One-Loop Massive Scattering Amplitudes and Ward Identities in String Theory

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We calculate bosonic open string one-loop massive scattering amplitudes for some low-lying string states. By using the periodicity relations of Jacobi theta functions, we explicitly prove an infinite number of one-loop type I stringy Ward identities derived from type I zero-norm states in the old covariant first quantized (OCFQ) spectrum of open bosonic string. The subtlety in the proofs of one-loop type II stringy Ward identities is discussed by comparing them with those of string-tree cases. High-energy limit of these stringy Ward identities can be used to fix the proportionality constants between one-loop massive high-energy scattering amplitudes of different string states with the same momenta. These proportionality constants *cannot* be calculated directly from sample calculation as we did previously in the cases of string-tree scattering amplitudes.

§1. Introduction

Recently it was discovered that[1),2)] the high-energy limit $\alpha' \to \infty$ of stringy Ward identities, or massive gauge invariances, derived from the decoupling of two types of zero-norm states imply an infinite number of linear relations[3)] among high energy scattering amplitudes of different string states with the same momenta. The calculation was first done for mass levels $m^2 = 4, 6$ and was soon generalized to arbitrary mass levels.[4),5)] These linear relations can be used to fix the proportionality constants between high energy scattering amplitudes of different string states algebraically at each fixed mass level. These proportionality constants were found to be independent of the scattering angle ϕ_{CM} and the loop order χ of string perturbation theory. Thus there is only one independent component of high-energy string scattering amplitudes for each fixed mass level. For the case of string-tree amplitudes, a general formula can even be given to express all high-energy stringy scattering amplitudes at arbitrary mass levels in terms of those of tachyons.[1),6)] Other approaches of stringy symmetries can be found in Refs. 7) and 8).

The importance of zero-norm states and their implication on stringy symmetries were first pointed out in the context of massive σ-model approach of string theory.[9)] On the other hand, zero-norm states were also shown[10)] to carry the spacetime ω_∞ symmetry charges of 2D string theory. Some implications of stringy Ward identities on the scattering amplitudes were also discussed in Ref. 11). All the above zero-norm

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state calculations are independent of the high-energy saddle-point calculations of Gross and Mende,[12] Gross[3] and Gross and Manes.[13] In fact, the results of saddle-point calculations by those authors were found[1,2,6] to be inconsistent with stringy Ward identities, which are valid to all energy, and thus could threat the validity of unitarity of string perturbation theory. A corrected saddle-point calculation was given in Ref. 6) where the missing terms of the calculation in Refs. 3), 12) and 13) were identified to recover the stringy Ward identities.

In this paper, we shall first calculate bosonic open string one-loop massive scattering amplitudes for some low-lying string states which were not calculated in the literature. Bosonic open string tree massive scattering amplitudes were calculated in Ref. 11), and the tree-level massive gauge invariances were explicitly justified for the first few mass levels. General tree-level gauge invariances were proved by "the canceled propagator argument" in the operator approach in Ref. 14). On the other hand, bosonic closed string one-loop massless scattering amplitudes were calculated in Ref. 15), and the one-loop modular invariance was justified there. Here we are aiming to explicitly show the one-loop massive gauge invariances or Ward identities. High-energy limit of these stringy Ward identities can be used to fix the proportionality constants between one-loop massive high-energy scattering amplitudes of different string states with the same momenta.

Unlike the string-tree scattering amplitudes, which can be exactly integrated to calculate their high-energy limit, one-loop scattering amplitudes are not exactly integrable and their high-energy limit are difficult to calculate. Thus the determination of the proportionality constants between high-energy one-loop scattering amplitudes relies solely on the algebraic high-energy stringy Ward identities, and cannot be calculated directly from sample calculation as we did previously[1,2] in the cases of string-tree scattering amplitudes. This is one of the main motivations to explicitly prove one-loop stringy Ward identities in this paper. In §2 of this paper, we first give a new proof of tree-level stringy Ward identities, which will be useful for the proof of one-loop Ward identities in §3. In §3, by using the periodicity relations of Jacobi theta functions, we will show an infinite number of type I one-loop stringy Ward identities derived from type I zero-norm states in the old covariant first quantized (OCFQ) spectrum of open bosonic string. The subtlety in the proofs of one-loop type II stringy Ward identities will be discussed by comparing them with those of string-tree cases. The explicit proof of type II one-loop Ward identities seems to be much more involved and are, presumably, related to more advanced identities of Jacobi theta functions. In §4, high-energy limit of stringy Ward identities will be used to fix the proportionality constants between one-loop massive high-energy scattering amplitudes of different string states. These proportionality constants are otherwise difficult to calculate directly from sample calculation as in the cases of string-tree scattering amplitudes. We thus have explicitly justify Gross's conjecture,[3] for the first time, that the proportionality constants between high energy scattering amplitudes are independent of the scattering angle ϕ_{CM} and the loop order χ of string perturbation theory, at least for $\chi = 1, 0$. A brief conclusion is given in §5.

§2.   A new proof of tree-level stringy Ward identities

For illustration and setting up the notations, let us begin with simple examples of string tree-level massive scattering amplitudes of the first massive level. For the string-tree level $\chi = 1$, with one tensor v_2 and three tachyons $v_{1,3,4}$, all scattering amplitudes of mass level $m^2 = 2$ were calculated in Ref. 11). These are

$$\mathcal{T}^{\mu\nu} = \int \prod_{i=1}^{4} dx_i \langle e^{ik_1 X} \partial X^\mu \partial X^\nu e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} \rangle \tag{1}$$

$$= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} [t/2(t/2+1)k_1^\mu k_1^\nu$$
$$-2(s/2+1)(t/2+1)k_1^{(\mu} k_3^{\nu)} + s/2(s/2+1)k_3^\mu k_3^\nu], \tag{2}$$

$$\mathcal{T}^{\mu} = \int \prod_{i=1}^{4} dx_i \langle e^{ik_1 X} \partial^2 X^\mu e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} \rangle \tag{3}$$

$$= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} [-t/2(t/2+1)k_1^\mu - s/2(s/2+1)k_3^\mu], \tag{4}$$

where $s = -(k_1+k_2)^2, t = -(k_2+k_3)^2$ and $u = -(k_1+k_3)^2$ are the Mandelstam variables. In deriving Eqs. (2) and (4), we have made the $SL(2,R)$ gauge fixing and restricted to the $s-t$ channel of the amplitudes by choosing $x_1 = 0, 0 \leqq x_2 \leqq 1, x_3 = 1, x_4 = \infty$.

In the OCFQ spectrum of open bosonic string theory, the solutions of physical states conditions include positive-norm propagating states and two types of zero-norm states. The latter are[14]

$$\text{Type I} : L_{-1}|x\rangle, \text{ where } L_1|x\rangle = L_2|x\rangle = 0, L_0|x\rangle = 0; \tag{5}$$

$$\text{Type II} : \left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|\widetilde{x}\rangle, \text{ where } L_1|\widetilde{x}\rangle = L_2|\widetilde{x}\rangle = 0, (L_0+1)|\widetilde{x}\rangle = 0. \tag{6}$$

Equations (5) and (6) can be derived from Kac determinant in conformal field theory. While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at $D=26$. In the first quantized approach of string theory, the stringy *on-shell* Ward identities are proposed to be[11] (for simplicity we choose four-point amplitudes in this paper)

$$\mathcal{T}_\chi(k_i) = g_c^{2-\chi} \int \frac{Dg_{\alpha\beta}}{\mathcal{N}} DX^\mu \exp\left(-\frac{\alpha'}{2\pi} \int d^2\xi \sqrt{g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right) \prod_{i=1}^{4} v_i(k_i) = 0, \tag{7}$$

where at least one of the 4 vertex operators corresponds to the zero-norm state solution of Eq. (5) or (6). In Eq. (7) g_c is the closedstring coupling constant, $\mathcal{N}$ is the volume of the group of diffeomorphisms and Weyl rescalings of the worldsheet

metric, and $v_i(k_i)$ are the on-shell vertex operators with momenta k_i. The integral is over orientable open surfaces of Euler number χ parametrized by moduli $\overrightarrow{m}$ with punctures at ξ_i. For the first massive level, $m^2 = 2$, there are two zero-norm states, and the corresponding string tree-level $\chi = 1$ Ward identities were explicitly calculated to be[11]

$$k_\mu \theta_\nu \mathcal{T}^{\mu\nu} + \theta_\mu \mathcal{T}^\mu = 0, \tag{8}$$

$$\left(\frac{3}{2}k_\mu k_\nu + \frac{1}{2}\eta_{\mu\nu}\right)\mathcal{T}^{\mu\nu} + \frac{5}{2}k_\mu \mathcal{T}^\mu = 0, \tag{9}$$

where θ_ν is a transverse vector. In Eqs. (8) and (9), we have chosen, say, $v_2(k_2)$ to be the vertex operators constructed from zero-norm states and $k_\mu \equiv k_{2\mu}$. Note that Eq. (8) is the type I Ward identity while Eq. (9) is the type II Ward identity which is valid only at $D = 26$.

The proof of the decoupling theorem at string-tree amplitudes for general mass levels, without explicit calculations of massive scattering amplitudes, has been demonstrated in Ref. 14), where cyclic symmetry is used to show that both types of zero-norm states decouple from the on-shell correlation functions. Unfortunately, both approaches in Refs. 7) and 10) only work for string tree amplitudes, and one cannot extend similar arguments to stringy amplitudes at loop levels. For this reason, it is instructive to give a new proof of the decoupling of zero-norm states for the string-tree amplitudes. As we will see soon that the essential features of this new proof will be maintained in our proof for the decoupling theorem at one-loop level. Also this new proof illustrates some subtle features associated with the proof of the decoupling theorem for type II zero-norm states.

Our strategy for proving the decoupling theorem is to rewrite the stringy amplitudes as an integral of worldsheet total derivatives, and the boundary terms vanish due to the special properties of the string propagator. Taking the massless state in open bosonic string theory as an example

$$\mathcal{T}^\mu \equiv \int \prod_{i=1}^{4} dx_i \langle e^{ik_1 X} \partial X^\mu e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} \rangle, \tag{10}$$

$$\begin{aligned}
k_{2\mu} \mathcal{T}^\mu &= \int_0^1 dx \ x^{(1,2)}(1-x)^{(2,3)}\left[\frac{(1,2)}{x} - \frac{(2,3)}{1-x}\right] \\
&= \int_0^1 dx \ \frac{\partial}{\partial x}[x^{(1,2)}(1-x)^{(2,3)}] \\
&= [x^{(1,2)}(1-x)^{(2,3)}]|_0^1 = 0.
\end{aligned} \tag{11}$$

Notice that we have introduced a convention for inner products among external momenta, e.g., $(1,2) \equiv k_1 \cdot k_2$. This derivation has employed the $SL(2,R)$ gauge fixing to reduce the four point integrations into a single integral, and we list only the $s - t$ channel of the scattering amplitude. However, it should be easy to generalize our derivation to the case without doing this $SL(2,R)$ fixing. In particular, for the

one-loop open string amplitudes the residual gauge symmetry is a $U(1)$ symmetry, thus we cannot fix the positions of vertex operators (three out of four) as in the case of tree amplitudes. Still, as we shall show later, the total derivative argument can be applied at the one-loop level, at least in the case of type I zero norm state.

The fact that scattering amplitudes containing a vertex operator of the massless zero-norm state can be expressed as an integral of total derivative should come as no surprise, since the vertex operator for $m^2 = 0$ zero-norm state can be written as a worldsheet total derivative,

$$v(k, \zeta = k) = k \cdot \partial X \exp^{ikX} = -i\partial(\exp^{ikX}),\tag{12}$$

where the partial derivative means derivative with respect to the worldsheet time variable. Indeed, according to Eq. (5), all type I zero-norm states are generated by the L_{-1} Virasoro generator, which is a partial derivative on the holomorphoic coordinate of string worldsheet.

To illustrate this point further for the cases of massive scattering amplitudes, let us work out the case for type I singlet zero-norm state at $m^2 = 4$. The stringy Ward identity associated with this state is,

$$\left(\frac{17}{4}k_\mu k_\nu k_\lambda + \frac{9}{2}k_\mu \eta_{\nu\lambda}\right)\mathcal{T}^{(\mu\nu\lambda)} + (21k_\mu k_\nu + 9\eta_{\mu\nu})\mathcal{T}^{(\mu\nu)} + 25k_\mu \mathcal{T}^{\ \mu} = 0,\tag{13}$$

where we have defined the $m^2 = 4$ scattering amplitudes as

$$\mathcal{T}^{\mu\nu\lambda} \equiv \int \prod_{i=1}^{4} dx_i \langle e^{ik_1 X} \partial X^\mu \partial X^\nu \partial X^\lambda e^{ik_2 X} e^{ik_3 X} e^{ik_4 X}\rangle,\tag{14}$$

$$\mathcal{T}^{(\mu\nu)} \equiv \int \prod_{i=1}^{4} dx_i \langle e^{ik_1 X} \partial X^{(\mu} \partial^2 X^{\nu)} e^{ik_2 X} e^{ik_3 X} e^{ik_4 X}\rangle,\tag{15}$$

$$\mathcal{T}^{\mu} \equiv \int \prod_{i=1}^{4} dx_i \langle e^{ik_1 X} \partial^3 X^\mu e^{ik_2 X} e^{ik_3 X} e^{ik_4 X}\rangle.\tag{16}$$

After some algebraic manipulations, we can rewrite the sum of the left-hand side of Eq. (13) as an integral

$$\int_0^1 dx \quad \left[\frac{17}{4}\frac{\partial^3}{\partial x^3}M_{0,0} - \frac{33}{4}\frac{\partial^2}{\partial x^2}M_{1,0} + \frac{33}{4}\frac{\partial^2}{\partial x^2}M_{0,1} + \frac{3}{4}((1,2)+(2,3))\frac{\partial}{\partial x}M_{1,1}\right.$$
$$\left. + \frac{3}{4}\frac{\partial}{\partial x}M_{2,0} - 9\frac{\partial}{\partial x}M_{1,1} + \frac{3}{4}\frac{\partial}{\partial x}M_{0,2}\right] = 0.\tag{17}$$

Here we have introduced a notation for the factor in the stringy amplitudes

$$M_{p,q} \equiv x^{(1,2)-p}(1-x)^{(2,3)-q}.\tag{18}$$

Based on these observations and the fact that all type I zero-norm states can be written as a worldsheet total derivative, one can easily extend our proof to all type I zero-norm states, and conclude that all stringy scattering amplitudes for a type I

zero-norm state and any other physical states can be written as integrals of world-sheet total derivatives and thus vanish due to the boundary conditions.

Having shown that one can use the total derivative argument to prove decoupling theorem for type I zero-norm states, one natural question is that whether we can use the same trick to prove the decoupling theorem for type II zero-norm states. As we have emphasized before,[1),2)] the decoupling of type II zero-norm states is of crucial importance for demonstrating linear relations between stringy scattering amplitudes at high energies. Nevertheless, given the definition of type II zero-norm states, Eq. (6), it can be shown that, in general, type II zero-norm states cannot be written as worldsheet total derivatives, thus our proof for type I zero-norm states seems need to be modified. Fortunately, a detailed investigation shows that, at least in the case of tree amplitudes, one can still express the stringy amplitudes associated with type II zero-norm states as integrals of total derivatives, and the boundary terms vanish as before. For instance, at first massive level, $-k^2 = m^2 = 2$, we have one singlet type II zero-norm state,

$$\left[\left(\frac{3}{2}k_\mu k_\nu + \frac{1}{2}\eta_{\mu\nu}\right)\alpha^\mu_{-1}\alpha^\nu_{-1} + \frac{5}{2}k_\mu\alpha^\mu_{-2}\right]|0,k\rangle. \tag{19}$$

The amplitude of this state with three tachyons can be written as

$$\int_0^1 dx\left\{\frac{3}{2}\frac{\partial^2}{\partial x^2}M_{0,0} + \frac{\partial}{\partial x}[M_{0,1} - M_{1,0}]\right\} = \frac{3}{2}\frac{\partial}{\partial x}M_{0,0}|_0^1 + [M_{0,1} - M_{1,0}]|_0^1 = 0. \tag{20}$$

At the second massive level, $m^2 = 4$, we have two vector zero-norm states, D_1 and D_2, which we chose to be linear combinations of the original type I and type II vector zero-norm states, and the stringy Ward identities associated with them are[11)]

$$\left(\frac{5}{2}k_\mu k_\nu\theta_\lambda + \eta_{\mu\nu}\theta_\lambda\right)\mathcal{T}^{(\mu\nu\lambda)} + 9k_\mu\theta_\nu\mathcal{T}^{(\mu\nu)} + 6\theta_\mu\mathcal{T}^\mu = 0, \tag{21}$$

$$\left(\frac{1}{2}k_\mu k_\nu\theta'_\lambda + 2\eta_{\mu\nu}\theta'_\lambda\right)\mathcal{T}^{(\mu\nu\lambda)} + 9k_\mu\theta'_\nu\mathcal{T}^{[\mu\nu]} - 6\theta'_\mu\mathcal{T}^\mu = 0. \tag{22}$$

The amplitudes of these states with three tachyons can be written as

$$\int_0^1 dx\left\{(\theta \cdot k_1)\left[\frac{5}{2}\frac{\partial^2}{\partial x^2}M_{1,0} - \frac{3}{2}\frac{\partial}{\partial x}M_{2,0} + 2\frac{\partial}{\partial x}M_{1,1}\right]\right.$$
$$\left. + (\theta \cdot k_3)\left[-\frac{5}{2}\frac{\partial^2}{\partial x^2}M_{0,1} - \frac{3}{2}\frac{\partial}{\partial x}M_{0,2} + 2\frac{\partial}{\partial x}M_{1,1}\right]\right\} = 0, \tag{23}$$

$$\int_0^1 dx\left\{(\theta' \cdot k_1)\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}M_{1,0} + \frac{3}{2}\frac{\partial}{\partial x}M_{2,0} + 4\frac{\partial}{\partial x}M_{1,1}\right]\right.$$
$$\left. + (\theta' \cdot k_3)\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}M_{0,1} + \frac{3}{2}\frac{\partial}{\partial x}M_{0,2} + 4\frac{\partial}{\partial x}M_{1,1}\right]\right\} = 0. \tag{24}$$

Notice that in these derivations, one needs to use momentum conservation and on-shell conditions for vertex operators. Instead of scattering with three tachyons, the derivation here can be generalized to arbitrary three string states.

In the next section, we shall extend the total derivative argument of our proof for the string-tree amplitudes in this section to the string one-loop amplitudes, where subtlety arises for the calculation of type II zero-norm Ward identities.

§3. One-loop stringy Ward identities

It is believed that the decoupling of zero-norm states in string theory, or the stringy Ward identities, should hold true for all loop orders in string perturbation theory. Nevertheless, a mathematical proof of this assertion is non-existent and, as we will see soon, our investigation shows some subtleties associated with the proof of the decoupling theorem for type II zero-norm states. To begin with, we first discuss the decoupling theorem of type I zero-norm state at one-loop level, the stringy amplitude for one massless zero-norm state scatters with three tachyons is calculated to be

$$\mathcal{T}_\mu = g^4 \int \prod_{i=1}^{4} dx_i \langle \partial X_\mu e^{ik_1 X} e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} \rangle, \tag{25}$$

$$k_1^\mu \cdot \mathcal{T}_\mu = -g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \Big(\frac{-2\pi}{\ln\omega}\Big)^{13}$$
$$\psi_{12}^{(1,2)} \psi_{13}^{(1,3)} \psi_{14}^{(1,4)} \psi_{23}^{(2,3)} \psi_{24}^{(2,4)} \psi_{34}^{(3,4)} [(1,2)\eta_{12} + (1,3)\eta_{13} + (1,4)\eta_{14}], \tag{26}$$

where g is the open string coupling constant. Here we follow the notations of Green, Schwarz and Witten,[10] and the one-loop open string propagator is given by

$$\ln\psi_{rs} \equiv \ln\psi(c_{sr}, \omega) = \langle X^\mu(\rho_r) X^\nu(\rho_s) \rangle, \tag{27}$$

where

$$\ln\psi(c, \omega) = -\frac{1}{2}\ln c + \frac{\ln^2 c}{2\ln\omega} - \sum_{m=1}^{\infty} \frac{c^m + (\omega/c)^m - 2\omega^m}{m(1 - \omega^m)}. \tag{28}$$

Here ρ_r, ω and c_{sr} are related to the worldsheet time coordinates τ_1, τ_2, etc., as follows:

$$\rho_r \equiv e^{-(\tau_1 + \tau_2 + \dots + \tau_r)}, \quad r = 1, 2, 3, 4; \quad \omega \equiv \rho_4; \quad c_{sr} \equiv \frac{\rho_s}{\rho_r}. \tag{29}$$

In Eq. (28), the ψ function can be recast in terms of the Jacobi θ function

$$\psi(\rho, \omega) = \frac{1-\rho}{\sqrt{\rho}} \exp\Big(\frac{\ln^2\rho}{2\ln\omega}\Big) \prod_{n=1}^{\infty} \frac{(1 - \omega^n\rho)(1 - \omega^n/\rho)}{(1 - \omega^n)^2} \tag{30}$$

$$= -2\pi i e^{-i\pi\nu^2/\tau} \frac{\theta_1(-\nu/\tau)| - 1/\tau)}{\theta_1'(0| - 1/\tau)}, \tag{31}$$

where the Jacobi θ_1 function satisfies the important periodicity relations

$$\theta_1(\nu + 1|\tau) = -\theta_1(\nu|\tau), \tag{32}$$
$$\theta_1(\nu + \tau|\tau) = -e^{-i\pi\tau - 2i\pi\nu}\theta_1(\nu|\tau), \tag{33}$$

and ν and τ are defined to be

$$\nu \equiv \frac{\ln \rho}{\ln \omega}, \quad \tau \equiv -\frac{2\pi i}{\ln \omega}. \tag{34}$$

For the calculations of massive scattering amplitudes, we also need the following expressions which can be obtained by taking higher derivatives of one-loop string propagator $\ln \psi$ in Eq. (28)

$$\eta(c_{rs}, \omega) = \left\langle \frac{\partial}{\partial \tau_r} X^\mu(\rho_r) X^\mu(\rho_s) \right\rangle \tag{35}$$

$$= c_{sr} \frac{\partial}{\partial c_{sr}} \ln \psi(c_{sr}, \omega), \tag{36}$$

where

$$\eta(c, \omega) = -\frac{1}{2} + \left(\frac{\ln c}{\ln \omega}\right) - \frac{c}{1-c} + \sum_{n=1}^{\infty} \left(\frac{\omega^n/c}{1 - \omega^n/c} - \frac{c\omega^n}{1 - c\omega^n}\right); \tag{37}$$

and

$$\Omega(c_{rs}, \omega) = \left\langle -\frac{\partial^2}{\partial \tau_r^2} X^\mu(\rho_r) X^\nu(\rho_s) \right\rangle \tag{38}$$

$$= -c_{sr} \frac{\partial}{\partial c_{sr}} \eta(c_{sr}, \omega), \tag{39}$$

where

$$\Omega(c, \omega) = -\left(\frac{1}{\ln \omega}\right) + \frac{c}{(1-c)^2} + \sum_{n=1}^{\infty} \left(\frac{\omega^n/c}{(1 - \omega^n/c)^2} + \frac{c\omega^n}{(1 - c\omega^n)^2}\right). \tag{40}$$

Due to the residual conformal symmetry $U(1)$ of the one-loop open string world-sheet, in addition to the moduli parameter ω, there are three points of vertex operators we need to integrate over for a four-point scattering amplitude, and we have chosen the first vertex operator corresponding to the zero-norm state. To calculate the one-loop Ward identity for the massless zero-norm state, we make the following observation. Taking the first term in the square bracket of Eq. (26) as an example, we can rewrite

$$(1,2)\psi_{12}^{(1,2)}\eta_{12} = (1,2)\psi_{12}^{(1,2)-1} \frac{\partial}{\partial \ln c_{21}} \psi_{12}$$

$$= \frac{\partial}{\partial \ln c_{21}} \psi_{12}^{(1,2)}$$

$$= -\frac{\partial}{\partial \ln \rho_1} \psi_{12}^{(1,2)} = -\rho_1 \frac{\partial}{\partial \rho_1} \psi_{12}^{(1,2)}. \tag{41}$$

Similarly, the next two terms in the square bracket of Eq. (26) can be rewritten as a partial derivative acting on $\psi_{13}^{(1,3)}$ and $\psi_{14}^{(1,4)}$ with respect to the variable ρ_1. Putting

them all together, we get

$$k_1^\mu \cdot \mathcal{T}_\mu = g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \left(\frac{-2\pi}{\ln\omega}\right)^{13}$$
$$\rho_1 \frac{\partial}{\partial\rho_1} [\psi_{12}^{(1,2)} \psi_{13}^{(1,3)} \psi_{14}^{(1,4)} \psi_{23}^{(2,3)} \psi_{24}^{(2,4)} \psi_{34}^{(3,4)}]. \tag{42}$$

Now, we can perform the integration by parts for the variable ρ_1, and rewrite the integral as two surface terms

$$k_1^\mu \cdot \mathcal{T}_\mu = g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \left(\frac{-2\pi}{\ln\omega}\right)^{13}$$
$$[\psi_{12}^{(1,2)} \psi_{13}^{(1,3)} \psi_{14}^{(1,4)} \psi_{23}^{(2,3)} \psi_{24}^{(2,4)} \psi_{34}^{(3,4)}]|_{\rho_2}^1. \tag{43}$$

However, both terms vanish due to the periodicity properties of the ψ function, or Eqs. (32) and (33) of the Jacobi θ_1 function. For the upper limit $\rho_1 = 1$, we have

$$\psi_{14}|_{\rho_1=1} = \psi\left(\frac{\omega}{\rho_1}, \omega\right)|_{\rho_1=1} = \psi(\omega, \omega) = \psi(1, \omega) = 0. \tag{44}$$

On the other hand, for the lower limit $\rho_1 = \rho_2$, we have

$$\psi_{12}|_{\rho_1=\rho_2} = \psi\left(\frac{\rho_2}{\rho_1}, \omega\right)|_{\rho_1=\rho_2} = \psi(1, \omega) = 0. \tag{45}$$

Again, we have assumed that both $(1,2) \sim -\frac{s}{2}$ and $(1,4) \sim -\frac{t}{2}$ are positive numbers, and have extended the decoupling theorem to physical region via analytical continuation, as we have done in the proof for tree amplitudes.

The Ward identity, Eqs. (43)–(45), corresponding to the massless zero-norm state serves as a typical example of one-loop decoupling theorem for type I zero-norm state. We can follow the similar procedure and calculate the one-loop Ward identity for $m^2 = 2$ vector zero-norm state

$$(\theta_{\mu} k_{1\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + \theta_\mu \alpha_{-2}^\mu)|0, k_1\rangle, \qquad -k_1^2 = m^2 = 2. \tag{46}$$

First of all, we define the following one-loop amplitudes

$$\mathcal{T}^{\mu\nu} \equiv g^4 \int \prod_{i=1}^4 dx_i \langle \partial X^\mu \partial X^\nu e^{ik_1 X} e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} \rangle, \tag{47}$$

$$\mathcal{T}^\mu \equiv g^4 \int \prod_{i=1}^4 dx_i \langle \partial^2 X^\mu e^{ik_1 X} e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} \rangle. \tag{48}$$

These amplitudes are calculated to be

$$\theta^\mu k_1^\nu \mathcal{T}_{\mu\nu} = g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \left(\frac{-2\pi}{\ln\omega}\right)^{13}$$
$$\times \psi_{12}^{(1,2)} \psi_{13}^{(1,3)} \psi_{14}^{(1,4)} \psi_{23}^{(2,3)} \psi_{24}^{(2,4)} \psi_{34}^{(3,4)}$$

$$\times \left\{ (\theta \cdot k_2)[(1,2)\eta_{12}^2 + (1,3)\eta_{12}\eta_{13} + (1,4)\eta_{12}\eta_{14}] \right.$$
$$+ (\theta \cdot k_3)[(1,3)\eta_{13}^2 + (1,2)\eta_{12}\eta_{13} + (1,4)\eta_{13}\eta_{14}]$$
$$\left. + (\theta \cdot k_4)[(1,4)\eta_{14}^2 + (1,2)\eta_{12}\eta_{14} + (1,3)\eta_{13}\eta_{14}] \right\}, \tag{49}$$

$$\theta^\mu \cdot \mathcal{T}_\mu = -g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \left(\frac{-2\pi}{\ln \omega} \right)^{13}$$
$$\times \psi_{12}^{(1,2)} \psi_{13}^{(1,3)} \psi_{14}^{(1,4)} \psi_{23}^{(2,3)} \psi_{24}^{(2,4)} \psi_{34}^{(3,4)}$$
$$\times [(\theta \cdot k_2)\Omega_{12} + (\theta \cdot k_3)\Omega_{13} + (\theta \cdot k_4)\Omega_{14}]. \tag{50}$$

Based on the same trick in Eqs. (41)–(45), we can now combine these results to obtain

$$\theta^\mu k_1^\nu \mathcal{T}_{\mu\nu} + \theta^\mu \mathcal{T}_\mu = g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \left(\frac{-2\pi}{\ln \omega} \right)^{13}$$
$$\times \frac{\partial}{\partial \ln \rho_1} \{ \psi_{12}^{(1,2)} \psi_{13}^{(1,3)} \psi_{14}^{(1,4)} \psi_{23}^{(2,3)} \psi_{24}^{(2,4)} \psi_{34}^{(3,4)}$$
$$\times [(\theta \cdot k_2)\eta_{12} + (\theta \cdot k_3)\eta_{13} + (\theta \cdot k_4)\eta_{14}] \}. \tag{51}$$

It can then be shown that, upon integration by parts, Eq. (51) vanishes due to the periodicity properties Eqs. (32)–(33) of the Jacobi θ_1 function.

To calculate the one-loop stringy Ward identity for the type II zero-norm state at $m^2 = 2$, we first decompose the combination of stringy amplitudes into two terms

$$\left(\frac{3}{2}k_\mu k_\nu + \frac{1}{2}\eta_{\mu\nu} \right) \mathcal{T}^{\mu\nu} + \frac{5}{2} k_\mu \mathcal{T}^\mu = \frac{3}{2}[k_\mu k_\nu \mathcal{T}^{\mu\nu} + k_\mu \mathcal{T}^\mu] + \left[\frac{1}{2}\eta_{\mu\nu} \mathcal{T}^{\mu\nu} + k_\mu \mathcal{T}^\mu \right]. \tag{52}$$

The first term in the decomposition can be expressed as an integral of a worldsheet total derivative as following

$$(\mathrm{I}) \equiv \frac{3}{2}[k_\mu k_\nu \mathcal{T}^{\mu\nu} + k_\mu \mathcal{T}^\mu]$$
$$= \frac{3}{2} g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \left(\frac{-2\pi}{\ln \omega} \right)^{13}$$
$$\times \psi_{12}^{(1,2)} \psi_{13}^{(1,3)} \psi_{14}^{(1,4)} \psi_{23}^{(2,3)} \psi_{24}^{(2,4)} \psi_{34}^{(3,4)}$$
$$\times \left\{ (1,2)^2\eta_{12}^2 + (1,3)^2\eta_{13}^2 + (1,4)^2\eta_{14}^2 \right.$$
$$+ 2(1,2)(1,3)\eta_{12}\eta_{13} + 2(1,2)(1,4)\eta_{12}\eta_{14} + 2(1,3)(1,4)\eta_{13}\eta_{14}$$
$$\left. - (1,2)\Omega_{1,2} - (1,3)\Omega_{1,3} - (1,4)\Omega_{1,4} \right\}$$
$$= \frac{3}{2} g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \left(\frac{-2\pi}{\ln \omega} \right)^{13}$$
$$\times \frac{\partial^2}{\partial \ln \rho_1^2} [\psi_{12}^{(1,2)} \psi_{13}^{(1,3)} \psi_{14}^{(1,4)} \psi_{23}^{(2,3)} \psi_{24}^{(2,4)} \psi_{34}^{(3,4)}]. \tag{53}$$

The second term on the right-hand side of Eq. (52) can be further decomposed into two pieces

$$(\text{II}) \equiv \frac{1}{2}\eta_{\mu\nu}\mathcal{T}^{\mu\nu} + k_\mu\mathcal{T}^\mu$$

$$= g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \Big(\frac{-2\pi}{\ln\omega}\Big)^{13}$$

$$\times [\psi_{12}^{(1,2)}\psi_{13}^{(1,3)}\psi_{14}^{(1,4)}\psi_{23}^{(2,3)}\psi_{24}^{(2,4)}\psi_{34}^{(3,4)}]$$

$$\times \Big\{ \frac{k_2^2}{2}\eta_{12}^2 + \frac{k_3^2}{2}\eta_{13}^2 + \frac{k_4^2}{2}\eta_{14}^2 + (2,3)\eta_{12}\eta_{13} + (2,4)\eta_{12}\eta_{14} + (3,4)\eta_{13}\eta_{14}$$

$$- (1,2)\Omega_{1,2} - (1,3)\Omega_{1,3} - (1,4)\Omega_{1,4} \Big\} \tag{54}$$

$$= g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \Big(\frac{-2\pi}{\ln\omega}\Big)^{13}$$

$$\times \frac{\partial}{\partial\ln\rho_1}\Big\{ \psi_{12}^{(1,2)}\psi_{13}^{(1,3)}\psi_{14}^{(1,4)}\psi_{23}^{(2,3)}\psi_{24}^{(2,4)}\psi_{34}^{(3,4)}[\eta_{1,2} + \eta_{1,3} + \eta_{1,4}] \Big\} + \Delta, \tag{55}$$

where the extra piece Δ in Eq. (55) is given by

$$\Delta \equiv g^4 \int_0^1 \frac{d\omega}{\omega^2} \int_{\rho_2}^1 \frac{d\rho_1}{\rho_1} \int_{\rho_3}^1 \frac{d\rho_2}{\rho_2} \int_\omega^1 \frac{d\rho_3}{\rho_3} f(\omega)^{-24} \Big(\frac{-2\pi}{\ln\omega}\Big)^{13}$$

$$\times [\psi_{12}^{(1,2)}\psi_{13}^{(1,3)}\psi_{14}^{(1,4)}\psi_{23}^{(2,3)}\psi_{24}^{(2,4)}\psi_{34}^{(3,4)}]$$

$$\times \Big\{ (1 + (1,2))(\eta_{12}^2 - \Omega_{12}) + (1 + (1,3))(\eta_{13}^2 - \Omega_{13}) + (1 + (1,4))(\eta_{14}^2 - \Omega_{14}) \Big\}. \tag{56}$$

From the expressions above, one sees that while piece (I) can be written as an integral of a worldsheet total derivative, piece (II) fails to be an integral of a worldsheet total derivative. Consequently, it seems that we cannot apply the integration by parts to show the total amplitudes to be zero.

One might be curious about the difference between string-tree and one-loop calculations, and wonders why the simple total derivative argument cannot be applied at one-loop level. To see this, we replace the one-loop string propagator $\ln\psi$ by the tree level propagator $\ln(x_1 - x_2)$ in Eq. (56). The derivatives of the one-loop string propagator become $\eta = \frac{1}{x_1 - x_2}$ and $\Omega = \frac{1}{(x_1 - x_2)^2}$ respectively. After making these replacements, the extra terms in piece (II) , which are proportional to $\eta^2 - \Omega$, vanish identically.

Another observation is that in our proofs of type I one-loop Ward identities, the vanishings of Eqs. (43) and (51) are valid for all values of moduli parameter ω. That means one need not do the ω integration to prove the type I Ward identities. On the contrary, it may happen that an explicit ω integration is needed in order to prove the type II Ward identities. If this is the case, the proof of *closed* string type II Ward identities will be closely related to the $SL(2, Z)$ modular invariance of one-loop massive scattering amplitudes on torus. Although the proofs of modular invariance of bosonic closed string one-loop *massless* scattering amplitudes were given in Ref. 15) as we have mentioned in §1, the proofs for the *massive* cases are still lacking. It is

thus of interest to see the relations between type II one-loop Ward identities and the one-loop modular invariance of massive stringy scattering amplitudes.

§4. High energy limit of stringy Ward identities

The proof of stringy Ward identities for both types of zero-norm states at one-loop level is not only of interest for demonstrating one-loop stringy gauge invariances, but also important in extracting the high energy behavior of one-loop scattering amplitudes. As we have shown in the previous works[1),2)] that, by taking high energy limit of these Ward identities, one can obtain linear relations among scattering amplitudes of different string states with the same momenta. In particular, we can exactly determine the proportionality constants between high energy scattering amplitudes of different string states algebraically without any integration.

For the cases of string-tree scattering amplitudes, which can be exactly integrated to calculate their high-energy limits, these proportionality constants have been explicitly calculated through a set of sample calculations and were found to agree with the algebraic calculations based on the stringy Ward identities. For the cases of string one-loop scattering amplitudes, however, exact integrations for the sample calculations are not possible and their high-energy limit are difficult to calculate. Thus the determination of the proportionality constants between high-energy one-loop scattering amplitudes relies solely on the algebraic high-energy stringy Ward identities. This is one of the main motivations to explicitly prove one-loop stringy Ward identities in this paper.

To illustrate this point in a more concrete way, we shall use the simplest case as an example. We first discuss the string-tree case. At the first massive level $m^2 = 2$, we have two string-tree Ward identities, Eqs. (8) and (9), which have been proved by various methods discussed in §2. In order to take high energy limits of these Ward identities, we need to define the following orthonormal polarization vectors for the second string vertex $v_2(k_2)$

$$e_P = \frac{1}{m_2}(E_2, \mathrm{k}_2, 0) = \frac{k_2}{m_2}, \tag{57}$$

$$e_L = \frac{1}{m_2}(\mathrm{k}_2, E_2, 0), \tag{58}$$

$$e_T = (0, 0, 1) \tag{59}$$

in the CM frame contained in the plane of scattering. In the high energy limit, we define the following projections of stringy amplitudes

$$\mathcal{T}^{\alpha\beta} \equiv e_\mu^\alpha e_\nu^\beta \cdot \mathcal{T}^{\mu\nu}, \qquad\qquad \alpha, \beta = P, L, T, \tag{60}$$

$$\mathcal{T}^\alpha \equiv e_\mu^\alpha \cdot \mathcal{T}^\mu, \qquad\qquad \alpha, \beta = P, L, T, \tag{61}$$

where $\mathcal{T}^{\alpha\beta}$ and $\mathcal{T}^\alpha$ are defined similarly as in Eqs. (1) and (3), except that $v_{1,3,4}$ can now be any string vertex and we have conventionally put the zero-norm state at the second vertex. After taking the high energy limit of the stringy Ward identities and

identifying $\mathcal{T}^{\cdot\cdot P\cdot\cdot} = \mathcal{T}^{\cdot\cdot L\cdot\cdot}$ in Eq. (9),[6] Eqs. (8) and (9) reduce to

$$\sqrt{2}\mathcal{T}_{TP}^{3\to1} + \mathcal{T}_T^1 = 0, \tag{62}$$

$$\sqrt{2}\mathcal{T}_{LL}^{4\to2} + \mathcal{T}_L^2 = 0, \tag{63}$$

$$6\mathcal{T}_{LL}^{4\to2} + \mathcal{T}_{TT}^2 + 5\sqrt{2}\mathcal{T}_L^2 = 0. \tag{64}$$

In the above equations, we have denoted the naive power counting for orders in energy[1,2] in the superscript of each amplitude according to the following rules, $e_L \cdot k \sim E^2, e_T \cdot k \sim E^1$. Note that since $\mathcal{T}_{TP}^1$ is of subleading order in energy, in general $\mathcal{T}_{TP}^1 \neq \mathcal{T}_{TL}^1$. A simple calculation of Eqs. (62)–(64) shows that

$$\mathcal{T}_{TP}^1 : \mathcal{T}_T^1 = 1 : -\sqrt{2}, \tag{65}$$

$$\mathcal{T}_{TT}^2 : \mathcal{T}_{LL}^2 : \mathcal{T}_L^2 = 4 : 1 : -\sqrt{2}. \tag{66}$$

It is interesting to see that, in addition to the leading order amplitudes in Eq. (66), the subleading order amplitudes in Eq. (65) are also proportional to each other. This does not seem to happen at higher mass level.[2] Since the proportionality constants in Eqs. (65) and (66) are independent of particles chosen for vertex $v_{1,3,4}$, we will choose them, for example, to be tachyons to do the sample calculation. The string-tree level calculations by both energy expansion method[1,2] and the saddle-point method[6] give the same results

$$\mathcal{T}_T^1 = 4E^5 \sin\phi_{CM}\mathcal{T}(2) = -\sqrt{2}\mathcal{T}_{TP}^1, \tag{67}$$

$$\mathcal{T}_{TT}^2 = 4E^6 \sin^2\phi_{CM}\mathcal{T}(2) = 4\mathcal{T}_{LL}^2 = -2\sqrt{2}\mathcal{T}_L^2, \tag{68}$$

where $\mathcal{T}(2) = -\frac{1}{4}\sqrt{\pi}E^{-5}(\sin\frac{\phi_{CM}}{2})^{-3}(\cos\frac{\phi_{CM}}{2})\exp(-\frac{s\ln s + t\ln t - (s+t)\ln(s+t)}{2})$. Equations (67) and (68) agree with Eqs. (65) and (66) respectively as expected. We have also checked that $\mathcal{T}_{TP}^1 \neq \mathcal{T}_{TL}^1$.

We now discuss the string one-loop case. The calculations of Eqs. (57)–(66) go through except that we have no sample calculations, Eqs. (67) and (68), for the string one-loop case. This is due to the fact that exact integrations, apart from numerical calculations, for the one-loop amplitudes, Eqs. (49) and (50), are not possible and their high-energy limits are difficult to calculate. Also, the string one-loop saddle-point calculations of Refs. 3), 12) and 13) are not reliable.[1,2,6] For example, the calculation of Ref. 13) predicts $\mathcal{T}_{LL}^2$ to be of the subleading order in energy compared to $\mathcal{T}_{TT}^2$, which obviously violates the high-energy stringy Ward identity Eq. (66). *Thus, the one-loop stringy Ward identities calculations become, so far, the only way to determine the one-loop proportionality constants in Eqs. (65) and (66).* This is one of the main motivations to prove one-loop stringy Ward identities in this paper as we have stressed before.

Notice that Eq. (65) is obtained from the high energy one-loop Ward identity for type I zero-norm state, Eq. (51), which has been proved in the previous section. The validity of Eq. (66), or Eqs. (63) and (64), however, relies on the proof of high energy one-loop Ward identities for both Type I and Type II zero-norm states, Eqs. (51) and (52). Unfortunately, we are not able to explicitly prove Eq. (52) at this moment.

Thus, strictly speaking, only Eq. (65) is rigorously proved at one-loop level but not Eq. (66). Equation (65) is our first example to explicitly justify Gross's conjecture[3] that the proportionality constants between high energy scattering amplitudes are independent of the scattering angle ϕ_{CM} and the loop order χ of string perturbation theory, at least for $\chi = 1, 0$. Equation (65) is remarkable in the sense that although both T^1_{TL} and T^1_T are not one-loop exactly calculable, they are indeed proportional to each other and the ratio of them is determined by the one-loop stringy Ward identity, Eq. (51). While a complete proof of one-loop decoupling theorem for type II zero-norm states might require sophisticated use of non-trivial identities of Jacobi theta functions, the validity of these type II stringy Ward identities should be a reasonable consequence from stringy gauge symmetries and the unitarity of the theory. Thus, even though one cannot exactly integrate Eqs. (49) and (50), we do believe that these one-loop scattering amplitudes are proportional to each other in the high energy limit, and the proportionality constants can be determined exactly by simple algebraic means. This simple example of one-loop $m^2 = 2$ amplitudes calculations serve as an illustrative example for the power of zero-norm state approach,[1,2,6] and can be generalized to higher massive levels and higher genus amplitudes.

§5.   Summary and conclusion

In this paper, we have studied one-loop massive scattering amplitudes and their associated Ward identities in bosonic open string theory. A new proof of the decoupling of two types of zero-norm states at string-tree level is given which allows us to express the scattering amplitudes containing zero-norm states as integrals of worldsheet total derivatives. Based on the explicit one-loop calculations of four-point scattering amplitudes for some low-lying massive string states, we show that the same technique for proving string-tree level Ward identities can be generalized to the case of type I zero-norm states. However, the one-loop Ward identities for type II zero-norm states cannot be proved in the same way. The subtlety in the proofs of one-loop type II stringy Ward identities are discussed by comparing them with those of string-tree cases. Finally, as an example, high-energy limit of $m^2 = 2$ stringy Ward identities are used to fix the proportionality constants between one-loop massive high-energy scattering amplitudes at mass level $m^2 = 2$. It is interesting to see that, in addition to the leading order amplitudes, the subleading order amplitudes are also proportional to each other. This does not seem to happen at higher mass level. These proportionality constants cannot be calculated directly from sample calculations as we did in the cases of string-tree scattering amplitudes.

It should be clear from our study in this paper that the explicit proof of the decoupling theorem for type II zero-norm states at one-loop level is of crucial importance. Presumably, one needs some higher identities of Jacobi theta functions.

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