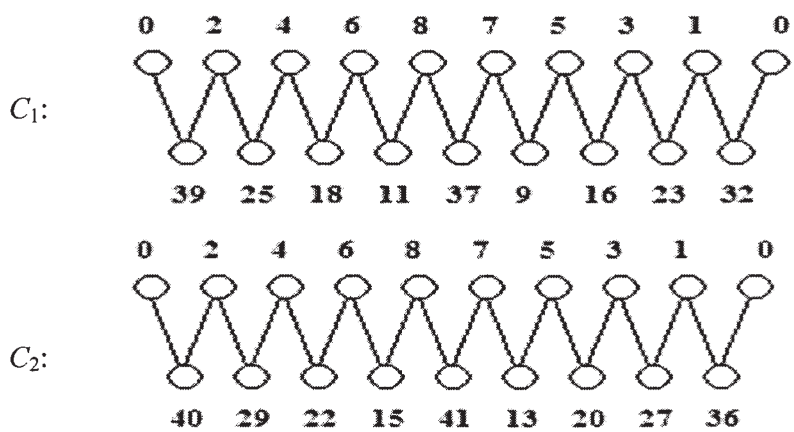

FIGURE 1.

$$c_{i,2j+1} = \begin{cases} (2k-1)(4p+3)+4i+3, & \text{for } j=0, \\ (2k-1-j)(4p+1)+4k+4i, & \text{for } 1 \leq j \leq 2k-1, \\ (2k+1)(4p+5)+i, & \text{for } 1 \leq 2 \text{ and } j=2k, \\ (2k+1)(4p+5)+1+i+2\left\lfloor\frac{(i-1)}{2}\right\rfloor, & \text{for } i>2 \text{ and } j=2k, \\ 2(4kp+5k+p+2)+2i, & \text{for } j=2k+1, \\ (j-2k-2)(4p+1)+4k+4i+2, & \text{for } 2k+2 \leq j \leq 4k, \\ (2k-1)(4p+3)+4i+5, & \text{for } j=4k+1, \\ 2k(4p+3)+3, & \text{for } i=1 \text{ and } j=4k+2, \\ 2k(4p+3)+2i+3, & \text{for } i>1 \text{ and } j=4k+2. \end{cases}$$

If $p=1$, then $\partial C = \pm\{4+7j, \ldots, 7(j+1) | j=0,1,\ldots,2k-2\} \cup \pm\{14k-2, \ldots, 14k+3, 14k+6, 14k+7, 14k+8, 14k+10\}$; if $p>1$, then $\bigcup_{i=1}^{p} \partial C_i = \pm\{4+(4p+3)j, \ldots, (4p+3)(j+1) | j=0,1,\ldots,2k-2\} \cup \pm\{(2k-1)(4p+3)+5+4j(p+1), \ldots, 2k(4p+3)+3+4j(p+1) | j=0,1\} \cup \pm\{(2k+1)(4p+3)+6, \ldots, p(8k+6)+6k+4\}$.

The short $(8k+6)$-cycles are: for $j=0,1,\ldots,k-1$, $C_j{}^* = [0, 4p+4+j(8p+6)]_{4p+3}$, $C_{k+j}{}^* = [0, 4p+5+j(8p+6)]_{4p+3}$, $C_{2k+j}{}^* = [0, 4p+6+j(8p+6)]_{4p+3}$, $C_{3k}{}^* = [0, 2k(4p+3)+4]_{4p+3}$, and $C_{3k+1}{}^*[0, (2k+1)(4p+3)+5]_{4p+3}$.

We have $\bigcup_{j=0}^{3k+1} \partial C_j{}^* = \pm\{1+i(8p+6), 2+i(8p+6), 3+i(8p+6), 4p+4+i(8p+6), 4p+5+i(8p+6), 4p+6+i(8p+6) | i=0,1,\ldots,k-1\} \cup \pm\{(2k-1)(4p+3)+4, 2k(4p+3)+4, 2k(4p+3)+5, (2k+1)(4p+3)+5\}$.

It can be checked that $(\bigcup_{i=1}^{p} \partial C_i) \cup (\bigcup_{j=0}^{3k+1} \partial C_j{}^*) = Z_v - \{0\}$, as desired. □

By virtue of Propositions 2.2, 2.4-(4), (5), 2.5, 2.6, and Theorem 1.1, we have:

Theorem 2.7. *If m is a prime power, then there exists a cyclic $2m$-cycle system of order v with the only definite exception of $v=3m$ when $m \equiv 3$* (mod 4).

In next section, we shall deal with the m-cycle systems for m not greater than 32. Since the constructions are different between odd cycles and even cycles, we classify the m-cycle systems into two cases: odd and even.

3. ODD CASES

We begin with introducing two results that are important for constructing odd cycles.

Lemma 3.1 [8]. *Let a, b, c, and r be positive integers with $c=a+b$ and $r>c$. Then there exists a cycle C of length $4s+3$ with the set of differences $\pm\{a, b, c, r, r+1, \ldots, r+4s-1\}$.*

Proof. We claim that the cycle $C = (v_0, v_1, \ldots, v_{2s+1}, v_{2s+1}{}', v_{2s}{}', \ldots, v_1{}')$ of length $4s+3$ exists according to the following two cases.

Case 1. Either a or b is odd, say b.

The vertices of C are defined as:

$v_0 = 0$; for $j=0,1,\ldots,s$, $v_{2j+1} = a+2j$, $v_{2j+1}{}' = c+2j$; for $j=1,2,\ldots,s$, $v_{2j} = a-r-2(j-1)$, $v_{2j}{}' = c+r+4s-2j+1$.

Case 2. Both a and b are even.

If r is even, then the vertices of C are given by

$$v_0 = 0; v_1 = a, v_1{}' = r + 4s - 2;\ \text{for } j = 1, 2, \ldots, s, v_{2j} = a + r + 4s - 2j + 1, v_{2j}{}' = c + r + 4s + 2j - 4, v_{2j+1} = a + 2j,\ \text{and } v_{2j+1}{}' = c + 4s - 2j.$$

If r is odd, then

$$v_0 = 0; v_1 = a, v_1{}' = r + 4s - 1;\ \text{for } j = 1, 2, \ldots, s, v_{2j} = a + r + 4s - 2j, v_{2j+1} = a + 2j, v_{2j}{}' = c + r + 4s + 2j - 3,\ \text{and } v_{2j+1}{}' = c + 4s - 2j.$$

A routine verification can show in each case that $\partial C = \pm\{a, b, c, r, r + 1, \ldots, r + 4s - 1\}$. □

As an example, we use the method stated above to construct a 15-cycle with the set of differences $\pm\{1, 2, 3, 6, \ldots, 17\}$ and $a=2$, $b=1$, $c=3$, $r=6$, and $s=3$. See Figure 2.

Next, we consider cycles of length $4s + 1$. Note that Lemma 3.2-(1) is also known in [8], but for completeness, we give a short proof here.

Lemma 3.2. *Let a, b, c, and r be positive integers with $c = a + b \pm 1$ and $r > c$.*

(1) *There exists a cycle C of length $4s + 1$ with the set of differences $\pm\{a, b, c, r, r + 1, \ldots, r + 4s - 3\}$.*
(2) *There exists a cycle C of length $4s + 1$ with the set of differences $\pm\{a, b, c, r, r + 1, r + 2k + 3, r + 2k + 4, \ldots, r + 2k + 4s - 2\}$ where $k \geq 0$.*

Proof.

(1) Let $C = (v_0, v_1, \ldots, v_{2s}, v_{2s}{}', v_{2s-1}{}', \ldots, v_1{}')$ be a cycle of length $4s + 1$ whose vertices are defined as $v_0 = 0, v_1 = a, v_1{}' = c, v_{2s} = c + 2s - 3, v_{2s}{}' = c + r + 2s - 2 + \varepsilon$, where $\varepsilon = 0$ or 1 according as $c = a + b + 1$ or $a + b - 1$; and for $i = 1, 2, \ldots, s - 1$, $v_{2i} = c + 2i - 3, v_{2i}{}' = c + r + 4s - 1 - 2i, v_{2i+1} = c - r - 2i - 1$, and $v_{2i+1}{}' = c + 2i$.
(2) Using the same method of construction stated above, we can obtain the desired result. □

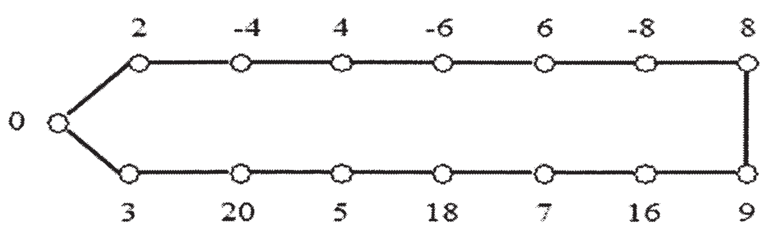

FIGURE 2.

For the case of odd cycle, we also need the crucial help of Skolem sequences and hooked Skolem sequences.

A Skolem sequence of order p is a collection of ordered pairs $\{(s_i, t_i) \mid 1 \leq i \leq p, t_i - s_i = i\}$ with $\bigcup_{i=1}^{p} \{s_i, t_i\} = \{1, 2, \ldots, 2p\}$; and a hooked Skolem sequence of order p is still a collection of ordered pairs $\{(s_i, t_i) \mid 1 \leq i \leq p, t_i - s_i = i\}$ with $\bigcup_{i=1}^{p} \{s_i, t_i\} = \{1, 2, \ldots, 2p-1, 2p+1\}$.

Theorem 3.3 [15]**.**

(1) *A Skolem sequence of order p exists if and only if* $p \equiv 0$ *or* 1 (mod 4).
(2) *A hooked Skolem sequence of order p exists if and only if* $p \equiv 2$ *or* 3 (mod 4).

In what follows, we will assume $\{(s_i, t_i) \mid 1 \leq i \leq p, t_i - s_i = i\}$ to be a (hooked) Skolem sequence of order p.

Proposition 3.4. *There exists a cyclic* 15*-cycle system of order v with* $v \equiv 21$ *or* 25 (mod 30) *with* $v > 25$.

Proof. By Proposition 2.2, we see that the value of v must be greater than 25. The proof is divided into two parts: $v \equiv 21$ or 25 (mod 30).

Part 1. $v \equiv 21$ (mod 30).

Let $v = 30p + 21$ for $p \geq 1$. If $p \equiv 0$ or 1 (mod 4), by Theorem 3.3-(1), there exists a Skolem sequence of order p so that $\bigcup_{i=1}^{p} \{i, s_i + p, t_i + p\} = \pm\{1, 2, \ldots, 3p\}$; and if $p \equiv 2$ or 3 (mod 4), by Theorem 3.3-(2), there exists a hooked Skolem sequence of order p so that $\bigcup_{i=1}^{p} \{i, s_i + p, t_i + p\} = \pm\{1, 2, \ldots, 3p - 1, 3p + 1\}$. This means that if distinct consecutive integers, say $\pm\{d + 1, d + 2, \ldots, d + 12r\}$ for some integers d and r, are in the set of differences from a 15-cycle, then we can repeatedly utilize Lemma 3.1 and hence, p full 15-cycles are obtained. It is therefore enough to show that there exist short 15-cycles $C_j^*(1 \leq j \leq s)$ such that $Z_v - \bigcup_{j=1}^{s} \partial C_j^* - \{0\} - \pm\{1, 2, \ldots, 3p\} (or \pm\{1, 2, \ldots, 3p - 1, 3p + 1\})$ constitutes the desired situation as stated above. This can be done as follows.

Case 1. $p \equiv 1$ (mod 4).

If $p = 1$, then $C_1^* = [0, 5, 1, 8, 2]_{17}$, $C_2^* = [0, 11, 3, 13, 4]_{17}$, and $C = (0, 2, 20, 4, 26, 6, 32, 8, 9, 30, 7, 24, 5, 17, 3)$.

We have that $\partial C_1^* \cup \partial C_2^* = \pm\{4, \ldots, 11, 13, 15\}$ and $\partial C = \pm\{1, 2, 3, 12, 14, 16, \ldots, 25\}$.

If $p > 1$, then we split the proof into the following three subcases.

Subcase 1: $p \equiv 5$ (mod 12), say $p = 12k + 5$ for $k \geq 0$.

$C_1^* = [0, 120k + 54, 156k + 71, 120k + 55, 300k + 141]_{120k+57}$ and $C_2^* = [0, 120k + 55, 300k + 136, 120k + 56, 300k + 139]_{120k+57}$.

$\partial C_1^* \cup \partial C_2^* = \pm\{36k + 16, 36k + 17, 120k + 54, 120k + 55, 180k + 80, \ldots, 180k + 85\}$.

Subcase 2: $p \equiv 9$ (mod 12), say $p = 12k + 9$ for $k \geq 0$.

$C_1^* = [0, 120k + 102, 300k + 243, 120k + 101, 300k + 244]_{120k+97}$ and $C_2^* = [0, 120k + 103, 156k + 131, 120k + 102, 300k + 242]_{120k+97}$.

$\partial C_1{}^* \cup \partial C_2{}^* = \pm\{36k+28,\ 36k+29,\ 120k+102,\ 120k+103,\ 180k+140, \ldots, 180k+145\}$.

Subcase 3: $p \equiv 1 \pmod{12}$, say $p = 12k+13$ for $k \geq 0$.

$C_1{}^* = [0,\ 120k+138,\ 156k+179,\ 120k+139,\ 300k+343]_{120k+137}$ and $C_2{}^* = [0,\ 120k+139,\ 300k+339,\ 120k+138,\ 300k+340]_{120k+137}$.

$\partial C_1{}^* \cup \partial C_2{}^* = \pm\{36k+40,\ 36k+41,\ 120k+138,\ 120k+139,\ 180k+200, \ldots, 180k+205\}$.

Case 2. $p \equiv 2 \pmod 4$.

Subcase 1: $p \equiv 2 \pmod{12}$, say $p = 12k+2$ for $k \geq 0$.

$C_1{}^* = [0,\ 120k+21,\ 300k+61,\ 120k+25,\ 300k+64]_{120k+27}$ and $C_2{}^* = [0,\ 120k+22,\ 300k+60,\ 120k+25,\ 156k+33]_{120k+27}$.

$\partial C_1{}^* \cup \partial C_2{}^* = \pm\{36k+6,\ 36k+8,\ 120k+21,\ 120k+22,\ 180k+35, \ldots, 180k+40\}$.

Subcase 2: $p \equiv 6 \pmod{12}$, say $p = 12k+6$ for $k \geq 0$.

$C_1{}^* = [0,\ 120k+69,\ 300k+166,\ 120k+68,\ 300k+167]_{120k+67}$ and $C_2{}^* = [0,\ 120k+70,\ 300k+165,\ 120k+69,\ 156k+87]_{120k+67}$.

$\partial C_1{}^* \cup \partial C_2{}^* = \pm\{36k+18,\ 36k+20,\ 120k+69,\ 120k+70,\ 180k+95, \ldots, 180k+100\}$.

Subcase 3: $p \equiv 10 \pmod{12}$, say $p = 12k+10$ for $k \geq 0$.

$C_1{}^* = [0,\ 120k+105,\ 300k+261,\ 120k+106,\ 300k+264]_{120k+107}$ and $C_2{}^* = [0,\ 120k+106,\ 300k+265,\ 120k+105,\ 156k+137]_{120k+107}$.

$\partial C_1{}^* \cup \partial C_2{}^* = \pm\{36k+30,\ 36k+32,\ 120k+105,\ 120k+106,\ 180k+155, \ldots, 180k+160\}$.

Case 3. $p \equiv 3 \pmod 4$.

Subcase 1: $p \equiv 3 \pmod{12}$, say $p = 12k+3$ for $k \geq 0$.

$C_1{}^* = [0,\ 120k+38,\ 156k+47,\ 120k+36,\ 156k+49]_{120k+37}$ and $C_2{}^* = [0,\ 120k+39,\ 300k+91,\ 120k+38,\ 300k+92]_{120k+37}$.

$\partial C_1{}^* \cup \partial C_2{}^* = \pm\{36k+9,\ 36k+11,\ 36k+12,\ 36k+13,\ 120k+38,\ 120k+39,\ 180k+52, \ldots, 180k+55\}$.

Subcase 2: $p \equiv 7 \pmod{12}$, say $p = 12k+7$ for $k \geq 0$.

$C_1{}^* = [0, 120k+74, 156k+97, 120k+76, 156k+101]_{120k+77}$ and $C_2{}^* = [0, 120k+75, 300k+188, 120k+76, 300k+191]_{120k+77}$.

$\partial C_1{}^* \cup \partial C_2{}^* = \pm\{36k+21, 36k+23, 36k+24, 36k+25, 120k+74, 120k+75, 180k+112, \ldots, 180k+115\}$.

Subcase 3: $p \equiv 11 \pmod{12}$, say $p = 12k+11$ for $k \geq 0$.

$C_1{}^* = [0,\ 120k+110,\ 156k+146,\ 120k+113,\ 300k+289]_{120k+117}$ and $C_2{}^* = [0, 120k+111, 156k+148, 120k+113, 300k+290]_{120k+117}$.

$\partial C_1{}^* \cup \partial C_2{}^* = \pm\{36k+33,\ 36k+35,\ 36k+36,\ 36k+37,\ 120k+110,\ 120k+111, 180k+172, \ldots, 180k+175\}$.

Part 2. $v \equiv 25 \pmod{30}$.

Let $v = 30p + 25$ for $p \geq 1$. Similarly, unless otherwise stated, we just consider here the construction of short 15-cycles.

Case 1. $p \equiv 0 \pmod 4$.

For $i = 1, 2, 3, 4$, $C_i^* = [0, 6p + i, 21p + j]_{6p+5}$, where $j = 11, 10, 12$, or 16 according as $i = 1, 2, 3$, or 4, and $\bigcup_{i=1}^{4} \partial C_i^* = \pm\{6p+1, \ldots, 6p+4, 15p+5, \ldots, 15p+12\}$.

Case 2. $p \equiv 1 \pmod 4$.

For $i = 1, 2, 3$, $C_i^* = [0, 6p + 1 + i, 21p + j]_{6p+5}$, where $j = 10, 14$, or 11 according as $i = 1, 2$, or 3, and $C_4^* = [0, 6p + 7, 21p + 17]_{6p+5}$ and so $\bigcup_{i=1}^{4} \partial C_i^* = \pm\{6p+2, 6p+3, 6p+4, 6p+7, 15p+5, \ldots, 15p+12\}$.

For $i = 1, 2, \ldots, p$, let C_i be the full 15-cycles. Let $C_1 = (0, t_1+1, t_1+6p+14, t_1+2, t_1+6p+13, t_1+3, t_1+6p+12, t_1+4, s_1-6p-1, -6p-2, -3p-1, -6p-3, -3p, -6p-4, 1)$ and then $\partial C_1 = \pm\{1, s_1+1, t_1+1, 3p+1, \ldots, 3p+4, 6p+5, 6p+6, 6p+8, \ldots, 6p+13\}$. The rest of the full 15-cycles are constructed by the same method described in Part 1.

The proofs of the cases when $p \equiv 2 \pmod 4$ and $p \equiv 3 \pmod 4$ are analogous to that in Case 2, so we omit the details.

By routine computation, it can be verified in each case that the union of differences of the short and full 15-cycles is equal to $Z_v - \{0\}$, and the proof then follows from Lemma 2.3. □

Proposition 3.5. *There exists a cyclic* 21*-cycle system of order* v *with* $v \equiv 7$ *or* 15 (mod 42).

Proof. Let $v = 42p + 7$ or $42p + 15$ for $p \geq 1$. If $p \equiv 0 \pmod 2$, for $i = 1, 3, \ldots, p-1$, let

$$(a_i, b_i, c_i) = (i+1, s_i + p, t_i + p)$$

and

$$(a_{i+1}, b_{i+1}, c_{i+1}) = (i, s_{i+1} + p, t_{i+1} + p).$$

Suppose that $p \equiv 1 \pmod 4$, say $p = 4k + 1$. If $k = 1$, let

$$(a_1, b_1, c_1) = (2, 13, 14), (a_2, b_2, c_2) = (1, 6, 8), (a_3, b_3, c_3) = (4, 9, 12),$$

$$(a_4, b_4, c_4) = (3, 7, 11), \text{ and } (a_5, b_5, c_5) = (5, 10, 16); \text{ and if } k > 1,$$

then let

$$(a_i, b_i, c_i) = (i+1, s_i + p, t_i + p) \text{ for odd } i \leq 4k - 3,$$

$$(a_{i+1}, b_{i+1}, c_{i+1}) = (i, s_{i+1} + p, t_{i+1} + p) \text{ for odd } i \leq 4k - 3,$$

$$(a_{4k-1}, b_{4k-1}, c_{4k-1}) = (4k-1, 4k+1, 8k+1),$$

$$(a_{4k}, b_{4k}, c_{4k}) = (4k, 8k+3, 12k+4),$$

and

$$(a_{4k+1}, b_{4k+1}, c_{4k+1}) = (4k+2, 6k+2, 10k+3).$$

Refer to [15, p. 458].

Assume that $p \equiv 3 \pmod 4$, say $p = 4k-1$. If $k = 1$, then let

$$(a_1, b_1, c_1) = (1, 5, 7), (a_2, b_2, c_2) = (2, 8, 9), \text{ and } (a_3, b_3, c_3) = (3, 4, 6);$$

and if $k > 1$, then let

$$(a_i, b_i, c_i) = (i+1, s_i + p, t_i + p) \text{ for odd } i \leq 4k-3,$$
$$(a_{i+1}, b_{i+1}, c_{i+1}) = (i, s_{i+1} + p, t_{i+1} + p) \text{ for odd } i \leq 4k-3,$$

and

$$(a_{4k-1}, b_{4k-1}, c_{4k-1}) = (4k-1, 8k-1, 12k-3).$$

Clearly, $c_i = a_i + b_i + 1$ or $a_i + b_i - 1$ for $1 \leq i \leq p$. Furthermore, it is easy to check that $\bigcup_{i=1}^{p} \{a_i, b_i, c_i\} = \pm\{1, 2, \ldots, 3p\}$, if $p \equiv 0$ or $3 \pmod 4$ and $\bigcup_{i=1}^{p} \{a_i, b_i, c_i\} = \pm\{1, 2, \ldots, 3p-1, 3p+1\}$, if $p \equiv 1$ or $2 \pmod 4$.

Part 1. $v = 42p + 7$ for $p \geq 1$.

The proof is divided into four cases, depending on whether $p \equiv 1, 3, 2$, or $0 \pmod 4$.

Case 1. $p \equiv 1 \pmod 4$.

$C^* = [0, 3p, 24p+4]_{6p+1}$ and $\partial C^* = \pm\{3p, 18p+3, 21p+4\}$. The construction of full 21-cycles is the following.

Subcase 1: $p \equiv 1 \pmod{12}$.

$C_1 = (0, a_1, a_1 + b_1,\ a_1 + b_1 - 3p - 2,\ a_1 + b_1 + 2,\ a_1 + b_1 - 3p - 4,\ a_1 + b_1 + 4, a_1 + b_1 - 3p - 6,\ a_1 + b_1 + 6,\ a_1 + b_1 - 3p - 8,\ a_1 + b_1 + 8, c_1 + 18p + 12 + \varepsilon,$ $c_1 + 8,\ c_1 + 3p + 11,\ c_1 + 6,\ c_1 + 3p + 13,\ c_1 + 4,\ c_1 + 3p + 15,\ c_1 + 2, c_1 + 3p + 17, c_1)$, where $\varepsilon = 0$ or 1 according as $c_1 = a_1 + b_1 + 1$ or $a_1 + b_1 - 1$, and we have $\partial C_1 = \pm\{a_1, b_1, c_1, 3p+2, \ldots, 3p+17, 18p+4, 18p+5\}$. Moreover, by Lemma 3.2-(1), the remaining $p - 1$ full 21-cycles $C_2, \ldots, C_p$ follows.

Notice that the construction of the full 21-cycles $C_2, \ldots, C_p$ in the remainder of Part 1 is the same as that stated above, so we just indicate the construction of the full 21-cycle C_1.

Subcase 2: $p \equiv 5 \pmod{12}$.

$C_1 = (0, a_1, a_1 + b_1, a_1 + b_1 - 3p - 2,\ a_1 + b_1 + 2, a_1 + b_1 - 18p - 2,\ a_1 + b_1 + 4,\ a_1 + b_1 - 18p - 4,\ a_1 + b_1 + 6,\ a_1 + b_1 - 18p - 6, a_1 + b_1 + 8, c_1 + 18p + 24 + \varepsilon,\ c_1 + 8,\ c_1 + 3p + 11,\ c_1 + 6,\ c_1 + 18p + 11,\ c_1 + 4,\ c_1 + 18p + 13,\ c_1 + 2,\ c_1 + 18p + 15, c_1)$, where $\varepsilon = 0$ or 1 according as $c_1 = a_1 + b_1 + 1$ or $a_1 + b_1 - 1$, and we have $\partial C_1 = \pm\{a_1, b_1, c_1, 3p+2, \ldots, 3p+5, 18p+4, \ldots, 18p+17\}$.

Subcase 3: $p \equiv 9 \pmod{12}$.

$C_1 = (0,\ a_1,\ a_1 + b_1,\ a_1 + b_1 - 3p - 2,\ a_1 + b_1 + 2,\ a_1 + b_1 - 3p - 4, a_1 + b_1 + 4,\ a_1 + b_1 - 18p,\ a_1 + b_1 + 6,\ a_1 + b_1 - 18p - 2,\ a_1 + b_1 + 8,\ c_1 + 3p + 18 + \varepsilon,$

$c_1 + 8$, $c_1 + 3p + 11$, $c_1 + 6$, $c_1 + 3p + 13$, $c_1 + 4, c_1 + 18p + 9, c_1 + 2, c_1 + 18p + 11, c_1)$, where $\varepsilon = 0$ or 1 according as $c_1 = a_1 + b_1 + 1$ or $a_1 + b_1 - 1$, and it follows that $\partial C_1 = \pm\{a_1, b_1, c_1, 3p + 2, \ldots, 3p + 11, 18p + 4, \ldots, 18p + 11\}$.

Case 2. $p \equiv 3 \pmod 4$.

The proof is similar to that in Case 1 and omitted.

Case 3. $p \equiv 2 \pmod 4$.

$C^* = [0, 3p, 9p + 3]_{6p+1}$ and $\partial C^* = \pm\{3p, 3p + 2, 6p + 3\}$.

Subcase 1: $p \equiv 2 \pmod{12}$.

$C_1 = (0, a_1, a_1 + b_1, a_1 + b_1 - 3p - 4, a_1 + b_1 + 2, a_1 + b_1 - 6p - 2$, $a_1 + b_1 + 4, a_1 + b_1 - 6p - 4$, $a_1 + b_1 + 6$, $a_1 + b_1 - 6p - 6$, $a_1 + b_1 + 8$, $c_1 + 3p + 15 + \varepsilon$, $c_1 + 8$, $c_1 + 3p + 11$, $c_1 + 6$, $c_1 + 6p + 11$, $c_1 + 4$, $c_1 + 6p + 13, c_1 + 2$, $c_1 + 6p + 15, c_1)$, where $\varepsilon = 0$ or 1 according as $c_1 = a_1 + b_1 + 1$ or $a_1 + b_1 - 1$, and so $\partial C_1 = \pm\{a_1, b_1, c_1, 3p + 3, \ldots, 3p + 8, 6p + 4, \ldots, 6p + 15\}$.

Subcase 2: $p \equiv 6 \pmod{12}$.

By directly and repeatedly using Lemma 3.2-(1), we then have p full 21-cycles $C_1, \ldots, C_p$.

Subcase 3: $p \equiv 10 \pmod{12}$.

$C_1 = (0, a_1, a_1 + b_1, a_1 + b_1 - 3p - 4, a_1 + b_1 + 2, a_1 + b_1 - 3p - 6, a_1 + b_1 + 4, a_1 + b_1 - 3p - 8, a_1 + b_1 + 6, a_1 + b_1 - 6p + 2, a_1 + b_1 + 8, c_1 + 6p + 16 + \varepsilon$, $c_1 + 8, c_1 + 3p + 11, c_1 + 6$, $c_1 + 3p + 13$, $c_1 + 4$, $c_1 + 3p + 15$, $c_1 + 2$, $c_1 + 6p + 7, c_1)$, where $\varepsilon = 0$ or 1 according as $c_1 = a_1 + b_1 + 1$ or $a_1 + b_1 - 1$, and $\partial C_1 = \pm\{a_1, b_1, c_1, 3p + 3, \ldots, 3p + 14, 6p + 4, \ldots, 6p + 9\}$.

Case 4. $p \equiv 0 \pmod 4$.

The proof can be obtained by a method similar to that in Case 3.
Part 2. $v = 42p + 15$ for $p \geq 1$.

Case 1. $p \equiv 1 \pmod 4$, say $p = 4k + 1$.

$C^* = [0, 3p, 6p + 2, 9p + 6, 6p + 3, 27p + 11, 6p + 5]_{14p+5}$ and $\partial C^* = \pm\{3p, 3p + 2, 3p + 3, 3p + 4, 8p, 21p + 6, 21p + 7\}$.
By Lemma 3.2-(2), we have k full 21-cycles $C_1, \ldots, C_k$ with $\bigcup_{i=1}^{k} \partial C_i = \pm\{3p + 5, \ldots, 3p + 2k + 4, 8p + 1, \ldots, 12p - 4\}$, and by Lemma 3.2-(1), there exist $3k + 1$ full 21-cycles $C_{k+1}, \ldots, C_{4k+1}$ with $\bigcup_{i=k+1}^{4k+1} \partial C_i = Z_v - \partial C^* - \bigcup_{i=1}^{k} \partial C_i - \{0\}$.

Case 2. $p \equiv 2 \pmod 4$, say $p = 4k + 2$.

$C^* = [0, 3p, 6p + 2, 9p + 7, 6p + 4, 9p + 10, 6p + 6]_{14p+5}$ and $\partial C^* = \pm\{3p, 3p + 2, 3p + 3, 3p + 4, 3p + 5, 3p + 6, 8p - 1\}$.
Similarly, by Lemma 3.2-(2) and Lemma 3.2-(1), there exist $k + 1$ full 21-cycles $C_1, \ldots, C_{k+1}$ with $\bigcup_{i=1}^{k+1} \partial C_i = \pm\{3p + 7, \ldots, 3p + 2k + 8, 8p, \ldots, 12p + 7\}$ and $3k + 1$ full 21-cycles $C_{k+1}, \ldots, C_{4k+1}$ with $\bigcup_{i=k+2}^{4k+2} \partial C_i = Z_v - \partial C^* - \bigcup_{i=1}^{k+1} \partial C_i - \{0\}$.

Case 3. $p \equiv 3 \pmod 4$.

$C^* = [0, 3p+1, 6p+3, 9p+8, 6p+5, 9p+11, 6p+7]_{14p+5}$ and $\partial C^* = \pm\{3p+1, \ldots, 3p+6, 8p-2\}$.

The p full 21-cycles can be obtained by the analogous method as mentioned in Case 1, so we omit the details.

Case 4. $p \equiv 0 \pmod 4$.

$C^* = [0, 3p+1, 6p+3, 9p+6, 6p+2, 14p+7, 35p+13]_{14p+5}$ and $\partial C^* = \pm\{3p+1, 3p+2, 3p+3, 3p+4, 8p+5, 21p+6, 21p+7\}$.

It is still similar to Case 1, and omitted. □

4. EVEN CASES

Proposition 4.1. *There exists a cyclic* 12*-cycle system of order* v *with* $v \equiv 9 \pmod{24}$.

Proof. Let $v = 24p+9$ for $p \geq 1$.

$C^* = [0,\ 1,\ 3,\ 12p+8]_{8p+3}$ and for $i = 1, \ldots, p$, $C_i = (0,\ 10p-4i+7,\ 2,\ 10p-4i+8,\ 4,\ 12p-2i+9,\ 5,\ 4p-4i+10,\ 3,\ 4p-4i+9,\ 1,\ t)$, where $t = -3$, if $i = 1$ and $t = 6p-2i+7$, if $i > 1$.

Proposition 4.2. *There exists a cyclic* 20*-cycle system of order* v *with* $v \equiv 25$ (mod 40) *and* $v > 25$.

Proof. Note that by Proposition 2.2, there does not exist a cyclic 20-cycle system of order 25. Let $v = 40p+25$ for $p \geq 1$.

${C_1}^* = [0,\ 2,\ 1,\ 8p+8]_{8p+5}$, ${C_2}^* = [0,\ 6,\ 1,\ 8p+9]_{8p+5}$, and ${C_3}^* = [0,\ 12p+11,\ 1,\ 20p+14]_{8p+5}$.

For $i = 1, \ldots, p$, $C_i = (0, 20p-8i+19, 2, 20p-8i+20, 4, 20p-8i+18, 6, 20p-8i+21, 8, 12p-4i+18, 9, 8p-8i+17, 7, 8p-8i+14, 5, 8p-8i+16, 3, 8p-8i+15, 1, 12p-4i+12)$. □

Proposition 4.3. *There exists a cyclic* 24*-cycle system of order* v *with* $v \equiv 33 \pmod{48}$.

Proof. Similarly, we just consider the case when $v = 48p+33$ for $p \geq 1$.

${C_1}^* = [0, 4p+15, 4p+11, 8p+12, 12p+14, 12p+13, 12p+11, 16p+14]_{16p+11}$ and ${C_2}^* = [0, 8p+31, 4p+11, 12p+28, 8p+11, 16p+29, 24p+48, 20p+30]_{16p+11}$.

$C_1 = (0, 8p+25, 2, 8p+26, 4, 8p+33, 6, 8p+34, 8, 8p+40, 10, 8p+31, 11, 4p+25, 9, 4p+19, 7, 4p+18, 5, 4p+11, 3, 4p+10, 1, -4p-4)$ and for $i = 2, \ldots, p$, $C_i = (0, 8p-4i+24, 2, 8p-4i+25, 4, 18p-4i+34, 6, 18p-4i+35, 8, 22p-4i+33, 10, 24p-2i+30, 11, 22p-4i+35, 9, 14p-4i+40, 7, 14p-i+39, 5, 4p-4i+10, 3, 4p-4i+9, 1, 10p-2i+34)$. □

Proposition 4.4. *There exists a cyclic* 28*-cycle system of order* v *with* $v \equiv 49$ (mod 56) *and* $v > 49$.

Proof. As mentioned previously, we see that $v > 49$. Let $v = 56p+49$ for $p \geq 1$.

If $p = 1$, then ${C_1}^* = [0,\ 2,\ 1,\ 18]_{15}$, ${C_i}^* = [0,\ 2i+6,\ 1,\ i+17]_{15}$ for $2 \leq i \leq 5$, ${C_6}^* = [0,\ 32,\ 1,\ 23]_{15}$, and $C = (0, 48, 2, 49, 4, 47, 6, 50, 8, 38, 10, 39, 12, 62, 13, 36, 11, 35, 9, 43, 7, 40, 5, 42, 3, 41, 1, 52)$.

If $p > 1$, then $C_1^* = [0, 2, 1, 8p + 10]_{8p+7}$, $C_i^* = [0, 8p + 12 + 2i, 1, 8p + 9 + i]_{8p+7}$ for $2 \leq i \leq 6$, and for $i = 1, \ldots, p$, $C_i = (0, 24p - 16i + 40, 2, 24p - 16i + 41, 4, 24p - 16i + 39, 6, 24p - 16i + 42, 8, 8p - 8i + 24, 10, 8p - 8i + 25, 12, 28p - 4i + 38, 13, 8p - 8i + 22, 11, 8p - 8i + 21, 9, 24p - 16i + 35, 7, 24p - 16i + 32, 5, 24p - 16i + 34, 3, 24p - 16i + 33, 1, 28p - 4i + 28)$.

Proposition 4.5. *There exists a cyclic* 30-*cycle system of order v with* $v \equiv 21, 25,$ *or* 45 (mod 60).

Proof.

Part 1. $v = 60p + 21$ for $p \geq 1$.

$C^* = [0, 4p + 2, 4p + 1, 8p + 4, 8p + 2, 12p + 6, 12p + 3, 16p + 8, 16p + 3, 20p + 11]_{20p+7}$.

If $p = 1$, then $C = (0, 30, 2, 31, 4, 38, 6, 39, 8, 46, 10, 47, 12, 53, 14, 24, 13, 36, 11, 35, 9, 28, 7, 27, 5, 20, 3, 19, 1, 14)$.

If $p > 1$, then $C_1 = (0, 28p + 2, 2, 28p + 3, 4, 28p + 10, 6, 28p + 11, 8, 28p + 18, 10, 28p + 19, 12, t_1, 14, 4p + 23, 13, 16p + 20, 11, 16p + 19, 9, 16p + 12, 7, 16p + 11, 5, 16p + 4, 3, 16p + 3, 1, 4p + 7)$, and for $i = 2, \ldots, p$, $C_i = (0, 28p - 12i + 14, 2, 28p - 12i + 15, 4, 28p - 12i + 22, 6, 28p - 12i + 23, 8, 28p - 12i + 30, 10, 28p - 12i + 31, 12, t_i, 14, 4i + 12, 13, 16p - 12i + 32, 11, 16p - 12i + 31, 9, 16p - 12i + 24, 7, 16p - 12i + 23, 5, 16p - 12i + 16, 3, 16p - 12i + 15, 1, 4i + 1)$, where for $j = 1, \ldots, \left\lfloor \frac{(p+1)}{2} \right\rfloor$, $t_{2j-1} = 30p + 26 - 44j + \varepsilon_1$, where $\varepsilon_1 = 0$ or 1 according as $p \equiv 0$ or 1 (mod 2) and for $j = 1, \ldots, \lfloor p/2 \rfloor$, $t_{2j} = t_{2j-1} - \varepsilon_2$, where $\varepsilon_2 = 1$ or 3 according as $p \equiv 0$ or 1 (mod 2).

Part 2. $v = 60p + 25$ for $p \geq 1$.

If $p = 1$, then $C_1^* = [0, 14, 11, 16, 15, 19]_{17}$, $C_2^* = [0, 32, 11, 34, 15, 37]_{17}$, and $C = (0, 27, 2, 28, 4, 35, 6, 36, 8, 46, 10, 47, 12, 55, 14, 53, 13, 23, 11, 22, 9, 15, 7, 14, 5, 20, 3, 19, 1, 34)$.

If $p > 1$, then $C_1^* = [0, 4p + 10, 4p + 7, 8p + 8, 8p + 7, 12p + 7]_{12p+5}$, $C_2^* = [0, 8p + 20, 4p + 7, 12p + 18, 8p + 7, 16p + 17]_{12p+5}$.
$C_1 = (0, 8p + 15, 2, 8p + 16, 4, 8p + 23, 6, 8p + 24, 8, 26p + 20, 10, 26p + 21, 12, t_1, 14, 28p + 25, 13, 26p + 18, 11, 26p + 17, 9, 4p + 15, 7, 4p + 14, 5, 4p + 7, 3, 4p + 6, 1, 26p + 4)$ and for $i = 2, \ldots, p$, $C_i = (0, 8p - 4i + 17, 2, 8p - 4i + 18, 4, 26p - 8i + 18, 6, 26p - 8i + 19, 8, 26p - 8i + 26, 10, 26p - 8i + 27, 12, t_i, 14, 28p - 2i + 27, 13, 18p - 8i + 36, 11, 18p - 8i + 30, 9, 18p - 8i + 33, 7, 18p - 8i + 27, 5, 4p - 4i + 9, 3, 4p - 4i + 8, 1, 10p - 2i + 22)$, where for $j = 1, \ldots, \lfloor \frac{(p+1)}{2} \rfloor$, $t_{2j-1} = 30p - 4j + 28 + \varepsilon_1$, where $\varepsilon_1 = 0$ or 1 according as $p \equiv 0$ or 1 (mod 2) and for $j = 1, \ldots, \lfloor p/2 \rfloor$, $t_{2j} = t_{2j-1} - \varepsilon_2$, where $\varepsilon_2 = 1$ or 3 according as $p \equiv 0$ or 1 (mod 2).

Part 3. $v = 60p + 45$ for $p \geq 0$.

Note that by Proposition 2.6, we know that there exists a cyclic 30-cycle system of order v with $v = 60p + 45$ for $p \geq 1$. It is therefore enough to prove that there exists a cyclic 30-cycle system of order 45. This can be easily given as follows:

$C_1^* = [0, 8]_3$, $C_2^* = [0, 16]_3$, $C_3^* = [0, 20]_3$, $C_4^* = [0, 1, 5, 8, 6, 21]_9$, and
$C_5^* = [0, 14, 3, 13, 6, 27, 9, 31, 40, 34]_{15}$. □

We now have the main result, which is obtained by combining the known results [5–13,16] and the propositions proved in Sections 2, 3, and 4.

Theorem 4.6. *If* $3 \leq m \leq 32$, *then there exists a cyclic m-cycle system of order v for all possible values of v with exceptions of* $(m, v) = (3, 9)$, (6, 9), (9, 9), (14, 21), (15, 15), (15, 21), (15, 25), (20, 25), (22, 33), (24, 33), (25, 25), (27, 27), *and* (28, 49).

5. CONCLUDING REMARK

Reviewing the construction of above-mentioned cyclic m-cycle systems, it is clear that the construction of the case when m is even is much easier than that of m odd. We expect that the cyclic m-cycle systems with m even can be solved in the near future. Furthermore, in view of Proposition 2.2, we believe that for any admissible value of v such that $m < v < 2m + 1$ and $\gcd(m, v)$ is not a prime power, then there exists a cyclic m-cycle system of order v.

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